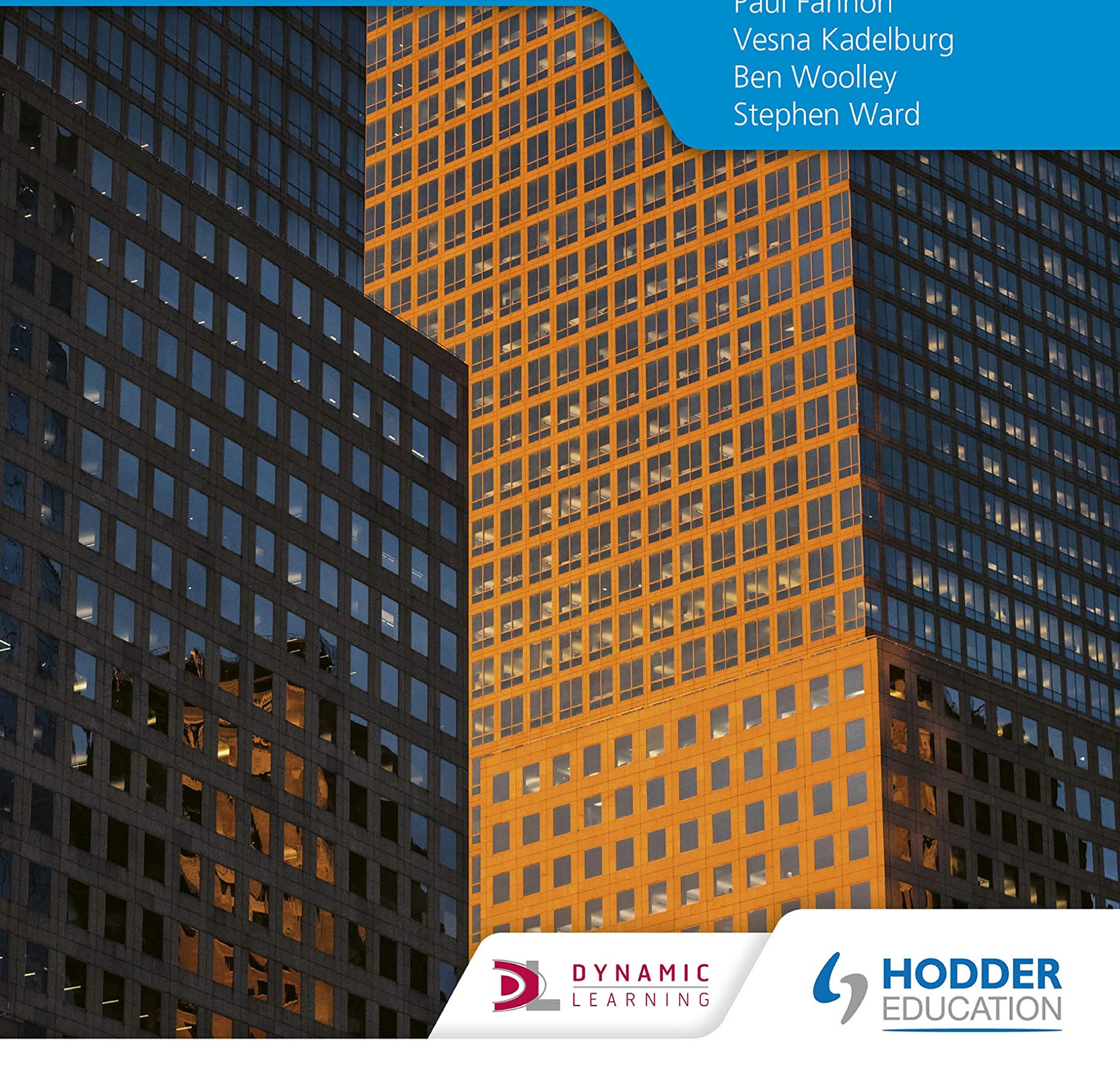
FOR THE
IB DIPLOMA

Mathematics

APPLICATIONS AND INTERPRETATION HL

Paul Fannon
Vesna Kadelburg
Ben Woolley
Stephen Ward



DYNAMIC LEARNING

HODDER EDUCATION

FOR THE
IB DIPLOMA

Mathematics

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Contents

Introduction

Welcome to your coursebook for Mathematics for the IB Diploma: applications and interpretation HL. The structure and content of this coursebook follow the structure and content of the 2019 IB Mathematics: applications and interpretation guide, with headings that correspond directly with the content areas listed in the guide.

This is the second book required by students taking the higher level course. Students should be familiar with the content of Mathematics for the IB Diploma: applications and interpretation SL before moving on to this book.

Using this book

Special features of the chapters include:

ESSENTIAL UNDERSTANDINGS

Each chapter begins with a summary of the key ideas to be explored and a list of the knowledge and skills you will learn. These are revisited in a checklist at the end of each chapter.

CONCEPTS

The IB guide identifies 12 concepts central to the study of mathematics that will help you make connections between topics, as well as with the other subjects you are studying. These are highlighted and illustrated with examples at relevant points throughout the book. The concepts are: Approximation, Change, Equivalence, Generalization,Modelling, Patterns, Relationships, Space, Systems and Validity.

KEY POINTS

Important mathematical rules and formulae are presented as Key Points, making them easy to locate and refer back to when necessary.

WORKED EXAMPLES

There are many Worked Examples in each chapter, demonstrating how the Key Points and mathematical content described can be put into practice. Each Worked Example comprises two columns:

On the left, how to **think** about the problem and what tools or methods will be needed at each step	On the right, what to **write**, prompted by the left column, to produce a formal solution to the question.

Exercises

Each section of each chapter concludes with a comprehensive exercise so that you can test your knowledge of the content described and practise the skills demonstrated in the Worked Examples. Each exercise contains the following types of questions:

- **Drill questions:** These are clearly linked to particular Worked Examples and gradually increase in difficulty. Each of them has two parts – **a** and **b** – designed such that if you get **a** wrong, **b** is an opportunity to have another go at a very similar question. If you get **a** right, there is no need to do **b** as well.
- **Problem-solving questions:** These questions require you to apply the skills you have mastered in the drill questions to more complex, exam-style questions. They are colour-coded for difficulty.

1 Green questions are closely related to standard techniques and require a small number of processes. They should be approachable for all candidates.

2 Blue questions require students to make a small number of tactical decisions about how to apply the standard methods and they will often require multiple procedures. Candidates targeting the medium HL grades should find these questions challenging but achievable.

3 Red questions often require a creative problem-solving approach and extended, technical procedures. Candidates targeting the top HL grades should find these questions challenging.

4 Black questions go beyond what is expected in IB examinations, but provide an enrichment opportunity for the very best students.

The questions in the Mixed Practice section at the end of each chapter are similarly colour-coded, and contain questions taken directly from past IB Diploma Mathematics exam papers. There are also three practice examination papers at the end of the book plus guidance on how to approach Paper 3.

Answers to all exercises can be found at the back of the book.

A calculator symbol is used where we want to remind you that there is a particularly important calculator trick required in the question.

A non-calculator icon suggests a question is testing a particular skill that you should be able to do without the use of a calculator. Although remember that you will always have a calculator in the exam.

The guide places great emphasis on the importance of technology in mathematics and expects you to have a high level of fluency with the use of your calculator and other relevant forms of hardware and software. Therefore, we have included plenty of screenshots and questions aimed at raising awareness and developing confidence in these skills, within the contexts in which they are likely to occur. This icon is used to indicate topics for which technology is particularly useful or necessary.

Making connections: Mathematics is all about making links. You might be interested to see how something you have just learned will be used elsewhere in the course and in different topics, or you may need to go back and remind yourself of a previous topic.

Be the Examiner

These are activities that present you with three different worked solutions to a particular question or problem. Your task is to determine which one is correct and to work out where the other two went wrong.

Proof

Proofs are set out in a similar way to Worked Examples, helping you to gain a deeper understanding of the mathematical rules and statements you will be using and to develop the thought processes required to write your own proofs.

TOOLKIT

There are questions, investigations and activities interspersed throughout the chapters to help you develop mathematical thinking skills, building on the introductory Toolkit chapter from the Mathematics for the IB Diploma: applications and interpretation SL book in relevant contexts. Although the ideas and skills presented will not be examined, these features are designed to give you a deeper insight into the topics that will be. Each Toolkit box addresses one of the following three key topics: proof, modelling and problem-solving.

International mindedness

These boxes explore how the exchange of information and ideas across national boundaries has been essential to the progress of mathematics and to illustrate the international aspects of the subject.

You are the Researcher

This feature prompts you to carry out further research into subjects related to the syllabus content. You might like to use some of these ideas as starting points for your mathematical exploration or even an extended essay. Remember that the way you use mathematics in your Higher Level exploration should be sophisticated and rigorous. Refer back to the Mathematics: applications and interpretation SL book for more information about the assessment criteria.

LEARNER PROFILE

Opportunities to think about how you are demonstrating the attributes of the IB Learner Profile are highlighted at the beginning of appropriate chapters.

Tips

There are short hints and tips provided in the margins throughout the book.

TOK Links

Links to the interdisciplinary Theory of Knowledge element of the IB Diploma course are made throughout the book.

Links to: Other subjects

Links to other IB Diploma subjects are made at relevant points, highlighting some of the real-life applications of the mathematical skills you will learn.

Topics that have direct real-world applications are indicated by this icon.

There is a glossary at the back of the book. Glossary terms are **purple**.

These features are designed to promote the IB's inquiry-based approach, in which mathematics is not seen as a collection of facts to be learned, but a set of skills to be developed.

About the authors

The authors are all Cambridge University graduates and have a wide range of expertise in pure mathematics and in applications of mathematics, including economics, epidemiology, linguistics, philosophy and natural sciences.

Between them they have considerable experience of teaching IB Diploma Mathematics at Standard and Higher Level, and two of them currently teach at the University of Cambridge.

1 Exponents and logarithms

ESSENTIAL UNDERSTANDINGS

- Number and algebra allow us to represent patterns, show equivalences and make generalizations.
- Algebra is an abstraction of numerical concepts and employs variables to solve mathematical problems.
- Creating different representations of functions to model the relationships between variables, visually and symbolically as graphs, equations and/or tables represents different ways to communicate mathematical ideas.

In this chapter you will learn…

- how to extend the laws of exponents to general rational exponents
- how to use the laws of logarithms
- how to find the sum to infinity of a geometric series
- how to interpret graphs with logarithmic scales
- how to linearize data to infer the parameters of models.

CONCEPTS

The following concepts will be addressed in this chapter:

- Numbers and formulae can appear in different but **equivalent** forms, or **representations**, which can help us to establish identities.
- **Patterns** in numbers inform the development of algebraic tools that can be applied to find unknowns.
- **Generalization** provides an insight into variation and allows us to access ideas such as half-life and scaling logarithmically to adapt theoretical models and solve complex real-life problems.

LEARNER PROFILE – Reflective

Is being really good at arithmetic the same as being really good at mathematics?

Figure 1.1 Why are logarithms and exponential functions used to describe these phenomena?

PRIOR KNOWLEDGE

Before starting this chapter, you should already be able to complete the following:

1 Simplify the following:

a $3x^2y^4 \times 5x^7y$ **b** $\dfrac{8c^2d^3}{2c^3d}$ **c** $(3a^4b^{-2})^2$

2 Find the value of y for which $\log_{10}(y) = 2$.

3 Solve the equation $3^{x-5} = \left(\dfrac{1}{9}\right)^x$.

4 Solve the equation $e^x = 11$.

5 Find the 5th term of the geometric sequence with first term 4 and common ratio −2.

6 Use technology to find the Pearson's product-moment correlation coefficient and the equation of the y-on-x regression line for the data in the table.

x	1	2	2	5
y	2	6	4	8

You have already seen how the laws of exponents allow you to manipulate exponential expressions, and how this can be useful for solving some types of exponential equation.

In the same way, it is useful to have some laws of logarithms, which will enable you to solve some more complicated exponential equations and equations involving different logarithm terms. It should be no surprise, given the relationship between exponents and logarithms, that these laws of logarithms follow from the laws of indices.

Laws of logarithms and indices can also be applied to extending your knowledge of geometric series and used to turn graphs of functions into straight lines, thereby making it easier to estimate parameters.

Starter Activity

Look at the pictures in Figure 1.1. Investigate how exponential and logarithmic functions can be used to measure or model these different phenomena.

Now look at this problem:

By trying different positive values of x and y, suggest expressions for the following in terms of $\ln x$ and $\ln y$:

a $\ln(xy)$ **b** $\ln\left(\dfrac{x}{y}\right)$ **c** $\ln(x^y)$

Do your suggested relationships work for $\log_{10}$ as well?

1A Laws of exponents with rational exponents

In order to extend the laws of exponents for integer exponents that you already know to rational exponents that are not integers, you need a new law.

For a reminder of the laws of exponents for integer exponents see Section 1A of the Mathematics: applications and interpretation SL book.

KEY POINT 1.1

$a^{\frac{1}{n}} = \sqrt[n]{a}$

Proof 1.1

Explain why $a^{\frac{1}{n}} = \sqrt[n]{a}$.

We need a defining feature of $\sqrt[n]{a}$

$\sqrt[n]{a}$ is the number which equals a when raised to the exponent n.

Use the fact that $(x^a)^b \equiv x^{ab}$

We know that $\left(a^{\frac{1}{n}}\right)^n \equiv a^{\left(\frac{1}{n} \times n\right)} \equiv a^1$.

Therefore, $a^{\frac{1}{n}}$ has the defining property of $\sqrt[n]{a}$.

CONCEPTS – REPRESENTATION

You might ask why writing the same thing in a different notation has any benefit. Using an exponent **representation** of $\sqrt[n]{a}$ has a distinct advantage as it allows us to use the laws of exponents on these expressions.

WORKED EXAMPLE 1.1

Evaluate $16^{\frac{1}{4}}$.

Use $a^{\frac{1}{n}} = \sqrt[n]{a}$

$$16^{\frac{1}{4}} = \sqrt[4]{16}$$
$$= 2$$

Combining the law in Key Point 1.1 with the law that states that $(a^m)^n = a^{mn}$ allows us to cope with any rational exponent.

WORKED EXAMPLE 1.2

Evaluate $27^{-\frac{2}{3}}$.

Use $(a^m)^n = a^{mn}$ to split the exponent

$$27^{-\frac{2}{3}} = \left(27^{\frac{1}{3}}\right)^{-2}$$

Remember that a negative exponent turns into 1 divided by the same expression with a positive exponent. $27^{\frac{1}{3}} = \sqrt[3]{27}$

$$= \frac{1}{\left(\sqrt[3]{27}\right)^2}$$

You should know small perfect squares and cubes. You can recognize that $\sqrt[3]{27} = 3$

$$= \frac{1}{3^2} = \frac{1}{9}$$

You will also need to be able use this new law in an algebraic context.

WORKED EXAMPLE 1.3

Write $\frac{4x}{\sqrt[3]{x}}$ in the form kx^n.

Use $a^{\frac{1}{n}} = \sqrt[n]{a}$ on the denominator

$$\frac{4x}{\sqrt[3]{x}} = \frac{4x}{x^{\frac{1}{3}}}$$

Then use $\frac{a^m}{a^n} = a^{m-n}$

$$= 4x^{1-\frac{1}{3}}$$

$$= 4x^{\frac{2}{3}}$$

Exercise 1A

Although in the exam you will have a calculator, to develop understanding we recommend trying this whole exercise without a calculator.

For questions 1 to 3, use the method demonstrated in Worked Example 1.1 to evaluate these without a calculator.

1 **a** $8^{\frac{1}{3}}$
b $27^{\frac{1}{3}}$

2 **a** $64^{\frac{1}{3}}$
b $625^{\frac{1}{4}}$

3 **a** $49^{\frac{1}{2}}$
b $25^{\frac{1}{2}}$

For questions 4 to 8, use the methods demonstrated in Worked Example 1.2 to evaluate these without a calculator.

4 **a** $8^{\frac{2}{3}}$
b $16^{\frac{3}{4}}$

5 **a** $625^{\frac{3}{4}}$
b $125^{\frac{2}{3}}$

6 **a** $100^{-\frac{1}{2}}$
b $1000^{-\frac{1}{3}}$

7 **a** $8^{-\frac{2}{3}}$
b $27^{-\frac{2}{3}}$

8 **a** $32^{-\frac{2}{5}}$
b $100\,000^{-\frac{3}{5}}$

For questions 9 to 13, use the methods demonstrated in Worked Example 1.3 to simplify each expression, giving your answers in the form ax^p.

9 **a** $x^2\sqrt{x}$
b $x\sqrt[3]{x}$

10 **a** $\dfrac{x^2}{\sqrt[3]{x}}$
b $\dfrac{x^2}{\sqrt{x}}$

11 **a** $\dfrac{4\sqrt{x}}{x^2}$
b $\dfrac{5\sqrt{x}}{x^3}$

12 **a** $\dfrac{x^2}{5\sqrt{x}}$
b $\dfrac{x}{3\sqrt[3]{x}}$

13 **a** $\sqrt[3]{x}\sqrt{x}$
b $x\sqrt[3]{x^2}$

14 Find the exact value of $8^{-\frac{4}{3}}$.

15 Find the exact value of $\left(\dfrac{1}{4}\right)^{-\frac{1}{2}}$.

16 Find the exact value of $\left(\dfrac{4}{9}\right)^{\frac{3}{2}}$.

17 Write $\sqrt[3]{x^2} \times \sqrt[4]{x}$ in the form x^p.

18 Write $\dfrac{3}{\sqrt[3]{x}} + 2\sqrt{x}$ in the form $ax^p + bx^q$.

19 Write $\dfrac{1}{3\sqrt{x^3}}$ in the form ax^p.

20 Solve the equation $x^{\frac{3}{2}} = \dfrac{1}{8}$.

21 Solve the equation $x^{-\frac{1}{2}} = \dfrac{2}{5}$.

22 Write in the form x^p: $\dfrac{x\sqrt{x}}{\sqrt[3]{x}}$

23 Write in the form x^p: $\dfrac{x}{x^2\sqrt{x}}$

24 Write $\left(x \times \sqrt[3]{x}\right)^2$ in the form x^k.

25 Write $\left(\dfrac{1}{2\sqrt{x}}\right)^3$ in the form ax^p.

26 Write in the form $x^a + x^b$: $\dfrac{x^2 + \sqrt{x}}{x\sqrt{x}}$

27 Write in the form $x^a - x^b$: $\dfrac{(x+\sqrt{x})(x-\sqrt{x})}{\sqrt{x}}$

28 Write in the form ax^k: $\dfrac{1}{3x\sqrt{x}}$

29 Write in the form $ax^p + bx^q$: $\dfrac{x^2 + 3\sqrt{x}}{2x}$

30 Given that $y = 2\sqrt[3]{x^2}$, write y^4 in the form ax^k.

31 Given that $y = 27\sqrt{x}$, write $\sqrt[3]{y}$ in the form ax^k.

32 Given that $\sqrt{x} = \sqrt[3]{y}$, write y in the form x^k.

33 Given that $y = \dfrac{2}{3\sqrt{x}}$, write y^3 in the form ax^k.

34 Solve $\left(\sqrt{3}\right)^x = 9^{x-1}$.

35 Solve $\left(\sqrt{2}\right)^x = 4^{x+2}$.

36 Solve $\left(\sqrt[3]{2}\right)^{2x} = 8^{x+1}$.

37 Solve $\left(\sqrt[3]{3}\right)^{4x} = 9^{x-3}$.

38 Given that $y = x\sqrt{x}$, express x in terms of y.

39 Solve the equation $x^{\frac{2}{3}} = 9$.

40 Solve the equation $\sqrt{x} = 2\sqrt[3]{x}$.

1B Logarithms

You met logarithms in the Mathematics: applications and interpretation SL book. The logarithm to the base a of b is defined as the power a must be raised to if you want the result to be b. We can write this as:

$b = a^x$ is equivalent to $\log_a b = x$

You will see in Chapter 4 that this means that logarithms and exponents are inverse functions – one undoes the other.

WORKED EXAMPLE 1.4

Find the exact value of y if $\log_{10}(y-3) = -1$.

Use $b = a^x$ is equivalent to $\log_a b = x$ .... $y - 3 = 10^{-1} = \frac{1}{10}$

$y = 3.1$

Remember that logarithms to the base e are given a special notation, $\ln x$.

Tip

Remember that most equations like this can actually be solved graphically, using technology. However, solving them manually can aid understanding.

WORKED EXAMPLE 1.5

Solve $2 + \ln x = 0$.

Isolate the $\ln x$ term .... $\ln x = -2$

Use $b = a^x$ is equivalent to $\log_a b = x$. Remember that $\ln x$ is equivalent to $\log_e x$ .... $x = e^{-2} \approx 0.135$

Laws of logarithms

The laws of exponents lead to a set of laws of logarithms.

Tip

Be careful not to invent similar looking rules of logarithms. For example, many students claim that $\log(x+y) = \log x + \log y$ or that $\log(xy) = \log x \times \log y$. Neither of these are true in general.

KEY POINT 1.2

- $\log_a xy = \log_a x + \log_a y$
- $\log_a \frac{x}{y} = \log_b x - \log_a y$
- $\log_a x^m = m \log_a x$

where $a, x, y > 0$

CONCEPTS – PATTERNS

Many rules like this are explored and discovered by systematically looking at **patterns** in numbers.

For example:

$\log_{10} 2 = 0.30103$ $\log_{10} 20 = 1.30103$ $\log_{10} 200 = 2.30103$ $\log_{10} 2000 = 3.30103$

The first law of logarithms is proved here. The others are proved similarly from the corresponding law of exponents.

Proof 1.2

Prove that $\log_a xy = \log_a x + \log_a y$.

Start by using the related law of exponents $a^m \times a^n = a^{m+n}$ with $m = \log_a x$ and $n = \log_a y$

$$a^{\log_a x} \times a^{\log_a y} = a^{\log_a x + \log_a y}$$

Use $a^{\log_a x} = x$ on both terms of the product on the LHS

$$xy = a^{\log_a x + \log_a y}$$

Take $\log_a$ of both sides

$$\log_a xy = \log_a a^{\log_a x + \log_a y}$$

Use $\log_a a^x = x$ on the RHS

$$\log_a xy = \log_a x + \log_a y$$

CONCEPTS – EQUIVALENCE

The ability to go easily between representing equations using logs and using exponents allows us to turn our old rules into **equivalent** new rules. This is a very common and powerful technique in many areas of mathematics – for example looking at how rules in differentiation apply to integration.

WORKED EXAMPLE 1.6

If $p = \log a$ and $q = \log b$, express $\log\left(\frac{a^3}{b}\right)$ in terms of p and q.

Use $\log_a \frac{x}{y} = \log_a x - \log_a y$. The question does not specify what base to use, as it actually does not matter

$$\log\left(\frac{a^3}{b}\right) = \log a^3 - \log b$$

Then use $\log_a x^m = m\log_a x$ on the first term

$$= 3\log a - \log b$$

Now replace $\log_a$ with p and $\log_b$ with q

$$= 3p - q$$

One common application of the laws of logarithms is in solving log equations. The usual method is to combine all log terms into one.

WORKED EXAMPLE 1.7

Solve $\log_{10}(3x-2)-\log_{10}(x-4)=1$.

	$\log_{10}(3x-2)-\log_{10}(x-4)=1$
Combine the log terms using $\log_a x-\log_a y=\log_a \frac{x}{y}$	$\log_{10}\left(\frac{3x-2}{x-4}\right)=1$
Remove the log using $\log_a b=x$ is equivalent to $b=a^x$	$\frac{3x-2}{x-4}=10^1$
Solve for x	$3x-2=10(x-4)$
	$3x-2=10x-40$
	$38=7x$
	$x=\frac{38}{7}\approx 5.43$

Be the Examiner 1.1

Solve $\log_{10}(x+10)+\log_{10}2=2$.

Which is the correct solution? Identify the errors in the incorrect solutions.

Solution 1	Solution 2	Solution 3
$\log_{10}(x+12)=2$	$\log_{10}(2x+10)=2$	$\log_{10}(2x+20)=2$
$x+12=100$	$2x+10=100$	$2x+20=100$
$x=88$	$x=45$	$x=40$

TOOLKIT: Problem Solving

In the exam you will only be asked about logs to the base of 10 and e. However, you might find solving the following equations helps you to really understand logarithms.

a $x=\log_3 27$ **b** $4=\log_2 x$ **c** $3=\log_x 64$

Solving exponential equations

One common use of logarithms is to solve exponential equations – these are equations where the unknowns are in the powers. The technique is to take logarithms of both sides (to any convenient base), apply the laws of logarithms to turn the equation into a linear equation and then solve for x.

WORKED EXAMPLE 1.8

Solve the equation $7^{x-2} = 5^{x+3}$ giving your answer in terms of natural logarithms.

Take natural logs of both sides: $\ln(7^{x-2}) = \ln(5^{x+3})$

Use $\ln a^m = m \ln a$: $(x-2)\ln 7 = (x+3)\ln 5$

Expand the brackets: $x\ln 7 - 2\ln 7 = x\ln 5 + 3\ln 5$

Group the x terms and the number terms (remember that $\ln 5$ and $\ln 7$ are just numbers): $x\ln 7 - x\ln 5 = 3\ln 5 + 2\ln 7$

$x(\ln 7 - \ln 5) = 3\ln 5 + 2\ln 7$

Factorize the left-hand side and divide: $x = \dfrac{3\ln 5 + 2\ln 7}{\ln 7 - \ln 5}$

Tip

The final answer can be written in a different form, by using laws of logarithms on the top and the bottom:

$$x = \frac{\ln(5^3 \times 7^2)}{\ln(7 \div 5)} = \frac{\ln(6125)}{\ln(1.4)}$$

Exercise 1B

For questions 1 to 4, use the method demonstrated in Worked Example 1.4 to solve the equations.

1 **a** $\log_{10} x = -2$ **b** $\log_{10} x = -1$

2 **a** $\log_{10}(y-1) = -1$ **b** $\log_{10}(y-2) = -2$

3 **a** $\log_{10}(x+2) = 2$ **b** $\log_{10}(x+3) = 3$

4 **a** $\log_{10}(t-1) = 0$ **b** $\log_{10}(r+3) = 0$

For questions 5 to 7, use the method demonstrated in Worked Example 1.5 to solve the equations.

5 **a** $\ln x = 2$ **b** $\ln x = 5$

6 **a** $2\ln x - 1 = 0$ **b** $3\ln x + 1 = 0$

7 **a** $\ln(2x+4) = 2$ **b** $\ln(3x-1) = 4$

For questions 8 to 11, use the methods demonstrated in Worked Example 1.6. Write each given expression in terms of p and q, where $p = \log_{10} a$ and $q = \log_{10} b$.

8 **a** $\log_{10}\left(\dfrac{a^2}{b}\right)$ **b** $\log_{10}\left(\dfrac{b^3}{a}\right)$

9 **a** $\log_{10}\left(\dfrac{a^2}{b^3}\right)$ **b** $\log_{10}\left(\dfrac{b^4}{a^2}\right)$

10 **a** $\log_{10}\sqrt{a^3b}$ **b** $\log_{10}\sqrt{a^4b^3}$

11 **a** $\log_{10}\left(\dfrac{100a}{b^2}\right)$ **b** $\log_{10}\left(\dfrac{10a^2}{b^5}\right)$

For questions 12 to 15, use the methods demonstrated in Worked Example 1.7 to solve the equations, giving your answers in an exact form.

12 **a** $\log_{10} x + \log_{10} 2 = 3$ **b** $\log_{10} x + \log_{10} 5 = 1$

13 **a** $\log_{10}(x+3) + \log_{10} 2 = 2$ **b** $\log_{10}(x-1) + \log_{10} 4 = 1$

14 **a** $\log_{10}(x+1) - \log_{10}(x-2) = 1$ **b** $\log_{10}(x+1) - \log_{10}(x-1) = 2$

15 **a** $\ln(x-3) - \ln(x+5) = 4$ **b** $\ln(x+2) - \ln(x-1) = 3$

For questions 16 and 17, use the method demonstrated in Worked Example 1.8 to solve the equations, giving your answer in terms of natural logarithms.

16 **a** $3^{x-2} = 2^{x+1}$ **b** $3^{x-1} = 2^{x+2}$

17 **a** $7^{2x-5} = 2^{x+3}$ **b** $7^{3x+1} = 2^{x+8}$

18 Given that $x = \log_{10} a$, $y = \log_{10} b$ and $z = \log_{10} c$, write the following in terms of x, y and z.

a $\log_{10}(ab^4)$

b $\log_{10}\left(\frac{a^2 b}{c^5}\right)$

c $\log_{10}(10a^2 b^3)$

19 Given that $x = \log_{10} a$, $y = \log_{10} b$ and $z = \log_{10} c$, write the following in terms of x, y and z.

a $\log_{10}(100\sqrt{a})$

b $\log_{10}\left(\frac{b}{10c^5}\right)$

20 Write $2\ln a + 6\ln b$ as a single logarithm.

21 Write $\frac{1}{3}\ln x - \frac{1}{2}\ln y$ as a single logarithm.

22 Solve $\log_{10}(x+3) = 3$.

23 Solve the equation $\log_{10}(2x-4) = 1$.

24 Use logarithms to solve these equations.

a $5^x = 10$

b $2 \times 3^x + 6 = 20$

25 Solve the equation $3 \times 1.1^x = 20$.

26 a Find the exact value of $\log_{10}\left(\frac{1}{\sqrt{10}}\right)$.

b Solve the equation $\log_x 27 = -3$.

27 Solve the equation $\log_x 32 = 5$.

28 Solve the equation $\log_x 64 = 3$.

29 Given that $5 \times 6^x = 12 \times 3^x$,

a write down the exact value of 2^x

b hence find the value of x.

30 Solve the equation $8^{3x+1} = 4^{x-3}$.

31 Solve the equation $5^{2x+3} = 9^{x-5}$, giving your answer in terms of natural logarithms.

32 The radioactivity (R) of a substance after a time t days is modelled by $R = 10 \times 0.9^t$.

a Find the initial (i.e. $t = 0$) radioactivity.

b Find the time taken for the radioactivity to fall to half of its original value.

33 The population of bacteria (B) at time t hours after being added to an agar dish is modelled by $B = 1000 \times 1.1^t$.

a Find the number of bacteria

i initially

ii after 2 hours.

b Find an expression for the time it takes to reach 2000. Use technology to evaluate this expression.

34 The population of penguins (P) after t years on two islands is modelled as follows.

First island: $P = 200 \times 1.1^t$

Second island: $P = 100 \times 1.2^t$

How many years are required before the populations of penguins on both islands are equal?

35 Solve the simultaneous equations

$\log_{10} x + \log_{10} y = 3$

$\log_{10} x - 2\log_{10} y = 0$

36 Solve the simultaneous equations

$\log_{10} x + \ln y = 1$

$\log_{10} x^2 + \ln y^3 = 4$

37 Solve the equation $2^{5-3x} = 3^{2x-1}$, giving your answer in the form $\frac{\ln p}{\ln q}$, where p and q are integers.

38 Moore's law states that the density of transistors on an integrated circuit doubles every 2 years. Find the time taken for the density to multiply by 10.

39 a If $\log_a(x^2) = b$, find the product of all possible values of x.

b If $(\log_a x)^2 = b$, find the product of all possible values of x.

You met the sum of a geometric sequence in Section 2B of the Mathematics: applications and interpretation SL book.

You met the idea of limits in Section 9A of the Mathematics: applications and interpretation SL book.

TOOLKIT: Problem Solving

This is a great opportunity to use some technology to explore how geometric series converge for different values of r.

1C Sum of infinite convergent geometric sequences

Geometric sequences are closely related to exponential functions. The main difference is that the domain of exponential functions can be all real numbers, but the domain of geometric sequences is normally restricted to positive integers.

You know that you can find the sum of the first n terms of a geometric sequence using the formula $S_n = \frac{u_1(1-r^n)}{1-r}$. Sometimes this sum will just increase (or decrease if negative) the more terms you add.

However, if r is between 1 and −1 then as n gets very large, r^n tends towards 0.

As a result, $\frac{u_1(1-r^n)}{1-r}$ tends towards $\frac{u_1(1-0)}{1-r} = \frac{u_1}{1-r}$.

So, in this situation, the sum of infinitely many terms converges to a finite limit – this is called the **sum to infinity** S_∞ of the geometric sequence.

KEY POINT 1.3

For a geometric sequence with common ratio r,

$$S_\infty = \frac{u_1}{1-r} \quad \text{if } |r| < 1$$

WORKED EXAMPLE 1.9

Find the value of the infinite geometric series

$$2 + \frac{4}{3} + \frac{8}{9} + \frac{16}{27} + \ldots$$

This is a geometric series with first term 2 and common ratio $\frac{2}{3}$

$$u_1 = 2,\ r = \frac{2}{3}$$

Use $S_\infty = \frac{u_1}{1-r}$

$$S_\infty = \frac{2}{1-\frac{2}{3}} = 6$$

You are the Researcher

The ancient Greeks had some real difficulties working with limits of series. Some of them thought incorrectly that if each term in the sum got smaller and smaller, then the series would reach a limit. However, this is not always true. You might like to explore the Harmonic series, $\frac{1}{2} + \frac{1}{3} + \frac{1}{4} \ldots$, and see why that does not converge.

There are many tests that mathematicians use to decide if a series will converge, for example the ratio test. You might like to learn more about these and how they work.

WORKED EXAMPLE 1.10

The geometric series $(x+4)+(x+4)^2+(x+4)^3+\dots$ converges.

Find the range of possible values of x.

State the common ratio

$r = x + 4$

Since the series converges,

$|x+4| < 1$

You know that $|r| < 1$

$-1 < x + 4 < 1$

$-5 < x < -3$

Be the Examiner 1.2

Find the sum to infinity of the geometric series

$$\frac{1}{2}-\frac{3}{4}+\frac{9}{8}-\frac{27}{16}+\dots$$

Which is the correct solution? Identify the errors made in the incorrect solutions.

Solution 1	Solution 2	Solution 3
$S_\infty = \dfrac{1}{1-\frac{1}{2}}$ $= 2$	$S_\infty = \dfrac{\frac{1}{2}}{1-\left(-\frac{3}{2}\right)}$ $= \dfrac{1}{5}$	$r = -\dfrac{3}{4} \div \dfrac{1}{2} = -\dfrac{3}{2}$ so S_∞ does not exist.

TOK Links

The idea of convergence has led to many paradoxes, most famously Zeno's paradoxes. You might want to explore these. How do we resolve situations where there is a conflict between reason and intuition?

TOOLKIT: Problem Solving

You may not have realized it, but you have already met infinite geometric series in your previous work. Explain how you can find the following sum using methods from your prior learning. Confirm the answer by using the formula for the geometric series.

$$\sum_{r=1}^{\infty} \frac{3}{10^r}$$

What happens to your argument when applied to the sum below?

$$\sum_{r=1}^{\infty} \frac{9}{10^r}$$

Exercise 1C

For questions 1 to 5, use the method demonstrated in Worked Example 1.9 to find the sum of these infinite geometric series.

1 **a** $\frac{1}{4}+\frac{1}{16}+\frac{1}{64}+\dots$

b $\frac{1}{2}+\frac{1}{4}+\frac{1}{8}+\dots$

2 **a** $\frac{1}{3}+\frac{2}{9}+\frac{4}{27}+\dots$

b $\frac{1}{3}+\frac{2}{15}+\frac{4}{75}+\dots$

3 **a** $4+1+\frac{1}{4}+\dots$

b $6+2+\frac{2}{3}+\dots$

4 a $\frac{1}{2}-\frac{1}{4}+\frac{1}{8}-\frac{1}{16}+\ldots$

b $\frac{1}{3}-\frac{1}{9}+\frac{1}{27}-\frac{1}{81}+\ldots$

5 a $15-9+\frac{27}{5}-\frac{81}{25}+\ldots$

b $16-12+9-\frac{27}{4}+\ldots$

For questions 6 to 10 use the method demonstrated in Worked Example 1.10 to find the range of values of x for which the infinite geometric series converges.

6 a $(x-2)+(x-2)^2+(x-2)^3+\ldots$

b $(x+3)+(x+3)^2+(x+3)^3+\ldots$

7 a $1+2x+4x^2+\ldots$

b $1+3x+9x^2+\ldots$

8 a $1+\frac{x}{2}+\frac{x^2}{4}+\ldots$

b $1+\frac{x}{5}+\frac{x^2}{25}+\ldots$

9 a $(x+4)-(x+4)^2+(x+4)^3-\ldots$

b $(x-1)-(x-1)^2+(x-1)^3-\ldots$

10 a $1-\frac{3x}{2}+\frac{9x^2}{4}-\ldots$

b $1-\frac{4x}{3}+\frac{16x^2}{9}-\ldots$

11 A geometric series has first term 3 and common ratio $\frac{1}{4}$. Find the sum to infinity of the series.

12 Find the sum to infinity of the geometric series with first term 5 and common ratio $-\frac{1}{4}$.

13 Find the sum to infinity of the series $2-\frac{2}{3}+\frac{2}{9}-\frac{2}{27}+\ldots$

14 An infinite geometric series has the first term 8 and its sum to infinity is 6. Find the common ratio of the series.

15 The first term of a geometric series is 3. Given that the sum to infinity of the series is 4, find the value of the common ratio.

16 The sum to infinity of a geometric series is 3 and the common ratio is $\frac{1}{3}$. Find the first three terms of the series.

17 The second term of a geometric series is 2 and the sum to infinity is 9. Find two possible values of the common ratio.

18 A geometric series is given by $\sum_{r=0}^{\infty}\left(\frac{2}{5}\right)^r$.

a Write down the first three terms of the series.

b Find the sum of the series.

19 Evaluate $\sum_{r=0}^{\infty}\frac{2}{3^r}$.

20 a Find the range of values of x for which the geometric series $3-\frac{x}{3}+\frac{x^2}{27}-\frac{x^3}{243}+\ldots$ converges.

b Find the sum of the series when $x=-2$.

21 A geometric series is given by $5+5(x-3)+5(x-3)^2+\ldots$

a Find the range of values of x for which the series converges.

b Find the expression, in terms of x, for the sum of the series.

22 For the geometric series $2+4x+8x^2+\ldots$

a Find the range of values of x for which the series converges.

b Find the expression, in terms of x, for the sum to infinity of the series.

23 The second term of a geometric series is $-\frac{6}{5}$ and the sum to infinity is 5. Find the first term of the series.

24 Given that x is a positive number,

a find the range of values of x for which the geometric series $x+4x^3+16x^5+\ldots$ converges

b find an expression, in terms of x, for the sum to infinity of the series.

25 a Find the value of x such that $\sum_{r=0}^{\infty}\frac{x^{r+1}}{2^r}=3$.

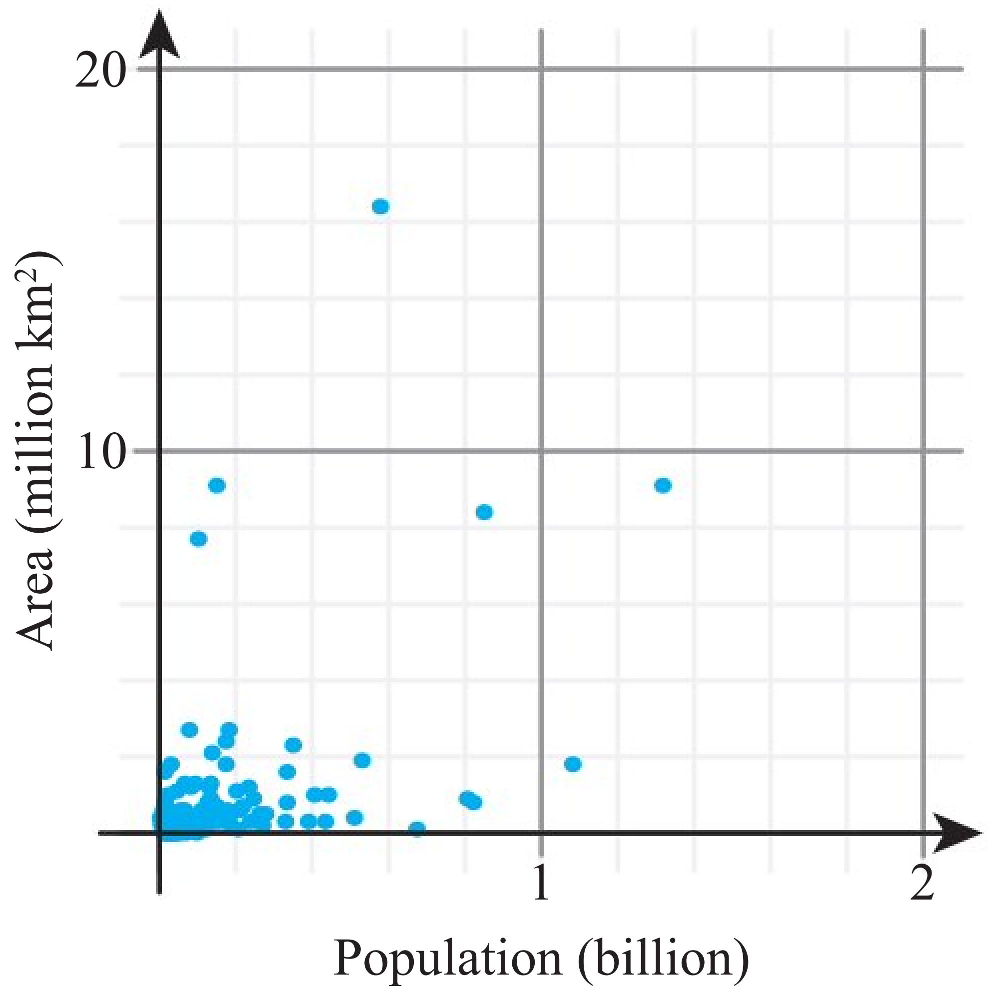
b Find the range of values of x for which the series converges.

26 An infinite geometric series has sum to infinity of 27 and sum of the first three terms equal to 19. Find the first term.

1D Using logarithmic scales on graphs

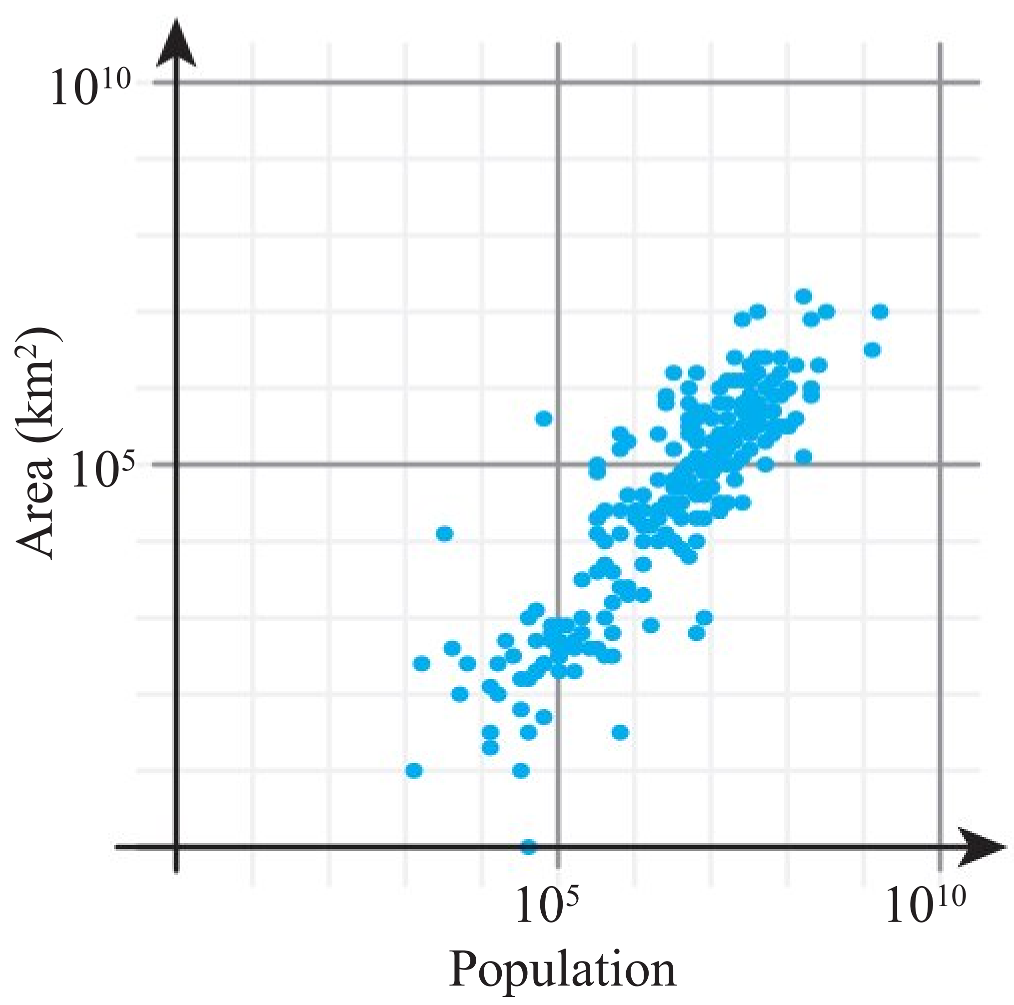
Scaling very large or very small numbers using logarithms

Sometimes you might be interested in displaying data which covers a very wide range of values, for example, the areas and populations of different countries. A raw plot of this data is not very helpful:



This is because some of the largest countries – China, India and Russia – are so much larger than the other countries that most get squashed into the bottom left corner, so no detail can be seen.

A much better way to represent data like this is to use a logarithmic scale. This is where the logs of the original numbers are plotted instead (although sometimes, as here, the original numbers are still labelled on the axes).

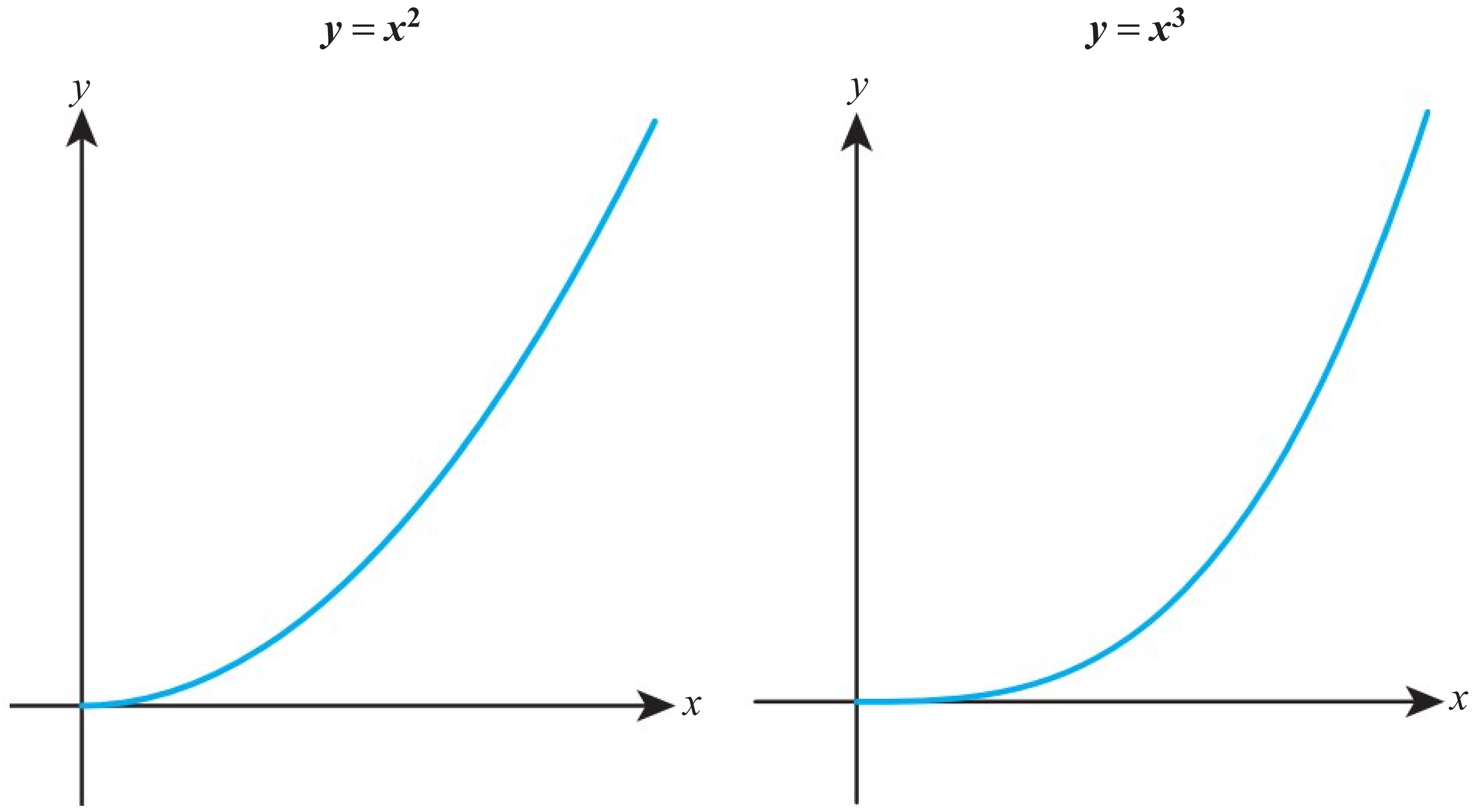


TOOLKIT: Modelling

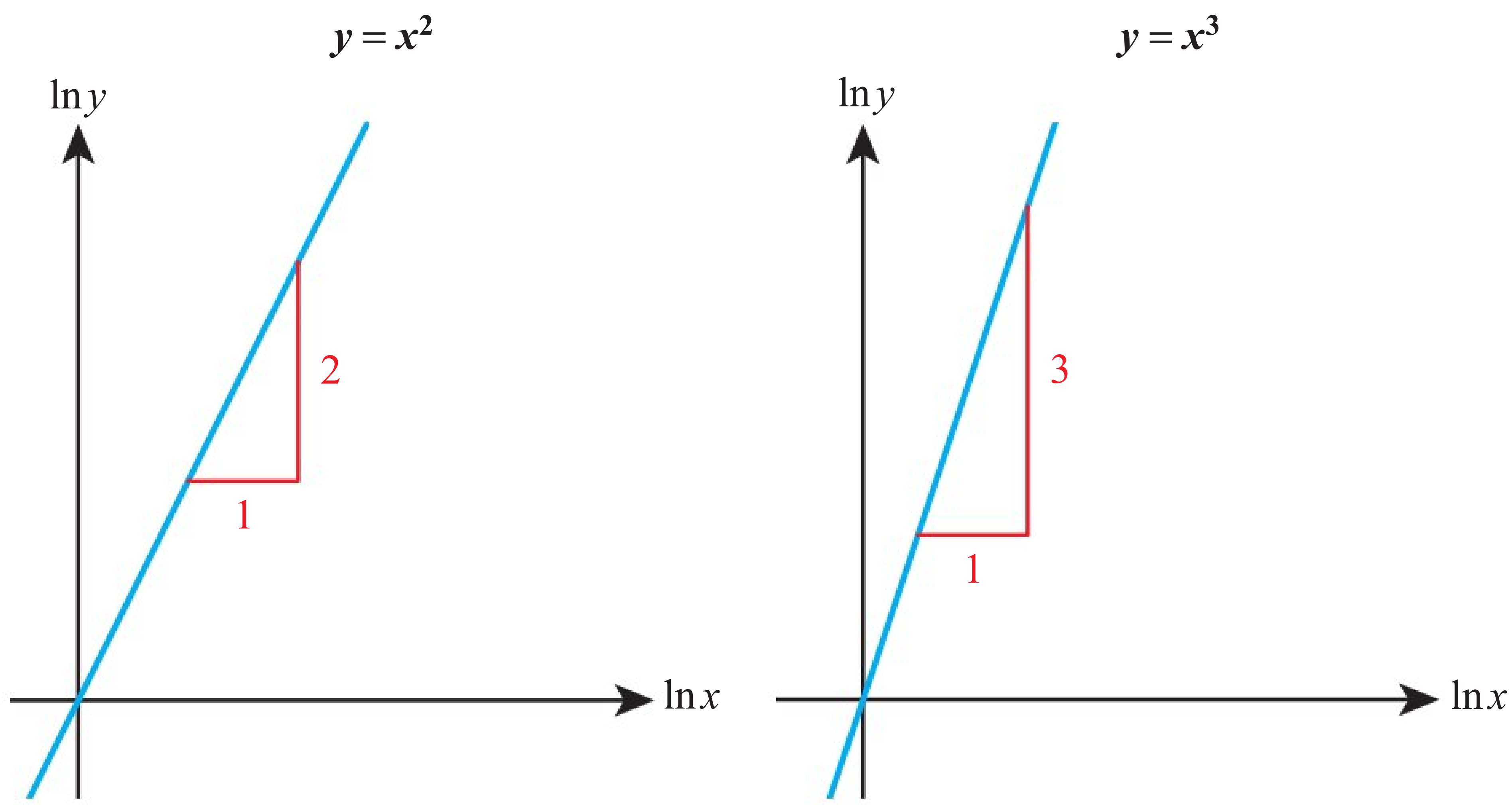
See if you can think of any other data which might cover several orders of magnitude. Try to find this data and plot it with and without logarithmic scales.

Linearizing data using logarithms

If you are just looking at positive values of x, it can be very tricky to visually tell the difference between the graphs of $y = x^2$ and $y = x^3$.



However, if we plot the logarithm of y against the logarithm of x we find something very interesting:



The graphs become straight lines through the origin with different gradients. To explain this, look at what happens if we investigate the general power relationship $y = ax^n$. Taking logs of both sides we find that

$$\log y = \log(ax^n)$$

Notice that it does not actually matter what base we use. Applying the laws of logarithms,

$$\log y = n \log x + \log a$$

This has the same form as the straight line graph

$$Y = mX + c$$

KEY POINT 1.4

If $y = ax^n$, the graph of $\log y$ against $\log x$ will be a straight line with gradient n and y-intercept $\log a$.

This is called **linearizing** the function $y = ax^n$. Plotting $\log y$ against $\log x$ is called creating a **log–log graph**.

WORKED EXAMPLE 1.11

Linearize the relationship $y = 3x^4$ and describe the resulting graph.

Take logs of both sides. It does not matter what base you use

$$\ln y = \ln(3x^4)$$

Apply the laws of logarithms

$$= \ln 3 + \ln x^4$$
$$= \ln 3 + 4\ln x$$

So, the graph of $\ln y$ against $\ln x$ will be a straight line with gradient 4 and y-intercept $\ln 3$.

If a function obeys an exponential law, $y = ka^x$, then taking logs of both sides results in

$$\log y = (\log a)x + \log k$$

This also has the same form as the straight line graph

$$Y = mX + c$$

KEY POINT 1.5

If $y = ka^x$ the graph of $\log y$ against x will be a straight line with gradient $\log a$ and y-intercept $\log k$.

Plotting $\log y$ against x is called creating a **semi-log graph**.

WORKED EXAMPLE 1.12

Linearize the relationship $y = 2 \times 4^x$ and describe the resulting graph.

Take logs of both sides. It does not matter what base you use

$$\log_{10} y = \log_{10}(2 \times 4^x)$$

Apply the laws of logarithms

$$= \log_{10}(2) + \log_{10}(4^x)$$
$$= \log_{10}(2) + x\log_{10}(4)$$

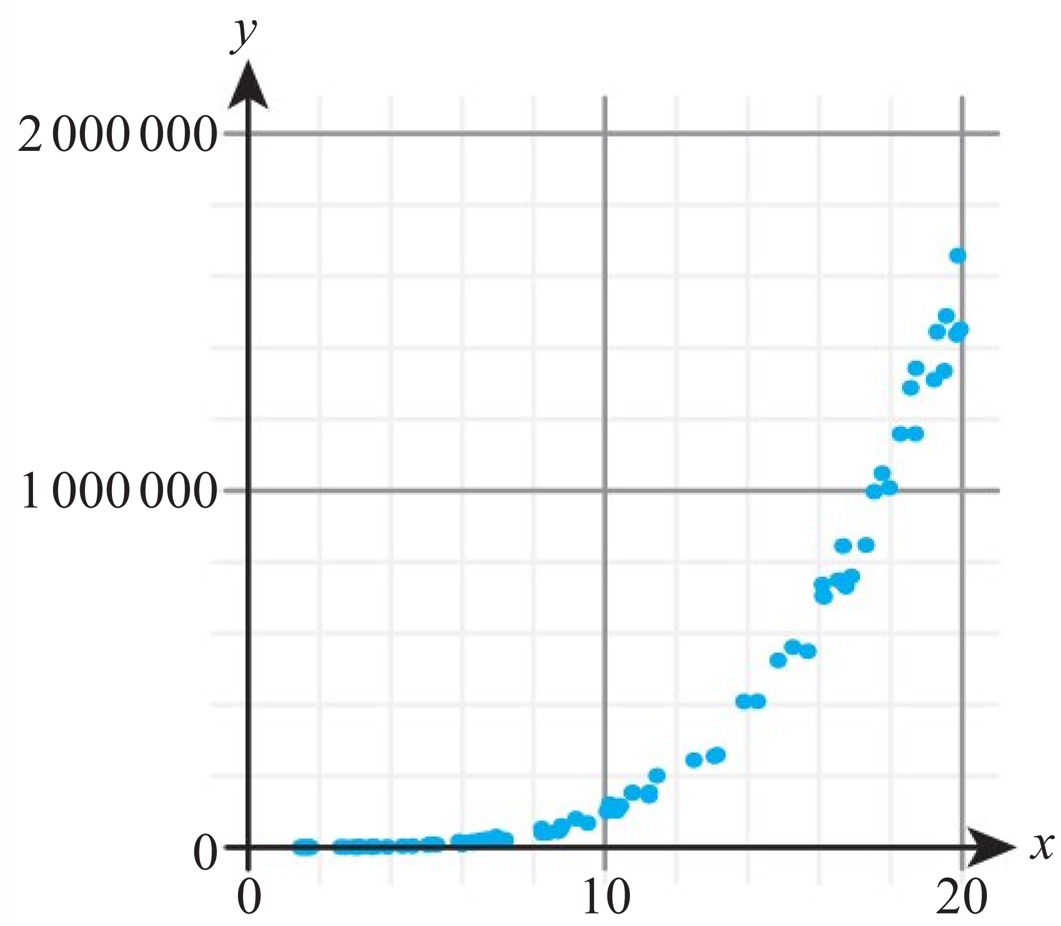
So, the graph of $\log_{10} y$ against x will be a straight line with gradient $\log_{10} 4$ and intercept $\log_{10} 2$.

Interpretation of log–log and semi-log graphs

Commonly, log–log and semi-log graphs are not used on perfect functions. They are applied to data with the aim of inferring which sort of relationship might exist.

WORKED EXAMPLE 1.13

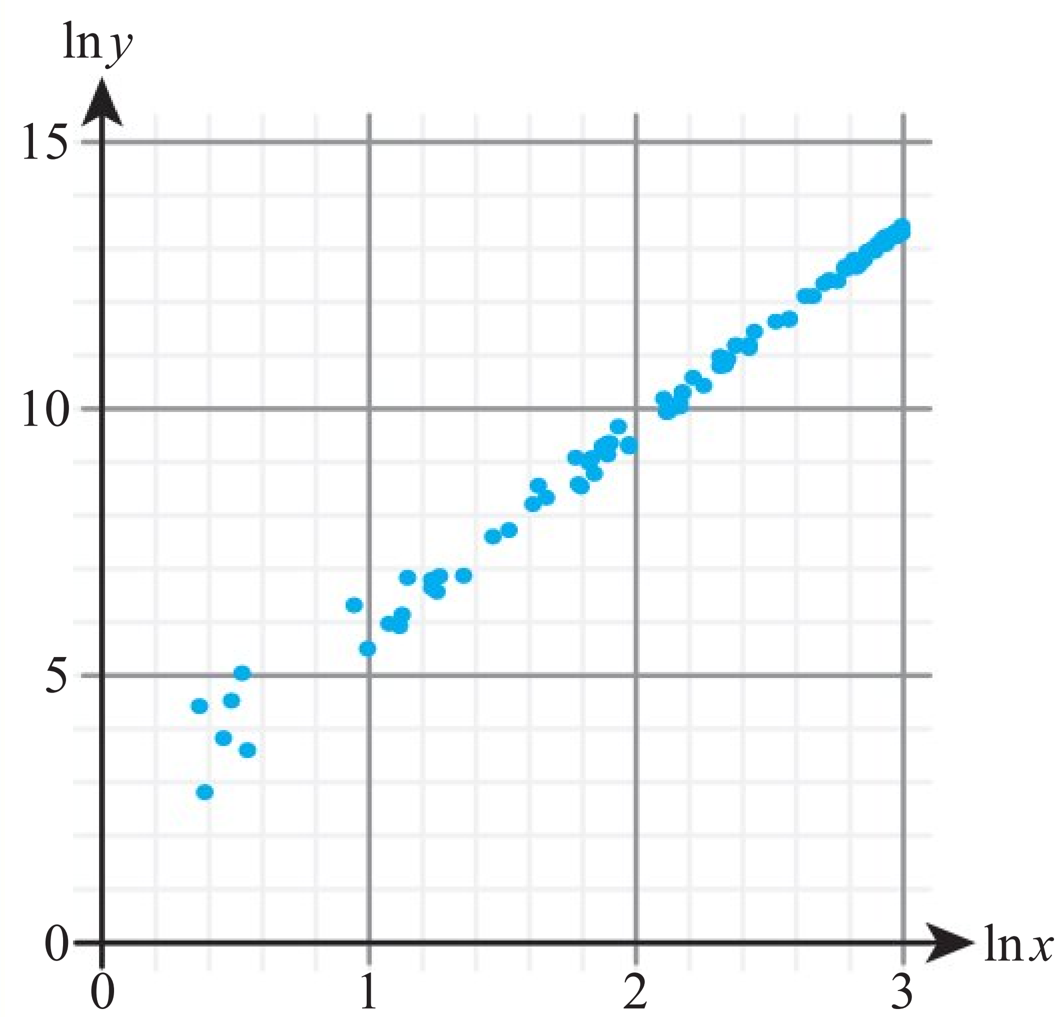
The graph below shows some collected data.

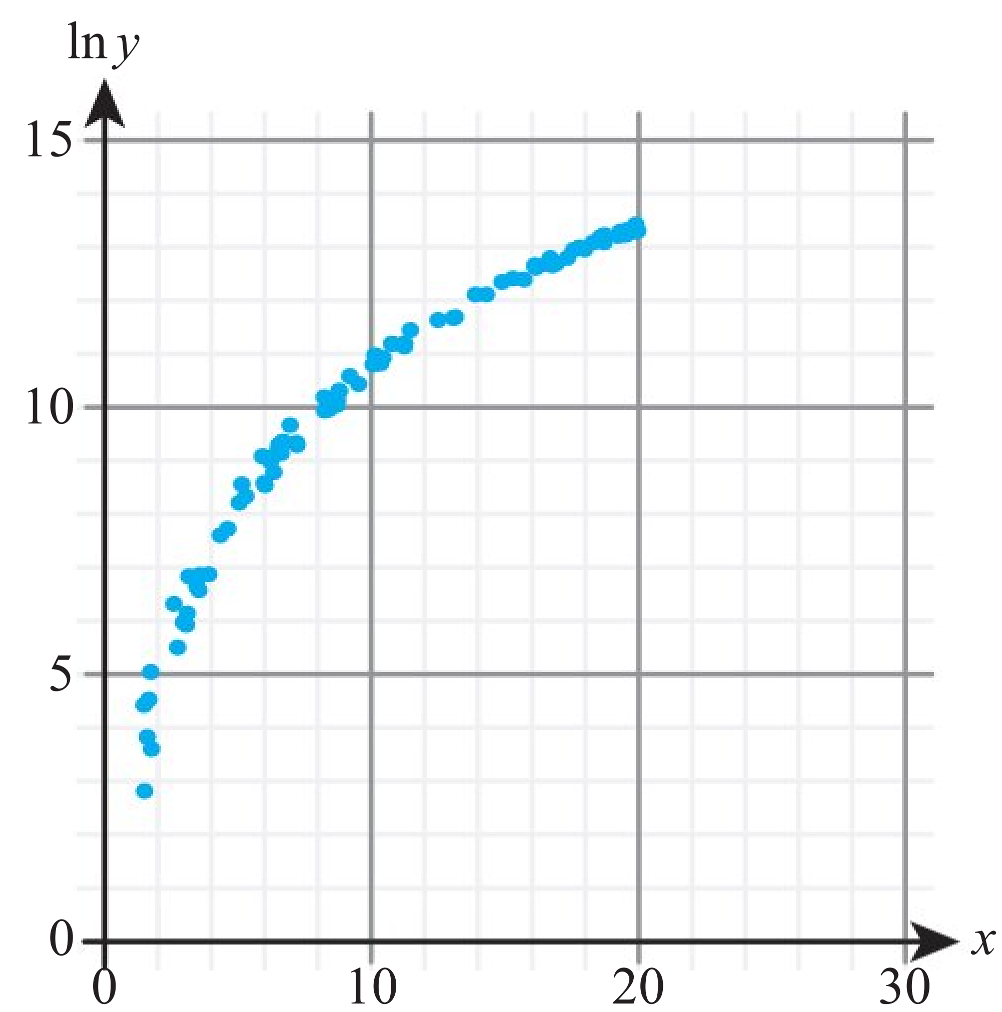


Tip

You can use your knowledge of linear regression and correlation to find the line of best fit in a more rigorous way.

The log–log and semi-log plots are shown below.



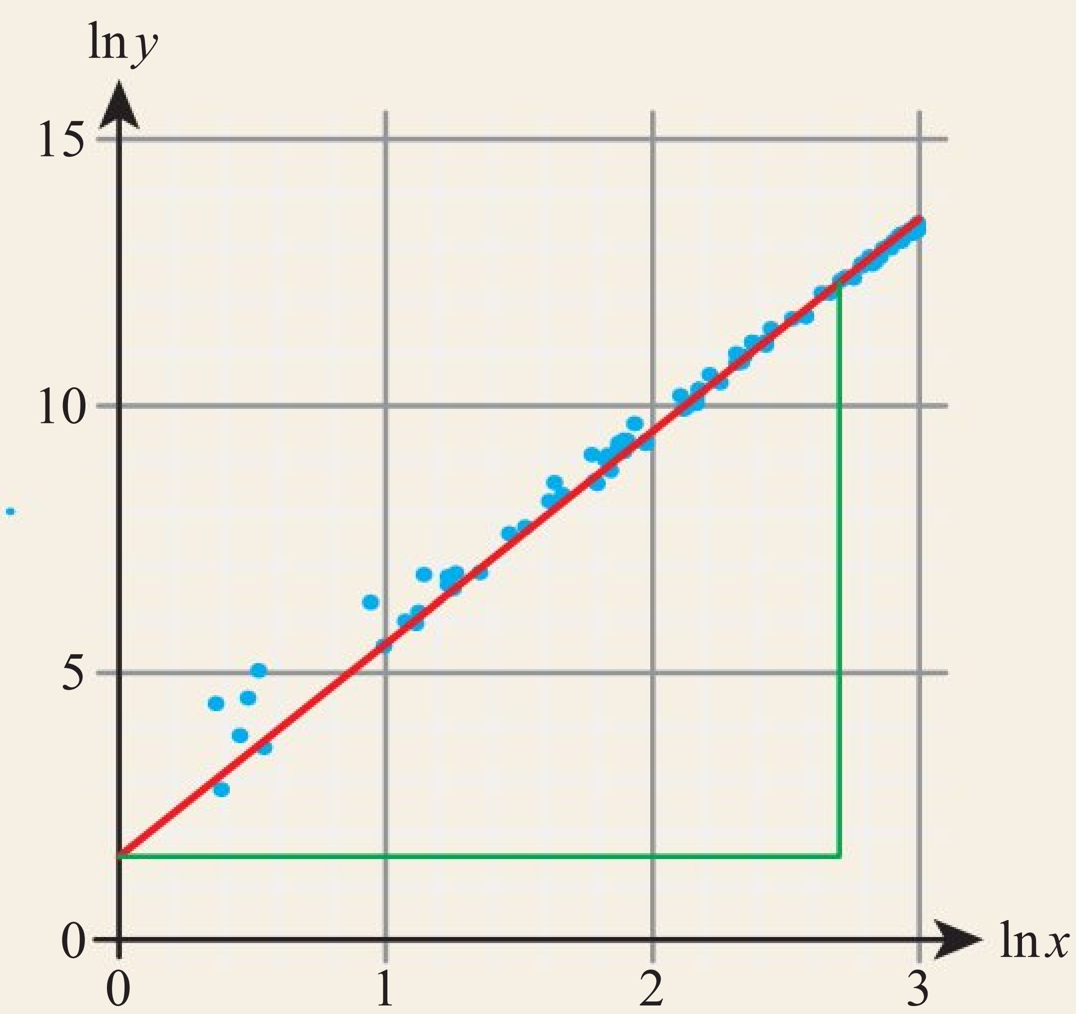


Determine the relationship between x and y.

Decide which of the graphs is plausibly linear. It does not have to be perfect

The log–log plot looks closer to a straight line than the semi-log plot, so it is a power relationship.

Add a line of best fit by eye to the graph



Find the gradient and intercept of the line of best fit

gradient $\approx \dfrac{12.4 - 1.6}{2.7 - 0} = 4$

y-intercept ≈ 1.6

Use Key Point 1.4. Since the intercept is $\ln a$, then $a = e^{\text{intercept}}$

So, $y = ax^n$ with $n \approx 4$ and $a \approx e^{1.6} \approx 4.95$

Exercise 1D

For questions 1 to 4, use the methods demonstrated in Worked Example 1.11 to linearize the given functions and describe the resulting graph.

1 **a** $y = x^4$
b $y = x^5$

2 **a** $y = 3x^4$
b $y = 5x^2$

3 **a** $y = 4\sqrt{x}$
b $y = 2\sqrt[3]{x}$

4 **a** $y = \dfrac{1}{x}$
b $y = \dfrac{2}{\sqrt{x}}$

For questions 5 to 8, use the methods demonstrated in Worked Example 1.12 to linearize the given functions and describe the resulting graph.

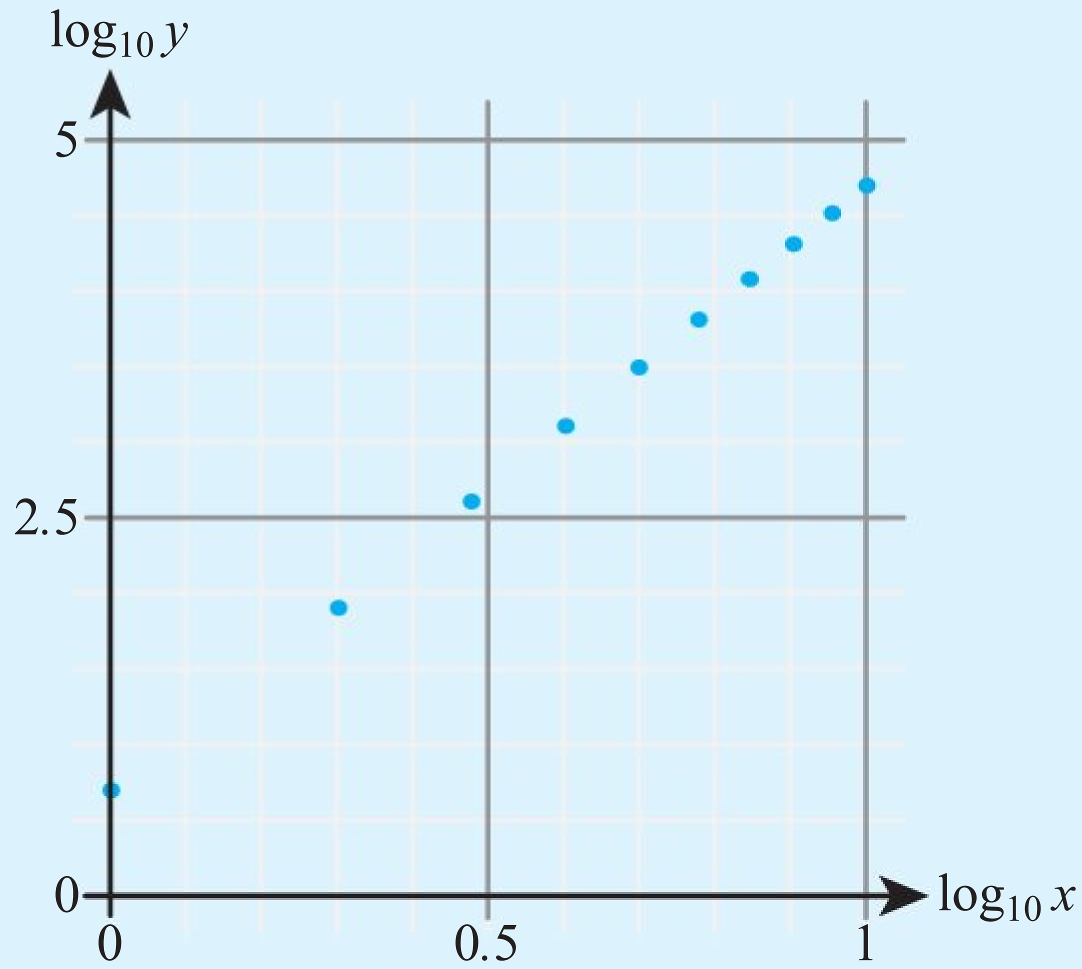
5 **a** $y = 2^x$
b $y = 5^x$

6 **a** $y = 3 \times 2^x$
b $y = 2 \times 3^x$

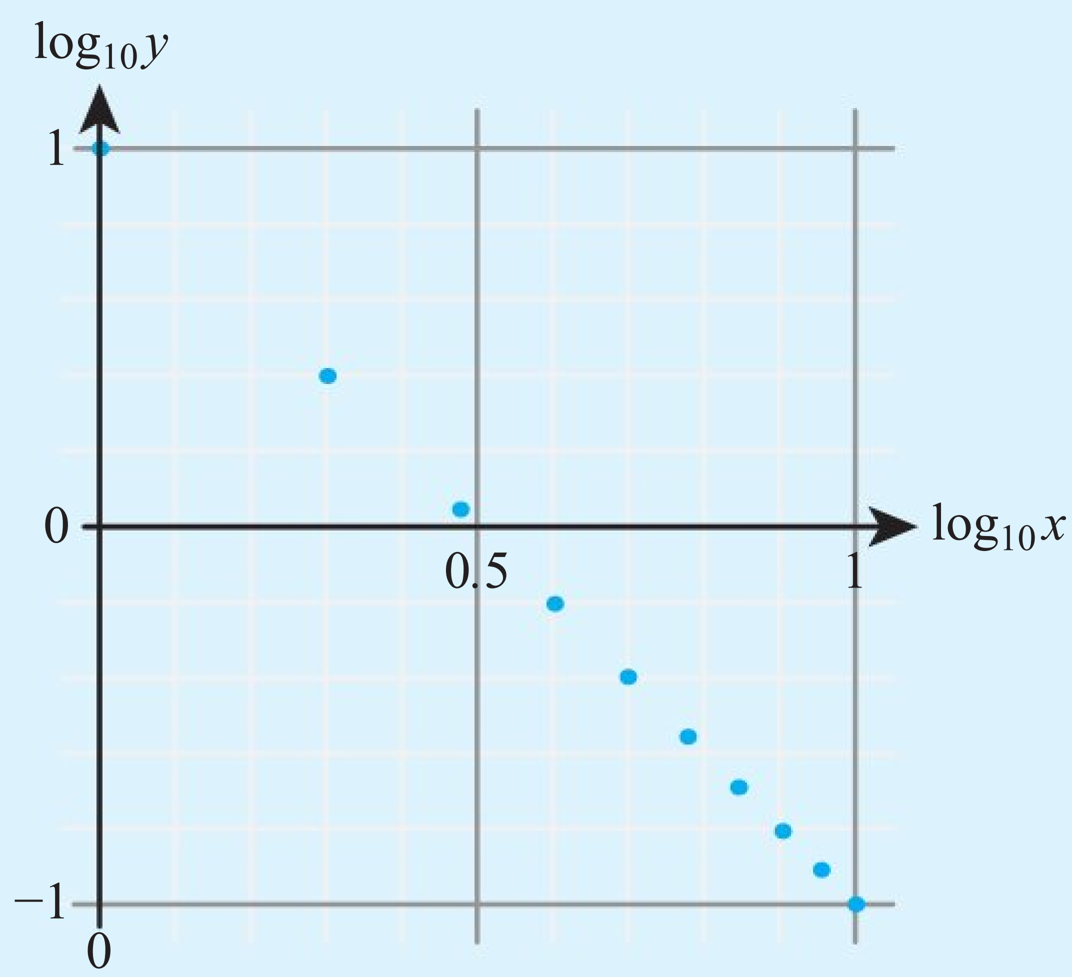
7 **a** $y = \dfrac{4}{7^x}$
b $y = \dfrac{6}{4^x}$

8 **a** $y = 2e^{3x}$
b $y = 10e^{-x}$

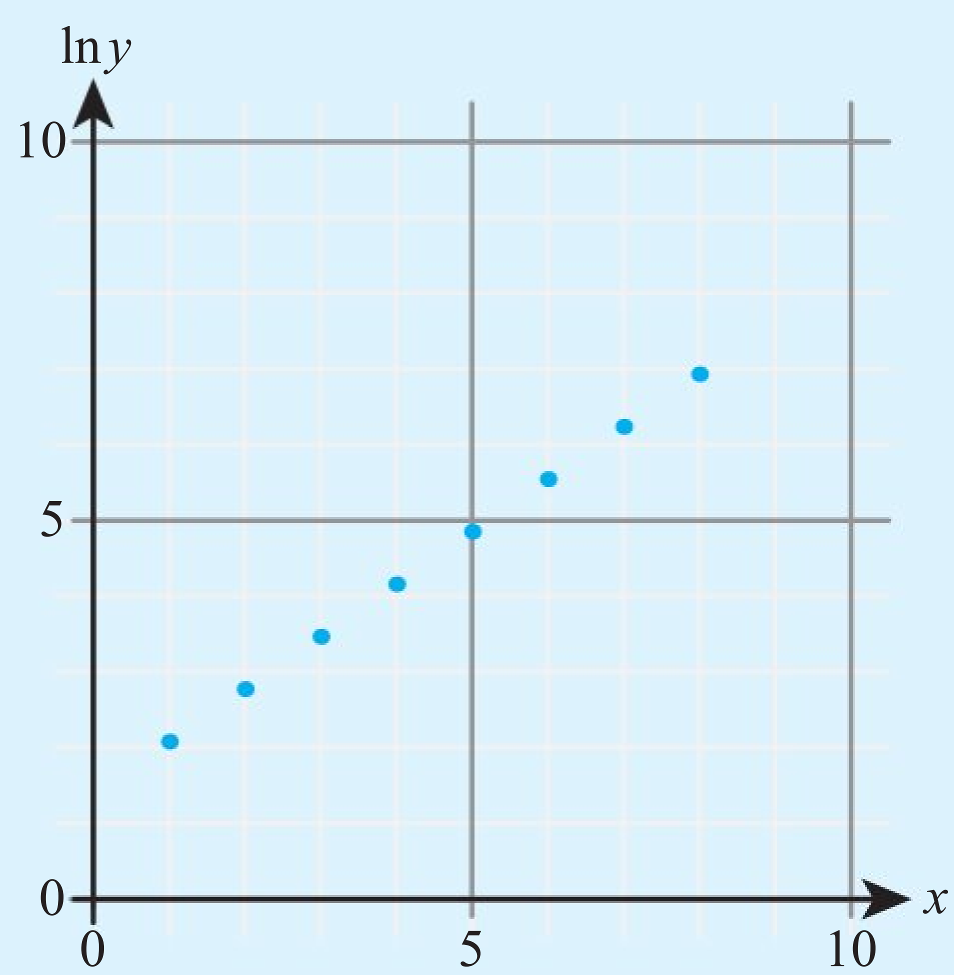
9 The graph below shows a log–log graph. Use this to suggest the relationship between x and y.



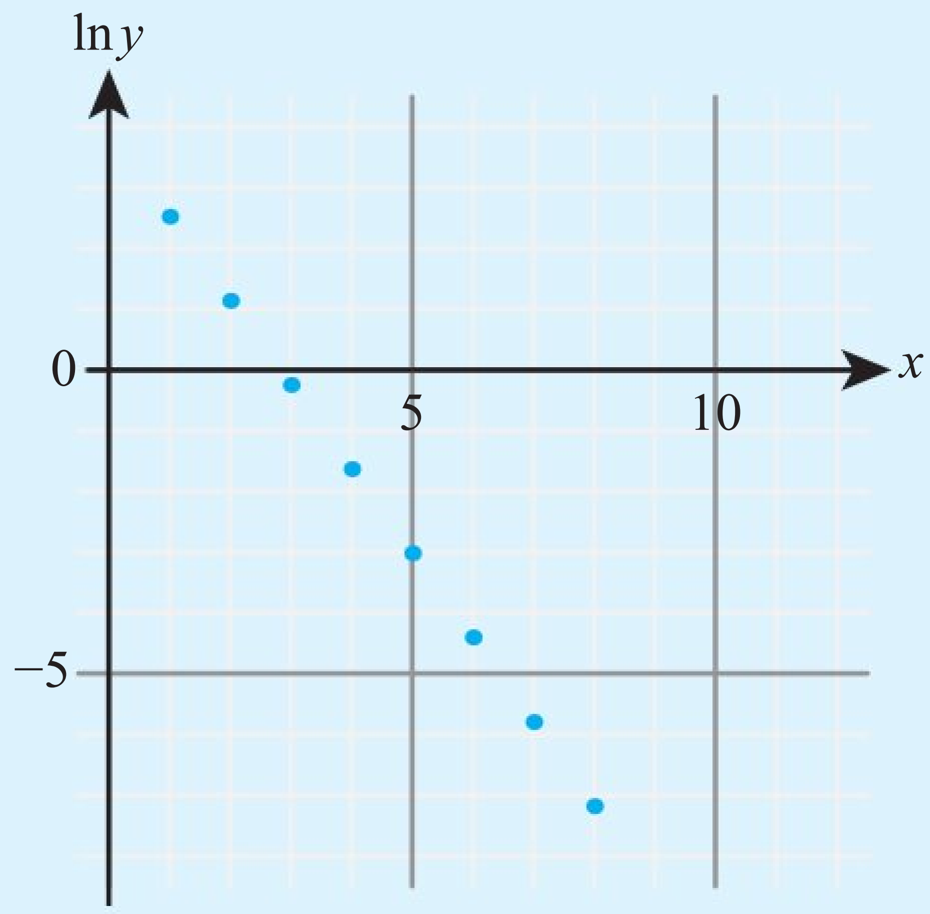
10 The graph below shows a log–log graph. Use this to suggest the relationship between x and y.



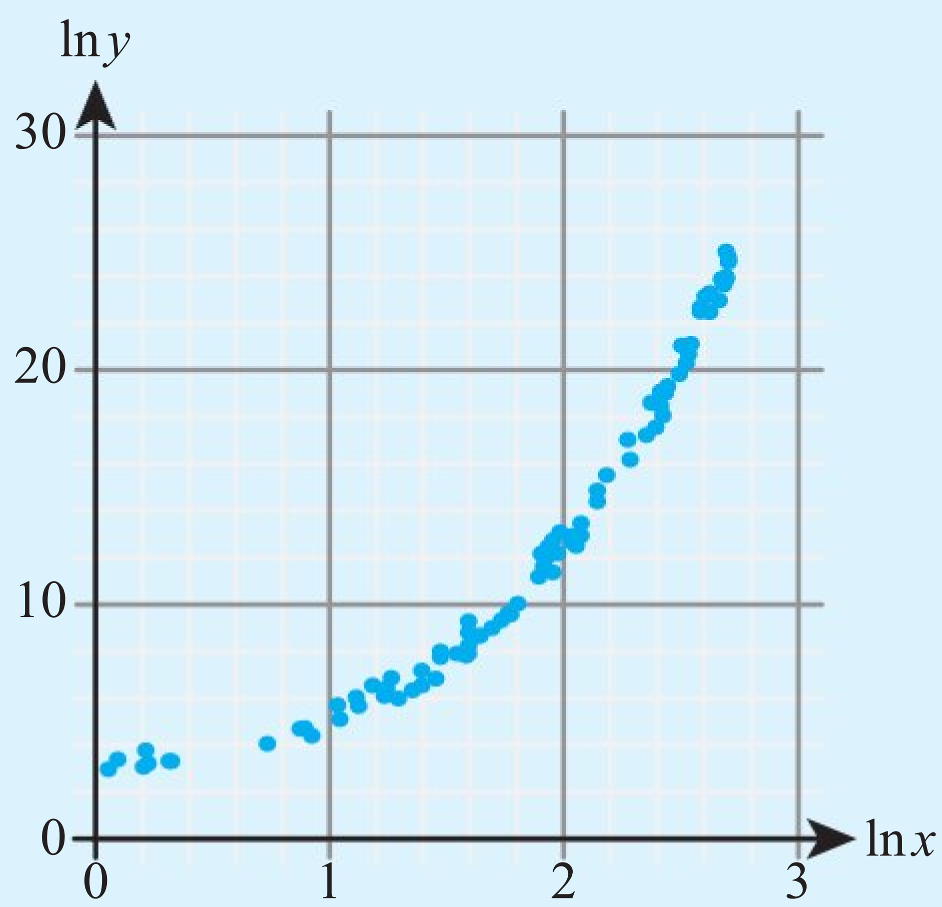
11 The graph below shows a semi-log graph. Use this to suggest the relationship between x and y.

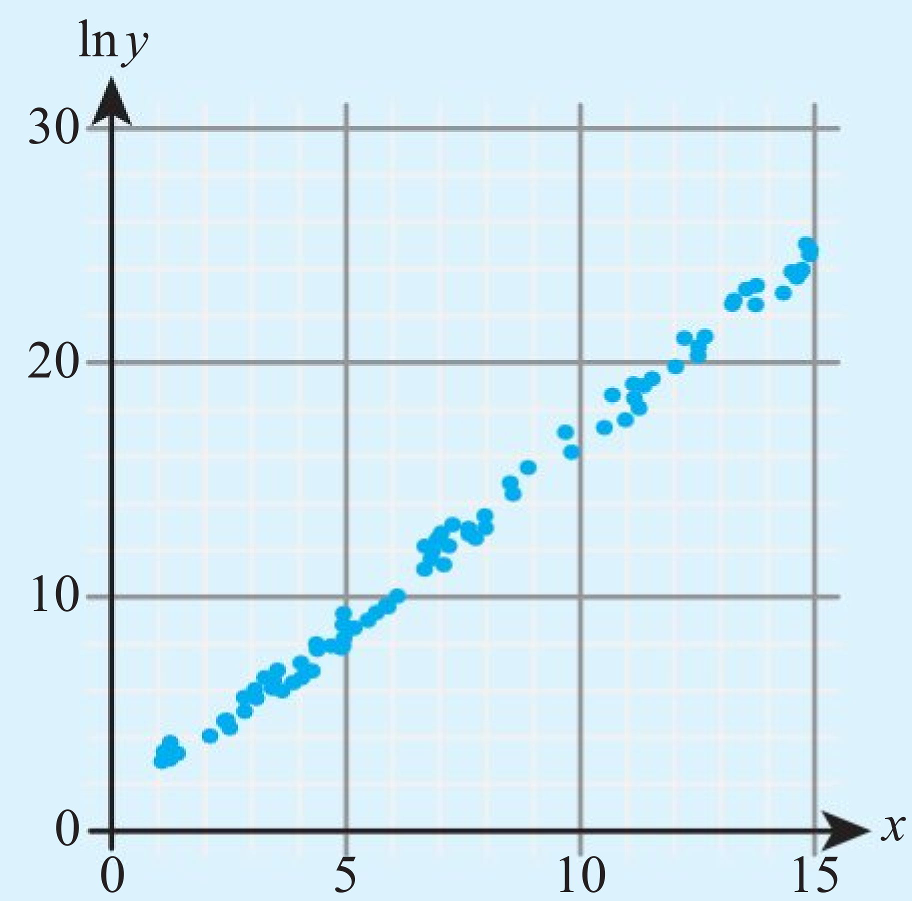


12 The graph below shows a semi-log graph. Use this to suggest the relationship between x and y.



13 The scatterplots below show a semi-log and log–log plot for a set of data.
Use these plots to suggest a relationship between x and y, giving the values of any parameters to one significant figure.





14 The amount of a radioactive substance, X grams, at a time t minutes after it is created is modelled by $X = X_0 e^{-kt}$.

a Linearize this relationship.

b The following measurements were taken:

t	1	2	3	4
X	12	7	4	3

Use linear regression to estimate the value of X_0 and k.

c Use your model to estimate the time taken for the amount of substance to halve.

15 The force, F Newtons, between two magnets a distance d metres apart is modelled by the equation $F = kd^n$.

a Linearize this relationship.

b The following measurements were taken.

d	1	2	3	4
F	0.15	0.0375	0.0167	0.00938

Use linear regression to find the values of k and n, giving your answer to two significant figures.

16 The following data show the length of railway added in the United States by year.

Year	1843	1844	1845	1846	1847	1848	1849	1850
Miles added	159	192	256	297	668	398	1369	1656

A model is created of the form $m = ab^y$, where y is the number of years after 1842 and m is the number of miles added.

a Use linear regression to estimate the values of a and b.

b What is the average percentage increase in the number of miles of railway added?

c Hence estimate the number of miles added in 1860.

17 The data in the table shows the average distance from the sun, d million km, and year length, y days, of several planets in the solar system.

Planet	d	y
Mercury	58	88
Venus	108	225
Earth	150	365
Mars	228	687
Jupiter	779	4380
Saturn	1434	10 585

a Use linear regression to form a model of the form $y = ad^n$, giving the parameters to one decimal place.

b Hence estimate the year length, to the nearest earth year, of Uranus which is 2871 million km from the sun.

18 Lotka's law states that the number of authors making a contribution to x published scientific papers is proportional to $\frac{1}{x^n}$ where n is an integer.

The data in the table are taken from a scientific journal.

x	1	2	3	4
Frequency	89	28	10	6

Use linear regression to find the integer value of n.

19 Becky records the population (P) in millions of six cities and the rank (R) of its population size within Becky's country. Here are her results.

R	1	2	3	4	5	6
P	6.5	3.4	2.1	1.6	1.2	1.1

a Using the natural logarithm find the value of Pearson's product-moment correlation coefficient for

i the log–log graph

ii the semi-log graph.

b Hence suggest a form for the relationship between R and P, justifying your answer.

c Use linear regression to estimate the value of the parameters in your suggested model, giving your answers to two significant figures.

You are the Researcher

The model that you found in part **b** is one actually observed in most countries. It is called the rank-size distribution. Investigate whether this rule applies in your country.

20 The data below is taken from the 19th century story 'Treasure Ireland' by Robert Louis Stevenson (Source: Oxford text archive). It shows the 16 most common words used along with their rank (R) and frequency (F).

Word	R	F
the	1	4355
and	2	2872
I	3	1748
a	4	1745
of	5	1665
to	6	1520
was	7	1135
in	8	968
he	9	901
that	10	835
you	11	824
had	12	738
it	13	706
his	14	651
as	15	623
my	16	618

From Oxford Text Archive, part of the Bodleian Libraries

a Using natural logarithms find the value of Pearson's product-moment correlation coefficient for

i the log–log graph

ii the semi-log graph.

b Hence suggest a form for the relationship between R and F, justifying your answer.

c Use linear regression to estimate the value of the parameters in your suggested model, giving your answers to two significant figures.

d Assuming that the least common word in Treasure Island is used 10 times, use your model to estimate how many *different* words are used in Treasure Island.

The relationship you found in **b** is called Zipf's law after the American linguist George Zipf (1902–1950), although it was also noticed by the French stenographer Jean-Baptiste Estoup (1868–1950) and the German physicist Felix Auerbach (1856-1933). Remarkably, it seems to apply across all known languages.

You are the Researcher

Word frequency analysis such as this is one of the tools used by forensic linguists to answer questions such as 'Did Shakespeare write all of his plays himself?', or 'Is a section of a piece of work plagiarized?'. It is also used by cryptographers (along with letter frequency analysis) to crack codes. You might want to find out more about how they use these techniques. Remember that your Higher Level mathematical exploration needs to demonstrate a sophisticated and rigorous use of mathematics.

21 The variables x and y in the table below are thought to be modelled by a relationship of the form $y = ax^n$, where n is an integer.

x	−3	−2	−1	1	2	3
y	−45.6	−19.8	−5.1	−4.7	−19.2	−44.2

Use linear regression to find the values of a and n.

22 Newton's law of cooling suggests that the temperature of a cooling body (T°C) at a time t hours will follow $T = Ae^{-kt} + c$.

A body is found at 12 noon. This point is chosen for $t = 0$. A forensic scientist takes the following measurements:

t	1	2	3	4
T	27	24.5	23	22

a Given that the background room temperature is assumed to be a constant 20 °C, linearize the given relationship and use linear regression to find A and k.

b Assuming that body temperature is normally 37 °C, use the model to estimate the time of death, giving your answer to the nearest 10 minutes.

23 The population of rabbits (P thousands) on an island at a time t years after they are introduced is modelled by the logistic function

$$P = \frac{10}{1 + Je^{-\beta t}}$$

a Find the value of P as t gets very large.

b Linearize the relationship.

c An ecologist finds the following measurements:

t	2	3	4	5
P	6	6.5	6.9	7.3

Use linear regression to estimate the value of β and J.

d Estimate how long it will take from the rabbit population being introduced for the rabbit population to reach 9000.

e Estimate the number of rabbits that were initially introduced to the island.

f Explain why your answers to d and e may not be accurate.

24 In biochemistry, the rate at which a reaction occurs, v, in the presence of an enzyme depends on the concentration of the reactant, [S] according to:

$$v = \frac{v_{\max}[S]}{K_m + [S]}$$

where K_m is a constant reflecting the efficiency of the enzyme and $v_{\max}$ is the rate in an excess of reactant.

a What is the rate when $[S] = K_m$?

b Sketch the graph of v against [S].

c Show that a plot of $\frac{1}{v}$ against $\frac{1}{[S]}$ linearizes the relationship, and state the gradient and y-intercept of the resulting line.

d Nasar measures the rate of reaction at different reactant concentrations. His results are shown below.

$[S]$	1	2	3	4
v	2	3	3.5	3.7

i Transform the data as suggested in part c and use least squares regression to find the equation of the regression line.

ii Hence estimate the values of $v_{\max}$ and K_m.

e i Show that a plot of v against $\frac{v}{[S]}$ is linear, and state the gradient and intercept.

ii Use Nasar's data to estimate the values of $v_{\max}$ and K_m using the linearization in e i.

Links to: Biology

The model of the rate of enzymatic reaction or 'enzyme kinetics' in question 24 is called Michaelis–Menten kinetics and it is very important in biology. The plot in part **c** is called a Lineweaver–Burk plot and the plot in part **e** is called an Eadie–Hofstee plot. You might wonder why we need two different plots. If the model was perfect and the data accurate then both plots would give the same answer; however neither of those conditions is true. The two different plots magnify different types of error. In modern biology both plots have been superseded by non-linear regression techniques, which you will meet in Chapter 9.

Checklist

- You should be able to extend the laws of exponents to general rational exponents:
  - $a^{\frac{1}{n}} = \sqrt[n]{a}$
- You should know the laws of logarithms:
  - $\log_a xy = \log_a x + \log_a y$
  - $\log_a \dfrac{x}{y} = \log_b x - \log_a y$
  - $\log_a x^m = m \log_a x$

  where $a, x, y > 0$.
- You should be able to find the sum of infinite geometric sequences:

  For a geometric sequence with common ratio r,

  $$S_\infty = \frac{u_1}{1-r} \quad \text{if } |r| < 1$$
- If $y = ax^n$, the graph of $\log y$ against $\log x$ will be a straight line with gradient n and y-intercept $\log a$.
- If $y = ka^x$, the graph of $\log y$ against x will be a straight line with gradient $\log a$ and y-intercept $\log k$.

Mixed Practice

1 **a** Find the exact value of $\left(\frac{4}{9}\right)^{-\frac{3}{2}}$.

b Find $\log_{10}\left(\frac{1}{1000}\right)$.

2 Write $\ln 4 + 2\ln 3$ in the form $\ln k$.

3 Solve the equation $2 \times 3^{x-2} = 54$.

4 Use technology to solve $1.05^x = 2$.

5 Solve the equation $100^{x+1} = 10^{3x}$.

6 Find the value of x such that $\log_{10}(5x+10) = 2$.

7 Find the exact solution of the equation $3\ln x + 2 = 2(\ln x - 1)$.

8 Given that $a = \log_{10} x$, $b = \log_{10} y$ and $c = \log_{10} z$, write the following in terms of a, b and c:

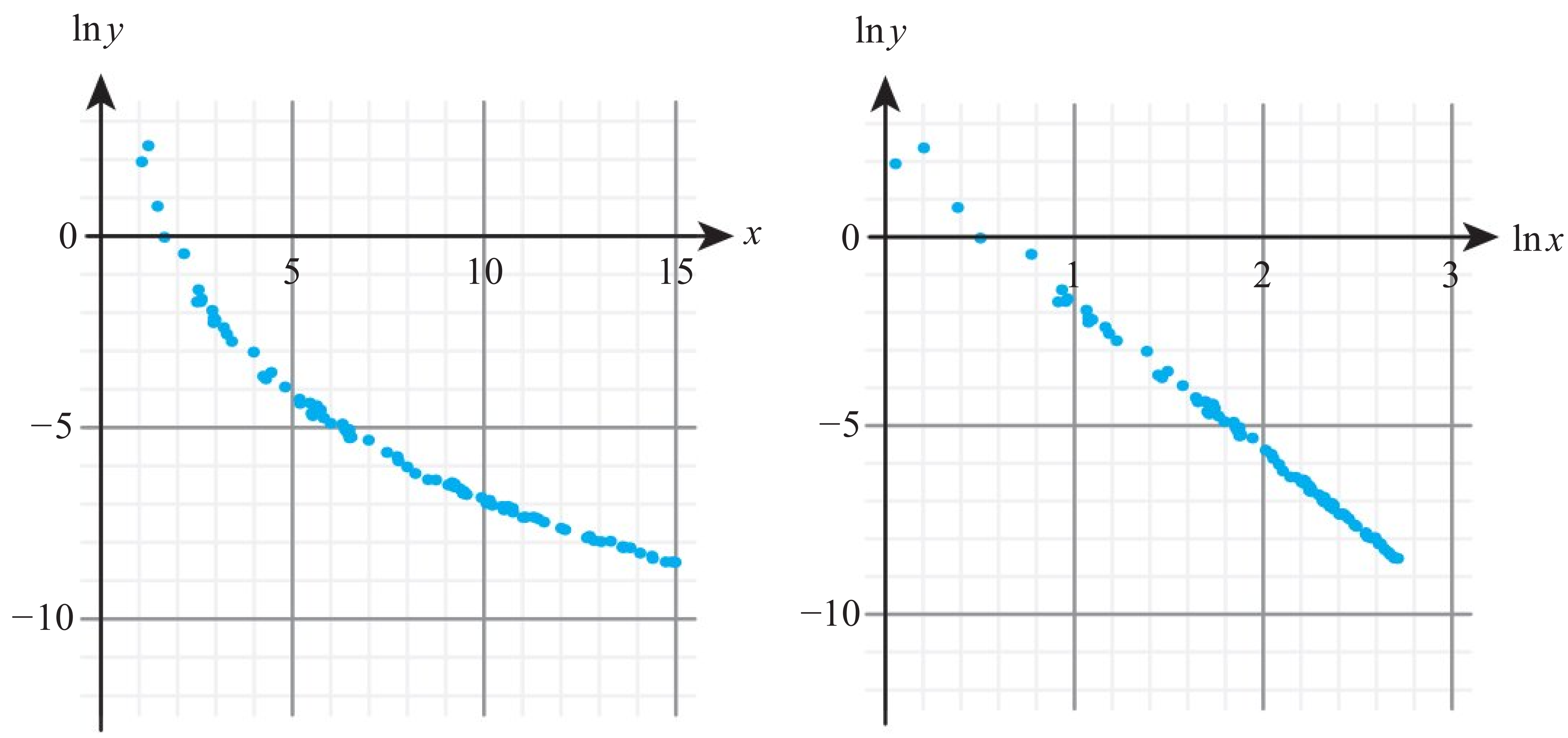
a $\log_{10}(x^2 y)$ **b** $\log_{10}\left(\frac{x}{yz^3}\right)$ **c** $\log_{10}\left(\sqrt{zx^3}\right)$

9 Find the sum of the infinite geometric series $\frac{1}{3}, \frac{1}{9}, \frac{1}{27}, \ldots$

10 A geometric series has first term 7 and common ratio $-\frac{5}{9}$. Find the sum to infinity of the series.

11 Find the first term of the geometric series with common ratio $\frac{3}{4}$ and sum to infinity 12.

12 The plots show a semi-log and log–log plot for a set of data.



Use these plots to suggest a relationship between x and y, giving the values of any parameters to one significant figure.

13 Given that $x = \log_{10} a$ and $y = \log_{10} b$, write the following in terms of x and y.

a $\log_{10}\left(\frac{a}{\sqrt{b}}\right)$ **b** $\log_{10}\left(\frac{a^2}{1000b^3}\right)$

14 Given that $x = \ln 2$ and $y = \ln 5$, write the following in terms of x and y.

a $\ln 10$ **b** $\ln 50$ **c** $\ln 0.08$

15 Write $3 + 2\log 5 - 2\log 2$ as a single logarithm.

16 Find the exact solution of the equation $3\ln x + \ln 8 = 5$, giving your answer in the form Ae^k, where A and k are fractions.

17 Solve the equation $4^{3x+5} = 8^{x-1}$.

18 Solve the simultaneous equations

$$\log_{10} x + \log_{10} y = 5$$

$$\log_{10} x - 2\log_{10} y = -1$$

19 Find the exact value of x such that $\log_x 8 = 6$.

20 Solve the equation $3^{2x} = 2e^x$, giving your answer in terms of natural logarithms.

21 Solve the equation $5^{2x+1} = 7^{x-3}$.

22 Find the solution of the equation $12^{2x} = 4 \times 3^{x+1}$ in the form $\frac{\log p}{\log q}$, where p and q are positive integers.

23 The number of cells in a laboratory experiment satisfies the equation $N = 150e^{1.04t}$, where t hours is the time since the start of the experiment.

a What was the initial number of cells?

b How many cells will there be after 3 hours?

c How long will it take for the number of cells to reach 1000?

24 Simplify $e^{1+3\ln x}$.

25 Given that $\log_{10}\left(\frac{100^x}{10^y}\right)$ can be written as $px + qy$, find the value of p and of q.

26 Let $\log_{10} p = 1.5$ and $\log_{10} q = 2.5$.

a Find $\log_{10} p^2$.

b Find $\log_{10}\left(\frac{p}{q}\right)$.

c Find $\log_{10}(10q)$.

27 A geometric sequence has first term 15 and common ratio 1.2. One term of the sequence equals 231, correct to the nearest integer. Which term is it?

28 An infinite geometric series is given by

$$(2 - 3x) + (2 - 3x)^2 + (2 - 3x)^3 + \ldots$$

a Find the range of values of x for which the series converges.

b Given that the sum of the series is $\frac{1}{2}$, find the value of x.

c Show that the sum of the series cannot equal $-\frac{2}{3}$.

29 The sum to infinity of a geometric series is three times as large as the first term. Find the common ratio of the series.

30 The amount of carbon dioxide absorbed by the Amazon rainforest (C billion tonnes) each year is shown below.

Year	2005	2008	2011	2015	2018
C	3.0	2.7	2.5	2.2	2.0

Consider the relationship between the variable y the number of years after 2004 (so that for example in 2005, $y = 1$) and C.

a Using logarithms, find the value of Pearson's product-moment correlation coefficient for
i the semi-log graph
ii the log–log graph.

b Hence suggest a form for the relationship between C and y, justifying your answer.

c Use linear regression to estimate the value of the parameters in your suggested model, giving your answers to two significant figures.

d Estimate the total amount of carbon dioxide that will be absorbed by the Amazon rain forest after the year 2004, assuming that the trends observed continue.

31 The sum of the first three terms of a geometric sequence is 62.755, and the sum of the infinite sequence is 440. Find the common ratio.

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32 Solve the equation $8^{x-1} = 6^{3x}$. Express your answer in terms of $\ln 2$ and $\ln 3$.

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33 Solve the equation $3^{x+1} = 3^x + 18$.

34 The Arrhenius equation suggests that the rate constant, k seconds^{-1}, of an equation depends on the temperature, T in degrees Kelvin according to the model

$$k = Ae^{-\frac{E_a}{RT}}$$

where E is the activation energy (in kJ mol^{-1} per mole) and R is the ideal gas constant, 8.3×10^{-3} kJ mol^{-1} K^{-1}.

a Linearize this equation.

T	280	290	300	310	320
K	4.9×10^{-10}	2.1×10^{-9}	8.5×10^{-9}	3.1×10^{-8}	1×10^{-7}

b Use linear regression to estimate the value of the activation energy.

35 Write in the form $k \ln x$, where k is an integer:

$$\ln x + \ln x^2 + \ln x^3 + \ldots + \ln x^{20}$$

36 Find the exact value of

$$\log_3\left(\frac{1}{3}\right) + \log_3\left(\frac{3}{5}\right) + \log_3\left(\frac{5}{7}\right) + \ldots + \log_3\left(\frac{79}{81}\right)$$

37 Evaluate

$$\sum_{r=0}^{\infty} \frac{3^n + 4^n}{5^n}$$

38 **a** Explain why the geometric series $e^{-x} + e^{-2x} + e^{-3x} + \ldots$ converges for all positive values of x.

b Find an expression for the sum to infinity of the series.

c Given that the sum to infinity of the series is 2, find the exact value of x.

39 A geometric series has sum to infinity 27, and the sum from (and including) the fourth term to infinity is 1. Find the common ratio of the series.

40 Let $\{u_n\}$, $n \in \mathbb{Z}^+$, be an arithmetic sequence with first term equal to a and common difference of d, where $d \neq 6$. Let another sequence $\{v_n\}$, $n \in \mathbb{Z}^+$, be defined by $v_n = 2^{u_n}$.

a **i** Show that $\frac{v_{n+1}}{v_n}$ is a constant.

ii Write down the first term of the sequence $\{v_n\}$.

iii Write down a formula for v_n in terms of a, d and n.

Let S_n be the sum of the first n terms of the sequence $\{v_n\}$.

b **i** Find S_n in terms of a, d and n.

ii Find the values of d for which $\sum_{i=1}^{\infty} v_i$ exists.

You are now told that $\sum_{i=1}^{\infty} v_i$ does exist and is denoted by S_∞.

iii Write down S_∞ in terms of a and d.

iv Given that $S_\infty = 2^{a+1}$, find the value of d.

Let $\{w_n\}$, $n \in \mathbb{Z}^+$, be a geometric sequence with first term equal to p and common ratio q, where p and q are both greater than zero. Let another sequence $\{z_n\}$ be defined by $z_n = \ln w_n$.

c Find $\sum_{i=1}^{n} z_i$ giving your answer in the form $\ln k$ with k in terms of n, p and q.

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2 Vectors

ESSENTIAL UNDERSTANDINGS

- Geometry allows us to quantify the physical world, enhancing our spatial awareness in two and three dimensions.
- This topic provides us with the tools for analysis, measurement and transformation of quantities, movements and relationships.

In this chapter you will learn…

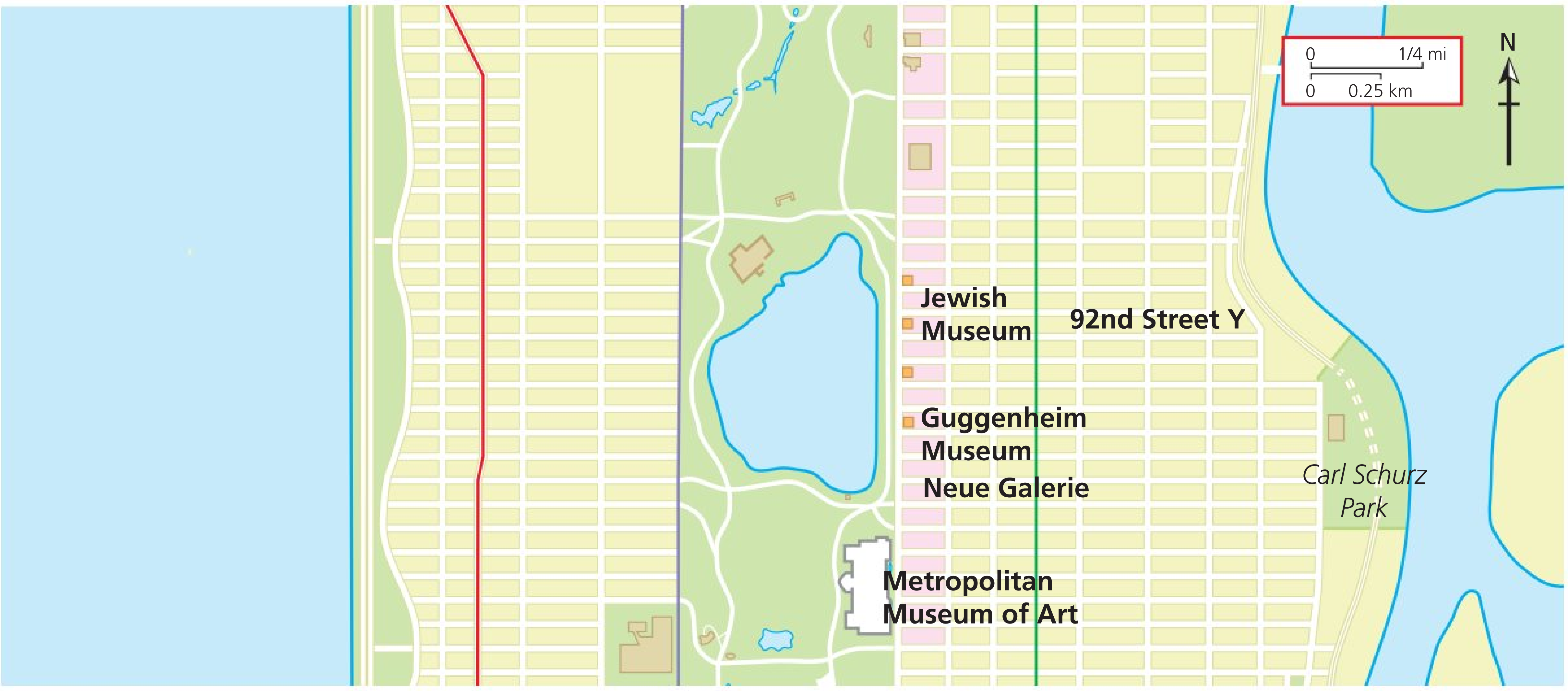
- about the concept of a vector and a scalar
- about different ways of representing vectors and how to add, subtract and multiply vectors by a scalar
- about the resultant of two or more vectors
- how to identify when vectors are parallel
- how to find the magnitude of a vector
- how to find a unit vector in a given direction
- about position vectors and displacement vectors
- how to find the vector equation of a line in two and three dimensions and how to convert to parametric form
- how to determine whether two lines intersect and find the point of intersection
- how to model linear motion with constant velocity in two and three dimensions
- how to use the scalar product to find the angle between two vectors
- how to identify when two vectors are perpendicular
- how to find the angle between two lines
- how to use the vector product to find perpendicular directions and areas
- how to find components of vectors in given directions.

CONCEPTS

The following concepts will be addressed in this chapter:

- The properties of shapes are highly dependent on the dimensions they occupy in **space**.
- Vectors allow us to **quantify** position, **change** of position (movement) and force in two- and three-dimensional **space**.

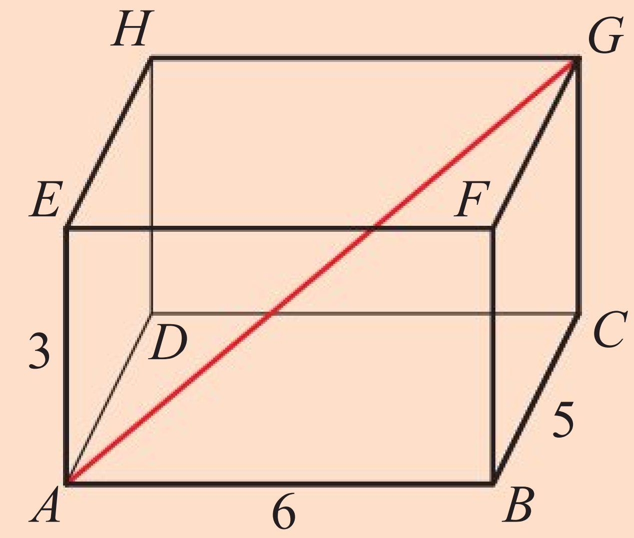
Figure 2.1 What information do you need to get from one place to another?



PRIOR KNOWLEDGE

Before starting this chapter, you should already be able to complete the following:

1. In the cuboid on the right, find the length AG.
2. Find the equation of straight line through the points $(4, 3)$ and $(-1, 5)$.
3. Four points have coordinates $A(3, 2)$, $B(-1, 5)$, $C(1, 6)$ and $D(9, k)$. Find the value of k for which AB and CD are parallel.
4. Solve the simultaneous equations
$$\begin{cases} 3x - 2y = 14 \\ 4x + 5y = 11 \end{cases}$$



You have probably met the distinction between scalar and vector quantities in physics. Scalar quantities, such as mass or time, can be described using a single number. Vector quantities need more than one piece of information to describe them. For example, velocity is described by its direction and magnitude (speed).

In mathematics we use vectors to describe positions of points and displacements between them. Some of this chapter is concerned with using vectors to solve geometrical problems, but there is also a focus on using vectors to describe the motion of object – an area of applied mathematics called kinematics.

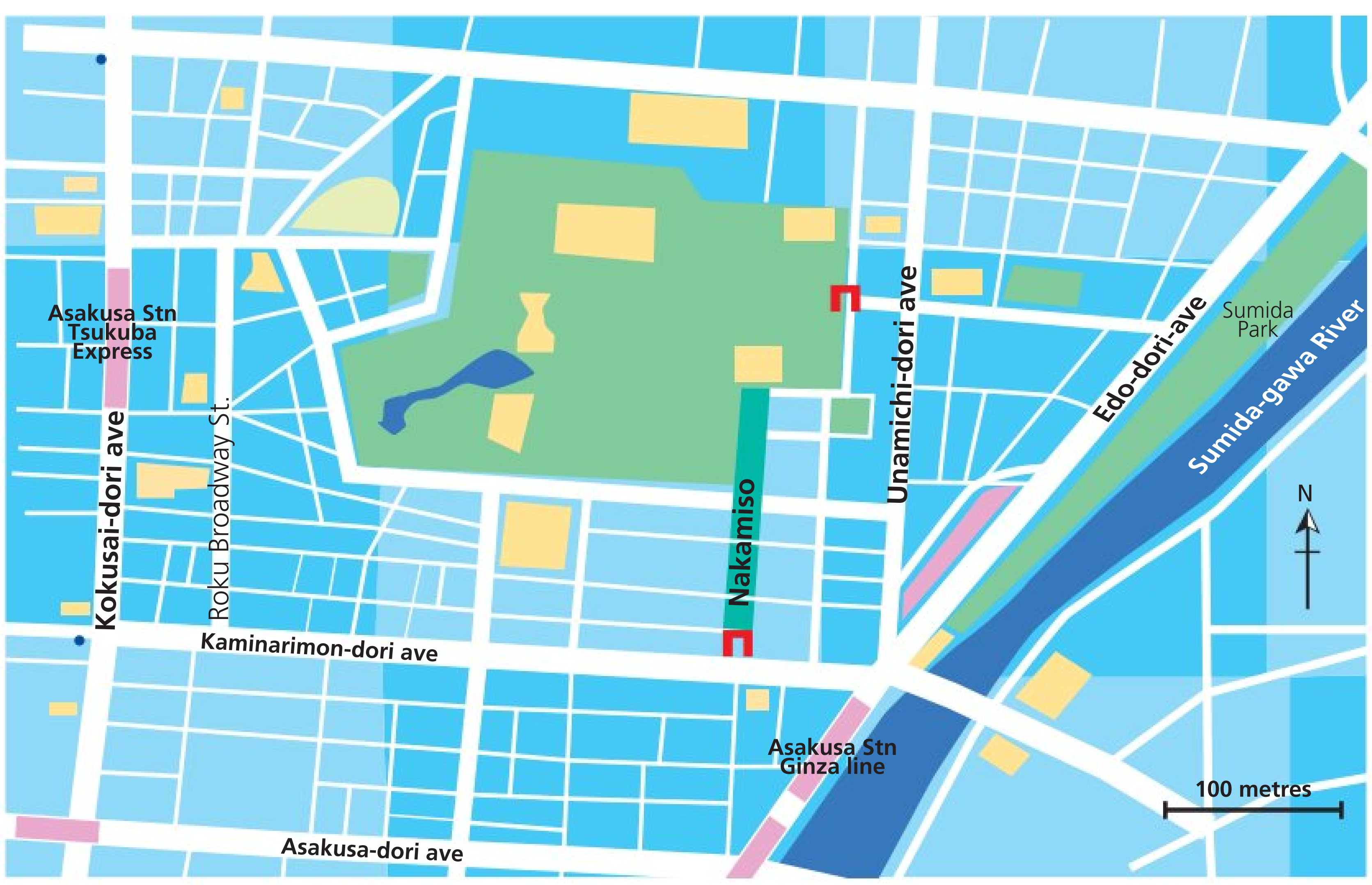
Vector equations can describe lines in two- and three-dimensional space. Vector methods enable us to use calculations to determine properties of shapes, such as angles, lengths and areas, in situations which may be difficult to visualize and solve geometrically.

Starter Activity

Look at the maps in Figure 2.1. In small groups, discuss how you would best give directions from getting from one marked location to another (for example, from the Metropolitan Museum of Art to 92nd Street Y).

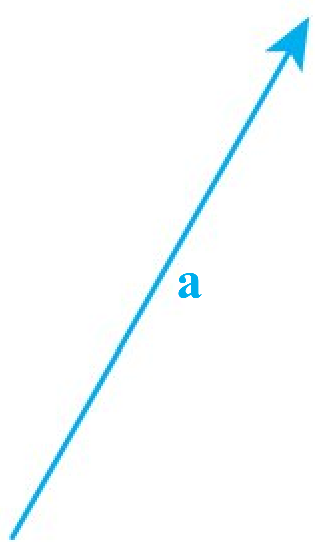
Now look at this problem:

Find the size of the angle between two diagonals of a cube.



You are the Researcher

It turns out that the given definition of scalars and vectors is slightly simplified. A more formal definition is based on an area of mathematics called tensor analysis, which looks at how quantities change when the frame of reference is rotated.



2A Introduction to vectors

Concept of a vector and a scalar

A **vector** is a quantity that has both magnitude and direction, for example force or displacement. This can be represented in several different ways, either graphically or using numbers. The most useful representation depends on the precise application, but you will often need to switch between different representations within the same problem.

A **scalar** is a quantity that has magnitude but no direction, for example time or energy.

Representation of vectors

A vector is labelled using either a bold lower case letter, for example **a**, or an underlined lower case letter $\underline{a}$.

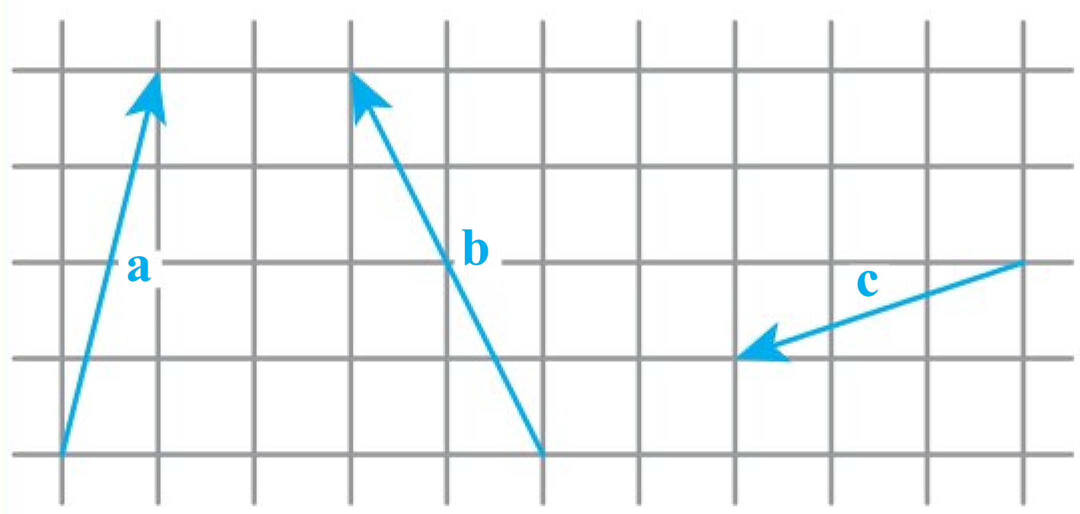
The simplest way to represent a vector is as a directed line segment, with the arrow showing the direction and the length representing the magnitude, as shown in the margin.

You will see that some problems can be solved using this diagrammatic representation, but sometimes you will also want to do numerical calculations. It that case, it may be useful to represent a vector using its **components**.

In two dimensions, you can represent any vector by two numbers. These horizontal and vertical components give the number of units in the two directions required to get from the 'tail' to the 'head' of the arrow. The components are written as a **column vector**: for example, $\begin{pmatrix} 3 \\ 2 \end{pmatrix}$ means 3 units to the right and 2 units up.

WORKED EXAMPLE 2.1

Write the following as column vectors (each grid space represents one unit).



The line labelled **a** goes 1 unit to the right and 4 units up

$$\mathbf{a} = \begin{pmatrix} 1 \\ 4 \end{pmatrix}$$

The line labelled **b** goes two units to the left and four units up

$$\mathbf{b} = \begin{pmatrix} -2 \\ 4 \end{pmatrix}$$

The line labelled **c** goes three units to the left and one unit down

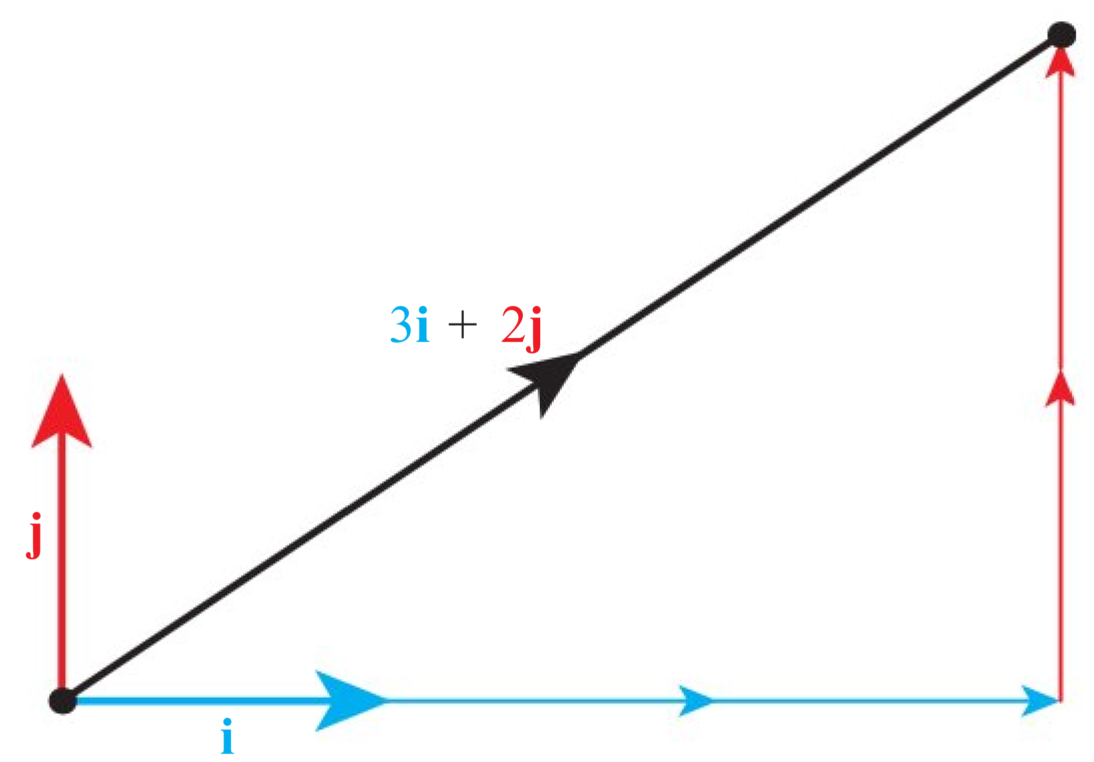
$$\mathbf{c} = \begin{pmatrix} -3 \\ -1 \end{pmatrix}$$

Tip

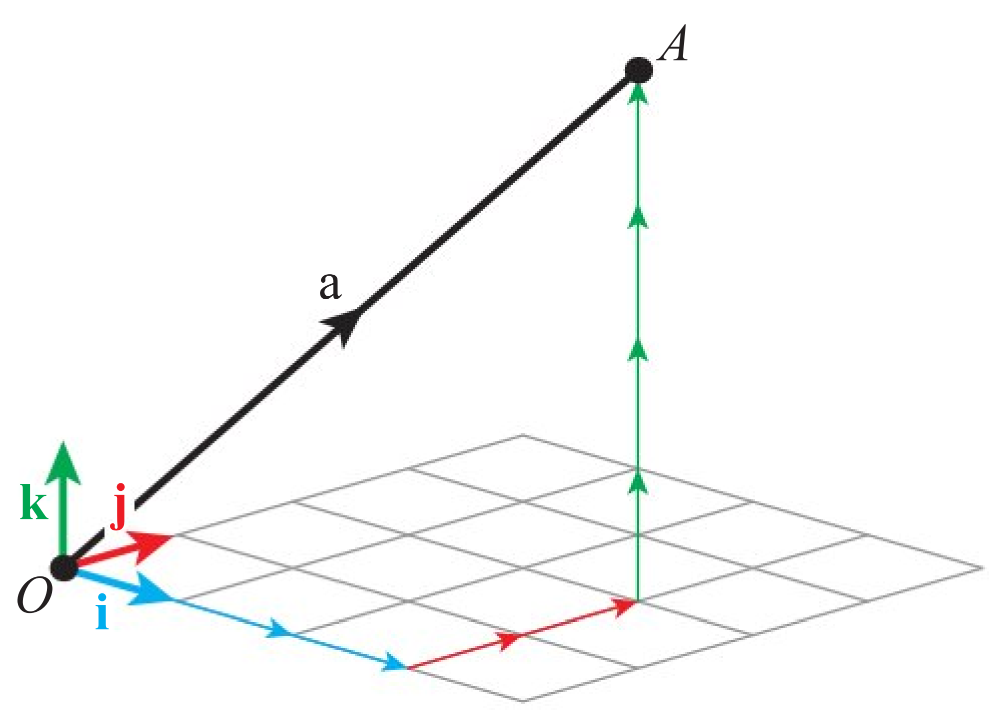
We often denote a vector by a single letter, just like variables in algebra. In printed text the letter is usually bold; when writing you should underline it (for example, $\underline{a}$) or use an arrow (for example, $\vec{a}$) to distinguish between vectors and scalars.

Base vectors

Another way to write a vector in components is to use **base vectors**, denoted **i** and **j** in two dimensions. These are vectors of length 1 unit in the directions 'right' and 'up'. For example, the vector $\begin{pmatrix} 3 \\ 2 \end{pmatrix}$ can be written $3\mathbf{i} + 2\mathbf{j}$.



This approach can be extended to three dimensions. We need three base vectors, called **i**, **j**, **k**, all perpendicular to each other.



$$\mathbf{a} = \begin{pmatrix} 3 \\ 2 \\ 4 \end{pmatrix} = 3\mathbf{i} + 2\mathbf{j} + 4\mathbf{k}$$

Tip

The base vectors written as column vectors are

$$\mathbf{i} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \mathbf{j} = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \text{ and } \mathbf{k} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

Tip

You must be familiar with both base vector and column vector notation, as both are frequently used. When you write your answers, you can use whichever notation you prefer.

WORKED EXAMPLE 2.2

a Write the following using base vectors.

$$\mathbf{a} = \begin{pmatrix} 3 \\ -2 \end{pmatrix}, \quad \mathbf{b} = \begin{pmatrix} -1 \\ 4 \\ 5 \end{pmatrix}, \quad \mathbf{c} = \begin{pmatrix} 4 \\ 0 \\ -2 \end{pmatrix}$$

b Write the following as column vectors with three components.

$$\mathbf{d} = 4\mathbf{i} + \mathbf{j} - 2\mathbf{k}, \quad \mathbf{e} = 2\mathbf{j} - \mathbf{k}, \quad \mathbf{f} = 3\mathbf{k} - \mathbf{i}$$

a

The components are the coefficients of **i** and **j**

$\mathbf{a} = 3\mathbf{i} - 2\mathbf{j}$

The components are coefficients of **i**, **j** and **k**

$\mathbf{b} = -\mathbf{i} + 4\mathbf{j} + 5\mathbf{k}$

Notice that the **j**-component is zero

$\mathbf{c} = 4\mathbf{i} - 2\mathbf{k}$

b

The coefficients are the components of the vectors

$$\mathbf{d} = \begin{pmatrix} 4 \\ 1 \\ -2 \end{pmatrix}$$

Notice that the **i**-component is missing

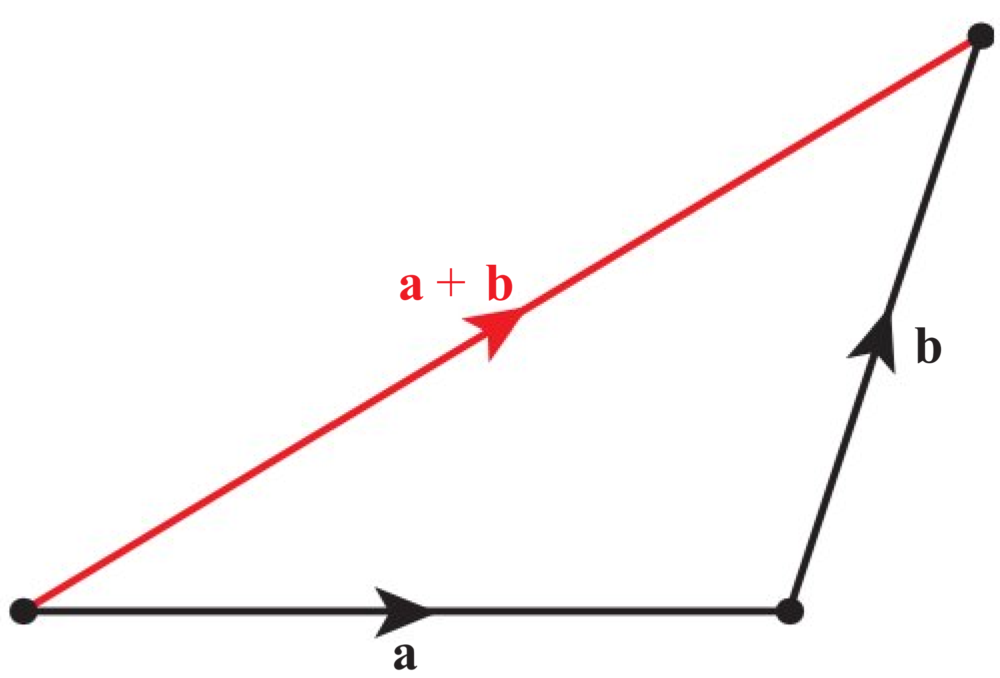
$$\mathbf{e} = \begin{pmatrix} 0 \\ 2 \\ -1 \end{pmatrix}$$

Be careful – the components are not in the correct order!

$$\mathbf{f} = \begin{pmatrix} -1 \\ 0 \\ 3 \end{pmatrix}$$

Addition and subtraction of vectors

On a diagram, vectors are added by joining the starting point of the second vector to the end point of the first. In component form, you just add the corresponding components.



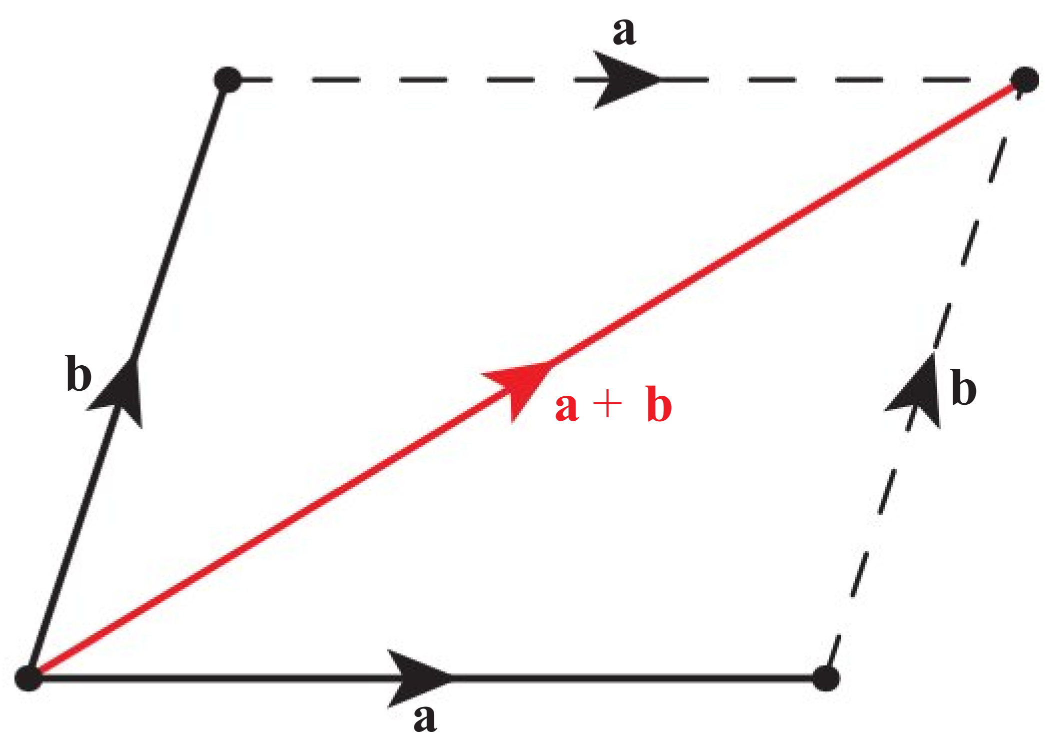
$$\begin{pmatrix} 3 \\ 1 \\ -2 \end{pmatrix} + \begin{pmatrix} 2 \\ 2 \\ 6 \end{pmatrix} = \begin{pmatrix} 5 \\ 3 \\ 4 \end{pmatrix}$$

The vector that results from the sum of two or more vectors is known as the **resultant vector**. So, in the above diagram, the resultant of the vectors $\begin{pmatrix} 3 \\ 1 \\ -2 \end{pmatrix}$ and $\begin{pmatrix} 2 \\ 2 \\ 6 \end{pmatrix}$ is $\begin{pmatrix} 5 \\ 3 \\ 4 \end{pmatrix}$.

Another way of visualizing vector addition is as a diagonal of the parallelogram formed by the two vectors.

Tip

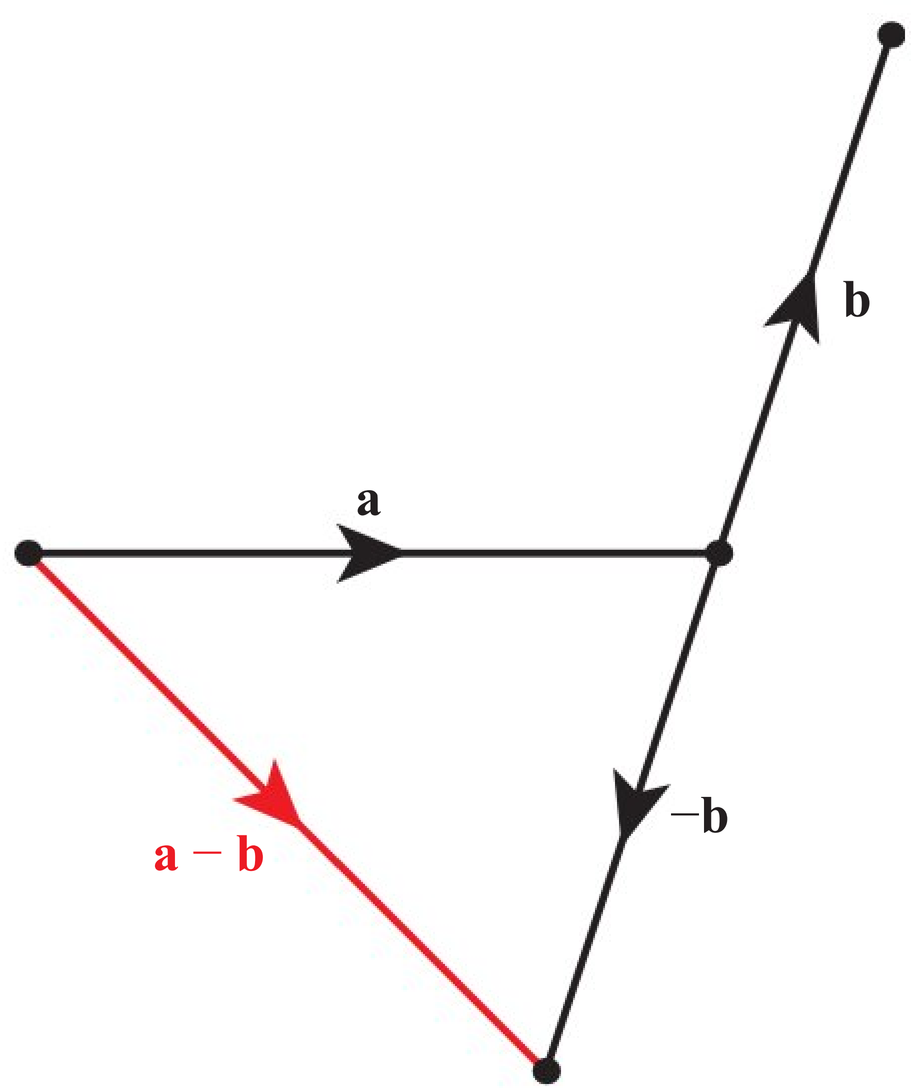
Equal vectors have the same magnitude and direction; they don't need to start or end at the same point.



To subtract vectors, reverse the direction of the second vector and add it to the first. Notice that subtracting a vector is the same as adding its negative. For example, if $\mathbf{a} = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$, then $-\mathbf{a} = \begin{pmatrix} -1 \\ -2 \end{pmatrix}$.

Tip

Subtracting a vector from itself gives the zero vector: $\mathbf{a} - \mathbf{a} = \mathbf{0}$.



$$\begin{pmatrix} 5 \\ 1 \\ -2 \end{pmatrix} - \begin{pmatrix} 3 \\ 3 \\ -3 \end{pmatrix} = \begin{pmatrix} 5 \\ 1 \\ -2 \end{pmatrix} + \begin{pmatrix} -3 \\ -3 \\ 3 \end{pmatrix} = \begin{pmatrix} 2 \\ -2 \\ 1 \end{pmatrix}$$

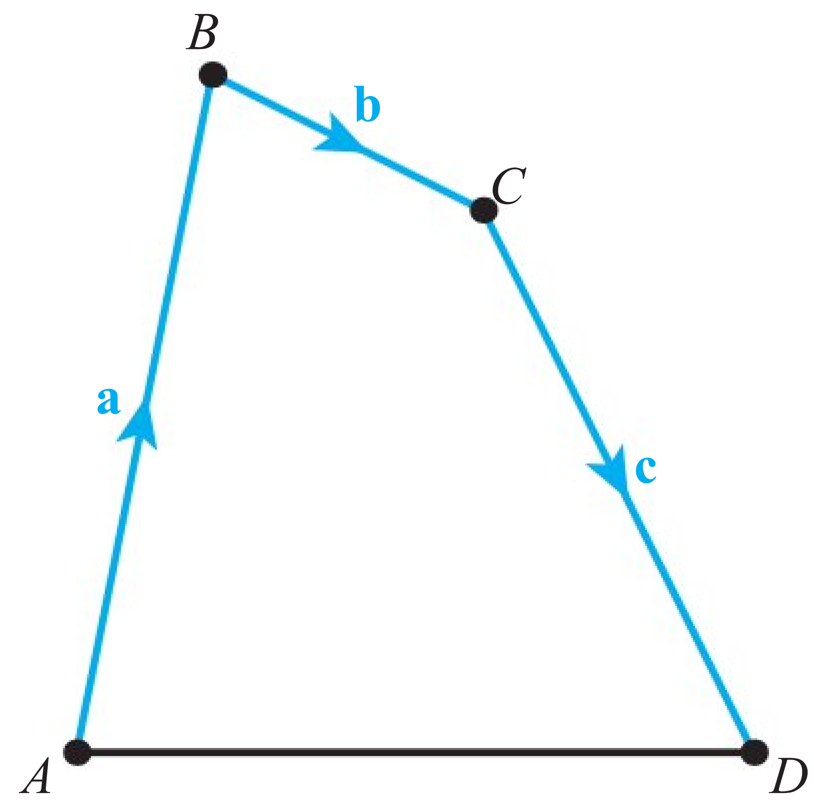
Another way of writing vectors is to label the end-points with capital letters. For example, $\overrightarrow{AB}$ is the vector in the direction from A to B, with magnitude equal to the length AB.

WORKED EXAMPLE 2.3

Express the following in terms of vectors **a**, **b** and **c**.

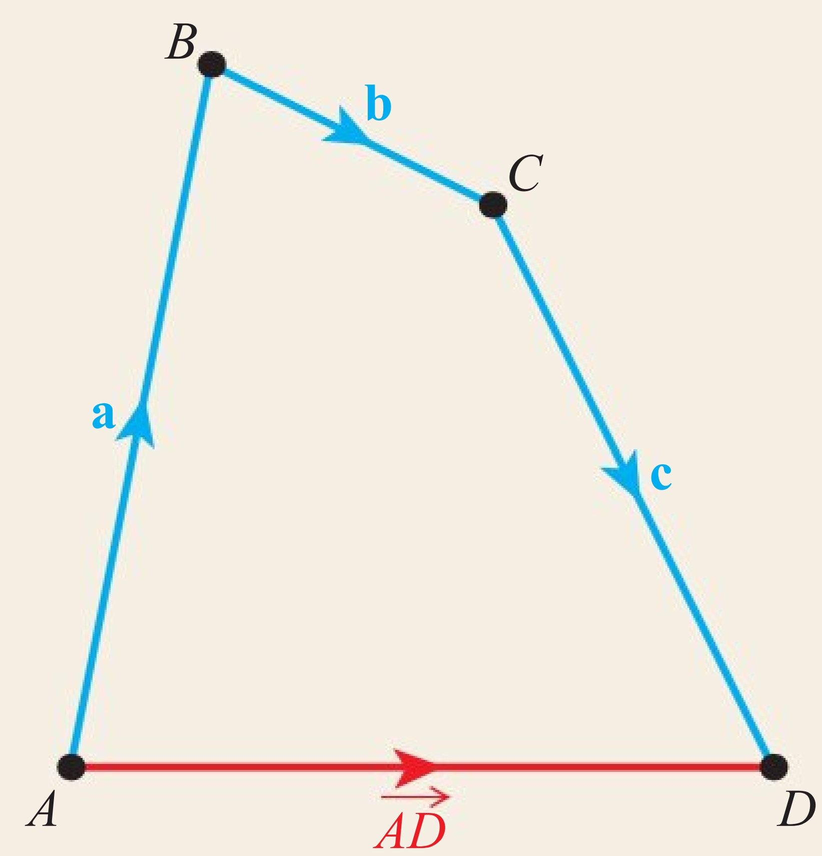
a $\overrightarrow{AD}$

b $\overrightarrow{DB}$



The vectors **a**, **b** and **c** are joined 'head to tail', starting at A and finishing at D

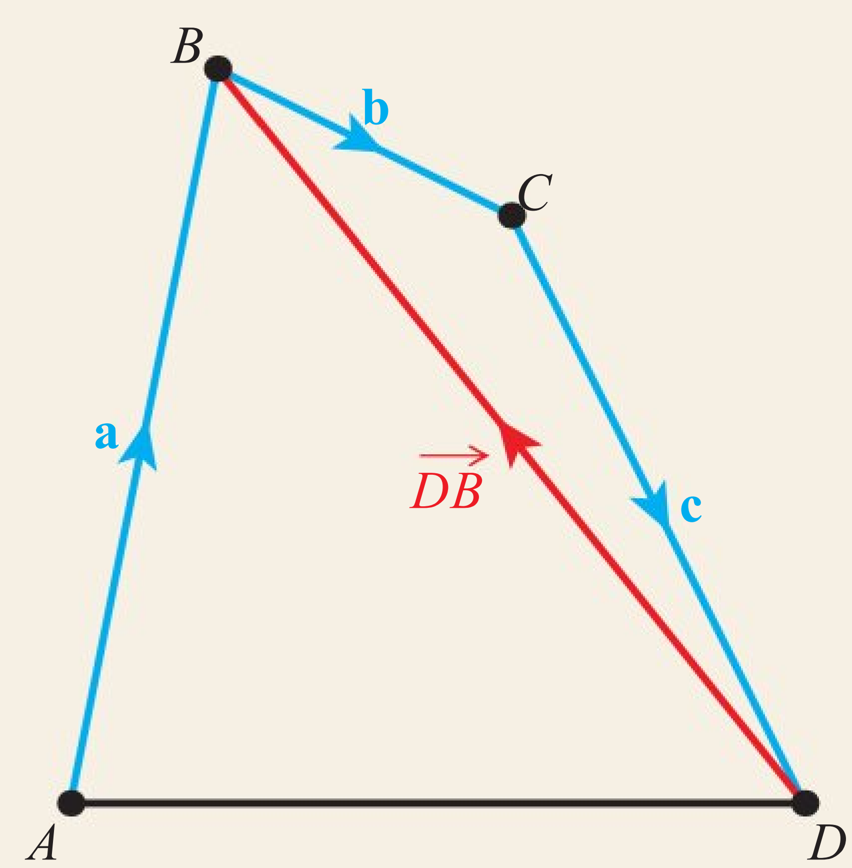
a



$\overrightarrow{AD} = \mathbf{a} + \mathbf{b} + \mathbf{c}$

You can see that $\mathbf{b} + \mathbf{c} = \overrightarrow{BD}$, and $\overrightarrow{DB}$ is the negative of this

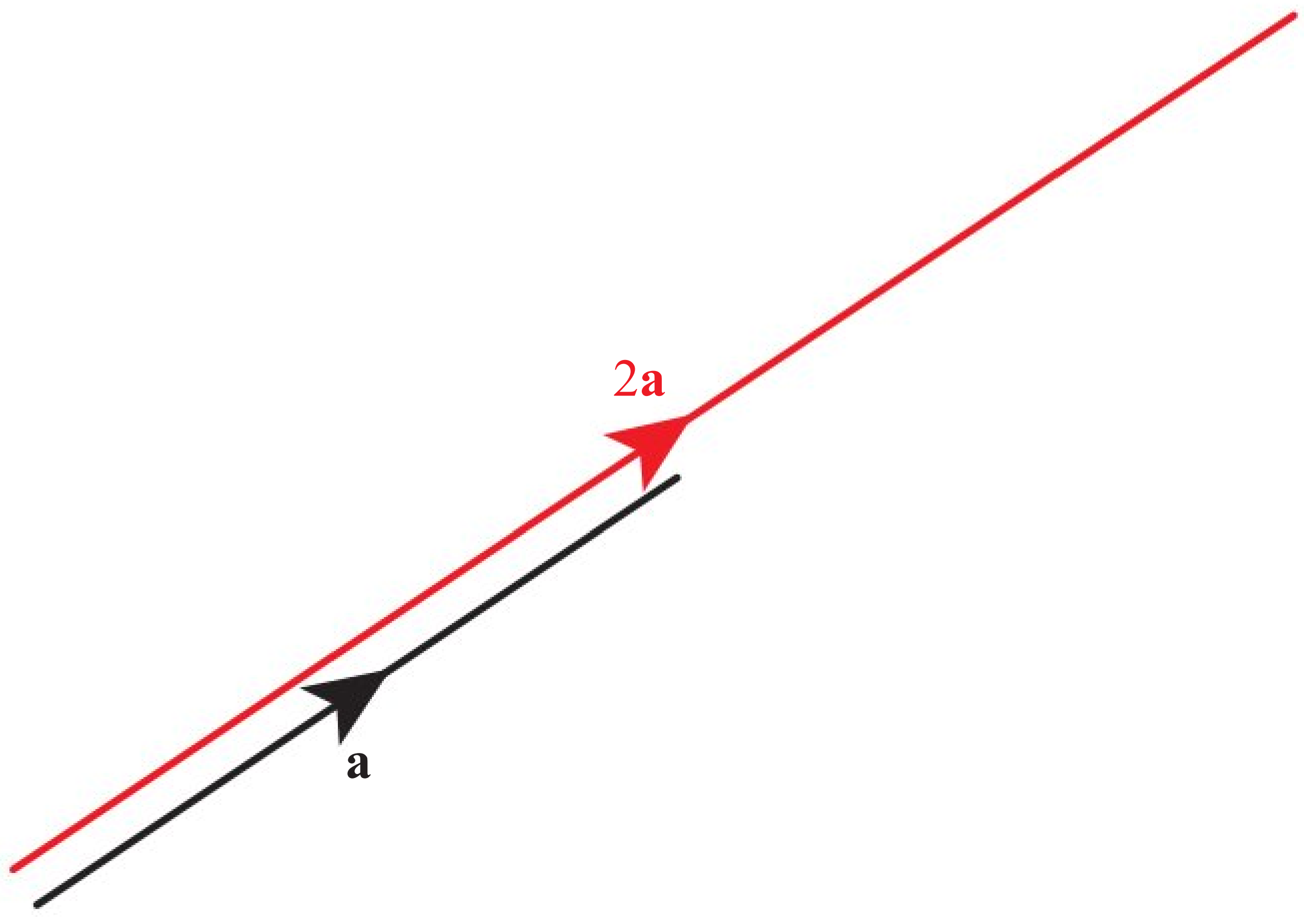
b



$$
\begin{aligned}
\overrightarrow{DB} &= -\overrightarrow{BD} \\
&= -(\mathbf{b} + \mathbf{c}) \\
&= -\mathbf{b} - \mathbf{c}
\end{aligned}
$$

Scalar multiplication and parallel vectors

Multiplying by a scalar changes the magnitude (length) of the vector, leaving the direction the same. In component form, each component is multiplied by the scalar.



$$2\begin{pmatrix} 3 \\ -5 \\ 0 \end{pmatrix} = \begin{pmatrix} 6 \\ -10 \\ 0 \end{pmatrix}$$

Tip

Multiplying by a negative scalar reverses the direction.

WORKED EXAMPLE 2.4

Given vectors $\mathbf{a} = \begin{pmatrix} 1 \\ 2 \\ 7 \end{pmatrix}$ and $\mathbf{b} = \begin{pmatrix} -3 \\ 4 \\ 2 \end{pmatrix}$, find $2\mathbf{a} - 3\mathbf{b}$.

$$2\mathbf{a} - 3\mathbf{b} = 2\begin{pmatrix} 1 \\ 2 \\ 7 \end{pmatrix} - 3\begin{pmatrix} -3 \\ 4 \\ 2 \end{pmatrix}$$

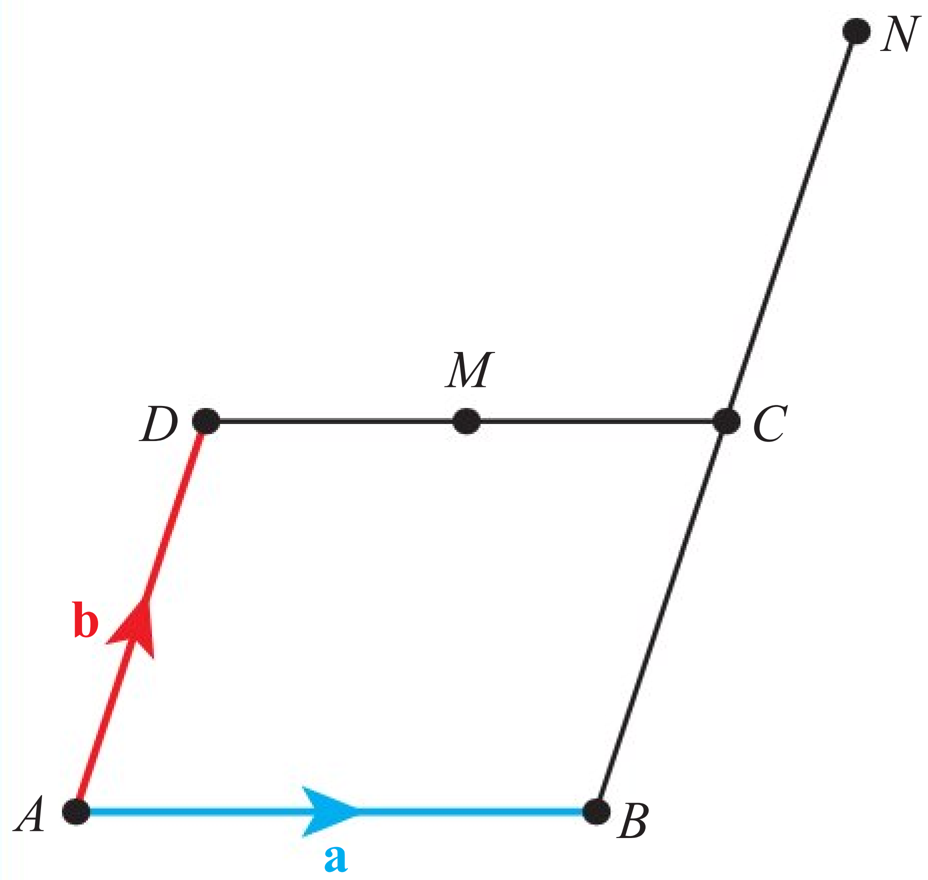
Multiply the scalar by each element of the relevant vector then subtract corresponding components of the vectors

$$= \begin{pmatrix} 2 \\ 4 \\ 14 \end{pmatrix} - \begin{pmatrix} -9 \\ 12 \\ 6 \end{pmatrix}$$

$$= \begin{pmatrix} 11 \\ -8 \\ 8 \end{pmatrix}$$

WORKED EXAMPLE 2.5

The diagram shows a parallelogram $ABCD$. Let $\overrightarrow{AB} = \mathbf{a}$ and $\overrightarrow{AD} = \mathbf{b}$. M is the midpoint of CD and N is the point on the extended line BC such that $CN = BC$.



a Express vectors $\overrightarrow{CM}$ and $\overrightarrow{BN}$ in terms of **a** and **b**.

b Express $\overrightarrow{MN}$ in terms of **a** and **b**.

$\overrightarrow{CM}$ has the same direction as $\overrightarrow{BA}$, but half the length

a $\overrightarrow{CM} = \frac{1}{2}\overrightarrow{BA}$

$\overrightarrow{BA} = -\overrightarrow{AB}$

$= -\frac{1}{2}\mathbf{a}$

$\overrightarrow{BN}$ is twice $\overrightarrow{BC}$…

$\overrightarrow{BN} = 2\overrightarrow{BC}$

… which is the same as $\overrightarrow{AD}$

$= 2\mathbf{b}$

b $\overrightarrow{MN} = \overrightarrow{MC} + \overrightarrow{CN}$

We can get from M to N by going from M to C and then from C to N

$= \frac{1}{2}\mathbf{a} + \mathbf{b}$

Two vectors are parallel if they have the same (or opposite) direction. This means that one is a scalar multiple of the other.

Tip

The scalar can be positive or negative.

KEY POINT 2.1

If vectors **a** and **b** are parallel, we can write $\mathbf{b} = t\mathbf{a}$ for some scalar t.

CONCEPTS – QUANTITY

We can do much more powerful mathematics with **quantities** than with drawings. Although the concept of parallel lines is a geometric concept, it is useful to quantify it in order to use it in calculations and equations. Key Point 2.1 gives us an equation to express the geometric statement 'two lines are parallel'.

WORKED EXAMPLE 2.6

Given vectors $\mathbf{a} = \begin{pmatrix} 1 \\ 2 \\ 7 \end{pmatrix}$ and $\mathbf{c} = \begin{pmatrix} -2 \\ p \\ q \end{pmatrix}$, find the values of p and q such that $\mathbf{c}$ is parallel to $\mathbf{a}$.

If two vectors are parallel, we can write $v_1 = tv_2$

$\mathbf{c} = t\mathbf{a}$ for some scalar t.

$$\begin{pmatrix} -2 \\ p \\ q \end{pmatrix} = t\begin{pmatrix} 1 \\ 2 \\ 7 \end{pmatrix} = \begin{pmatrix} t \\ 2t \\ 7t \end{pmatrix}$$

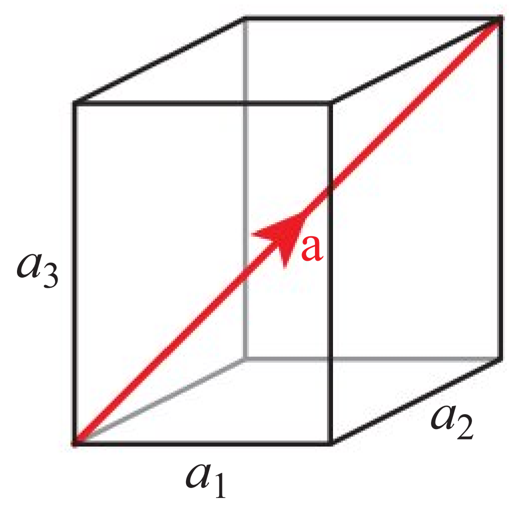
If two vectors are equal, then all their components are equal

$$\begin{cases} -2 = t \\ p = 2t \\ q = 7t \end{cases}$$

$t = -2,\ p = -4,\ q = -14$

Magnitude of a vector and unit vectors

The magnitude of a vector can be found from its components, using Pythagoras' theorem. The symbol for the magnitude is the same as the symbol for absolute value (modulus).



KEY POINT 2.2

The magnitude of a vector $\mathbf{a} = \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix}$ is $|\mathbf{a}| = \sqrt{a_1^2 + a_2^2 + a_3^2}$.

In some applications it is useful to make vectors have length 1. These are called **unit vectors**. The base vectors **i**, **j**, **k** are examples of unit vectors, but you can create a unit vector in any direction. You can take any vector in that direction and divide it by its magnitude; this will keep the direction the same but change the magnitude to 1.

KEY POINT 2.3

The unit vector in the same direction as vector **a** is $\frac{1}{|\mathbf{a}|}\mathbf{a}$.

Tip

Note that there are two possible answers to part **b**, as you could multiply **a** by $-\frac{1}{3}$ instead of $\frac{1}{3}$.

WORKED EXAMPLE 2.7

a Find the magnitude of the vector $\mathbf{a} = \begin{pmatrix} 2 \\ -2 \\ 1 \end{pmatrix}$.

b Find a unit vector parallel to **a**.

Use Pythagoras's theorem

a $|\mathbf{a}| = \sqrt{2^2 + 2^2 + 1^2} = 3$

Divide by the magnitude of **a** to create a vector of length 1

b Unit vector is $\frac{1}{3}\mathbf{a} = \begin{pmatrix} \frac{2}{3} \\ -\frac{2}{3} \\ \frac{1}{3} \end{pmatrix}$

Position and displacement vectors

In Section 2C, you will find out how to use vectors in kinematics.

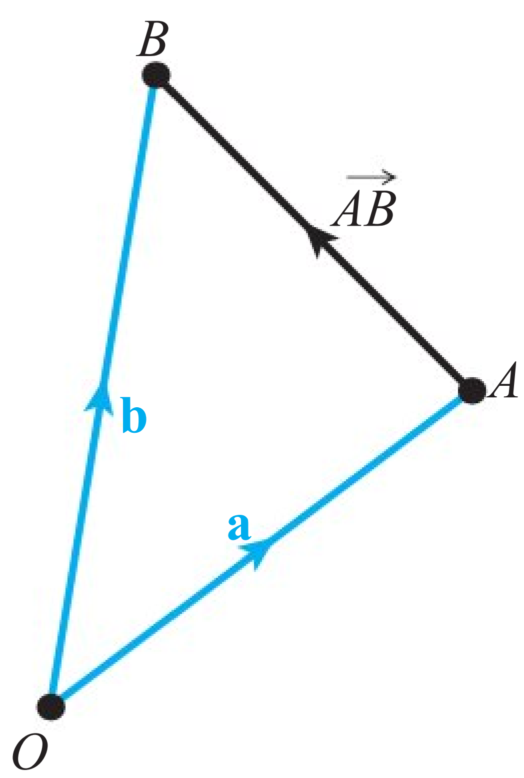
Vectors can be used to represent many different quantities, such as force, velocity or acceleration. They always obey the same algebraic rules you learnt in the previous section. One of the most common applications of vectors in pure mathematics is to represent positions of points in space.

You already know how to use coordinates to represent the position of a point, measured along the coordinate axes from the origin O. The vector from the origin to a point A is called the **position vector** of A.

KEY POINT 2.4

- The position vector of a point A is the vector $\mathbf{a} = \overrightarrow{OA}$, where O is the origin.
- The components of $\mathbf{a}$ are the coordinates of A.

Position vectors describe positions of points relative to the origin, but you sometimes want to know the position of one point relative to another. This is described by a **displacement vector**.



KEY POINT 2.5

If points A and B have position vectors $\mathbf{a}$ and $\mathbf{b}$, then the displacement vector from A to B is

$$\overrightarrow{AB} = \overrightarrow{OB} - \overrightarrow{OA} = \mathbf{b} - \mathbf{a}$$

Tip

You can think of the equation $\overrightarrow{AB} = \overrightarrow{OB} - \overrightarrow{OA}$ as saying: 'to get from A to B, go from A to O and then from O to B'.

Tip

The displacement vectors $\overrightarrow{AB}$ and $\overrightarrow{BA}$ have equal magnitude but opposite direction.

WORKED EXAMPLE 2.8

Points A and B have coordinates (3, −1, 2) and (5, 0, 3). Find the displacement vector $\overrightarrow{AB}$.

The components of the position vectors are the coordinates of the points

$$\mathbf{a} = \begin{pmatrix} 3 \\ -1 \\ 2 \end{pmatrix}, \quad \mathbf{b} = \begin{pmatrix} 5 \\ 0 \\ 3 \end{pmatrix}$$

$$\overrightarrow{AB} = \mathbf{b} - \mathbf{a}$$

The displacement is the difference between the position vectors (end − start)

$$= \begin{pmatrix} 5 \\ 0 \\ 3 \end{pmatrix} - \begin{pmatrix} 3 \\ -1 \\ 2 \end{pmatrix}$$

$$= \begin{pmatrix} 2 \\ 1 \\ 1 \end{pmatrix}$$

The distance between two points is equal to the magnitude of the displacement vector.

KEY POINT 2.6

The distance between the points A and B with position vectors $\mathbf{a}$ and $\mathbf{b}$ is

$$AB = \left|\overrightarrow{AB}\right| = |\mathbf{b} - \mathbf{a}|$$

WORKED EXAMPLE 2.9

Points A and B have position vectors $\mathbf{a} = 3\mathbf{i} - \mathbf{j} - 4\mathbf{k}$ and $\mathbf{b} = \mathbf{i} - 2\mathbf{j} + \mathbf{k}$. Find the exact distance AB.

First find the displacement vector

$$\begin{aligned}\overrightarrow{AB} &= \mathbf{b} - \mathbf{a}\\ &= (\mathbf{i} - 2\mathbf{j} + \mathbf{k}) - (3\mathbf{i} - \mathbf{j} - 4\mathbf{k})\\ &= -2\mathbf{i} - \mathbf{j} + 5\mathbf{k}\end{aligned}$$

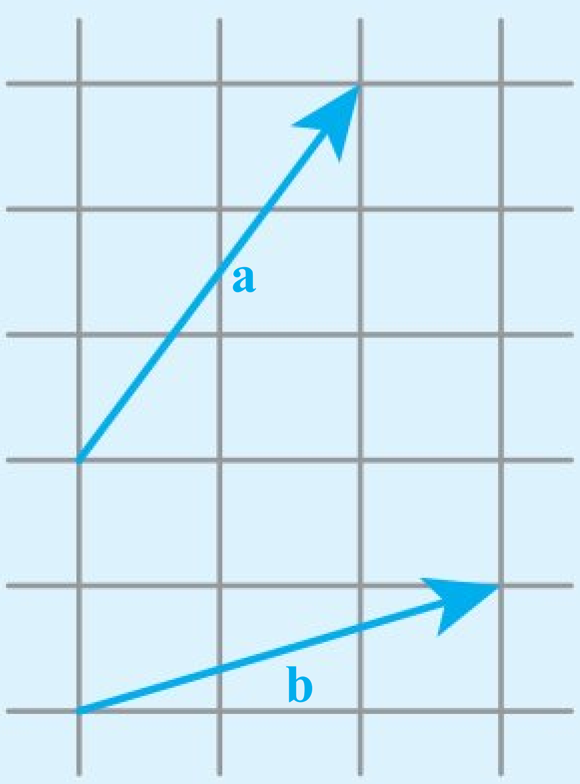
The distance is the magnitude of the displacement vector

$$\begin{aligned}\left|\overrightarrow{AB}\right| &= \sqrt{4 + 1 + 25}\\ &= \sqrt{30}\end{aligned}$$

Exercise 2A

For questions 1 to 4, use the method demonstrated in Worked Example 2.1 to write vectors **a** and **b** as column vectors.

1



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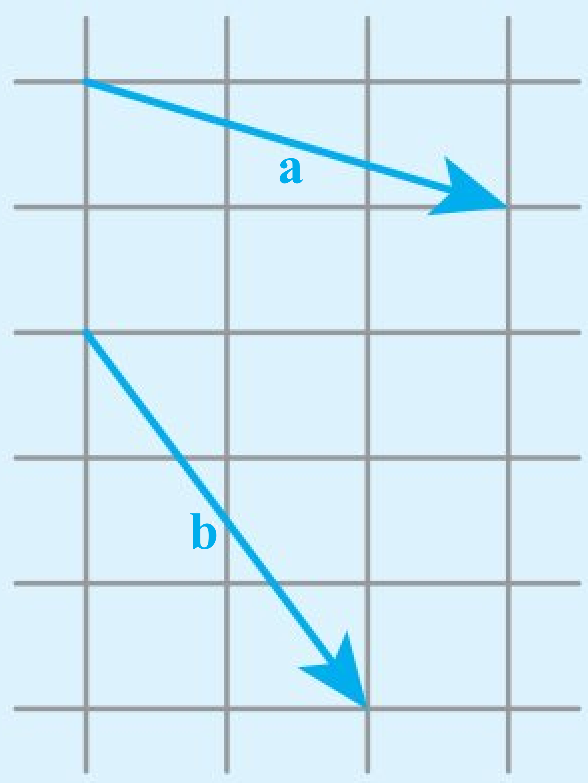

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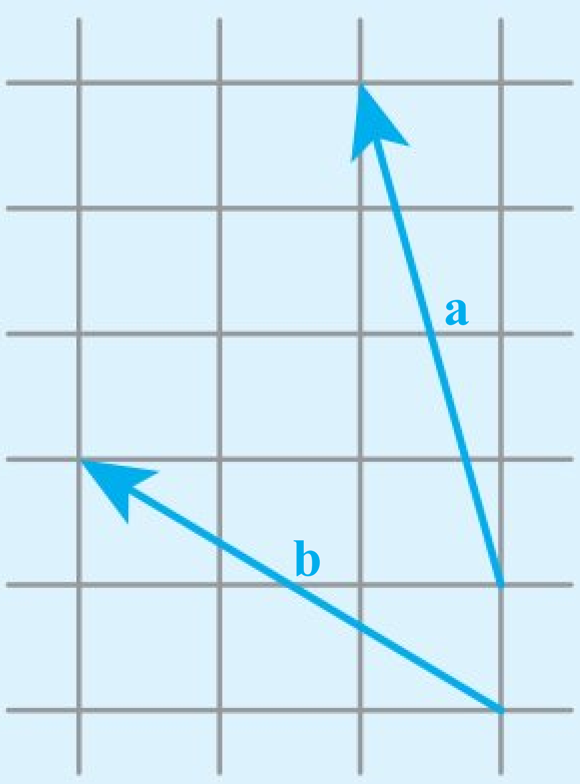

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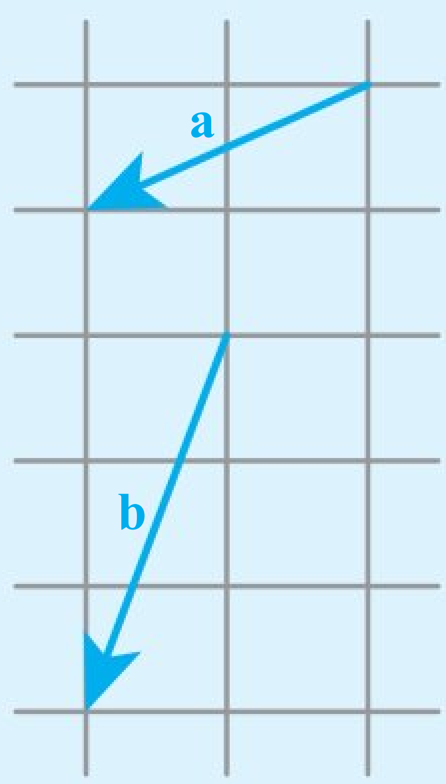


For questions 5 to 7, use the method demonstrated in Worked Example 2.2a to write the following using base vectors $\mathbf{i}$, $\mathbf{j}$, $\mathbf{k}$.

5 **a** $\begin{pmatrix}1\\2\\3\end{pmatrix}$ **6** **a** $\begin{pmatrix}-3\\1\\0\end{pmatrix}$ **7** **a** $\begin{pmatrix}3\\0\\0\end{pmatrix}$

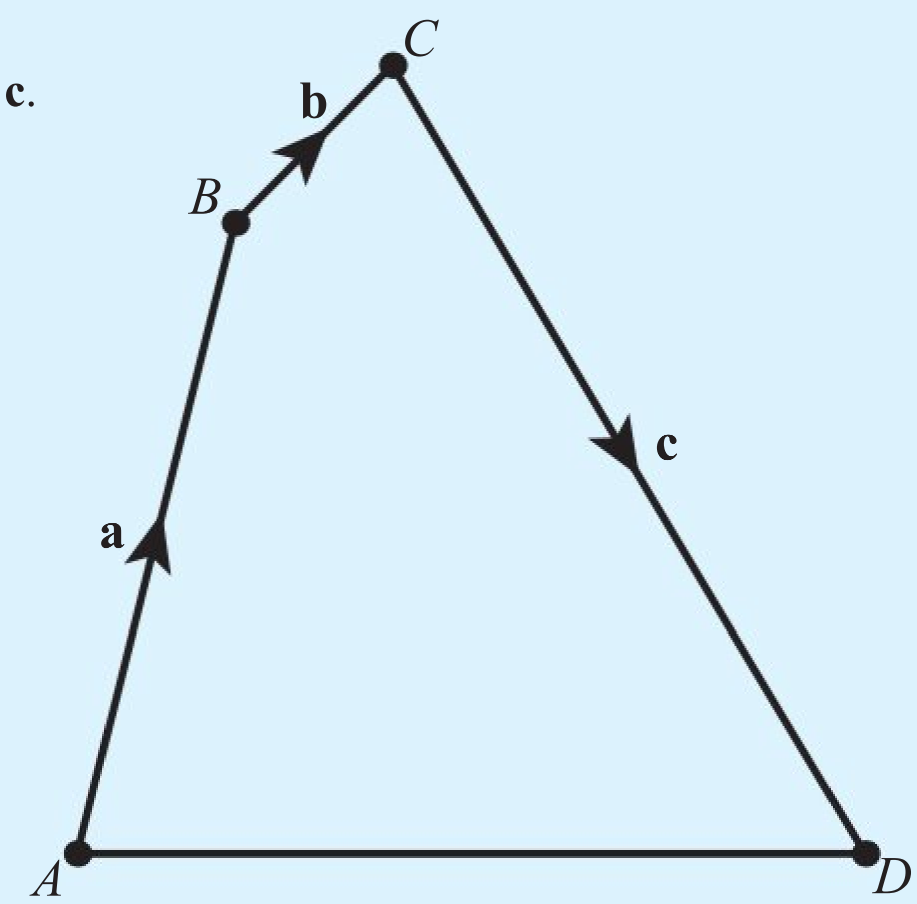
b $\begin{pmatrix}4\\5\\6\end{pmatrix}$ **b** $\begin{pmatrix}2\\-2\\0\end{pmatrix}$ **b** $\begin{pmatrix}0\\-5\\0\end{pmatrix}$

For questions 8 to 10, use the method demonstrated in Worked Example 2.2b to write the following as three-dimensional column vectors.

8 **a** $2\mathbf{i}-3\mathbf{j}+\mathbf{k}$ **9** **a** $\mathbf{i}+3\mathbf{k}$ **10** **a** $4\mathbf{j}-\mathbf{i}-2\mathbf{k}$

b $3\mathbf{i}+\mathbf{j}-4\mathbf{k}$ **b** $2\mathbf{j}-\mathbf{k}$ **b** $\mathbf{k}-3\mathbf{i}$

For questions 11 and 12, use the method demonstrated in Worked Example 2.3 to express the following vectors in terms of **a**, **b** and **c**.



11 **a** $\overrightarrow{AC}$ **12** **a** $\overrightarrow{DA}$

b $\overrightarrow{BD}$ **b** $\overrightarrow{DB}$

For questions 13 to 15, you are given vectors $\mathbf{a}=\begin{pmatrix}1\\2\\3\end{pmatrix}$ and $\mathbf{b}=\begin{pmatrix}4\\-5\\6\end{pmatrix}$.

Use the method demonstrated in Worked Example 2.4 to write the following as column vectors.

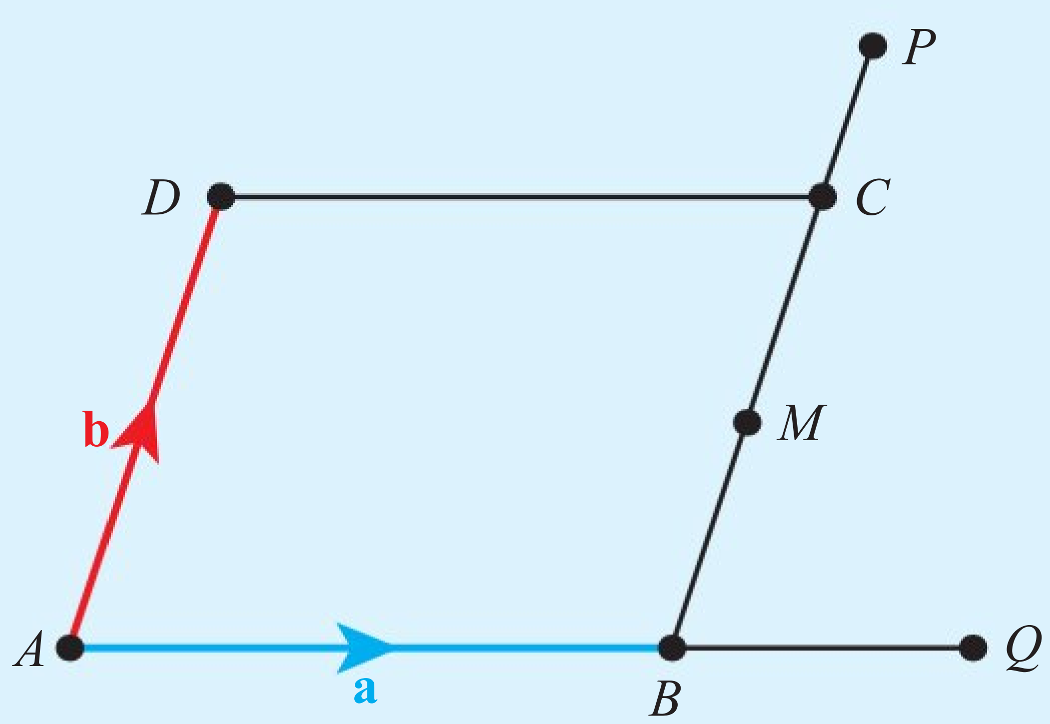
13 **a** $\mathbf{a}-\mathbf{b}$ **14** **a** $3\mathbf{a}+2\mathbf{b}$

b $\mathbf{b}-\mathbf{a}$ **b** $2\mathbf{a}+5\mathbf{b}$

15 **a** $3\mathbf{b}-\mathbf{a}$

b $\mathbf{b}-2\mathbf{a}$

For questions 16 to 18, $ABCD$ is a parallelogram, with $\overrightarrow{AB}=\overrightarrow{DC}=\mathbf{a}$ and $\overrightarrow{AD}=\overrightarrow{BC}=\mathbf{b}$. M is the midpoint of BC, Q is the point on the extended line AB such that $BQ=\frac{1}{2}AB$ and P is the point on the extended line BC such that $CP=\frac{1}{3}BC$, as shown on the diagram.



Use the method demonstrated in Worked Example 2.5 to write the following vectors in terms of **a** and **b**.

16 **a** $\overrightarrow{AP}$ **17** **a** $\overrightarrow{QD}$ **18** **a** $\overrightarrow{DQ}$

b $\overrightarrow{AM}$ **b** $\overrightarrow{MQ}$ **b** $\overrightarrow{PQ}$

For questions 19 to 23, use the method demonstrated in Worked Example 2.6 to find the values of p and q such that the vectors **a** and **b** are parallel.

19 a $\mathbf{a} = \begin{pmatrix} 2 \\ 1 \\ 5 \end{pmatrix}$ and $\mathbf{b} = \begin{pmatrix} 6 \\ p \\ q \end{pmatrix}$

b $\mathbf{a} = \begin{pmatrix} 1 \\ 1 \\ 4 \end{pmatrix}$ and $\mathbf{b} = \begin{pmatrix} 4 \\ p \\ q \end{pmatrix}$

20 a $\mathbf{a} = \begin{pmatrix} 3 \\ 2 \\ -1 \end{pmatrix}$ and $\mathbf{b} = \begin{pmatrix} -9 \\ p \\ q \end{pmatrix}$

b $\mathbf{a} = \begin{pmatrix} 1 \\ -1 \\ 4 \end{pmatrix}$ and $\mathbf{b} = \begin{pmatrix} -2 \\ p \\ q \end{pmatrix}$

21 a $\mathbf{a} = \begin{pmatrix} 2 \\ p \\ 6 \end{pmatrix}$ and $\mathbf{b} = \begin{pmatrix} q \\ 2 \\ 3 \end{pmatrix}$

b $\mathbf{a} = \begin{pmatrix} -3 \\ p \\ 6 \end{pmatrix}$ and $\mathbf{b} = \begin{pmatrix} q \\ 15 \\ 2 \end{pmatrix}$

22 a $\mathbf{a} = \mathbf{i} - 2\mathbf{j} + 6\mathbf{k}$ and $\mathbf{b} = p\mathbf{i} + 6\mathbf{j} + q\mathbf{k}$

b $\mathbf{a} = -2\mathbf{i} + \mathbf{j} - 3\mathbf{k}$ and $\mathbf{b} = p\mathbf{i} + q\mathbf{j} + 6\mathbf{k}$

23 a $\mathbf{a} = 2\mathbf{i} + p\mathbf{j} + 5\mathbf{k}$ and $\mathbf{b} = -4\mathbf{i} + 4\mathbf{j} + q\mathbf{k}$

b $\mathbf{a} = p\mathbf{i} + 6\mathbf{j} + 9\mathbf{k}$ and $\mathbf{b} = \mathbf{i} - 2\mathbf{j} + q\mathbf{k}$

For questions 24 to 27, use the method demonstrated in Worked Example 2.7 to find the unit vector in the same direction as vector **a**.

24 a $\mathbf{a} = \begin{pmatrix} 1 \\ 2 \\ 2 \end{pmatrix}$

b $\mathbf{a} = \begin{pmatrix} 3 \\ 0 \\ 4 \end{pmatrix}$

25 a $\mathbf{a} = \begin{pmatrix} -1 \\ 1 \\ 2 \end{pmatrix}$

b $\mathbf{a} = \begin{pmatrix} 1 \\ -2 \\ 3 \end{pmatrix}$

26 a $\mathbf{a} = 3\mathbf{i} + \mathbf{j} - \mathbf{k}$

b $\mathbf{a} = \mathbf{i} - 2\mathbf{j} + \mathbf{k}$

27 a $\mathbf{a} = \mathbf{i} - 4\mathbf{j}$

b $\mathbf{a} = 2\mathbf{j} - 3\mathbf{k}$

For questions 28 to 30, points A, B and C have position vectors $\mathbf{a} = \begin{pmatrix} 5 \\ -2 \\ 3 \end{pmatrix}$, $\mathbf{b} = \begin{pmatrix} 7 \\ 1 \\ 12 \end{pmatrix}$ and $\mathbf{c} = \begin{pmatrix} -1 \\ 1 \\ 2 \end{pmatrix}$, respectively.

Use the method demonstrated in Worked Example 2.8 to find the given displacement vectors.

28 a $\overrightarrow{AB}$

b $\overrightarrow{AC}$

29 a $\overrightarrow{CB}$

b $\overrightarrow{CA}$

30 a $\overrightarrow{BA}$

b $\overrightarrow{BC}$

For questions 31 to 33, use the method demonstrated in Worked Example 2.9 to find the exact distance between the points A and B with the given position vectors.

31 a $\mathbf{a} = 2\mathbf{i} + 4\mathbf{j} - 2\mathbf{k}$ and $\mathbf{b} = \mathbf{i} - 2\mathbf{j} - 6\mathbf{k}$

b $\mathbf{a} = 3\mathbf{i} - \mathbf{j} + 2\mathbf{k}$ and $\mathbf{b} = 2\mathbf{i} + \mathbf{j} - 3\mathbf{k}$

32 a $\mathbf{a} = \mathbf{i} + \mathbf{k}$ and $\mathbf{b} = \mathbf{i} + \mathbf{j}$

b $\mathbf{a} = \mathbf{i} + \mathbf{j}$ and $\mathbf{b} = \mathbf{j} - \mathbf{k}$

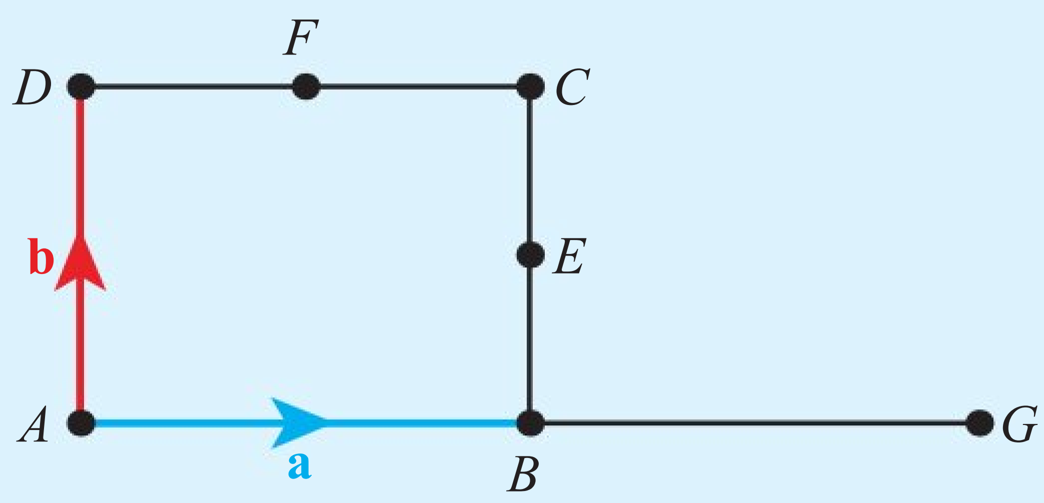
33 a $\mathbf{a} = \begin{pmatrix} 3 \\ 7 \\ -2 \end{pmatrix}$ and $\mathbf{b} = \begin{pmatrix} 1 \\ -2 \\ -5 \end{pmatrix}$

b $\mathbf{a} = \begin{pmatrix} 2 \\ 0 \\ -2 \end{pmatrix}$ and $\mathbf{b} = \begin{pmatrix} 0 \\ 0 \\ 5 \end{pmatrix}$

34 Given that $\begin{pmatrix} 1 \\ -2 \\ 5 \end{pmatrix}$ is the resultant of the vectors $\mathbf{a} = \begin{pmatrix} 12 \\ y \\ -3 \end{pmatrix}$, $\mathbf{b} = \begin{pmatrix} -5 \\ 2 \\ z \end{pmatrix}$ and $\mathbf{c} = \begin{pmatrix} x \\ 9 \\ 4 \end{pmatrix}$, find the value of x, y and z.

35 The diagram shows a rectangle $ABCD$. E is the midpoint of BC, F is the midpoint of CD and G is the point on the extension of the side AB such that $BG = AB$.

Define vectors $\mathbf{a} = \overrightarrow{AB}$ and $\mathbf{b} = \overrightarrow{AD}$. Express the following vectors in terms of $\mathbf{a}$ and $\mathbf{b}$.



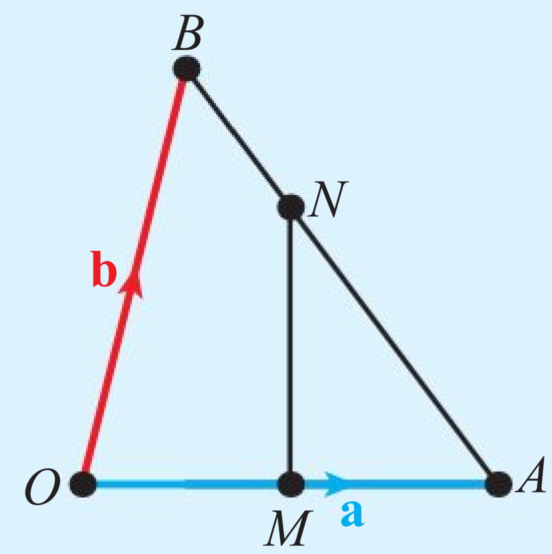
a $\overrightarrow{AE}$

b $\overrightarrow{EF}$

c $\overrightarrow{DG}$

36 In triangle OAB, $\overrightarrow{OA} = \mathbf{a}$ and $\overrightarrow{OB} = \mathbf{b}$. M is the midpoint of OA and N is the point on AB such that $BN = \frac{1}{3}BA$.

Express the following vectors in terms of $\mathbf{a}$ and $\mathbf{b}$.



a $\overrightarrow{BA}$

b $\overrightarrow{ON}$

c $\overrightarrow{MN}$

37 Given the vectors $\mathbf{a} = \begin{pmatrix} 2 \\ -1 \\ 3 \end{pmatrix}$, $\mathbf{b} = \begin{pmatrix} 0 \\ 1 \\ -5 \end{pmatrix}$ and $\mathbf{c} = \begin{pmatrix} -2 \\ 2 \\ 3 \end{pmatrix}$, find:

a $3\mathbf{a} - \mathbf{c} + 5\mathbf{b}$

b $|\mathbf{b} - 2\mathbf{a}|$

38 Find the possible values of the constant k such that the vector $\begin{pmatrix} 3k \\ -k \\ k \end{pmatrix}$ has magnitude 22.

39 The vector $2\mathbf{i} + 3t\mathbf{j} + (t-1)\mathbf{k}$ has magnitude 3. Find the possible values of t.

40 An object is acted on by three forces:

$$\mathbf{F}_1 = \begin{pmatrix} 2 \\ -1 \\ 5 \end{pmatrix}\text{N},\ \mathbf{F}_2 = \begin{pmatrix} 6 \\ 3 \\ -2 \end{pmatrix}\text{N},\ \mathbf{F}_3 = \begin{pmatrix} 0 \\ -4 \\ 7 \end{pmatrix}\text{N}$$

Find the magnitude of the resultant force acting on the object.

41 Given that $\mathbf{a} = \begin{pmatrix} -1 \\ 1 \\ 2 \end{pmatrix}$ and $\mathbf{b} = \begin{pmatrix} 5 \\ 3 \\ 3 \end{pmatrix}$, find vector $\mathbf{x}$ such that $3\mathbf{a} + 4\mathbf{x} = \mathbf{b}$.

42 Given that $\mathbf{a} = 3\mathbf{i} - 2\mathbf{j} + 5\mathbf{k}$, $\mathbf{b} = \mathbf{i} - \mathbf{j} + 2\mathbf{k}$ and $\mathbf{c} = \mathbf{i} + \mathbf{k}$, find the value of the scalar t such that $\mathbf{a} + t\mathbf{b} = \mathbf{c}$.

43 a Find a unit vector parallel to $6\mathbf{i} + 6\mathbf{j} - 3\mathbf{k}$.

b Find a vector of magnitude 10 in the same direction as $\begin{pmatrix} 4 \\ -1 \\ 2\sqrt{2} \end{pmatrix}$.

44 Given that $\mathbf{a} = \begin{pmatrix} 2 \\ 0 \\ 2 \end{pmatrix}$ and $\mathbf{b} = \begin{pmatrix} 3 \\ 1 \\ 3 \end{pmatrix}$ find the value of the scalar p such that $\mathbf{a} + p\mathbf{b}$ is parallel to the vector $\begin{pmatrix} 3 \\ 2 \\ 3 \end{pmatrix}$.

45 Given that $\mathbf{x} = 2\mathbf{i} + 3\mathbf{j} + \mathbf{k}$ and $\mathbf{y} = 4\mathbf{i} + \mathbf{j} + 2\mathbf{k}$, find the value of the scalar λ such that $\lambda\mathbf{x} + \mathbf{y}$ is parallel to vector $\mathbf{j}$.

46 Find a vector of magnitude 6 parallel to $\begin{pmatrix} 4 \\ -1 \\ 1 \end{pmatrix}$.

47 Let $\mathbf{a} = \begin{pmatrix} -2 \\ 0 \\ -1 \end{pmatrix}$ and $\mathbf{b} = \begin{pmatrix} 2 \\ -1 \\ 2 \end{pmatrix}$. Find the possible values of λ such that $|\mathbf{a} + \lambda\mathbf{b}| = 5\sqrt{2}$.

48 Points A and B are such that $\overrightarrow{OA} = \begin{pmatrix} -1 \\ -6 \\ 13 \end{pmatrix}$ and $\overrightarrow{OB} = \begin{pmatrix} 1 \\ -2 \\ 4 \end{pmatrix} + t\begin{pmatrix} 2 \\ 1 \\ -5 \end{pmatrix}$, where O is the origin.

Find the possible values of t such that $AB = 3$.

49 Points P and Q have position vectors $\mathbf{p} = 2\mathbf{i} - \mathbf{j} - 3\mathbf{k}$ and $\mathbf{q} = \mathbf{i} + 4\mathbf{j} - \mathbf{k}$.

a Find the position vector of the midpoint M of PQ.

b Point R lies on the line PQ such that $QR = QM$. Find the coordinates of R.

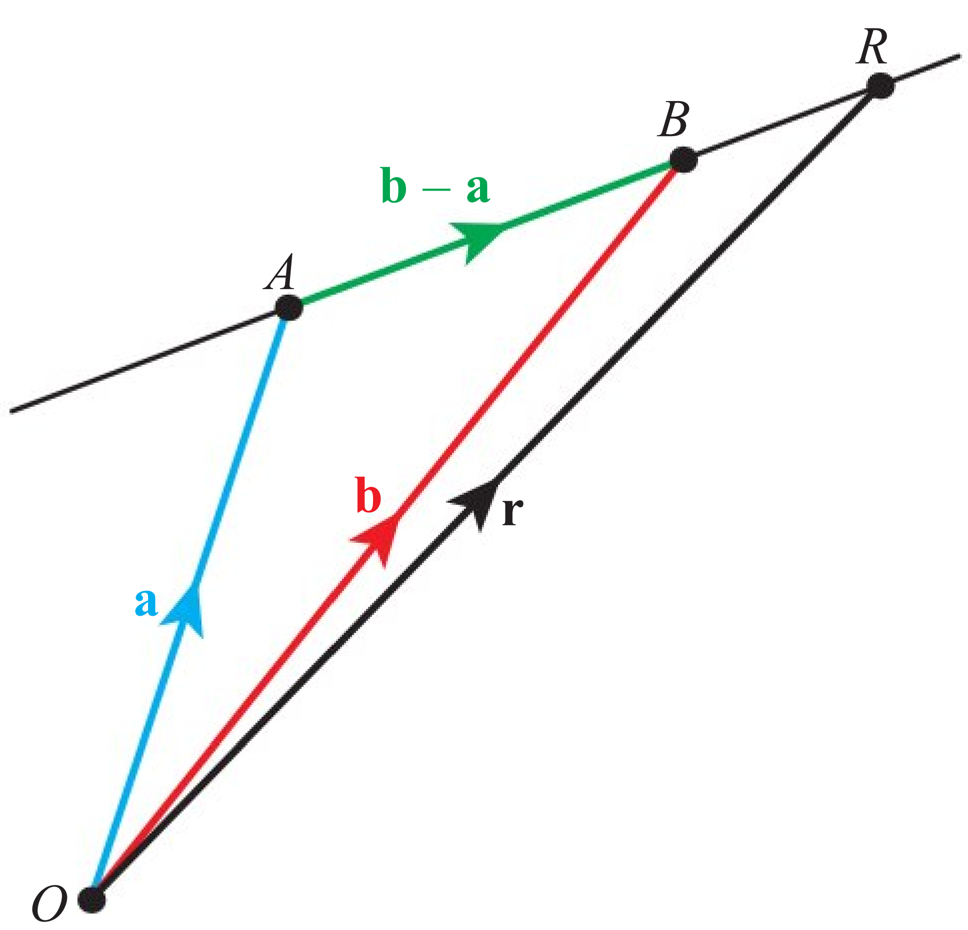
50 Find the smallest possible magnitude of the vector $(2t)\mathbf{i} + (t + 3)\mathbf{j} - (2t + 1)\mathbf{k}$, where t is a real constant.

2B Vector equation of a line

Consider a straight line through points A and B, with position vectors $\mathbf{a}$ and $\mathbf{b}$. For any other point R on the line, the vector $\overrightarrow{AR}$ is in the same direction as $\overrightarrow{AB}$, so you can write $\overrightarrow{AR} = \lambda\overrightarrow{AB}$ for some scalar λ. Using position vectors, this equation becomes

$$\mathbf{r} - \mathbf{a} = \lambda(\mathbf{b} - \mathbf{a})$$

This can be rearranged to express the position vector $\mathbf{r}$ in terms of $\mathbf{a}$, $\mathbf{b}$ and λ.



KEY POINT 2.7

The equation of the line through points with position vectors $\mathbf{a}$ and $\mathbf{b}$ is

$$\mathbf{r} = \mathbf{a} + \lambda(\mathbf{b} - \mathbf{a})$$

Tip

You can write $\mathbf{r} = \begin{pmatrix} x \\ y \\ z \end{pmatrix}$ to find the coordinates of a point on the line.

Different values of λ give position vectors of different points on the line. For example, you can check that $\lambda = 0$ gives point A, $\lambda = 1$ gives point B and $\lambda = 0.5$ gives $\mathbf{r} = \frac{1}{2}(\mathbf{a} + \mathbf{b})$, which is the midpoint of AB.

WORKED EXAMPLE 2.10

a Find the equation of the straight line through the point $A(2, -1, 3)$ and $B(1, 1, 5)$.

b Determine whether the point $C(1.5, 0, 4.5)$ lies on this line.

(a)

First find the vector $\mathbf{b} - \mathbf{a}$

$$\mathbf{b} - \mathbf{a} = \begin{pmatrix} 1 \\ 1 \\ 5 \end{pmatrix} - \begin{pmatrix} 2 \\ -1 \\ 3 \end{pmatrix} = \begin{pmatrix} -1 \\ 2 \\ 2 \end{pmatrix}$$

The equation of line is

Use $\mathbf{r} = \mathbf{a} + \lambda(\mathbf{b} - \mathbf{a})$

$$\mathbf{r} = \begin{pmatrix} 2 \\ -1 \\ 3 \end{pmatrix} + \lambda \begin{pmatrix} -1 \\ 2 \\ 2 \end{pmatrix}$$

(b)

Find λ such that

Is there a value of λ which gives $\mathbf{r} = \mathbf{c}$?

$$\begin{pmatrix} 2 \\ -1 \\ 3 \end{pmatrix} + \lambda \begin{pmatrix} -1 \\ 2 \\ 2 \end{pmatrix} = \begin{pmatrix} 1.5 \\ 0 \\ 4.5 \end{pmatrix}$$

This represents three equations, one for each component

$$\begin{cases} 2 - \lambda = 1.5 & (1) \\ -1 + 2\lambda = 0 & (2) \\ 3 + 2\lambda = 4.5 & (3) \end{cases}$$

The same value of λ needs to satisfy all three equations. Find λ from the first equation and check in the other two

(1): $\lambda = 0.5$

(2): $-1 + 2(0.5) = 0$

(3): $3 + 2(0.5) = 4 \neq 4.5$

There is no value of λ which satisfies all three equations

So, point C does not lie on the line.

You will learn more about recognizing which equations describe the same line in the next section.

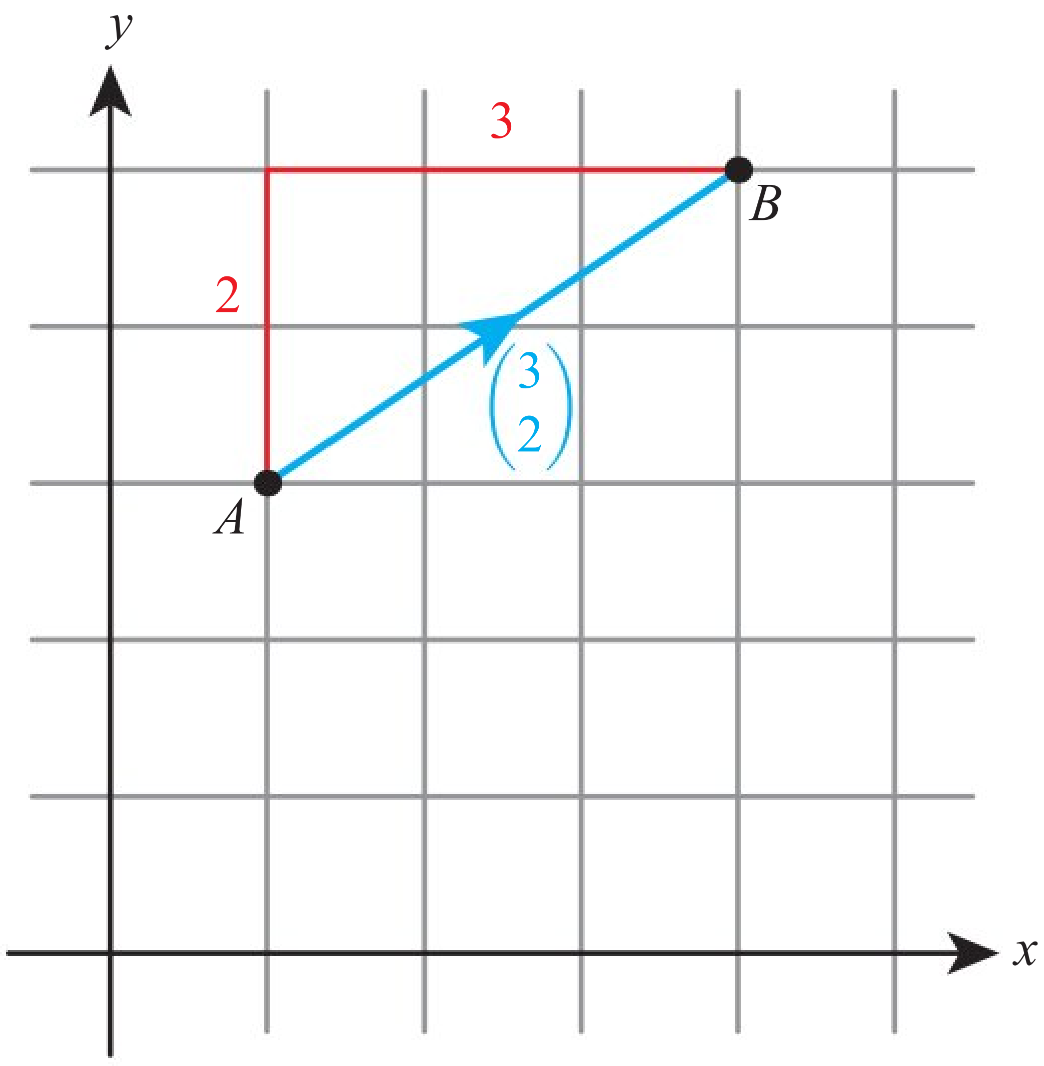
Tip

You could have used point B instead of A to write the equation in Worked Example 2.10 as $\mathbf{r} = \begin{pmatrix} 1 \\ 1 \\ 5 \end{pmatrix} + \lambda \begin{pmatrix} -1 \\ 2 \\ 2 \end{pmatrix}$. Or you could use vector $\overrightarrow{BA}$ instead of $\overrightarrow{AB}$ to get $\mathbf{r} = \begin{pmatrix} 2 \\ -1 \\ 3 \end{pmatrix} + \lambda \begin{pmatrix} 1 \\ -2 \\ -2 \end{pmatrix}$.

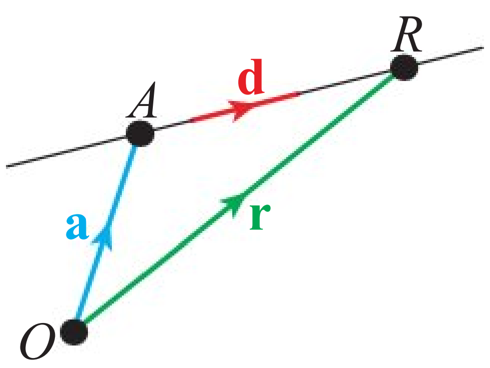
All of those equations represent the same line, but the value of λ for a given point on the line is different for different equations. For example, the point $(0, 3, 7)$ corresponds to $\lambda = 1$ in the first equation and $\lambda = -2$ in the second equation.

The vector equation of a line takes the same form in two dimensions. For example, the equation of the line through the points $(1, 3)$ and $(4, 5)$ can be written as $\mathbf{r} = \begin{pmatrix} 1 \\ 3 \end{pmatrix} + \lambda \begin{pmatrix} 3 \\ 2 \end{pmatrix}$.

If you draw a diagram, you can see how the vector **b** – **a** is related to the gradient of the line:



The vector **b** – **a** is a **direction vector** of the line. You can find the equation of a line if you know only one point and a direction vector.



KEY POINT 2.8

A vector equation of the line with direction vector **d** passing through the point **a** is

$$\mathbf{r} = \mathbf{a} + \lambda\mathbf{d}.$$

Tip

You can use any multiple of $\begin{pmatrix}4\\-1\\2\end{pmatrix}$ as a direction vector.

WORKED EXAMPLE 2.11

Write down a vector equation of the line with direction vector $\begin{pmatrix}4\\-1\\2\end{pmatrix}$ passing through the point (–3, 3, 5).

Use $\mathbf{r} = \mathbf{a} + \lambda\mathbf{d}$, where **a** is the position vector of a point on the line and **d** is a direction vector

$$\mathbf{r} = \begin{pmatrix}-3\\3\\5\end{pmatrix} + \lambda\begin{pmatrix}4\\-1\\2\end{pmatrix}$$

Parametric form of the equation of a line

You can rewrite the vector equation of a line as three separate equations for x, y and z.

To do this, just remember that $\mathbf{r} = \begin{pmatrix}x\\y\\z\end{pmatrix}$.

KEY POINT 2.9

The **parametric form** of the equation of a line is found by expressing x, y and z in terms of λ.

WORKED EXAMPLE 2.12

Write the parametric form of the equation of the line with vector equation $\mathbf{r} = \begin{pmatrix} 3 \\ -2 \\ 0 \end{pmatrix} + \lambda \begin{pmatrix} -4 \\ 0 \\ 5 \end{pmatrix}$.

Write r as $\begin{pmatrix} x \\ y \\ z \end{pmatrix}$

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 3 \\ -2 \\ 0 \end{pmatrix} + \lambda \begin{pmatrix} -4 \\ 0 \\ 5 \end{pmatrix}$$

Write a separate equation for each component

$$\Rightarrow \begin{cases} x = 3 - 4\lambda \\ y = -2 \\ z = 5\lambda \end{cases}$$

Tip

When working with two different lines, use two different letters (such as λ and μ) for the parameters.

Finding the point of intersection of two lines

Suppose two lines have vector equations $\mathbf{r}_1 = \mathbf{a} + \lambda \mathbf{d}_1$ and $\mathbf{r}_2 = \mathbf{b} + \mu \mathbf{d}_2$. If they intersect, then there is a point which lies on both lines. Remembering that the position vector of a point on the line is given by the vector $\mathbf{r}$, this means that we need to find the values of λ and μ which make $\mathbf{r}_1 = \mathbf{r}_2$.

WORKED EXAMPLE 2.13

Find the coordinates of the point of intersection of the following pair of lines.

$$\mathbf{r} = \begin{pmatrix} 0 \\ -4 \\ 1 \end{pmatrix} + \lambda \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix} \text{ and } \mathbf{r} = \begin{pmatrix} 1 \\ 3 \\ 5 \end{pmatrix} + \mu \begin{pmatrix} 4 \\ -2 \\ -2 \end{pmatrix}$$

You need to make $\mathbf{r}_1 = \mathbf{r}_2$

$$\begin{pmatrix} 0 \\ -4 \\ 1 \end{pmatrix} + \lambda \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ 3 \\ 5 \end{pmatrix} + \mu \begin{pmatrix} 4 \\ -2 \\ -2 \end{pmatrix}$$

Write three separate equations, one for each component

$$\Rightarrow \begin{cases} 0 + \lambda = 1 + 4\mu \\ -4 + 2\lambda = 3 - 2\mu \\ 1 + \lambda = 5 - 2\mu \end{cases}$$

$$\Rightarrow \begin{cases} \lambda - 4\mu = 1 & (1) \\ 2\lambda + 2\mu = 7 & (2) \\ \lambda + 2\mu = 4 & (3) \end{cases}$$

Pick two equations to solve, then check the answers in the third. In this case, subtracting (1) from (3) eliminates λ

$$(3) - (1) \Rightarrow 6\mu = 3$$

$$\mu = \frac{1}{2}, \lambda = 3$$

The values of λ and μ you have found must also satisfy equation (2)

$$(2): \ 2 \times 3 + 2 \times \frac{1}{2} = 7$$

The lines intersect.

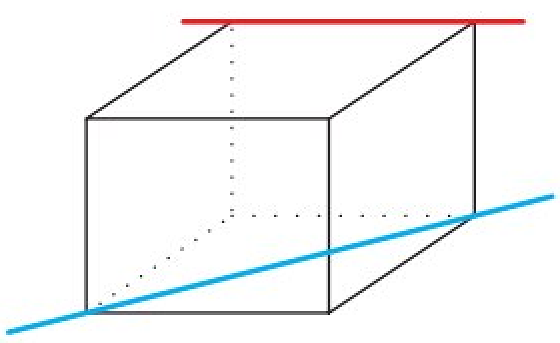
The position of the intersection point is given by the vector $\mathbf{r}_1$ (or $\mathbf{r}_2$ – they should be the same – you should always check this)

$$\mathbf{r}_1 = \begin{pmatrix} 0 \\ -4 \\ 1 \end{pmatrix} + 3\begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix} = \begin{pmatrix} 3 \\ 2 \\ 4 \end{pmatrix}$$

The lines intersect at the point (3, 2, 4).

CONCEPTS – SPACE

Some properties of objects depend on the dimension they occupy in **space**. One of the most interesting examples of this is diffusion, which is very important in physics and biology. If a large number of particles move randomly (performing a so-called random walk) in three dimensions, on average they will keep moving away from the starting point. This is, however, not the case in one or two dimensions where, on average, the particles will return to the starting point.



In two dimensions, two distinct lines either intersect or are parallel. In three dimensions it is possible for the lines to neither intersect nor be parallel.

If two lines do not intersect, it is impossible to find the values of λ and μ which solve all three equations.

WORKED EXAMPLE 2.14

Show that the lines with equations $\mathbf{r} = \begin{pmatrix} -4 \\ 1 \\ 3 \end{pmatrix} + t\begin{pmatrix} 1 \\ 1 \\ 4 \end{pmatrix}$ and $\mathbf{r} = \begin{pmatrix} 2 \\ 1 \\ 1 \end{pmatrix} + \lambda\begin{pmatrix} 2 \\ -3 \\ 2 \end{pmatrix}$ do not intersect.

Try to make $\mathbf{r}_1 = \mathbf{r}_2$ and then show that this is not possible

$$\begin{pmatrix} -4 \\ 3 \\ 3 \end{pmatrix} + t\begin{pmatrix} 1 \\ 1 \\ 4 \end{pmatrix} = \begin{pmatrix} 2 \\ 1 \\ 1 \end{pmatrix} + \lambda\begin{pmatrix} 2 \\ -3 \\ 2 \end{pmatrix}$$

$$\Rightarrow \begin{cases} t - 2\lambda = 6 & (1) \\ t + 3\lambda = -2 & (2) \\ 4t - 2\lambda = -2 & (3) \end{cases}$$

Find t and λ from the first two equations

$$(1) \text{ and } (2) \Rightarrow \lambda = -\frac{8}{5},\ t = \frac{14}{5}$$

The values found must also satisfy the third equation

$$(3):\ 4\left(\frac{14}{5}\right) - 2\left(-\frac{8}{5}\right) = \frac{72}{5} \neq -2$$

This tells you that it is impossible to find t and λ to make $\mathbf{r}_1 = \mathbf{r}_2$

The two lines do not intersect.

Exercise 2B

For questions 1 to 3, use the method demonstrated in Worked Example 2.10 to find the equation of the line through A and B, and determine whether point C lies on the line.

1 **a** $A(2, 1, 5)$, $B(1, 3, 7)$, $C(0, 5, 9)$

b $A(-1, 0, 3)$, $B(3, 1, 8)$, $C(-5, -1, 3)$

2 **a** $A(4, 0, 3)$, $B(8, 0, 6)$, $C(0, 0, 2)$

b $A(-1, 5, 1)$, $B(-1, 5, 8)$, $C(-1, 3, 8)$

3 **a** $A(4, 1)$, $B(1, 2)$, $C(5, -2)$

b $A(2, 7)$, $B(4, -2)$, $C(1, 11.5)$

For questions 4 to 6, use the method demonstrated in Worked Example 2.11 to write down a vector equation of the line with the give direction vector passing through the given point.

4 **a** Point $(1, 0, 5)$, direction $\begin{pmatrix} 1 \\ 3 \\ -3 \end{pmatrix}$

b Point $(-1, 1, 5)$, direction $\begin{pmatrix} 3 \\ -2 \\ 2 \end{pmatrix}$

5 **a** Direction $\mathbf{i} - 3\mathbf{k}$, point $(0, 2, 3)$

b Direction $2\mathbf{i} + 3\mathbf{j} - \mathbf{k}$, point $(4, -3, 0)$

6 **a** Direction $\begin{pmatrix} 1 \\ 4 \end{pmatrix}$, point $(4, -1)$

b Direction $\begin{pmatrix} 2 \\ -3 \end{pmatrix}$, point $(4, 1)$

For questions 7 to 9, use the method demonstrated in Worked Example 2.12 to write the parametric form of the equation of the line with the given vector equation.

7 **a** $\mathbf{r} = \begin{pmatrix} 3 \\ 5 \\ 2 \end{pmatrix} + \lambda \begin{pmatrix} 8 \\ 2 \\ 4 \end{pmatrix}$

b $\mathbf{r} = \begin{pmatrix} 4 \\ -1 \\ 2 \end{pmatrix} + \lambda \begin{pmatrix} -2 \\ 3 \\ 5 \end{pmatrix}$

8 **a** $\mathbf{r} = (2\mathbf{i} + 3\mathbf{j}) + \lambda(\mathbf{i} - 4\mathbf{k})$

b $\mathbf{r} = (3\mathbf{j} - \mathbf{k}) + \lambda(2\mathbf{i} + \mathbf{j})$

9 **a** $\mathbf{r} = \begin{pmatrix} -1 \\ 3 \end{pmatrix} + \lambda \begin{pmatrix} 0 \\ 5 \end{pmatrix}$

b $\mathbf{r} = \begin{pmatrix} 4 \\ 0 \end{pmatrix} + \lambda \begin{pmatrix} -3 \\ 2 \end{pmatrix}$

For questions 10 to 12, use the method demonstrated in Worked Example 2.13 to find the coordinates of the point of intersection of the two lines.

10 **a** $\mathbf{r} = \begin{pmatrix} 6 \\ 1 \\ 2 \end{pmatrix} + \lambda \begin{pmatrix} -1 \\ 2 \\ 1 \end{pmatrix}$ and $\mathbf{r} = \begin{pmatrix} 2 \\ 1 \\ -14 \end{pmatrix} + \mu \begin{pmatrix} 2 \\ -2 \\ 3 \end{pmatrix}$

b $\mathbf{r} = \begin{pmatrix} 1 \\ 3 \\ 7 \end{pmatrix} + \lambda \begin{pmatrix} 2 \\ 2 \\ 1 \end{pmatrix}$ and $\mathbf{r} = \begin{pmatrix} 5 \\ 1 \\ 4 \end{pmatrix} + \mu \begin{pmatrix} 3 \\ 0 \\ -1 \end{pmatrix}$

11 **a** $\mathbf{r} = \begin{pmatrix} 1 \\ 0 \\ 0.5 \end{pmatrix} + \lambda \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}$ and $\mathbf{r} = \begin{pmatrix} 2 \\ -2 \\ 1 \end{pmatrix} + \mu \begin{pmatrix} 3 \\ -4 \\ 0 \end{pmatrix}$

b $\mathbf{r} = \begin{pmatrix} 4 \\ -1 \\ 2 \end{pmatrix} + \lambda \begin{pmatrix} 1 \\ 2 \\ -4 \end{pmatrix}$ and $\mathbf{r} = \begin{pmatrix} 6 \\ -2 \\ 0 \end{pmatrix} + \mu \begin{pmatrix} 3 \\ -4 \\ 0 \end{pmatrix}$

12 **a** $\mathbf{r} = (3\mathbf{i} + \mathbf{j}) + \lambda(2\mathbf{j} + \mathbf{k})$ and $\mathbf{r} = (\mathbf{i} + \mathbf{j} + 3\mathbf{k}) + \mu(\mathbf{i} + \mathbf{j} - \mathbf{k})$

b $\mathbf{r} = (3\mathbf{i} - 2\mathbf{j} + \mathbf{k}) + \lambda(4\mathbf{i} + 4\mathbf{j} + 3\mathbf{k})$ and $\mathbf{r} = (8\mathbf{i} - \mathbf{j} + 2\mathbf{k}) + \mu(-\mathbf{i} + 3\mathbf{j} + 2\mathbf{k})$

For questions 13 and 14, use the method demonstrated in Worked Example 2.14 to show that the two lines do not intersect.

13 **a** $\mathbf{r} = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} + t \begin{pmatrix} -1 \\ 1 \\ 2 \end{pmatrix}$ and $\mathbf{r} = \begin{pmatrix} -4 \\ -4 \\ -11 \end{pmatrix} + s \begin{pmatrix} 5 \\ 1 \\ 2 \end{pmatrix}$

b $\mathbf{r} = \begin{pmatrix} 4 \\ 0 \\ 2 \end{pmatrix} + t \begin{pmatrix} 2 \\ 0 \\ 1 \end{pmatrix}$ and $\mathbf{r} = \begin{pmatrix} -1 \\ 2 \\ 3 \end{pmatrix} + s \begin{pmatrix} 1 \\ -2 \\ -2 \end{pmatrix}$

14 a $\mathbf{r} = (\mathbf{i} + 5\mathbf{j} + 4\mathbf{k}) + \lambda(2\mathbf{i} - \mathbf{j} + \mathbf{k})$ and $\mathbf{r} = (3\mathbf{j} + 5\mathbf{k}) + \mu(\mathbf{i} + \mathbf{k})$

b $\mathbf{r} = (2\mathbf{i} + 3\mathbf{j} - \mathbf{k}) + \lambda(\mathbf{i} + 2\mathbf{j} - 4\mathbf{k})$ and $\mathbf{r} = (3\mathbf{i} + 2\mathbf{k}) + \mu(3\mathbf{i} - 5\mathbf{j} + 2\mathbf{k})$

15 a Find a vector equation of the line passing through the points (3, −1, 5) and (−1, 1, 2).

b Determine whether the point (0, 1, 5) lies on the line.

16 Find the parametric form of the equation of the line passing through the point (−1, 1, 2) parallel to the line with vector equation $\mathbf{r} = (3\mathbf{i} - \mathbf{j} + 2\mathbf{k}) + \lambda(2\mathbf{i} - \mathbf{j} - 3\mathbf{k})$.

17 Determine whether the point $A(3, -2, 2)$ lies on the line with equation $x = 2\lambda - 1$, $y = 4 - 3\lambda$, $z = 1.5\lambda$.

18 a Find a vector equation of the line with parametric equation

$$x = 3 - 2\lambda,\ y = -1,\ z = \frac{4\lambda - 1}{3}$$

b Find the value of p so that the point $(2, -1, p)$ lies on the line.

19 A line is given by parametric equations $x = 3 - \lambda$, $y = 4\lambda$, $z = 2 + \lambda$.

The point $(0, p, q)$ lies on the line. Find the values of p and q.

20 Show that these two lines intersect, and find the coordinates of the intersection point.

$$\mathbf{r} = \begin{pmatrix} 1 \\ 0 \\ 5 \end{pmatrix} + \lambda \begin{pmatrix} -1 \\ 1 \\ 2 \end{pmatrix} \text{ and } \mathbf{r} = \begin{pmatrix} 7 \\ 2 \\ 7 \end{pmatrix} + \mu \begin{pmatrix} 2 \\ 2 \\ 3 \end{pmatrix}$$

21 Determine whether or not these two lines intersect.

l_1: $x = -3 - \lambda$, $y = 5 - 2\lambda$, $z = 2 - 4\lambda$

l_2: $x = 8 - \mu$, $y = 5 - 3\mu$, $z = 1 + 3\mu$

22 a Show that the equations $\mathbf{r} = \begin{pmatrix} -1 \\ 1 \\ 2 \end{pmatrix} + \lambda \begin{pmatrix} 2 \\ 2 \\ 1 \end{pmatrix}$ and $x = 5 + 6\mu$, $y = 7 + 6\mu$, $z = 5 + 3\mu$ represent the same straight line.

b Show that the equation $\mathbf{r} = (4t - 5)\mathbf{i} + (4t - 3)\mathbf{j} + (1 + 2t)\mathbf{k}$ represents a different straight line.

23 a Show that the points $A(4, -1, -8)$ and $B(2, 1, -4)$ lie on the line l with equation $\mathbf{r} = \begin{pmatrix} 2 \\ 1 \\ -4 \end{pmatrix} + t \begin{pmatrix} -1 \\ 1 \\ 2 \end{pmatrix}$.

b Find the coordinates of the point C on the line l such that $AB = BC$.

24 a Find the vector equation of line l through points $P(7, 1, 2)$ and $Q(3, -1, 5)$.

b Point R lies on l and $PR = 3PQ$. Find the possible coordinates of R.

25 a Write down the vector equation of the line l through the point $A(2, 1, 4)$ parallel to the vector $2\mathbf{i} - 3\mathbf{j} + 6\mathbf{k}$.

b Calculate the magnitude of the vector $2\mathbf{i} - 3\mathbf{j} + 6\mathbf{k}$.

c Find the possible coordinates of point P on l such that $AP = 35$.

26 Points A, B, C and D have coordinates $A(1, 0, 0)$, $B(6, 5, 5)$, $C(8, 3, 3)$ and $D(6, 3, 3)$. Find the point of intersection of the lines AB and CD.

27 a Find the coordinates of the point where the line with equation $\mathbf{r} = \begin{pmatrix} 6 \\ -1 \\ -9 \end{pmatrix} + \lambda \begin{pmatrix} 2 \\ 7 \\ -3 \end{pmatrix}$ intersects the y-axis.

b Show that the line does not intersect the z-axis.

28 Find the value of p for which the lines with equations $\mathbf{r} = (\mathbf{j} - \mathbf{k}) + \mu(\mathbf{i} + 2\mathbf{j} + \mathbf{k}$ and $\mathbf{r} = (\mathbf{i} + 7\mathbf{j} - 4\mathbf{k}) + \lambda(\mathbf{i} + p\mathbf{k})$ intersect. Find the point of intersection in this case.

29 Find the distance of the line with equation $\mathbf{r} = \begin{pmatrix} 1 \\ -2 \\ 2 \end{pmatrix} + \lambda \begin{pmatrix} 2 \\ 2 \\ 1 \end{pmatrix}$ from the origin.

30 Find the shortest distance from the point (−1, 1, 2) to the line with equation $\mathbf{r} = \begin{pmatrix} 1 \\ 0 \\ 2 \end{pmatrix} + t \begin{pmatrix} -3 \\ 1 \\ 1 \end{pmatrix}$.

2C Vector applications to kinematics

We can now apply the methods of the previous sections to allow us to model the motion of objects moving with constant velocity.

You have already met the idea of the position vector of an object being the vector that gives the coordinates of that object. As well as knowing where an object is at any given point in time, it is also useful to know the direction in which it is moving and how fast it is moving. This information is given by the **velocity vector**.

KEY POINT 2.10

An object with constant velocity $\mathbf{v}$ is moving parallel to the vector $\mathbf{v}$ with speed $|\mathbf{v}|$.

WORKED EXAMPLE 2.15

An object is moving with velocity $\mathbf{v} = \begin{pmatrix} 3 \\ -2 \end{pmatrix}$ m s^{-1}.

The unit vectors $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$ and $\begin{pmatrix} 0 \\ 1 \end{pmatrix}$ point east and north, respectively.

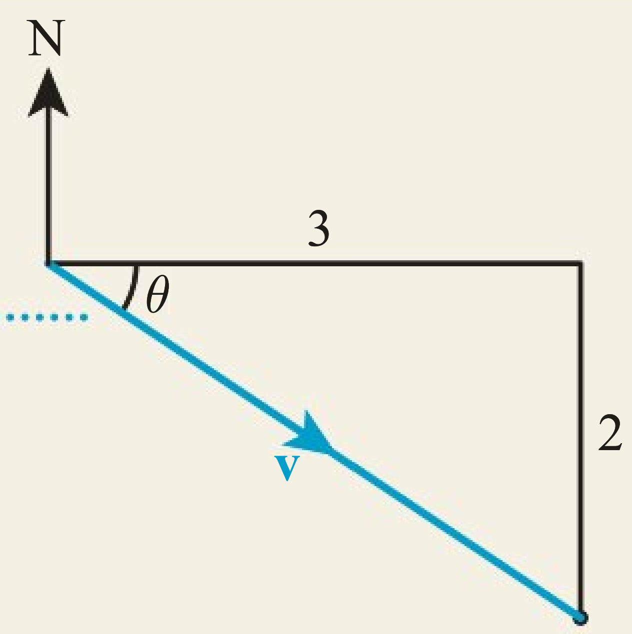
Find

a the object's direction of motion, as a bearing

b the object's speed.

It is helpful to draw a diagram

a



Find θ using trigonometry

$$\theta = \tan^{-1}\frac{2}{3} = 33.7°$$

Convert to a bearing

$$\text{Bearing} = 90 + 33.7 = 124°$$

b

$$\text{Speed} = \left|\begin{pmatrix} 3 \\ -2 \end{pmatrix}\right|$$

Speed = $|\mathbf{v}|$

$$= \sqrt{3^2 + (-2)^2}$$

$$= 3.61\,\text{m s}^{-1}$$

You saw how to find unit vectors, and vectors of given magnitude in a certain direction in Section 2A.

WORKED EXAMPLE 2.16

Find the velocity vector of an object moving in the direction $\begin{pmatrix}1\\3\\4\end{pmatrix}$ with speed $10\,\mathrm{m\,s^{-1}}$.

First find a unit vector in the direction of $\begin{pmatrix}1\\3\\4\end{pmatrix}$

$$\text{Unit vector} = \frac{1}{\sqrt{1^2+3^2+4^2}}\begin{pmatrix}1\\3\\4\end{pmatrix}$$

$$= \frac{1}{\sqrt{26}}\begin{pmatrix}1\\3\\4\end{pmatrix}$$

Then multiply by 10 so that $|\mathbf{v}| = 10$

So,

$$\mathbf{v} = \frac{10}{\sqrt{26}}\begin{pmatrix}1\\3\\4\end{pmatrix}\mathrm{m\,s^{-1}}$$

If an object is moving with constant velocity it will move in a straight line. The displacement from the starting position after time t will be $t\mathbf{v}$, where $\mathbf{v}$ is the velocity vector. The actual position of the object (relative to the origin) can be found by adding this displacement to the initial position vector.

KEY POINT 2.11

For an object moving with constant velocity $\mathbf{v}$ from the starting position $\mathbf{r}_0$, the position after time t is given by $\mathbf{r} = \mathbf{r}_0 + t\mathbf{v}$.

Notice that the equation in Key Point 2.11 represents a straight line with direction vector $\mathbf{v}$, where t plays the role of the parameter λ. It is indeed the equation of the line along which the object moves.

WORKED EXAMPLE 2.17

An object moves with constant velocity $\mathbf{v} = (2\mathbf{i} - \mathbf{j} + 5\mathbf{k})\,\mathrm{m\,s^{-1}}$. Its position vector when $t = 0$ is $\mathbf{r}_0 = (4\mathbf{i} + 3\mathbf{j} - 4\mathbf{k})\,\mathrm{m}$.

a Write down an equation for the position of the object at time t seconds.

b Find the distance of the object from the origin when $t = 5$ seconds.

Use $\mathbf{r} = \mathbf{r}_0 + t\mathbf{v}$

a $\mathbf{r} = (4\mathbf{i} + 3\mathbf{j} - 4\mathbf{k}) + t(2\mathbf{i} - \mathbf{j} + 5\mathbf{k})$

b When $t = 5$:

First find the position vector when $t = 5$

$$\mathbf{r} = (4\mathbf{i} + 3\mathbf{j} - 4\mathbf{k}) + 5(2\mathbf{i} - \mathbf{j} + 5\mathbf{k})$$
$$= (14\mathbf{i} - 2\mathbf{j} + 21\mathbf{k})$$

Distance is the magnitude of the displacement vector

$$\text{Distance} = |\mathbf{r}| = \sqrt{14^2 + 2^2 + 21^2}$$
$$= 25.3\,\mathrm{m}$$

TOOLKIT: Modelling

The velocity of an aeroplane is modelled by the constant vector $(p\mathbf{i} + q\mathbf{j} + 0\mathbf{k})\,\text{km h}^{-1}$.

- **a** Suggest suitable directions for the unit base vectors **i**, **j** and **k**.
- **b** What assumptions have been made in this model? Are those assumptions reasonable?
- **c** Can you suggest other ways of modelling the motion of an aeroplane?

Now that you have the equation of a straight line defining the paths of objects, you can use the methods of Section 2B to determine whether objects meet or if not how close they get to each other.

Tip

Whereas before you needed to use different parameters for the different lines, you now have the same parameter, t, for both lines. This means the lines could intersect but the particles do not collide as they are not both at the same point at the same time.

WORKED EXAMPLE 2.18

The position vectors (in km) of two model aircraft at time t hours are given by

$$\mathbf{r}_A = \begin{pmatrix} 2 \\ -6 \\ 0 \end{pmatrix} + t\begin{pmatrix} 1 \\ 4 \\ 3 \end{pmatrix} \quad \text{and} \quad \mathbf{r}_B = \begin{pmatrix} -1 \\ -3 \\ 1 \end{pmatrix} + t\begin{pmatrix} 6 \\ -1 \\ 2 \end{pmatrix}$$

a Show that they do not collide.

b Find the time at which they are closest together, and their distance apart at this point.

As before, for showing that two lines do not intersect, try to make $\mathbf{r}_A = \mathbf{r}_B$ and show that this is not possible

a $$\begin{pmatrix} 2 \\ -6 \\ 0 \end{pmatrix} + t\begin{pmatrix} 1 \\ 4 \\ 3 \end{pmatrix} = \begin{pmatrix} -1 \\ -3 \\ 1 \end{pmatrix} + t\begin{pmatrix} 6 \\ -1 \\ 2 \end{pmatrix}$$

The difference now is that you do not use a different parameter on each side of the equations. If they do intersect, this must happen at the same time, t

$$\Rightarrow \begin{cases} 2 + t = -1 + 6t & (1) \\ -6 + 4t = -3 - t & (2) \\ 3t = 1 + 2t & (3) \end{cases}$$

Solve each equation for t. The first two give the same value of t

$$(1): t = \frac{3}{5}$$

$$(2): t = \frac{3}{5}$$

However, the third gives a different value

$$(3): t = 1 \neq \frac{3}{5}$$

There is no time at which all three coordinates are equal

So, the model aircraft never collide.

The distance between A and B is $|\overrightarrow{AB}|$ so first find the displacement vector of A from B

b $AB = \mathbf{r}_B - \mathbf{r}_A$

$$= \begin{pmatrix} -1+6t \\ -3-t \\ 1+2t \end{pmatrix} - \begin{pmatrix} 2+t \\ -6+4t \\ 3t \end{pmatrix}$$

$$= \begin{pmatrix} -3+5t \\ 3-5t \\ 1-t \end{pmatrix}$$

So,

$$d = |\overrightarrow{AB}| = \sqrt{(-3+5t)^2 + (3-5t)^2 + (1-t)^2}$$

You could expand the expression to get the quadratic $d^2 = 51t^2 - 62t + 11$ and then complete the square or differentiate to find the minimum, but it is easier just to use your GDC

From GDC,

Min $d = 0.396$ km

and this occurs when $t = 0.608$ hours.

Exercise 2C

For questions 1 and 2, use the method demonstrated in Worked Example 2.15a to find the bearing on which the particle with the given velocity vector is moving.

1 **a** $\mathbf{v} = \begin{pmatrix} 5 \\ 4 \end{pmatrix}$

b $\mathbf{v} = 2\mathbf{i} + 3\mathbf{j}$

2 **a** $\mathbf{v} = -\mathbf{i} + 6\mathbf{j}$

b $\mathbf{v} = \begin{pmatrix} 3 \\ -8 \end{pmatrix}$

For questions 3 and 4, use the method demonstrated in Worked Example 2.15b to find the speed of the particle moving with the given velocity vector (all in $\mathrm{m\,s^{-1}}$).

3 **a** $\mathbf{v} = 5\mathbf{i} - 12\mathbf{j}$

b $\mathbf{v} = \begin{pmatrix} 7 \\ 3 \end{pmatrix}$

4 **a** $\mathbf{v} = \begin{pmatrix} 1 \\ 1 \\ -9 \end{pmatrix}$

b $\mathbf{v} = 6\mathbf{i} + 3\mathbf{j} - 2\mathbf{k}$

For questions 5 to 8, use the method demonstrated in Worked Example 2.16 to find the velocity vector of an object moving in the given direction with the given speed.

5 **a** Direction $3\mathbf{i} - 4\mathbf{j}$ with speed $12\,\mathrm{m\,s^{-1}}$

b Direction $1.2\mathbf{i} + 0.5\mathbf{j}$ with speed $6.5\,\mathrm{m\,s^{-1}}$

6 **a** Direction $\begin{pmatrix} 2 \\ 7 \end{pmatrix}$ with speed $8\,\mathrm{m\,s^{-1}}$

b Direction $\begin{pmatrix} -1 \\ 3 \end{pmatrix}$ with speed $5\,\mathrm{m\,s^{-1}}$

7 **a** Direction $\mathbf{i} - 5\mathbf{j} + 2\mathbf{k}$ with speed $10\,\mathrm{m\,s^{-1}}$

b Direction $2\mathbf{i} + \mathbf{j} - 6\mathbf{k}$ with speed $15\,\mathrm{m\,s^{-1}}$

8 **a** Direction $\begin{pmatrix} 0.4 \\ -0.3 \\ 1.2 \end{pmatrix}$ with speed $13\,\mathrm{m\,s^{-1}}$

b Direction $\begin{pmatrix} 6 \\ 2 \\ 3 \end{pmatrix}$ with speed $2.1\,\mathrm{m\,s^{-1}}$

For questions 9 to 12, use the method demonstrated in Worked Example 2.17 to find the position vector of an object moving from initial position $\mathbf{r}_0$ m with velocity $\mathbf{v}$ m s^{-1} for time t s.

9 a $\mathbf{r}_0 = 2\mathbf{i} - 7\mathbf{j}$, $\mathbf{v} = 0.6\mathbf{i} + 1.4\mathbf{j}$, $t = 6$

b $\mathbf{r}_0 = \mathbf{i}$, $\mathbf{v} = 3\mathbf{i} + 2.4\mathbf{j}$, $t = 3$

10 a $\mathbf{r}_0 = \begin{pmatrix} 0 \\ 2 \end{pmatrix}$, $\mathbf{v} = \begin{pmatrix} 3.2 \\ -5.5 \end{pmatrix}$, $t = 10$

b $\mathbf{r}_0 = \begin{pmatrix} -3 \\ 4 \end{pmatrix}$, $\mathbf{v} = \begin{pmatrix} -2 \\ 7 \end{pmatrix}$, $t = 2.4$

11 a $\mathbf{r}_0 = -3\mathbf{i} + 2\mathbf{k}$, $\mathbf{v} = 8\mathbf{i} - 3\mathbf{j} + 4\mathbf{k}$, $t = 7.5$

b $\mathbf{r}_0 = 4\mathbf{i} + 0.5\mathbf{j} - 3.2\mathbf{k}$, $\mathbf{v} = 8\mathbf{i} - 1.5\mathbf{j} + \mathbf{k}$, $t = 5$

12 a $\mathbf{r}_0 = \begin{pmatrix} -2 \\ 1 \\ 5 \end{pmatrix}$, $\mathbf{v} = \begin{pmatrix} 0 \\ 5 \\ -3 \end{pmatrix}$, $t = 2$

b $\mathbf{r}_0 = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$, $\mathbf{v} = \begin{pmatrix} 2.6 \\ -1.5 \\ 4.1 \end{pmatrix}$, $t = 8$

13 A particle moves with constant velocity $\mathbf{v} = (0.5\mathbf{i} + 2\mathbf{j} + 1.5\mathbf{k})$ m s^{-1}. At $t = 0$ seconds the particle is at the point with the position vector $(12\mathbf{i} - 5\mathbf{j} + 11\mathbf{k})$ m.

a Find the speed of the particle.

b Write down an equation for the position vector of the particle at time t seconds.

c Does the particle pass through the point (16, 8, 14)?

14 Two particles move so that their position vectors at time t seconds are given by

$$\mathbf{r}_1 = \begin{pmatrix} 10 \\ 5 \\ -3 \end{pmatrix} + t\begin{pmatrix} 1 \\ -1 \\ 4 \end{pmatrix} \text{ and } \mathbf{r}_2 = \begin{pmatrix} -1 \\ 0 \\ 0 \end{pmatrix} + t\begin{pmatrix} 0.5 \\ 2 \\ -0.5 \end{pmatrix}$$

The distance is measured in metres.

Find the distance between the particles when $t = 3$ seconds.

15 An object moves with a constant velocity. Its position vector at time t seconds is given by $\mathbf{r} = \begin{pmatrix} 3 \\ -1 \\ 4 \end{pmatrix} + t\begin{pmatrix} 1 \\ -1 \\ 3 \end{pmatrix}$, where distance is measured in metres.

a Find the initial position of the object.

b Find the speed of the object.

c Find the distance of the object from the origin after 3 seconds.

16 Ship A is initially at the point with position vector $(9\mathbf{i} + \mathbf{j})$ km and moves with velocity $(4\mathbf{i} + 5\mathbf{j})$ km h^{-1}. Ship B is initially at the point with position vector $(-3\mathbf{i} + 2\mathbf{j})$ km and moves with velocity $(6\mathbf{i} - 2\mathbf{j})$ km h^{-1}. Here $\mathbf{i}$ and $\mathbf{j}$ are unit vectors directed east and north, respectively.

a Find expressions for the position vectors of A and B after t hours.

b Calculate the distance of A from B after 2 hours.

c Find the time at which A is due north of B.

17 In this question, the distance is measured in km and the time in hours.

An aeroplane, initially at the point (2, 0, 0), moves with constant speed 894 km h^{-1} in the direction of the vector $(2\mathbf{i} - 2\mathbf{j} + \mathbf{k})$.

Find an equation for the position vector of the aeroplane at time t hours.

18 Two toy helicopters are flown, each in a straight line. The position vectors of the two helicopters at time t seconds are given by

$$\mathbf{r}_1 = \begin{pmatrix} 1 \\ 0 \\ 3 \end{pmatrix} + t\begin{pmatrix} 1 \\ -1 \\ 2 \end{pmatrix} \text{ and } \mathbf{r}_2 = \begin{pmatrix} 1 \\ -1 \\ 4 \end{pmatrix} + t\begin{pmatrix} 2 \\ -1 \\ 3 \end{pmatrix}.$$

Distance is measured in metres.

a Show that the paths of the helicopters cross.

b Determine whether the helicopters collide.

19 Two particles move so that their position vectors at time t are given by

$$\mathbf{r}_1 = (1+2t)\mathbf{i} + (t-3)\mathbf{j} + (3+7t)\mathbf{k}$$

and $\mathbf{r}_2 = (9-2t)\mathbf{i} + (t-2)\mathbf{j} + (22+2t)\mathbf{k}$

a Find the speed of each particle.

b Determine whether the particles meet.

20 Particles P and Q are initially at the points $(-1, -1, 2)$ and $(7, -2, 5)$, respectively.

P moves with velocity $\mathbf{i} + 4\mathbf{j} + 3\mathbf{k}$ and Q moves with velocity $2\mathbf{i} - 3\mathbf{j} - \mathbf{k}$.

a Write down the position vector of each particle at time t.

b Show that their paths cross but that they do not collide.

21 In this question, $\mathbf{i}$ and $\mathbf{j}$ are unit vectors directed east and north, respectively. Time is measured in hours and distance in miles.

Two particles move with constant velocity. At $t = 0$, Particle 1 starts from the point $(3, 0)$ and moves with velocity $(-2\mathbf{i} + 5\mathbf{j})$; Particle 2 starts from the point $(0, 5)$ and moves with velocity $(4\mathbf{i} + \mathbf{j})$.

a Write down the position vector of each particle at time t.

b Find and simplify an expression for the distance between the two particles at time t.

c Hence find the minimum distance between the particles.

d Find the time at which particle 2 is due east of particle 1.

22 In this question time is measures in metres and distance in seconds.

Two model aeroplanes move with constant velocity. Plane A starts from the point $(12, -10, 1)$ and moves with velocity $\begin{pmatrix}-4\\3\\1\end{pmatrix}\,\mathrm{m\,s^{-1}}$; plane B starts from the point $(5, 1, 4)$ and moves with velocity $\begin{pmatrix}2\\0\\1\end{pmatrix}\,\mathrm{m\,s^{-1}}$.

a Write down the position vector of each particle at time t.

b Find an expression for the distance between the two particles at time t.

c Hence find the minimum distance between the particles.

23 The position vectors of two drones at time t hours are given by $\mathbf{r}_1 = \begin{pmatrix}2\\4\\3\end{pmatrix} + t\begin{pmatrix}4\\5\\-2\end{pmatrix}$ and $\mathbf{r}_2 = \begin{pmatrix}5\\1\\0\end{pmatrix} + t\begin{pmatrix}-1\\10\\3\end{pmatrix}$.

a i Show that the drones will collide.

ii Determine the position vector of the point of collision.

In fact, after half an hour, drone 2 changes its velocity vector to $\begin{pmatrix}0\\8\\2\end{pmatrix}$ to avoid the collision.

b Find the distance between the two drones at the time at which they would have collided.

24 Two flies move so that their position vectors at time t seconds are given by

$$\mathbf{r}_1 = (0.36\mathbf{j} + 3.3\mathbf{k}) + t(1.2\mathbf{i} + 0.8\mathbf{j} - 0.1\mathbf{k})$$

and $\mathbf{r}_2 = (3.96\mathbf{i} + \mathbf{k}) + t(-\mathbf{i} + \mathbf{j} + 0.3\mathbf{k})$

where distance is measured in metres. The base vectors $\mathbf{i}$ and $\mathbf{j}$ are in the horizontal plane and vector $\mathbf{k}$ points upwards.

a Show that there is a time when one fly is vertically above the other.

b Find the distance between the flies at that time.

25 In this question, distance is measured in kilometres and time in hours.

A boat is moving with the constant velocity $(64\mathbf{i})\,\mathrm{km\,h^{-1}}$. At time $t = 0$, it is located at the origin. A small submarine is located at the point $(0, 0.5, -0.02)$. At time $t = 0$, it starts moving with a constant velocity in the direction of the vector $(40\mathbf{i} - 25\mathbf{j} + c\mathbf{k})$. Given that the submarine reaches the boat,

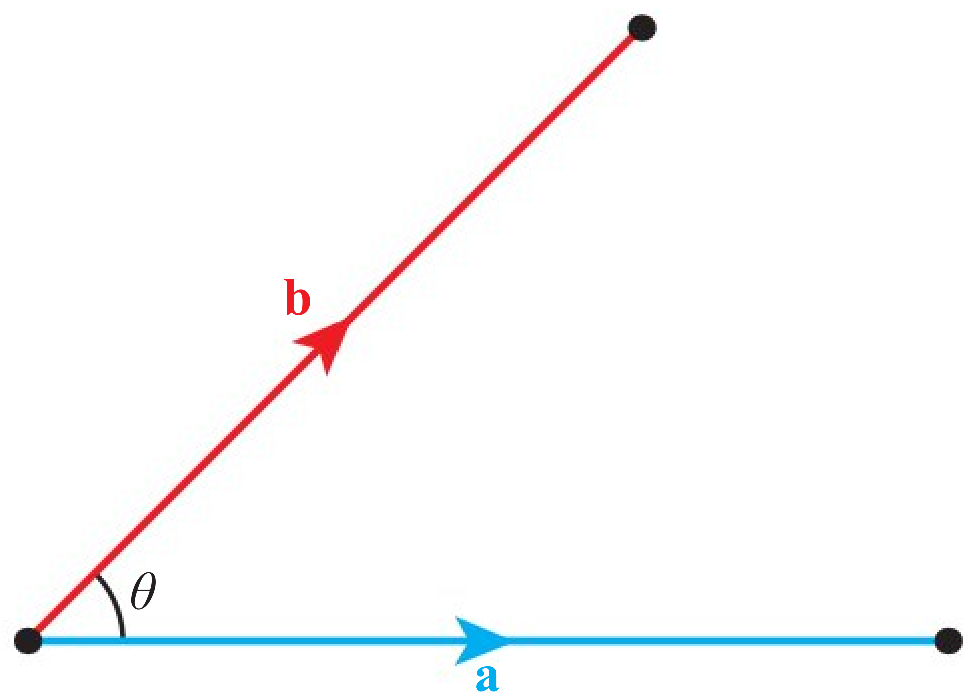
a find the value of c

b find the speed of the submarine.

2D Scalar and vector product

Definition and calculation of the scalar product of two vectors

The diagram shows two lines with angle θ between them. $\mathbf{a}$ and $\mathbf{b}$ are vectors in the directions of the two lines. Notice that both arrows are pointing away from the intersection point.



It turns out that $\cos\theta$ can be expressed in terms of the components of the two vectors.

Links to: Physics

The formula for the scalar product can be considered as the projection of one vector onto the other and it has many applications in physics. For example, if a force $\mathbf{F}$ acts on an object that moves from the origin to a point with position $\boldsymbol{x}$, then the work done is $\mathbf{F} \bullet \boldsymbol{x}$.

KEY POINT 2.12

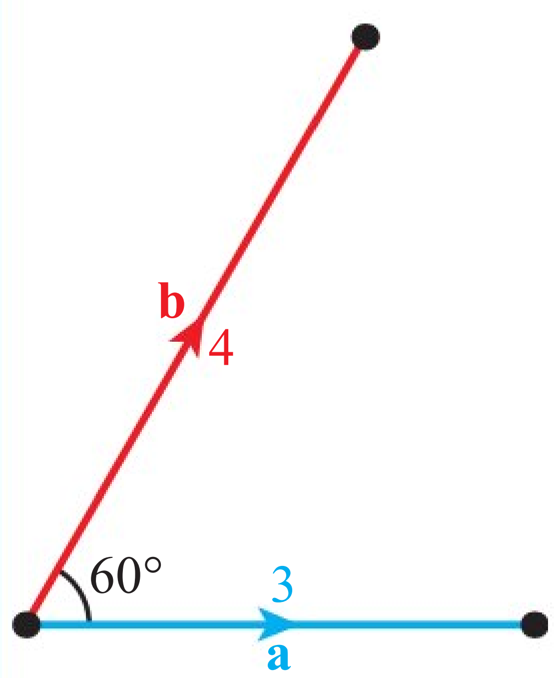
The **scalar product** (or **dot product**) of two vectors is defined by

$$\mathbf{a} \bullet \mathbf{b} = |\mathbf{a}||\mathbf{b}|\cos\theta$$

where θ is the angle between $\mathbf{a}$ and $\mathbf{b}$.

WORKED EXAMPLE 2.19

The diagram shows vectors $\mathbf{a}$ and $\mathbf{b}$. The numbers represent their lengths. Find the value of $\mathbf{a} \bullet \mathbf{b}$.



Use the formula for $\mathbf{a} \bullet \mathbf{b}$

$$\mathbf{a} \bullet \mathbf{b} = |\mathbf{a}| \bullet |\mathbf{b}| \cos\theta$$

The magnitudes are the lengths of the lines

$$= 3 \times 4 \times \cos 60°$$

$$= 6$$

Later in this section you will meet the cross product, for which the result is a vector.

Notice that the scalar product is a number (scalar).

Vectors are often given in terms of components, rather than by magnitude and direction. You can use the cosine rule to express the scalar product in terms of the components of the two vectors.

KEY POINT 2.13

If $\mathbf{a} = \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix}$ and $\mathbf{b} = \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix}$, then $\mathbf{a} \bullet \mathbf{b} = a_1b_1 + a_2b_2 + a_3b_3$.

Tip

The value of the scalar product can be negative.

WORKED EXAMPLE 2.20

Given that $\mathbf{a} = \begin{pmatrix} 3 \\ -2 \\ 1 \end{pmatrix}$ and $\mathbf{b} = \begin{pmatrix} -3 \\ 1 \\ 4 \end{pmatrix}$, calculate $\mathbf{a} \bullet \mathbf{b}$.

Use $\mathbf{a} \bullet \mathbf{b} = a_1b_1 + a_2b_2 + a_3b_3$

$$\begin{aligned} \mathbf{a} \bullet \mathbf{b} &= (3)(-3) + (-2)(1) + (1)(4) \\ &= -9 - 2 + 4 \\ &= -7 \end{aligned}$$

The angle between two vectors

Combining the results from Key Points 2.12 and 2.13 gives a formula for calculating the angle between two vectors given in component form.

Tip

The same formula can be used to find the angle between vectors in two dimensions – just set $a_3 = b_3 = 0$.

KEY POINT 2.14

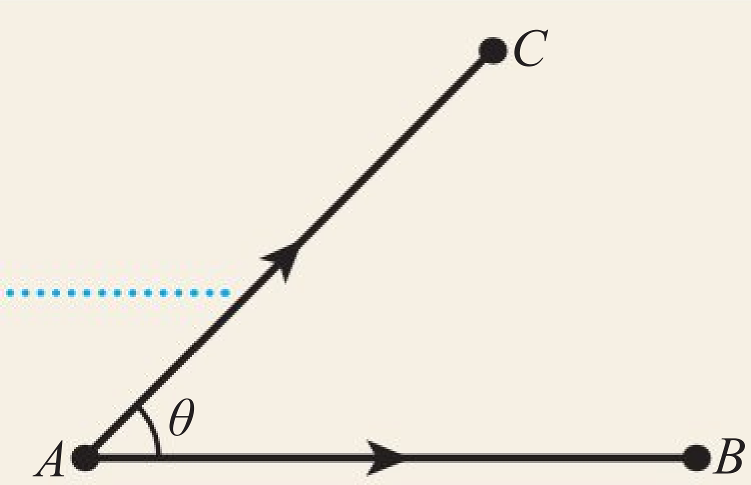
If θ is the angle between vectors $\mathbf{a} = \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix}$ and $\mathbf{b} = \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix}$, then $\cos\theta = \dfrac{\mathbf{a} \bullet \mathbf{b}}{|\mathbf{a}||\mathbf{b}|}$

where $\mathbf{a} \bullet \mathbf{b} = a_1b_1 + a_2b_2 + a_3b_3$.

WORKED EXAMPLE 2.21

Given points $A(3, -5, 2)$, $B(4, 1, 1)$ and $C(-1, 1, 2)$, find the size of angle $B\hat{A}C$ in degrees.

It is always a good idea to draw a diagram to see which vectors you need to use



You can see that the required angle is between vectors $\overrightarrow{AB}$ and $\overrightarrow{AC}$

Let $\theta = B\hat{A}C$.

$$\cos\theta = \frac{\overrightarrow{AB} \bullet \overrightarrow{AC}}{|\overrightarrow{AB}||\overrightarrow{AC}|}$$

You need to find the components of vectors $\overrightarrow{AB}$ and $\overrightarrow{AC}$

$$\overrightarrow{AB} = \begin{pmatrix} 4 \\ 1 \\ 1 \end{pmatrix} - \begin{pmatrix} 3 \\ -5 \\ 2 \end{pmatrix} = \begin{pmatrix} 1 \\ 6 \\ -1 \end{pmatrix}$$

$$\overrightarrow{AC} = \begin{pmatrix} -1 \\ 1 \\ 2 \end{pmatrix} - \begin{pmatrix} 3 \\ -5 \\ 2 \end{pmatrix} = \begin{pmatrix} -4 \\ 6 \\ 0 \end{pmatrix}$$

$$\cos\theta = \frac{1 \times (-4) + 6 \times 6 + (-1) \times 0}{\sqrt{1^2 + 6^2 + 1^2}\sqrt{4^2 + 6^2 + 0^2}}$$

$$= \frac{32}{\sqrt{38}\sqrt{52}} = 0.7199$$

$$\theta = \cos^{-1}(0.7199) = 44.0°$$

Be the Examiner 2.1

Given points $A(-1, 4, 2)$, $B(3, 3, 1)$ and $C(2, -5, 3)$, find the size of angle $A\hat{B}C$ in degrees.

Which is the correct solution? Identify the errors made in the incorrect solutions.

Solution 1	Solution 2	Solution 3
$\overrightarrow{AB} = \begin{pmatrix} 4 \\ -1 \\ -1 \end{pmatrix}$	$\overrightarrow{BA} = \begin{pmatrix} 4 \\ -1 \\ -1 \end{pmatrix}$	$\overrightarrow{BA} = \begin{pmatrix} -4 \\ 1 \\ 1 \end{pmatrix}$
$\overrightarrow{BC} = \begin{pmatrix} -1 \\ -8 \\ 2 \end{pmatrix}$	$\overrightarrow{BC} = \begin{pmatrix} 1 \\ 8 \\ -2 \end{pmatrix}$	$\overrightarrow{BC} = \begin{pmatrix} -1 \\ -8 \\ 2 \end{pmatrix}$
$\cos\theta = \frac{-4+8-2}{\sqrt{16+1+1}\sqrt{1+64+4}}$	$\cos\theta = \frac{4-8+2}{\sqrt{16+1+1}\sqrt{1+64+4}}$	$\cos\theta = \frac{4-8+2}{\sqrt{16+1+1}\sqrt{1+64+4}}$
$= 0.142$	$= -0.142$	$= -0.142$
$\theta = \cos^{-1}(0.142) = 81.8°$	$\theta = \cos^{-1}(-0.142) = 98.2°$	$\theta = \cos^{-1}(-0.142) = 98.2°$
	angle $= 180 - 98.2 = 81.8°$	

TOOLKIT: Problem Solving

There is more than one solution to $\cos x = 0.7199$ in Worked Example 2.21, but we have only given one answer. What do the other solutions represent?

Perpendicular and parallel vectors

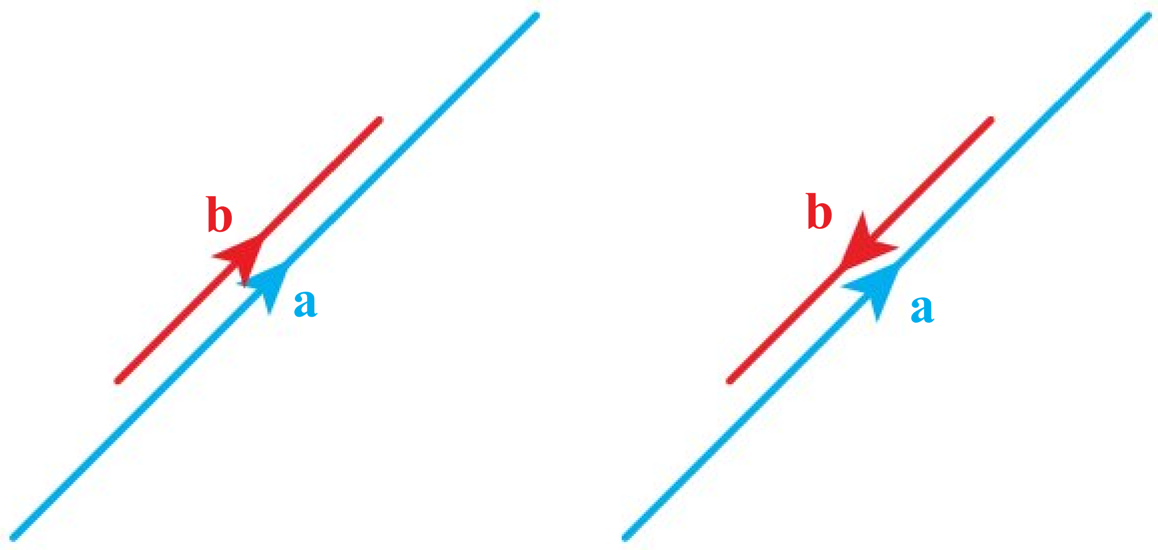
Two further important properties of the scalar product concern perpendicular and parallel vectors. They are derived using the facts that $\cos 90° = 0$, $\cos 0° = 1$ and $\cos 180° = -1$.

KEY POINT 2.15

- If $\mathbf{a}$ and $\mathbf{b}$ are perpendicular vectors, then $\mathbf{a} \bullet \mathbf{b} = 0$.
- If $\mathbf{a}$ and $\mathbf{b}$ are parallel vectors, then $|\mathbf{a} \bullet \mathbf{b}| = |\mathbf{a}||\mathbf{b}|$.

Tip

The angle between two parallel vectors can be either 0° or 180°.



WORKED EXAMPLE 2.22

Find the value of t such that the vector $\mathbf{a} = \begin{pmatrix} 3t \\ 1+t \\ 2-5t \end{pmatrix}$ is perpendicular to the vector $\mathbf{b} = \begin{pmatrix} 1 \\ -2 \\ 2 \end{pmatrix}$.

For perpendicular vectors, $\mathbf{a} \bullet \mathbf{b} = 0$

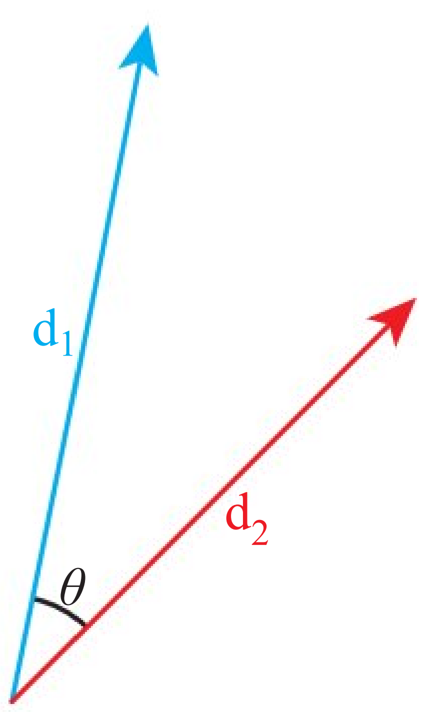
$$\begin{pmatrix} 3t \\ 1+t \\ 2-5t \end{pmatrix} \bullet \begin{pmatrix} 1 \\ -2 \\ 2 \end{pmatrix} = 0$$

Express the scalar product in terms of the components

$$(3t)(1) + (1+t)(-2) + (2-5t)(2) = 0$$

$$-9t + 2 = 0$$

$$t = \frac{2}{9}$$



Angle between two lines

You can find the angle between two lines by using their direction vectors.

KEY POINT 2.16

The angle between two lines is equal to the angle between their direction vectors.

WORKED EXAMPLE 2.23

Find the acute angle between the lines with equations

$$\mathbf{r} = \begin{pmatrix} 3 \\ -1 \\ 2 \end{pmatrix} + \lambda \begin{pmatrix} -4 \\ 0 \\ 3 \end{pmatrix} \quad \text{and} \quad \mathbf{r} = \begin{pmatrix} 1 \\ 0 \\ 3 \end{pmatrix} + \mu \begin{pmatrix} 1 \\ 1 \\ -2 \end{pmatrix}.$$

Identify the direction vectors

Direction vectors:

$$\mathbf{d}_1 = \begin{pmatrix} -4 \\ 0 \\ 3 \end{pmatrix}, \ \mathbf{d}_2 = \begin{pmatrix} 1 \\ 1 \\ -2 \end{pmatrix}$$

Use scalar product to find the angle between the direction vectors

$$\cos\theta = \frac{\mathbf{d}_1 \bullet \mathbf{d}_2}{|\mathbf{d}_1||\mathbf{d}_2|}$$

$$= \frac{(-4+0-6)}{\sqrt{4^2+0^2+3^2}\sqrt{1^2+1^2+2^2}}$$

$$= -0.816$$

$$\theta = \cos^{-1}(-0.816) = 144.7°$$

The question asks for the acute angle

$$180 - 144.7 = 35.3°$$

Be the Examiner 2.2

Find the acute angle between the lines with equations

$$\mathbf{r} = \begin{pmatrix} 5 \\ 0 \\ 2 \end{pmatrix} + \lambda \begin{pmatrix} -1 \\ 2 \\ 3 \end{pmatrix} \quad \text{and} \quad \mathbf{r} = \begin{pmatrix} 4 \\ -1 \\ 3 \end{pmatrix} + \mu \begin{pmatrix} 4 \\ -5 \\ 1 \end{pmatrix}.$$

Which is the correct solution? Identify the errors made in the incorrect solutions.

Solution 1	Solution 2	Solution 3
$\mathbf{d}_1 \bullet \mathbf{d}_2 = 20+0+6 = 26$	$\mathbf{d}_1 \bullet \mathbf{d}_2 = -4-10+3 = -11$	$\mathbf{d}_1 \bullet \mathbf{d}_2 = -4-10+3 = -11$
$\lvert\mathbf{d}_1\rvert = \sqrt{25+0+4} = \sqrt{29}$	$\lvert\mathbf{d}_1\rvert = \sqrt{1+4+9} = \sqrt{14}$	$\lvert\mathbf{d}_1\rvert = \sqrt{1+4+9} = \sqrt{14}$
$\lvert\mathbf{d}_2\rvert = \sqrt{16+1+9} = \sqrt{26}$	$\lvert\mathbf{d}_2\rvert = \sqrt{16+25+1} = \sqrt{42}$	$\lvert\mathbf{d}_2\rvert = \sqrt{16+25+1} = \sqrt{42}$
$\cos\theta = \frac{\mathbf{d}_1 \bullet \mathbf{d}_2}{\lvert\mathbf{d}_1\rvert\lvert\mathbf{d}_2\rvert}$	$\cos\theta = \frac{\mathbf{d}_1 \bullet \mathbf{d}_2}{\lvert\mathbf{d}_1\rvert\lvert\mathbf{d}_2\rvert}$	$\cos\theta = \frac{\mathbf{d}_1 \bullet \mathbf{d}_2}{\lvert\mathbf{d}_1\rvert\lvert\mathbf{d}_2\rvert}$
$= \frac{26}{\sqrt{29\times 26}} = 0.947$	$= \frac{-11}{\sqrt{14\times 42}} = -0.454$	$= \frac{-11}{\sqrt{14\times 42}} = -0.454$
$\theta = 18.8°$	$\theta = 117°$	$\theta = 117°$
	So, acute angle $= 180 - 117 = 63.0°$	So, acute angle $= 117 - 90 = 27.0°$

Tip

Remember that you can use the scalar product to identify perpendicular vectors. This is particularly useful for finding perpendicular distances.

WORKED EXAMPLE 2.24

Line l has equation $\mathbf{r} = \begin{pmatrix} 3 \\ -1 \\ 0 \end{pmatrix} + \lambda \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix}$ and point A has coordinates (3, 9, –2).

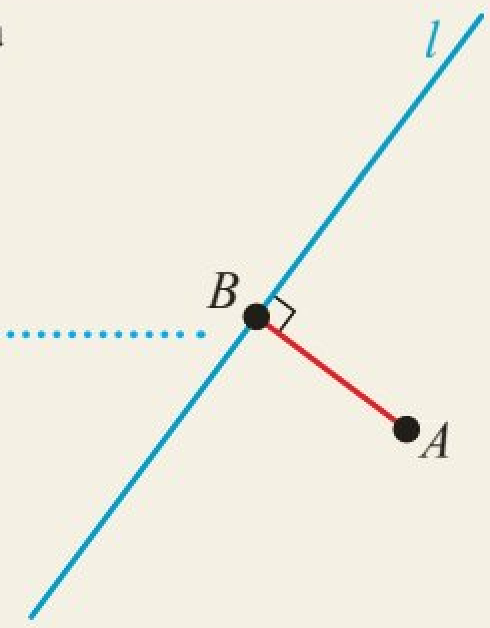
Point B lies on the line l and AB is perpendicular to l.

a Find the coordinates of B.

b Hence, find the shortest distance from A to l.

a

Draw a diagram. The line AB should be perpendicular to the direction vector of l



$$\overrightarrow{AB} \cdot \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix} = 0$$

We know that B lies on l, so its position vector is given by the equation for $\mathbf{r}$

$$\mathbf{b} = \begin{pmatrix} 3+\lambda \\ -1-\lambda \\ t \end{pmatrix}$$

$$\overrightarrow{AB} = \mathbf{b} - \mathbf{a}$$

$$= \begin{pmatrix} 3+\lambda \\ -1-\lambda \\ \lambda \end{pmatrix} - \begin{pmatrix} 3 \\ 9 \\ -2 \end{pmatrix} = \begin{pmatrix} \lambda \\ -10-\lambda \\ \lambda+2 \end{pmatrix}$$

We can now find the value of λ for which the two lines are perpendicular

$$\begin{pmatrix} \lambda \\ -10-\lambda \\ \lambda+2 \end{pmatrix} \cdot \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix} = 0$$

$$(\lambda) + (10+\lambda) + (\lambda+2) = 0$$

$$\lambda = -4$$

Using this value of λ in the equation of the line gives the position vector of B

$$\mathbf{r} = \begin{pmatrix} -1 \\ 3 \\ -4 \end{pmatrix}$$

B has coordinates (–1, 3, –4)

The shortest distance from a point to a line is the perpendicular distance, in other words, the distance AB

b $$\overrightarrow{AB} = \begin{pmatrix} -1 \\ 3 \\ -4 \end{pmatrix} - \begin{pmatrix} 3 \\ 9 \\ -2 \end{pmatrix} = \begin{pmatrix} -4 \\ -6 \\ -2 \end{pmatrix}$$

$$\left|\overrightarrow{AB}\right| = \sqrt{16+36+4} = 2\sqrt{14}$$

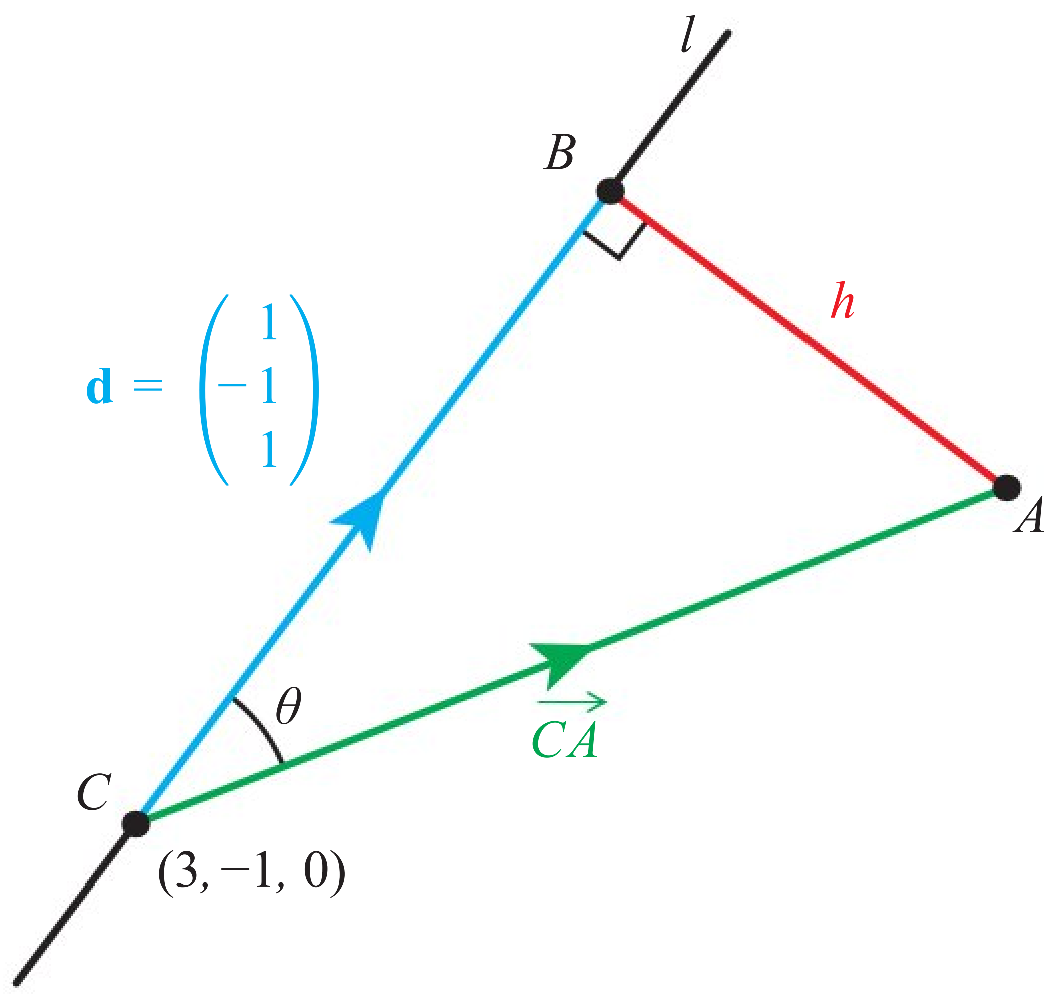
TOOLKIT: Problem Solving

Here are two alternative methods to solve the problem in Worked Example 2.24.

1 You found that $\overrightarrow{AB} = \begin{pmatrix} \lambda \\ -10-\lambda \\ \lambda+2 \end{pmatrix}$. Show that $AB^2 = 3\lambda^2 + 24\lambda + 104$.

Hence, find the smallest possible value of AB.

2 In the diagram below, h is the required shortest distance from A to the line. You know, from the equation of the line, that the point $C(3, -1, 0)$ lies on the line. θ is the angle between $\overrightarrow{CA}$ and the direction vector of the line.



a Find the length AC.

b Find the exact value of $\cos\theta$. Hence, show that $\sin\theta = \frac{6}{\sqrt{78}}$.

c Use the right-angled triangle ABC to find the length h.

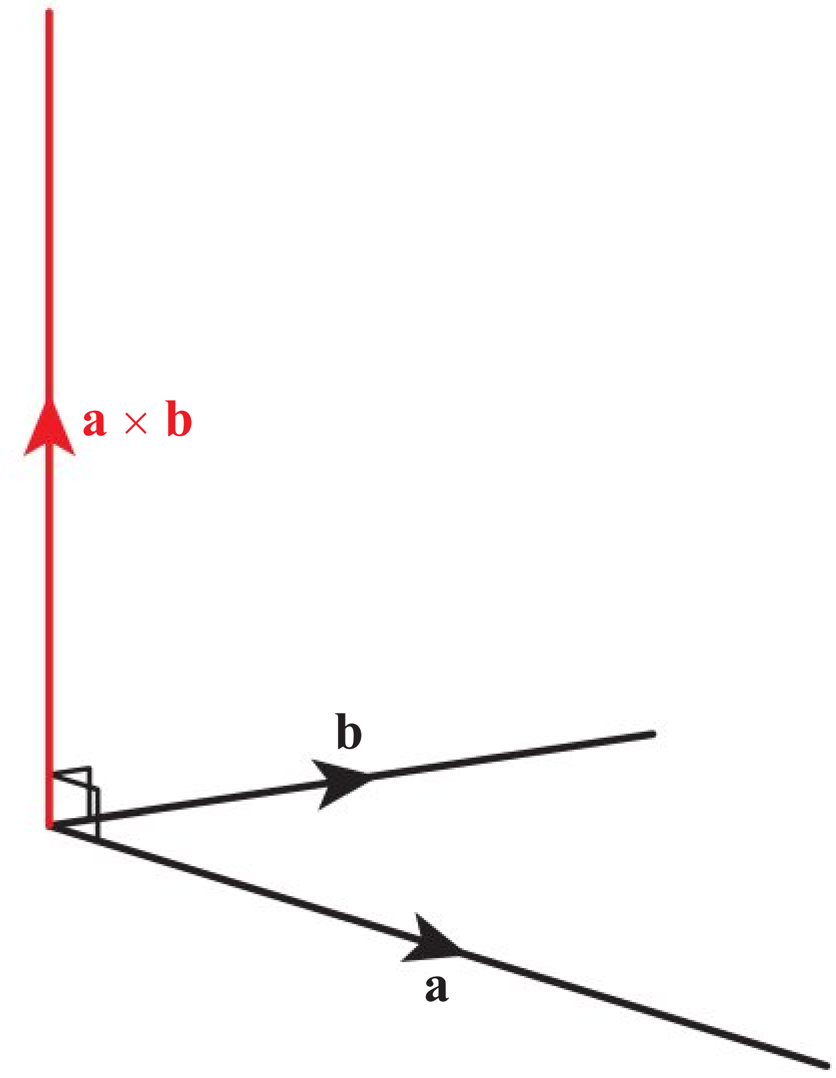
The definition and calculation of the vector product of two vectors

One way to define the vector product is to give its magnitude and direction.

KEY POINT 2.17

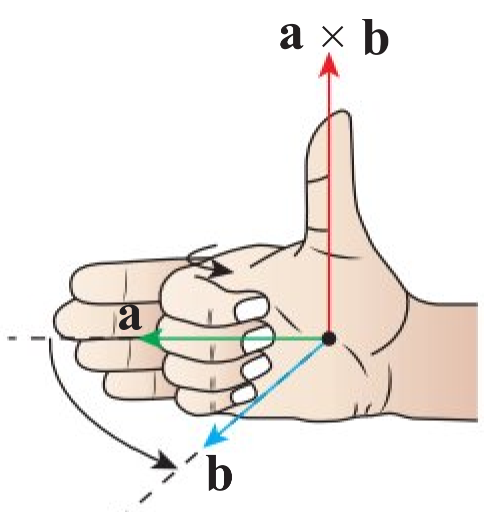
The **vector product** (or **cross product**) of vectors $\mathbf{a}$ and $\mathbf{b}$ is a vector denoted by $\mathbf{a} \times \mathbf{b}$.

- The magnitude is equal to $|\mathbf{a}||\mathbf{b}|\sin\theta$ (where θ is the angle between $\mathbf{a}$ and $\mathbf{b}$).
- The direction is perpendicular to both $\mathbf{a}$ and $\mathbf{b}$ (as shown in the diagram).



Tip

The direction of the vector $\mathbf{a} \times \mathbf{b}$ is given by the right-hand screw rule.

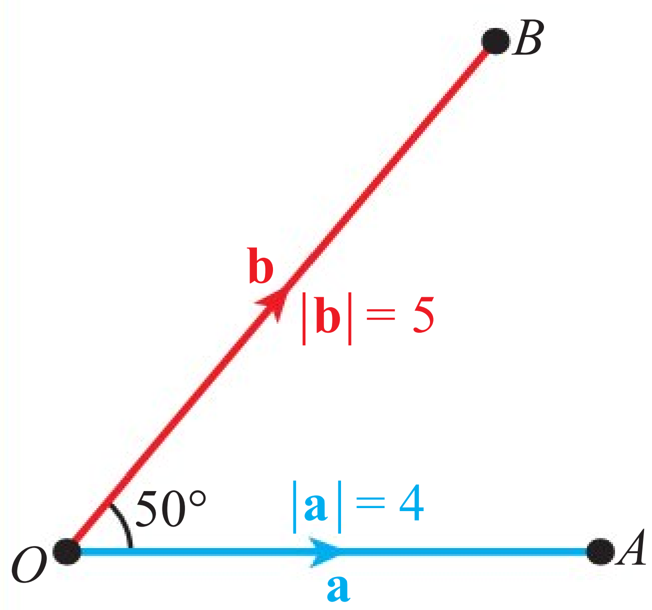


Links to: Physics

Vectors are used to model quantities in both mathematics and physics. In geometry, the vector product is used to find the direction which is perpendicular to two given vectors. In physics, it is used in many equations involving quantities that are modelled as vectors. For example, the angular momentum of a particle moving in a circle is given by $\mathbf{L} = m\mathbf{r} \times \mathbf{v}$, where $\mathbf{r}$ is the position vector and $\mathbf{v}$ is the velocity of the particle. In electrodynamics, the Lorentz force acting on a charge q moving with velocity $\boldsymbol{v}$ in magnetic field $\mathbf{B}$ is given by $\mathbf{F} = q\mathbf{v} \times \mathbf{B}$.

WORKED EXAMPLE 2.25

For the vectors **a** and **b** shown in the diagram, find the magnitude of $\mathbf{a} \times \mathbf{b}$.



Use $|\mathbf{a} \times \mathbf{b}| = |\mathbf{a}||\mathbf{b}|\sin\theta$

$$|\mathbf{a} \times \mathbf{b}| = (4)(5)\sin 50°$$
$$= 15.3$$

Tip

This formula is given in the Mathematics: applications and interpretation formula booklet. You could prove this by writing it out in terms of the unit vectors **i**, **j** and **k** then using the definition from Key Point 2.17.

The vector product can also be expressed in component form.

KEY POINT 2.18

If $\mathbf{a} = \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix}$ and $\mathbf{b} = \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix}$, then $\mathbf{a} \times \mathbf{b} = \begin{pmatrix} a_2b_3 - a_3b_2 \\ a_3b_1 - a_1b_3 \\ a_1b_2 - a_2b_1 \end{pmatrix}$.

WORKED EXAMPLE 2.26

Given that $\mathbf{a} = \begin{pmatrix} 1 \\ -3 \\ 2 \end{pmatrix}$ and $\mathbf{b} = \begin{pmatrix} 2 \\ 1 \\ 5 \end{pmatrix}$, find the vector $\mathbf{a} \times \mathbf{b}$.

Use the formula for the component form of $\mathbf{a} \times \mathbf{b}$

$$\mathbf{a} \times \mathbf{b} = \begin{pmatrix} (-3)(5) - (2)(1) \\ (2)(2) - (1)(5) \\ (1)(1) - (-3)(2) \end{pmatrix}$$

$$= \begin{pmatrix} -17 \\ -1 \\ 7 \end{pmatrix}$$

You can check that $\begin{pmatrix} -17 \\ -1 \\ 7 \end{pmatrix}$ is perpendicular to both $\begin{pmatrix} 1 \\ -3 \\ 2 \end{pmatrix}$ and $\begin{pmatrix} 2 \\ 1 \\ 5 \end{pmatrix}$.

Be the Examiner 2.3

Find $\begin{pmatrix} 2 \\ 5 \\ -2 \end{pmatrix} \times \begin{pmatrix} 1 \\ 3 \\ 4 \end{pmatrix}$.

Which is the correct solution? Identify the errors made in the incorrect solutions.

Solution 1	Solution 2	Solution 3
$\begin{pmatrix} 6-5 \\ 20+6 \\ 8+2 \end{pmatrix} = \begin{pmatrix} 1 \\ 26 \\ 10 \end{pmatrix}$	$\begin{pmatrix} 20-6 \\ 8-2 \\ 6-5 \end{pmatrix} = \begin{pmatrix} 14 \\ 6 \\ 1 \end{pmatrix}$	$\begin{pmatrix} 20+6 \\ -(8+2) \\ 6-5 \end{pmatrix} = \begin{pmatrix} 26 \\ -10 \\ 1 \end{pmatrix}$

It is worth remembering the special result for parallel and perpendicular vectors, which follows from the fact that $\sin 0° = \sin 180° = 0$ and $\sin 90° = 1$.

Tip

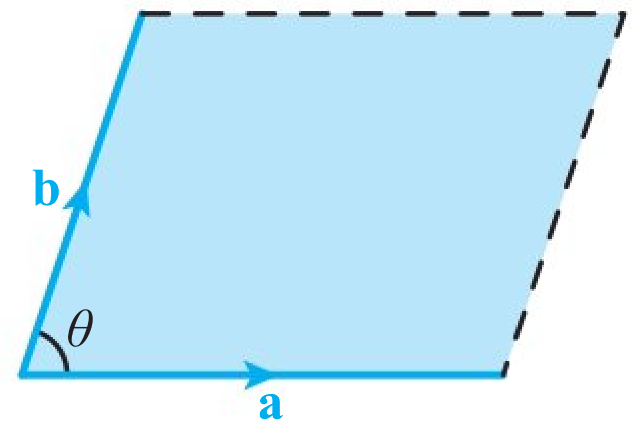
Notice that, since the vector product produces a vector, each zero in Key Point 2.19 is the zero *vector*, not a scalar value.

KEY POINT 2.19

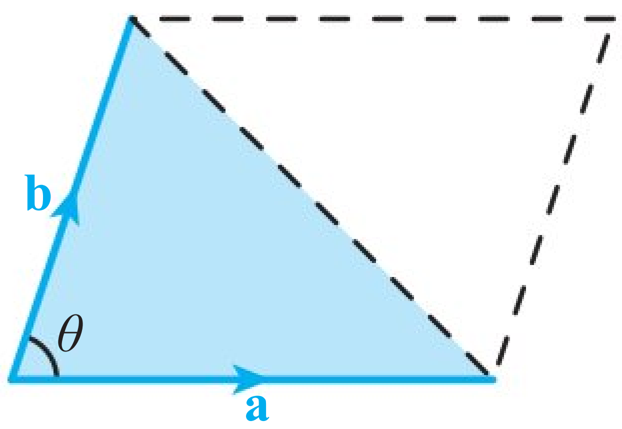
- If $\mathbf{a}$ and $\mathbf{b}$ are parallel vectors, then $\mathbf{a} \times \mathbf{b} = \mathbf{0}$.
- In particular, $\mathbf{a} \times \mathbf{a} = \mathbf{0}$.
- If $\mathbf{a}$ and $\mathbf{b}$ are perpendicular vectors, then $|\mathbf{a} \times \mathbf{b}| = |\mathbf{a}||\mathbf{b}|$.

Geometric interpretation

The magnitude of the vector product $\mathbf{a} \times \mathbf{b}$ is $|\mathbf{a}||\mathbf{b}| \sin\theta$. But this is also the area of the parallelogram determined by the vectors $\mathbf{a}$ and $\mathbf{b}$.



A parallelogram can be divided into two triangles, so you can also use the vector product to find the area of a triangle.



KEY POINT 2.20

The area of the triangle with two sides defined by vectors $\mathbf{a}$ and $\mathbf{b}$ is equal to $\frac{1}{2}|\mathbf{a} \times \mathbf{b}|$.

When you are given the coordinates of the vertices of a triangle, sketch a diagram to identify which two vectors to use.

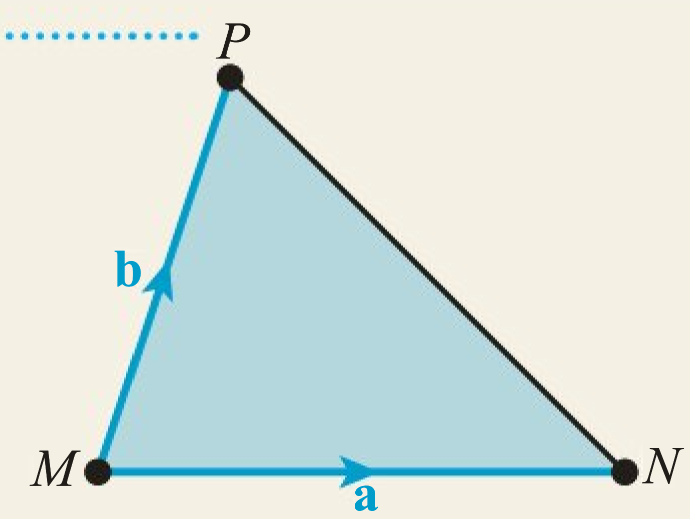
Tip

It does not matter which two sides of the triangle you use.

WORKED EXAMPLE 2.27

Find the area of the triangle with vertices $M(1, 4, 2)$, $N(3, -3, 0)$ and $P(-1, 8, 9)$.

Sketch a diagram to see which vectors to use



Two of the sides of the triangle are vectors $\overrightarrow{MN}$ and $\overrightarrow{MP}$

$$\mathbf{a} = \overrightarrow{MN} = \begin{pmatrix} 2 \\ -7 \\ -2 \end{pmatrix}$$

$$\mathbf{b} = \overrightarrow{MP} = \begin{pmatrix} -2 \\ 4 \\ 7 \end{pmatrix}$$

The area of the triangle is $\frac{1}{2}|\mathbf{a}\times\mathbf{b}|$. Find the vector $\mathbf{a}\times\mathbf{b}$ first …

$$\mathbf{a}\times\mathbf{b} = \begin{pmatrix}2\\-7\\-2\end{pmatrix}\times\begin{pmatrix}-2\\4\\7\end{pmatrix}$$

$$= \begin{pmatrix}-49+8\\4-14\\8-14\end{pmatrix} = \begin{pmatrix}-41\\-10\\-6\end{pmatrix}$$

… then find half of its magnitude

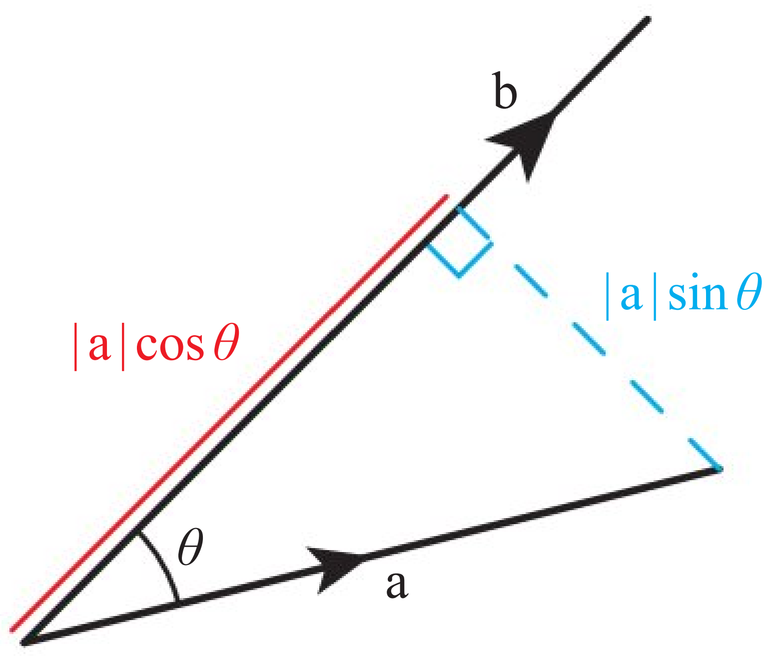
$$|\mathbf{a}\times\mathbf{b}| = \sqrt{41^2+10^2+6^2} = 42.62\ \ldots$$

$$\text{Area} = \frac{1}{2}|\mathbf{a}\times\mathbf{b}| = 21.3\ (3\text{ s.f.})$$

Components of vectors

A vector is given in terms of components in the direction of the base vectors. For example, given the force $\mathbf{F} = (2\mathbf{i} - 3\mathbf{j} + 4\mathbf{k})\,\text{N}$, we know that a 2 N force acts in the direction of the vector $\mathbf{i}$, a 3 N force acts in the direction of the vector $-\mathbf{j}$ and a 4 N force acts in the direction of the vector $\mathbf{k}$.

However, often we want to know what the component of a vector is in a direction of a vector that is not one of the base vectors. We can do this using trigonometry.



So, the component of $\mathbf{a}$ in the direction of $\mathbf{b}$ is $|\mathbf{a}|\cos\theta$.

We can also see that the component of $\mathbf{a}$ perpendicular to $\mathbf{b}$ is $|\mathbf{a}|\sin\theta$.

Since $\mathbf{a}\bullet\mathbf{b} = |\mathbf{a}||\mathbf{b}|\cos\theta$ and $|\mathbf{a}\times\mathbf{b}| = |\mathbf{a}||\mathbf{b}|\sin\theta$ we can express the results shown above in terms of the dot and cross product respectively.

KEY POINT 2.21

The component of $\mathbf{a}$ acting

- in the direction of $\mathbf{b}$ is $\dfrac{\mathbf{a}\bullet\mathbf{b}}{|\mathbf{b}|}$
- perpendicular to $\mathbf{b}$ is $\dfrac{|\mathbf{a}\times\mathbf{b}|}{|\mathbf{b}|}$

WORKED EXAMPLE 2.28

A tugboat pulls a barge into port. The tugboat starts pulling with a force of $\begin{pmatrix} 8 \\ 5 \\ -2 \end{pmatrix}$ kN and the direction back to port initially is $\begin{pmatrix} 11 \\ 4 \\ 0 \end{pmatrix}$.

Find the component of the force in the direction back to port.

The component of the force, F, acting in the direction of the port d is $\frac{\mathbf{F} \cdot \mathbf{d}}{|\mathbf{d}|}$

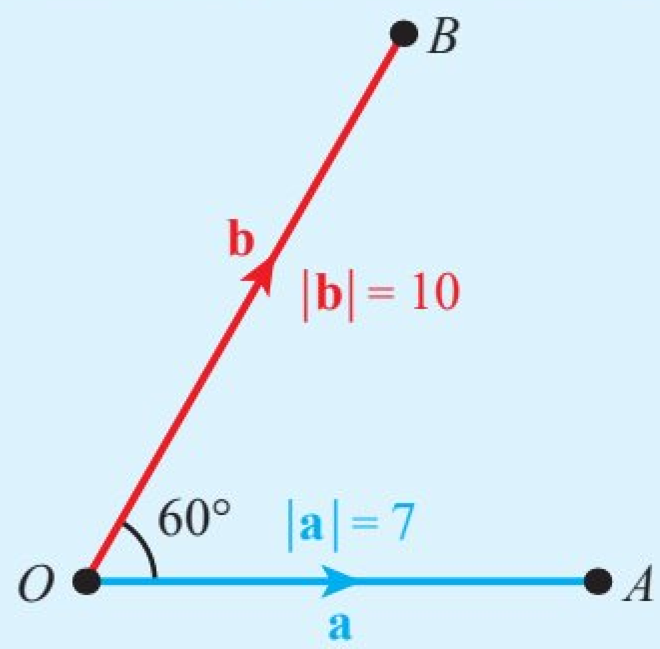
The component of $\begin{pmatrix} 8 \\ 5 \\ -2 \end{pmatrix}$ in the direction of $\begin{pmatrix} 11 \\ 4 \\ 0 \end{pmatrix}$ is

$$\frac{\begin{pmatrix} 8 \\ 5 \\ -2 \end{pmatrix} \cdot \begin{pmatrix} 11 \\ 4 \\ 0 \end{pmatrix}}{\left|\begin{matrix} 11 \\ 4 \\ 0 \end{matrix}\right|} = \frac{108}{\sqrt{11^2 + 4^2 + 0^2}} = 9.23\text{kN}$$

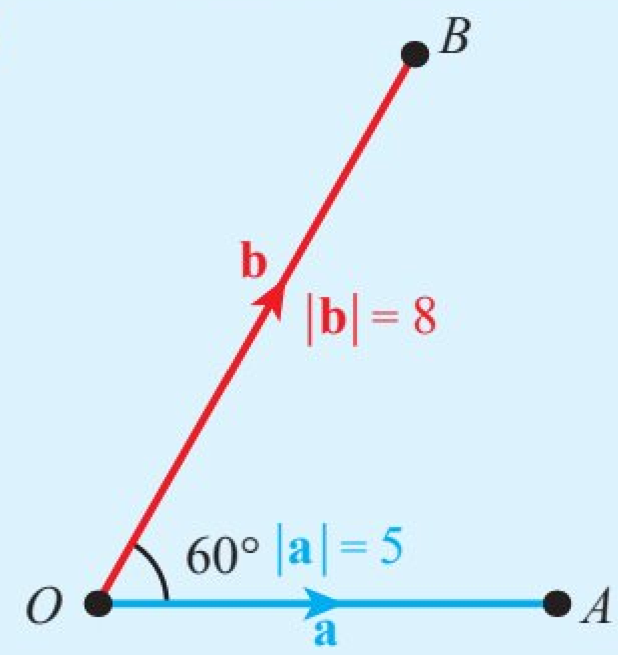
Exercise 2D

For questions 1 to 3, use the method demonstrated in Worked Example 2.19 to find $\mathbf{a} \cdot \mathbf{b}$ for the vectors in each diagram.

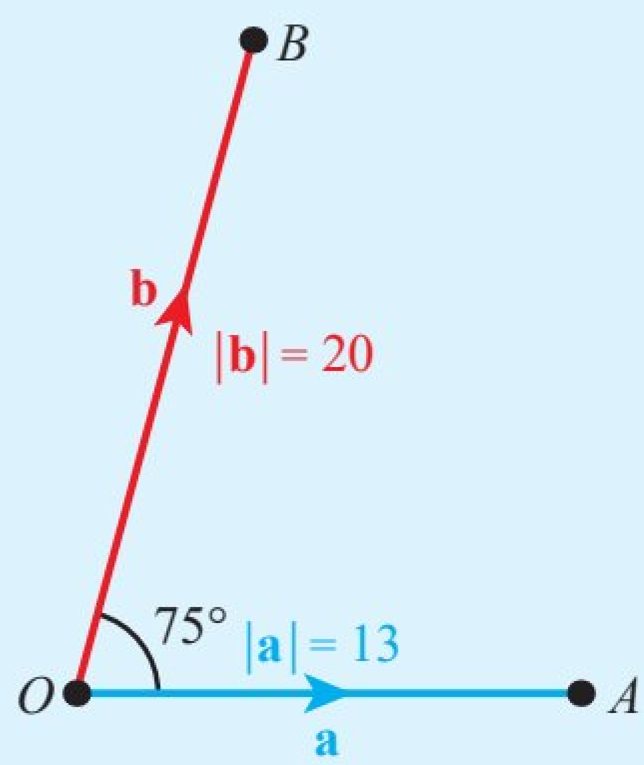
1 a



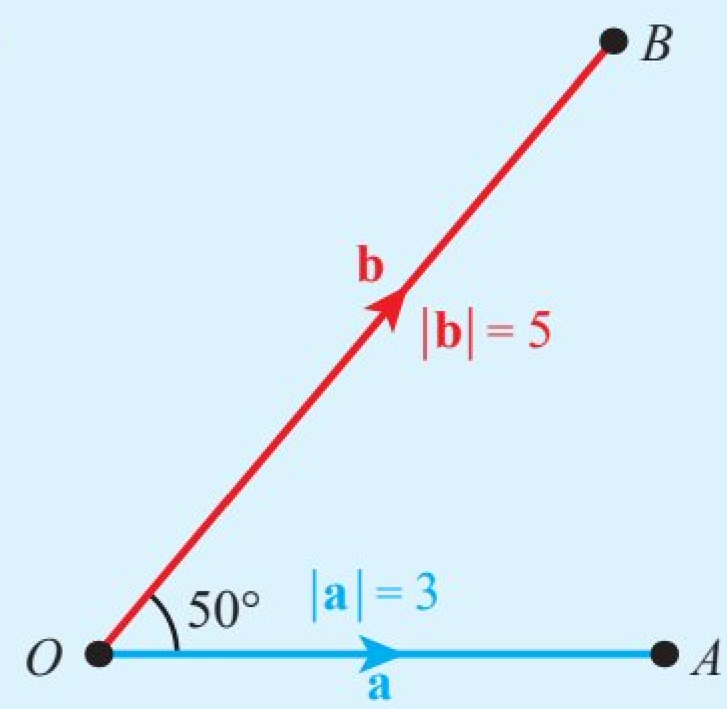
b

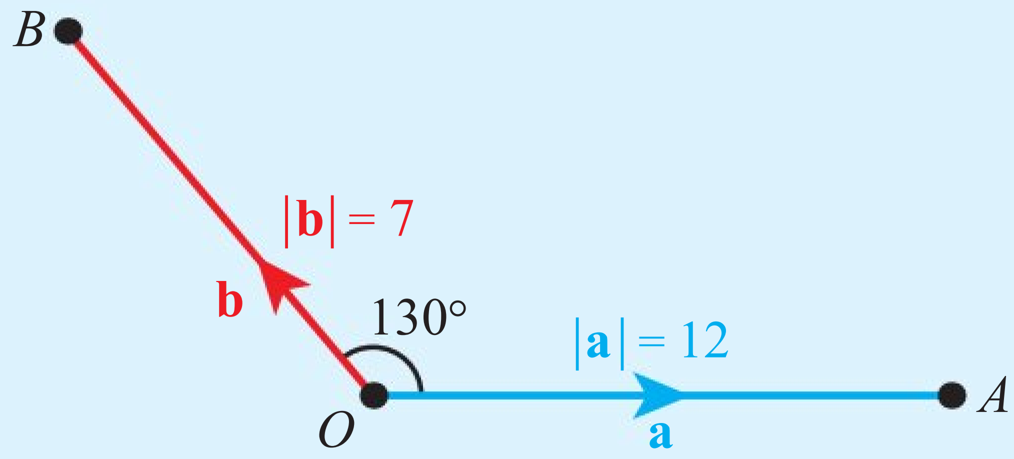


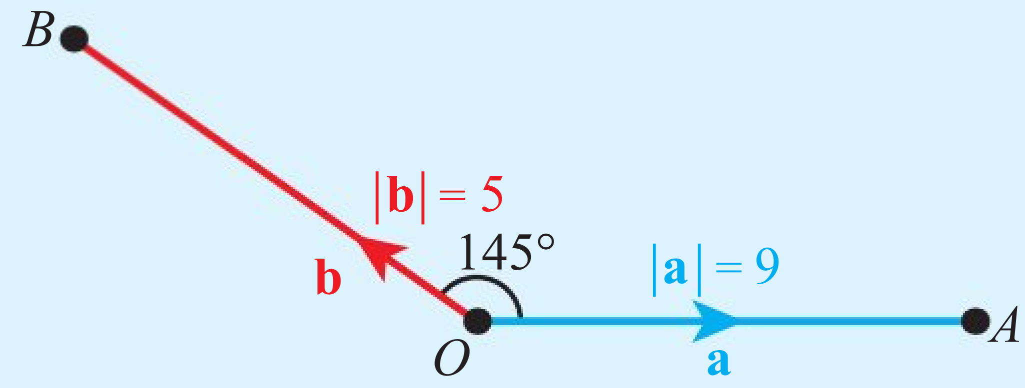
2 a



b



3 **a** 

b 

For questions 4 to 6, use the method demonstrated in Worked Example 2.20 to find $\mathbf{a} \bullet \mathbf{b}$ for the two given vectors.

4 **a** $\mathbf{a} = \begin{pmatrix} 2 \\ 1 \\ 2 \end{pmatrix}$ and $\mathbf{b} = \begin{pmatrix} 5 \\ 2 \\ 2 \end{pmatrix}$ **b** $\mathbf{a} = \begin{pmatrix} 3 \\ -1 \\ 2 \end{pmatrix}$ and $\mathbf{b} = \begin{pmatrix} -12 \\ 4 \\ -8 \end{pmatrix}$

5 **a** $\mathbf{a} = \begin{pmatrix} 3 \\ 0 \\ 2 \end{pmatrix}$ and $\mathbf{b} = \begin{pmatrix} 0 \\ 0 \\ -8 \end{pmatrix}$ **b** $\mathbf{a} = \begin{pmatrix} 0 \\ -3 \\ 2 \end{pmatrix}$ and $\mathbf{b} = \begin{pmatrix} 4 \\ 0 \\ 5 \end{pmatrix}$

6 **a** $\mathbf{a} = 4\mathbf{i} + 2\mathbf{j} + \mathbf{k}$ and $\mathbf{b} = \mathbf{i} + \mathbf{j} + 3\mathbf{k}$ **b** $\mathbf{a} = 4\mathbf{i} - 2\mathbf{j} + \mathbf{k}$ and $\mathbf{b} = \mathbf{i} - \mathbf{j} + 3\mathbf{k}$

For questions 7 to 10, use the method demonstrated in Worked Example 2.21 to find the required angle, giving your answer to the nearest degree.

7 **a** Angle between vectors $\begin{pmatrix} 5 \\ 1 \\ 2 \end{pmatrix}$ and $\begin{pmatrix} 1 \\ -2 \\ 3 \end{pmatrix}$ **b** Angle between vectors $\begin{pmatrix} 3 \\ 0 \\ 2 \end{pmatrix}$ and $\begin{pmatrix} 0 \\ -1 \\ 1 \end{pmatrix}$

8 **a** Angle between vectors $2\mathbf{i} + 2\mathbf{j} - \mathbf{k}$ and $\mathbf{i} - \mathbf{j} + 3\mathbf{k}$ **b** Angle between vectors $\mathbf{i} - \mathbf{j}$ and $2\mathbf{i} + 3\mathbf{j}$

9 **a** Angle $B\hat{A}C$, where $A(2, 1, 0)$, $B(3, 1, 2)$, $C(4, 4, 1)$ **b** Angle $B\hat{A}C$, where $A(2, 1, 0)$, $B(3, 0, 0)$, $C(2, -2, 4)$

10 **a** Angle $A\hat{B}C$, where $A(3, 6, 5)$, $B(2, 3, 6)$, $C(4, 0, 1)$ **b** Angle $A\hat{B}C$, where $A(8, -1, 2)$, $B(3, 1, 2)$, $C(0, -2, 0)$

For questions 11 to 14, use the method demonstrated in Worked Example 2.22 to find the value t such that the two given vectors are perpendicular.

11 **a** $\begin{pmatrix} t+1 \\ 2t-1 \\ 2t \end{pmatrix}$ and $\begin{pmatrix} 2 \\ 6 \\ 0 \end{pmatrix}$ **b** $\begin{pmatrix} 2t \\ 1 \\ -3t \end{pmatrix}$ and $\begin{pmatrix} 1 \\ -2 \\ 2 \end{pmatrix}$

12 **a** $\begin{pmatrix} t+1 \\ -2 \\ 1 \end{pmatrix}$ and $\begin{pmatrix} 1 \\ t \\ 2 \end{pmatrix}$ **b** $\begin{pmatrix} 1 \\ t+1 \\ 1 \end{pmatrix}$ and $\begin{pmatrix} -4 \\ 5 \\ 3-7t \end{pmatrix}$

13 **a** $(5-t)\mathbf{i} + 3\mathbf{j} - (10-t)\mathbf{k}$ and $-3\mathbf{i} + 6\mathbf{j} + 2\mathbf{k}$ **b** $(2t)\mathbf{i} + (t+1)\mathbf{j} - 5\mathbf{k}$ and $4\mathbf{i} - \mathbf{j} + 3\mathbf{k}$

14 **a** $5t\mathbf{i} - (2+t)\mathbf{j} + \mathbf{k}$ and $3\mathbf{i} + 4\mathbf{j} - t\mathbf{k}$ **b** $t\mathbf{i} - 3\mathbf{j}$ and $2\mathbf{i} + (t+4)\mathbf{j}$

For questions 15 to 17, use the method demonstrated in Worked Example 2.23 to find the acute angle between the two given lines.

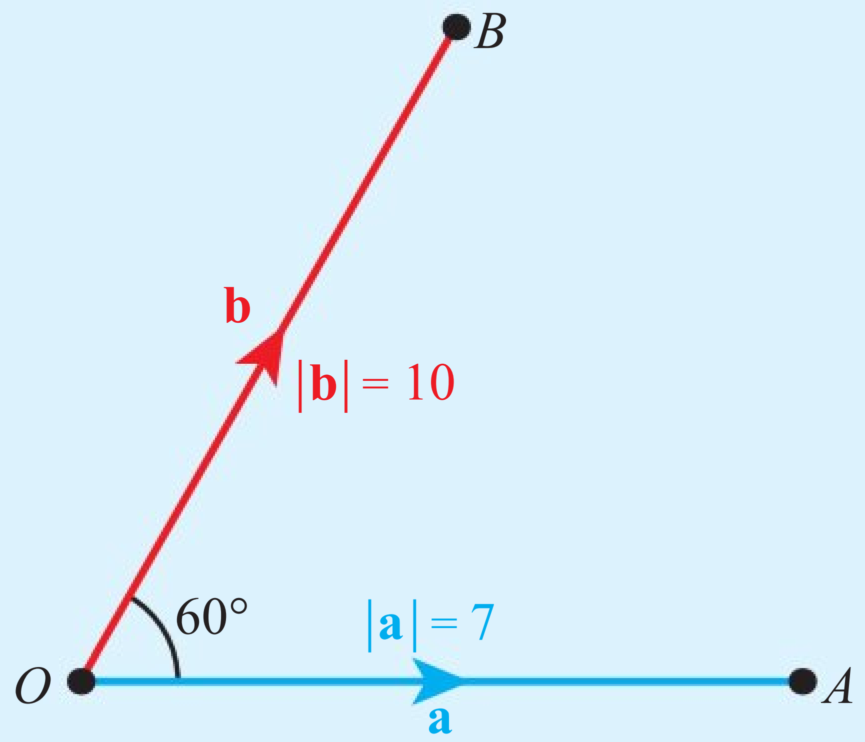
15 **a** $\mathbf{r} = \begin{pmatrix} 2 \\ 2 \\ -1 \end{pmatrix} + \lambda \begin{pmatrix} 1 \\ 4 \\ 3 \end{pmatrix}$ and $\mathbf{r} = \begin{pmatrix} 3 \\ 1 \\ 2 \end{pmatrix} + \lambda \begin{pmatrix} 2 \\ 2 \\ 5 \end{pmatrix}$ **b** $\mathbf{r} = \begin{pmatrix} 1 \\ 0 \\ 3 \end{pmatrix} + \lambda \begin{pmatrix} 2 \\ 2 \\ 1 \end{pmatrix}$ and $\mathbf{r} = \begin{pmatrix} -1 \\ 3 \\ 0 \end{pmatrix} + \lambda \begin{pmatrix} 4 \\ 1 \\ 3 \end{pmatrix}$

16 **a** $\mathbf{r} = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} + \lambda \begin{pmatrix} -1 \\ 1 \\ 3 \end{pmatrix}$ and $\mathbf{r} = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} + \lambda \begin{pmatrix} 2 \\ 2 \\ -5 \end{pmatrix}$ **b** $\mathbf{r} = \begin{pmatrix} 3 \\ 0 \\ 1 \end{pmatrix} + \lambda \begin{pmatrix} -5 \\ 4 \\ 3 \end{pmatrix}$ and $\mathbf{r} = \begin{pmatrix} 2 \\ 0 \\ 0 \end{pmatrix} + \lambda \begin{pmatrix} 1 \\ -1 \\ 2 \end{pmatrix}$

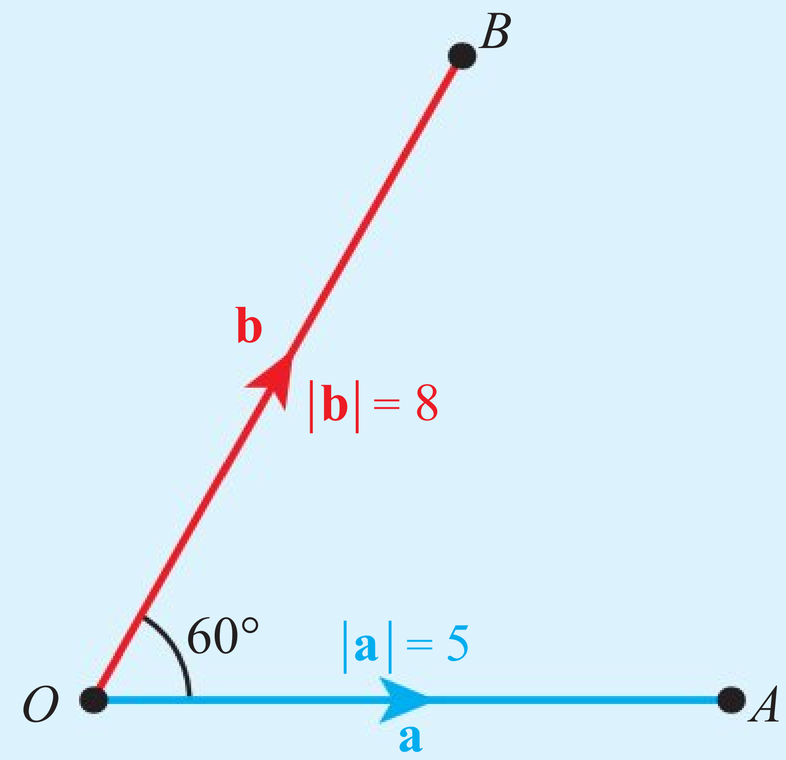
17 **a** $\mathbf{r} = \begin{pmatrix} 3 \\ -1 \end{pmatrix} + \lambda \begin{pmatrix} 5 \\ 4 \end{pmatrix}$ and $\mathbf{r} = \begin{pmatrix} 0 \\ 2 \end{pmatrix} + \lambda \begin{pmatrix} -1 \\ 1 \end{pmatrix}$ **b** $\mathbf{r} = \begin{pmatrix} 1 \\ 1 \end{pmatrix} + \lambda \begin{pmatrix} -2 \\ 3 \end{pmatrix}$ and $\mathbf{r} = \begin{pmatrix} 3 \\ 2 \end{pmatrix} + \lambda \begin{pmatrix} 1 \\ -2 \end{pmatrix}$

For questions 18 to 20, use the method demonstrated in Worked Example 2.25 to find the magnitude of $\mathbf{a} \times \mathbf{b}$ for the vectors in each diagram.

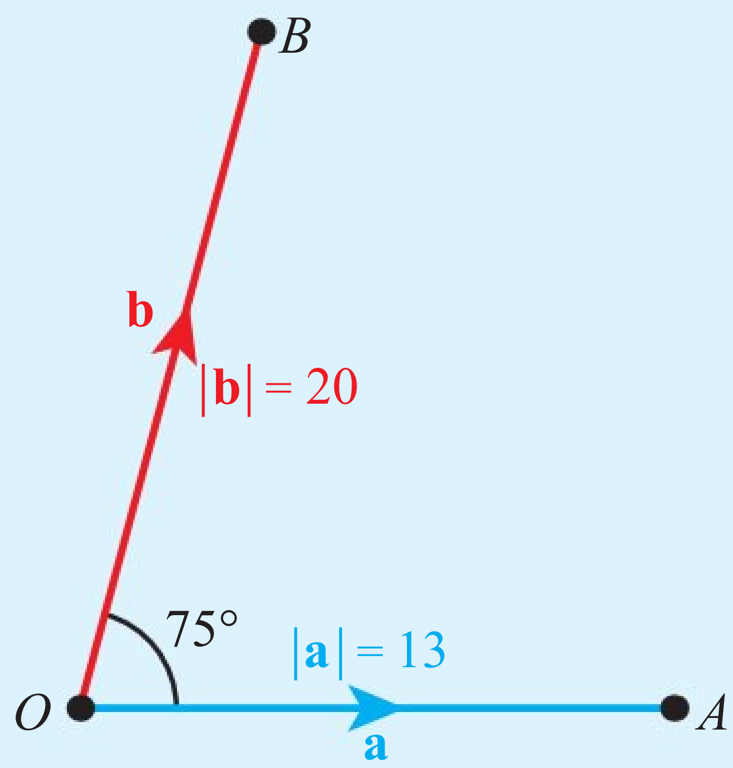
18 **a**



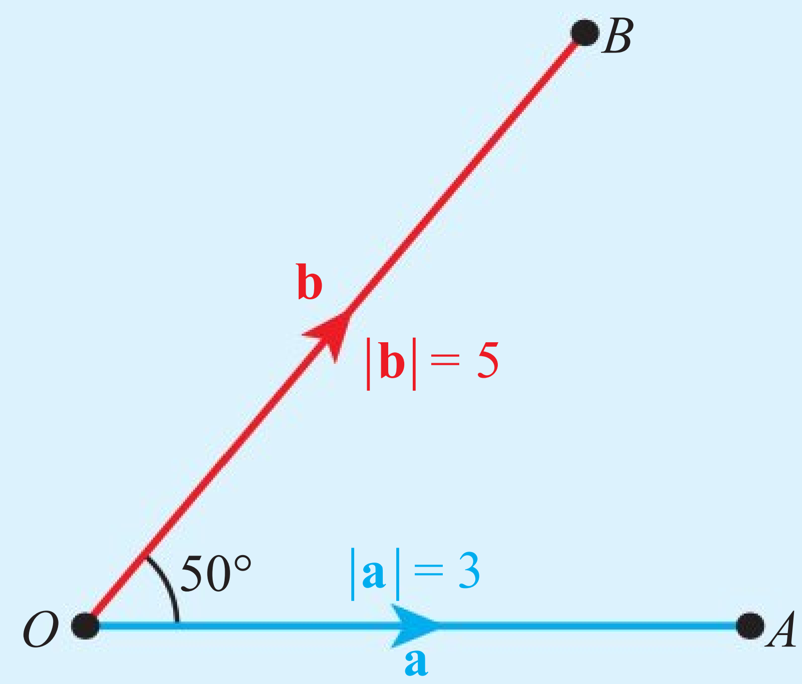
b



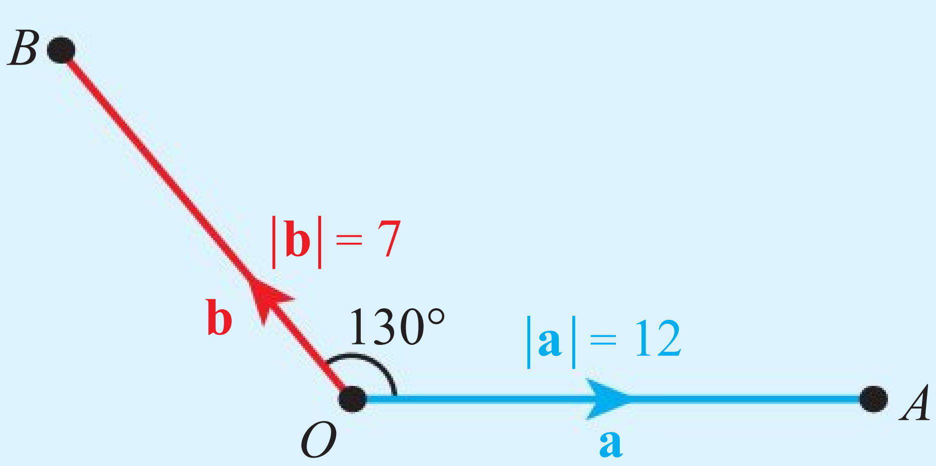
19 **a**



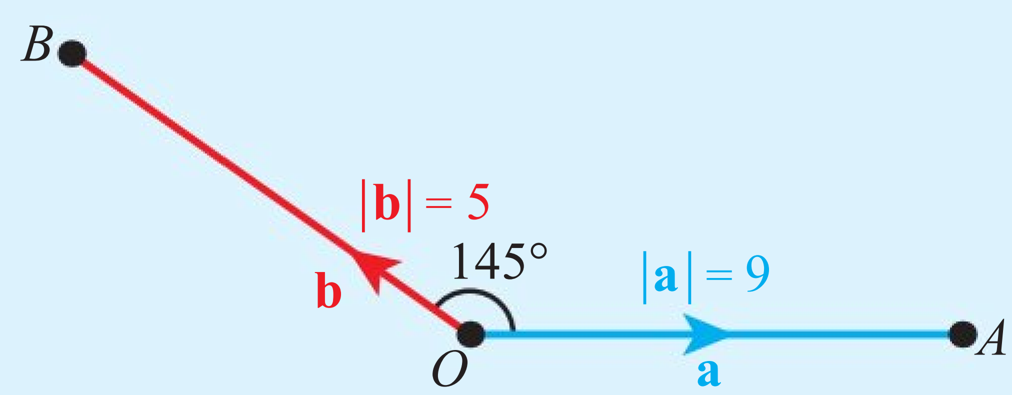
b



20 **a**



b



For questions 21 to 23, use the method demonstrated in Worked Example 2.26 to find $\mathbf{a} \times \mathbf{b}$ for the two given vectors.

21 **a** $\mathbf{a} = \begin{pmatrix} 2 \\ 1 \\ 2 \end{pmatrix}$ and $\mathbf{b} = \begin{pmatrix} 5 \\ 2 \\ 2 \end{pmatrix}$

b $\mathbf{a} = \begin{pmatrix} 4 \\ 1 \\ 2 \end{pmatrix}$ and $\mathbf{b} = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}$

22 **a** $\mathbf{a} = \begin{pmatrix} 3 \\ -1 \\ 4 \end{pmatrix}$ and $\mathbf{b} = \begin{pmatrix} -1 \\ 1 \\ 5 \end{pmatrix}$

b $\mathbf{a} = \begin{pmatrix} 1 \\ 7 \\ 2 \end{pmatrix}$ and $\mathbf{b} = \begin{pmatrix} -1 \\ 1 \\ -3 \end{pmatrix}$

23 **a** $\mathbf{a} = 4\mathbf{i} - 2\mathbf{j} + \mathbf{k}$ and $\mathbf{b} = \mathbf{i} - \mathbf{j} + 3\mathbf{k}$

b $\mathbf{a} = -3\mathbf{j} + \mathbf{k}$ and $\mathbf{b} = 2\mathbf{i} - 4\mathbf{k}$

For questions 24 to 26, use the method demonstrated in Worked Example 2.27 to find the area of the triangle with the given vertices.

24 **a** $(1, 3, 3)$, $(-1, 1, 2)$ and $(1, -2, 4)$

b $(3, -5, 1)$, $(-1, 1, 3)$ and $(-1, -5, 2)$

25 **a** $(-3, -5, 1)$, $(4, 7, 2)$ and $(-1, 2, 2)$

b $(4, 0, 2)$, $(4, 1, 5)$ and $(4, -3, 2)$

26 **a** $(1, 5, 2)$, $(8, 4, 6)$ and $(0, 6, 7)$

b $(2, 1, 2)$, $(3, 8, 4)$ and $(1, 3, -1)$

For questions 27 to 32, use the method demonstrated in Worked Example 2.28 to find the component of the vector **v** in the given direction.

27 **a** $\mathbf{v} = 3\mathbf{i} - 2\mathbf{j}$ in the direction of $4\mathbf{i} + 5\mathbf{j}$

b $\mathbf{v} = 5\mathbf{i} + 2\mathbf{j}$ in the direction of $3\mathbf{i} - \mathbf{j}$

28 **a** $\mathbf{v} = \begin{pmatrix} 4 \\ 1 \end{pmatrix}$ in the direction of $\begin{pmatrix} 2 \\ 3 \end{pmatrix}$

b $\mathbf{v} = \begin{pmatrix} -6 \\ 5 \end{pmatrix}$ in the direction of $\begin{pmatrix} -1 \\ 2 \end{pmatrix}$

29 **a** $\mathbf{v} = 7\mathbf{i} + 4\mathbf{j} - 2\mathbf{k}$ in the direction of $3\mathbf{j} - 4\mathbf{k}$

b $\mathbf{v} = \mathbf{i} - 3\mathbf{j} + 4\mathbf{k}$ in the direction of $\mathbf{i} + \mathbf{j} + \mathbf{k}$

30 **a** $\mathbf{v} = \begin{pmatrix} 6 \\ 2 \\ 3 \end{pmatrix}$ in the direction of $\begin{pmatrix} 1 \\ 5 \\ 2 \end{pmatrix}$

b $\mathbf{v} = \begin{pmatrix} -2 \\ 7 \\ 1 \end{pmatrix}$ in the direction of $\begin{pmatrix} 3 \\ 0 \\ 8 \end{pmatrix}$

31 **a** $\mathbf{v} = -3\mathbf{i} + \mathbf{j} + 5\mathbf{k}$ perpendicular to $\mathbf{i} - 2\mathbf{j} + \mathbf{k}$

b $\mathbf{v} = \mathbf{i} - 4\mathbf{j} + \mathbf{k}$ perpendicular to $2\mathbf{i} + \mathbf{j} + 3\mathbf{k}$

32 **a** $\mathbf{v} = \begin{pmatrix} 5 \\ 2 \\ 4 \end{pmatrix}$ perpendicular to $\begin{pmatrix} -2 \\ 0 \\ 1 \end{pmatrix}$

b $\mathbf{v} = \begin{pmatrix} -2 \\ 4 \\ 1 \end{pmatrix}$ perpendicular to $\begin{pmatrix} 3 \\ 1 \\ 6 \end{pmatrix}$

33 Points A and B have position vectors $\overrightarrow{OA} = \begin{pmatrix} 2 \\ 2 \\ 3 \end{pmatrix}$ and $\overrightarrow{OB} = \begin{pmatrix} -1 \\ 7 \\ 2 \end{pmatrix}$. Find the angle between $\overrightarrow{AB}$ and $\overrightarrow{OA}$.

34 Four points are given with coordinates $A(2, -1, 3)$, $B(1, 1, 2)$, $C(6, -1, 2)$ and $D(7, -3, 3)$. Find the angle between $\overrightarrow{AC}$ and $\overrightarrow{BD}$.

35 Given that $|\mathbf{a}| = 3$, $|\mathbf{b}| = 5$ and $\mathbf{a} \bullet \mathbf{b} = 10$ find, in degrees, the angle between **a** and **b**.

36 Given that $|\mathbf{c}| = 9$, $|\mathbf{d}| = 12$ and $\mathbf{c} \bullet \mathbf{d} = -15$ find, in degrees, the angle between **c** and **d**.

37 Given that $|\mathbf{a}| = 8$, $\mathbf{a} \bullet \mathbf{b} = 12$ and the angle between **a** and **b** is $60°$, find the value of $|\mathbf{b}|$.

38 Find the acute angle between lines with equations $\mathbf{r} = \begin{pmatrix} 4 \\ 1 \\ -2 \end{pmatrix} + t\begin{pmatrix} 1 \\ -1 \\ 3 \end{pmatrix}$ and $\mathbf{r} = \begin{pmatrix} 4 \\ 1 \\ -2 \end{pmatrix} + \lambda\begin{pmatrix} -1 \\ 4 \\ 1 \end{pmatrix}$.

39 Show that the lines with equations $\mathbf{r} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} + \lambda\begin{pmatrix} 2 \\ -1 \\ 3 \end{pmatrix}$ and $\mathbf{r} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} + \mu\begin{pmatrix} 2 \\ 1 \\ -1 \end{pmatrix}$ are perpendicular.

40 Given that $|\mathbf{a}| = 5$, $|\mathbf{b}| = 7$, and the angle between **a** and **b** is $30°$ find the exact value of $|\mathbf{a} \times \mathbf{b}|$.

41 Given that $|\mathbf{a}| = 2$, $|\mathbf{b}| = 5$ and $|\mathbf{a} \times \mathbf{b}| = 7$, find, in radians, the acute angle between the directions of vectors **a** and **b**.

42 Given that $|\mathbf{a}| = 7$, $|\mathbf{b}| = 1$ and $\mathbf{a} \times \mathbf{b} = 2\mathbf{i} - 3\mathbf{j} - 2\mathbf{k}$, find, in radians, the acute angle between the directions of vectors **a** and **b**.

43 Find a vector perpendicular to both $\mathbf{a} = 2\mathbf{i} - 3\mathbf{j} + \mathbf{k}$ and $\mathbf{b} = \mathbf{i} - \mathbf{j} + 3\mathbf{k}$.

44 A rowing boat is travelling at a speed of $7\,\text{km h}^{-1}$ in the direction $\begin{pmatrix}4\\1\end{pmatrix}$.

The river is flowing at $10\,\text{km h}^{-1}$ in the direction $\begin{pmatrix}2\\3\end{pmatrix}$.

a Write the velocity of the river as a column vector.

b Find the component of the water's velocity acting in the direction in which the boat is travelling.

45 Find, in degrees, the angles of the triangle with vertices $(1, 1, 3)$, $(2, -1, 1)$ and $(5, 1, 2)$.

46 The vertices of a triangle have position vectors $\mathbf{a} = \begin{pmatrix}2\\1\\2\end{pmatrix}$, $\mathbf{b} = \begin{pmatrix}4\\-1\\5\end{pmatrix}$ and $\mathbf{c} = \begin{pmatrix}7\\1\\-2\end{pmatrix}$.
Find, in degrees, the angles of the triangle.

47 Vertices of a triangle have position vectors $\mathbf{a} = \mathbf{i} - 2\mathbf{j} + 2\mathbf{k}$, $\mathbf{b} = 3\mathbf{i} - \mathbf{j} + 7\mathbf{k}$ and $\mathbf{c} = 5\mathbf{i}$.

a Show that the triangle is right angled.

b Calculate the other two angles of the triangle.

c Find the area of the triangle.

48 Given that $\mathbf{p}$ is a unit vector making a $45°$ angle with vector $\mathbf{q}$, and that $\mathbf{p} \cdot \mathbf{q} = 3\sqrt{2}$, find $|\mathbf{q}|$.

49 Given that $\mathbf{p} = \begin{pmatrix}4\\-1\\2\end{pmatrix}$ and $\mathbf{q} = \begin{pmatrix}2\\1\\1\end{pmatrix}$, find the value of the scalar t such that $\mathbf{p} + t\mathbf{q}$ is perpendicular to $\begin{pmatrix}3\\5\\1\end{pmatrix}$.

50 Given that the vectors $t\mathbf{i} - 3\mathbf{k}$ and $2t\mathbf{i} + \mathbf{j} + t\mathbf{k}$ are perpendicular, find the possible values of t.

51 A line has parametric equation

$$x = \frac{1}{2} + \frac{3}{2}\lambda,\ y = 7,\ z = 2 - 4\lambda$$

a Find a vector equation of the line.

b Find the angle that the line makes with the x-axis.

52 Find, in degrees, the acute angle between the lines

$$\begin{cases} x = 3 + 5\lambda \\ y = 2 + \lambda \\ z = 3 - 2\lambda \end{cases} \quad \text{and} \quad \begin{cases} x = 3\mu - 1 \\ y = 1 \\ z = 3 - \mu \end{cases}$$

53 Find a unit vector perpendicular to both $\begin{pmatrix}2\\1\\1\end{pmatrix}$ and $\begin{pmatrix}1\\1\\1\end{pmatrix}$.

54 Find a unit vector perpendicular to both $\begin{pmatrix}1\\1\\2\end{pmatrix}$ and $\begin{pmatrix}3\\1\\5\end{pmatrix}$.

55 a Explain why $(\mathbf{a} \times \mathbf{b}) \cdot \mathbf{a} = 0$.

b Evaluate $(\mathbf{a} \times \mathbf{b}) \cdot (\mathbf{a} - \mathbf{b})$.

56 Given points A, B and C with coordinates $(3, -5, 1)$, $(7, 7, 2)$ and $(-1, 1, 3)$,

a calculate $\mathbf{p} = \overrightarrow{AB} \times \overrightarrow{AC}$ and $\mathbf{q} = \overrightarrow{BA} \times \overrightarrow{BC}$.

b What can you say about vectors $\mathbf{p}$ and $\mathbf{q}$?

57 A quadrilateral has vertices $A(1, 4, 2)$, $B(3, -3, 0)$, $C(1, 1, 7)$ and $D(-1, 8, 9)$.

a Show that the quadrilateral is a parallelogram.

b Find the area of the quadrilateral.

58 Find the area of the triangle with vertices $(2, 1, 2)$, $(5, 0, 1)$ and $(-1, 3, 5)$.

59 The points $A(3, 1, 2)$, $B(-1, 1, 5)$ and $C(7, -2, 3)$ are vertices of a parallelogram $ABCD$.

a Find the coordinates of D.

b Calculate the area of the parallelogram.

60 Prove that for any two vectors **a** and **b**, $|\mathbf{a}\times\mathbf{b}|^2+(\mathbf{a}\bullet\mathbf{b})^2=|\mathbf{a}|^2|\mathbf{b}|^2$.

61 Times run by athletes in a 100 m sprint race can only be counted as official records if the wind speed in the direction of the track and assisting the runners is less than $2\,\mathrm{m\,s^{-1}}$.

A track is in the direction $\begin{pmatrix}1\\2\end{pmatrix}$ and a wind of $3\,\mathrm{m\,s^{-1}}$ is blowing in the direction $\begin{pmatrix}3\\4\end{pmatrix}$.

Determine whether a record time run in these conditions is valid or not.

62 $ABCD$ is a parallelogram with $AB \parallel DC$. Let $\overrightarrow{AB}=\mathbf{a}$ and $\overrightarrow{AD}=\mathbf{b}$.

a Express $\overrightarrow{AC}$ and $\overrightarrow{BD}$ in terms of **a** and **b**.

b Simplify $(\mathbf{a}+\mathbf{b})\bullet(\mathbf{b}-\mathbf{a})$.

c Hence show that if $ABCD$ is a rhombus then its diagonals are perpendicular.

63 Points A and B have position vectors $\begin{pmatrix}2\\1\\4\end{pmatrix}$ and $\begin{pmatrix}2\lambda\\\lambda\\4\lambda\end{pmatrix}$.

a Show that B lies on the line OA for all values of λ.

Point C has position vector $\begin{pmatrix}12\\2\\4\end{pmatrix}$.

b Find the value of λ for which CBA is a right angle.

c For the value of λ found above, calculate the exact distance from C to the line OA.

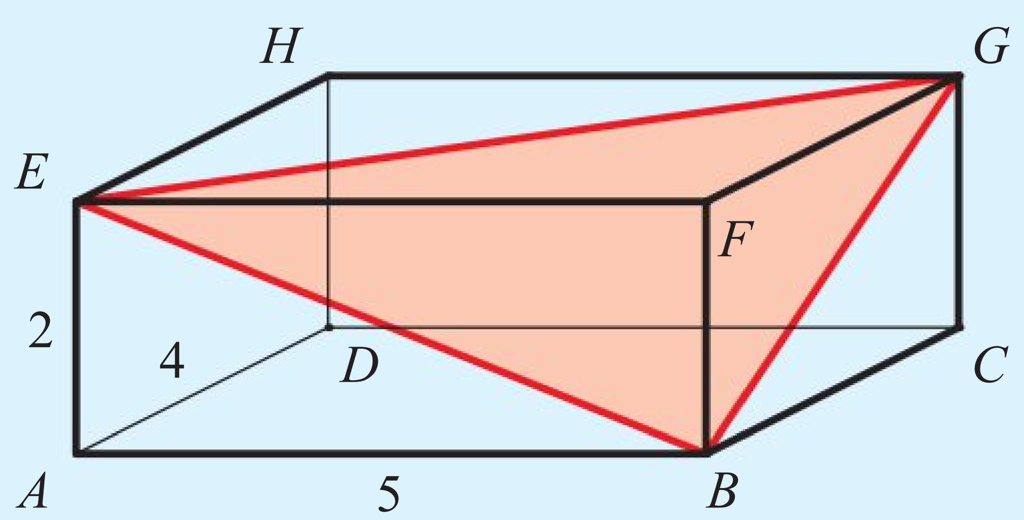
64 Given line $l:\mathbf{r}=\begin{pmatrix}5\\1\\2\end{pmatrix}+\lambda\begin{pmatrix}2\\-3\\3\end{pmatrix}$ and point $P(21, 5, 10)$,

a find the coordinates of point M on l such that PM is perpendicular to l

b show that the point $Q(15, -14, 17)$ lies on l.

c Point R, distinct from Q, lies on l such that $|PR|=|PQ|$. Find the coordinates of R.

65 A cuboid $ABCDEFGH$ is shown in the diagram. The coordinates of four of the vertices are $A(0, 0, 0)$, $B(5, 0, 0)$, $D(0, 4, 0)$ and $E(0, 0, 2)$.



a Find the coordinates of the remaining four vertices.

Face diagonals BE, BG and EG are drawn as shown.

b Find the area of the triangle BEG.

Checklist

- You should understand the concept of a vector and of a scalar.
  - A vector is a quantity that has both magnitude and direction.
  - A scalar is a quantity that has magnitude but no direction.
- You should know about different ways of representing vectors and how to add, subtract and multiply vectors by a scalar.
  - Vectors can be represented either as directed line segments or by their components (as column vectors or using **i**, **j**, **k** base vectors).
- You should be able to find the resultant of two or more vectors.
- You should be able to identify when vectors are parallel.
  - If vectors **a** and **b** are parallel, then $\mathbf{b} = t\mathbf{a}$ for some scalar t.
- You should be able to find the magnitude of a vector.
  - The magnitude of a vector $\mathbf{a} = \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix}$ is $|\mathbf{a}| = \sqrt{a_1^2 + a_2^2 + a_3^2}$.
- You should be able to find a unit vector in a given direction.
  - The unit vector in the same direction as vector **a** is $\frac{1}{|\mathbf{a}|}\mathbf{a}$.
- You should know about position vectors and displacement vectors.
  - The position vector of a point A is the vector $\mathbf{a} = \overrightarrow{OA}$, where O is the origin. The components of **a** are the coordinates of A.
  - If points A and B have position vectors **a** and **b** then:
    - — the displacement vector from A to B is $\overrightarrow{AB} = \overrightarrow{OB} = \overrightarrow{OA} = \mathbf{b} - \mathbf{a}$
    - — the distance between the points A and B is $AB = |\overrightarrow{AB}| = |\mathbf{b} - \mathbf{a}|$
- You should be able to find the vector equation of a line in two and three dimensions and convert to parametric form.
  - The vector equation of a line has the form $\mathbf{r} = \mathbf{a} + \lambda\mathbf{d}$, where **d** is the direction vector and **a** is the position vector of one point on the line.
  - The components of the position vector $\mathbf{r} = \begin{pmatrix} x \\ y \\ z \end{pmatrix}$ are the coordinates of a general point on the line.
  - The parametric form of the equation of a line is found by expressing x, y and z in terms of λ.
- You should be able to determine whether two lines intersect and find the point of intersection.
- You should be able to model linear motion with constant velocity in two and three dimensions.
  - An object with constant velocity **v**:
    - — is moving parallel to the vector **v** with speed $|\mathbf{v}|$
    - — has position vector after time t given by $\mathbf{r} = \mathbf{r}_0 + t\mathbf{v}$, where $\mathbf{r}_0$ is its initial position vector
- You should be able to use the scalar product to find the angle between two vectors.
  - If θ is the angle between vectors $\mathbf{a} = \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix}$ and $\mathbf{b} = \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix}$ then $\cos\theta = \frac{\mathbf{a} \bullet \mathbf{b}}{|\mathbf{a}||\mathbf{b}|}$ where $\mathbf{a} \bullet \mathbf{b} = a_1b_1 + a_2b_2 + a_3b_3$.
- You should be able to identify when two vectors are perpendicular.
  - If **a** and **b** are perpendicular vectors, then $\mathbf{a} \bullet \mathbf{b} = 0$.
- You should be able to find the angle between two lines.
  - The angle between two lines is equal to the angle between their direction vectors.

- You should be able to use the vector product to find perpendicular directions and areas.
  - The vector product (cross product) is a vector perpendicular to both **a** and **b**.
  - The component form is $\mathbf{a}\times\mathbf{b} = \begin{pmatrix} a_2b_3 - a_3b_2 \\ a_3b_1 - a_1b_3 \\ a_1b_2 - a_2b_1 \end{pmatrix}$.
  - The magnitude of the cross product is $|\mathbf{a}\times\mathbf{b}| = |\mathbf{a}||\mathbf{b}|\sin\theta$, which is equal to the area of the parallelogram formed by the vectors **a** and **b**.
  - If **a** and **b** are parallel vectors, then $\mathbf{a}\times\mathbf{b} = 0$.
- You should be able to find components of vectors in given directions.
  - The component of **a** acting:
    — in the direction of **b** is $\dfrac{\mathbf{a}\bullet\mathbf{b}}{|\mathbf{b}|} = |\mathbf{a}|\cos\theta$

    — perpendicular to **b** is $\dfrac{|\mathbf{a}\times\mathbf{b}|}{|\mathbf{b}|} = |\mathbf{a}|\sin\theta$

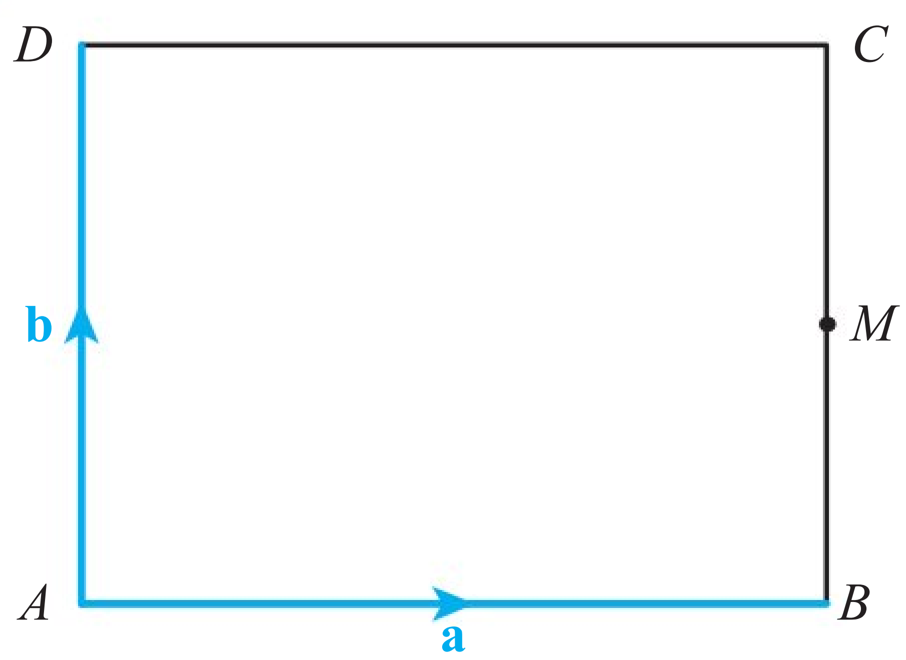
Mixed Practice

1 An aeroplane takes off from an airport and travels with displacement vector $(-22\mathbf{i} - 5\mathbf{j} + 8\mathbf{k})$ km. It changes direction and travels with displacement vector $(215\mathbf{i} - 73\mathbf{j} - 0.5\mathbf{k})$ km and then changes direction again and travels with displacement vector $(83\mathbf{i} + 16\mathbf{j})$ km.
Find

a the displacement vector it now needs to travel along to return to the airport

b the distance it needs to travel to return directly to the airport.

2 The diagram shows a rectangle $ABCD$, with $\overrightarrow{AB} = \mathbf{a}$ and $\overrightarrow{AD} = \mathbf{b}$. M is the midpoint of BC.



a Express $\overrightarrow{MD}$ in terms of **a** and **b**.

b N is the midpoint of DM. Express $\overrightarrow{AN}$ in terms of **a** and **b**.

c P is the point on the extension of the side BC such that $CP = CM$. Show that A, N and P lie on the same straight line.

3 Points A, B and D have coordinates (1, 1, 7), (−1, 6, 3) and (3, 1, k), respectively. AD is perpendicular to AB.

a Write down, in terms of k, the vector $\overrightarrow{AD}$.

b Show that $k = 6$.

Point C is such that $\overrightarrow{BC} = 2\overrightarrow{AD}$.

c Find the coordinates of C.

d Find the exact value of $\cos(A\hat{D}C)$

4 Line l_1 has equation $\mathbf{r} = \begin{pmatrix} 5 \\ 1 \\ 2 \end{pmatrix} + t\begin{pmatrix} -1 \\ 1 \\ 3 \end{pmatrix}$ and line l_2 has equation $\mathbf{r} = \begin{pmatrix} 5 \\ 4 \\ 9 \end{pmatrix} + s\begin{pmatrix} 2 \\ 1 \\ 1 \end{pmatrix}$.

Find the coordinates of the point of intersection of the two lines.

5 **a** Show that the lines

$$\mathbf{r} = \begin{pmatrix} 2 \\ -1 \\ 5 \end{pmatrix} + \lambda\begin{pmatrix} 4 \\ -1 \\ 2 \end{pmatrix} \text{ and } \mathbf{r} = \begin{pmatrix} -2 \\ 0 \\ 3 \end{pmatrix} + \mu\begin{pmatrix} 1 \\ -2 \\ -3 \end{pmatrix} \text{ are perpendicular.}$$

b Determine whether the lines intersect.

6 Show that the lines

$$\mathbf{r} = \begin{pmatrix} 4 \\ 1 \\ 2 \end{pmatrix} + \lambda\begin{pmatrix} -3 \\ 3 \\ 1 \end{pmatrix} \quad \text{and} \quad \begin{cases} x = 1 + 4\mu \\ y = -2 + 3\mu \\ z = 0.5 + 2\mu \end{cases}$$

do not intersect.

7 **a** Find a vector equation of the line through the points $A(1, -3, 2)$ and $B(2, 2, 1)$.

b Find the acute angle between this line and the line l_2 with equation $\mathbf{r} = \begin{pmatrix} 4 \\ 0 \\ 3 \end{pmatrix} + \lambda\begin{pmatrix} 1 \\ 1 \\ 7 \end{pmatrix}$.

c Find the value of k for which the point $C(7, 3, k)$ lies on l_2.

d Find the distance AC.

8 **a** Find the vector equation of the line with parametric equation $x = 3\lambda + 1,\ y = 4 - 2\lambda,\ z = 3\lambda - 1$.

b Find the unit vector in the direction of the line.

9 In this question, distance is measured in metres and time in seconds. The base vectors **i**, **j** and **k** point east, north and up, respectively.

An aeroplane takes off from the ground. It moves with constant velocity $\mathbf{v} = (116\mathbf{i} + 52\mathbf{j} + 12\mathbf{k})$.

a Find the speed of the aeroplane.

b How long does it take for the aeroplane to reach a height of 1 km?

10 In this question **i** and **j** are unit vectors directed east and north, respectively.

Ship A is initially at the point with position vector $\begin{pmatrix} 2 \\ -8 \end{pmatrix}$ km and moves with velocity $\begin{pmatrix} -5 \\ 3 \end{pmatrix}$ km h^{-1}.

Ship B is initially at the point with position vector $\begin{pmatrix} 3 \\ 4 \end{pmatrix}$ km and moves with velocity $\begin{pmatrix} 2 \\ -6 \end{pmatrix}$ km h^{-1}.

a Find expressions for the position vectors of A and B after t hours.

b Calculate the distance of A from B after 3 hours.

c Find the time at which A is due west of B.

11 In this question the base vectors **i**, **j** and **k** point east, north and up, respectively.

A drone takes off from a point with position vector $(2\mathbf{i} - 3\mathbf{j})$ m and moves with constant velocity $(10\mathbf{i} - 9\mathbf{j} + 7\mathbf{k})\,\text{m s}^{-1}$.

a Write down an expression for the position vector of the drone after time t seconds.

b Find the speed of the drone.

The drone's controller is located at the point with position vector $(67\mathbf{i} - 61.5\mathbf{j})$ m.

c Find the time at which it is directly overhead of the controller.

12 Find the value of x, with $0 < x < \pi$, such that the vectors $\begin{pmatrix} 3\sin x \\ 8 \\ 1 \end{pmatrix}$ and $\begin{pmatrix} 4\cos x \\ 1 \\ -2 \end{pmatrix}$ are perpendicular.

13 Given that $\mathbf{a} = 3\mathbf{i} - \mathbf{j} + \mathbf{k}$, $\mathbf{b} = \mathbf{i} + 5\mathbf{j} - 2\mathbf{k}$, $\mathbf{c} = 3\mathbf{i} + \mathbf{j} - \mathbf{k}$ and $\mathbf{d} = 4\mathbf{i} - \mathbf{j} + p\mathbf{k}$,

a find $(\mathbf{a} \times \mathbf{b}) \bullet \mathbf{c}$

b find the value of p such that **d** is perpendicular to $\mathbf{a} \times \mathbf{b}$.

14 Points A and C have position vectors $\mathbf{a} = 2\mathbf{j} - 5\mathbf{k}$ and $\mathrm{c} = \mathbf{i} + 3\mathbf{k}$.

a Find $\overrightarrow{OA} \times \overrightarrow{OC}$.

b Find the coordinates of the point B such that $OABC$ is a parallelogram.

c Find the exact area of $OABC$.

15 A javelin thrower aims to throw the javelin in the direction $\begin{pmatrix} 0 \\ 4 \\ 3 \end{pmatrix}$.

He releases the javelin with velocity $\begin{pmatrix} -2 \\ 21 \\ 15 \end{pmatrix}\text{m s}^{-1}$.

Find

a the speed of the javelin at the point of release

b the component of the velocity in the direction in which he was aiming

c the component of the velocity perpendicular to the direction in which he was aiming.

16 The position vectors of the points A, B and C are **a**, **b** and **c**, respectively, relative to an origin O. The following diagram shows the triangle ABC and points M, R, S and T.

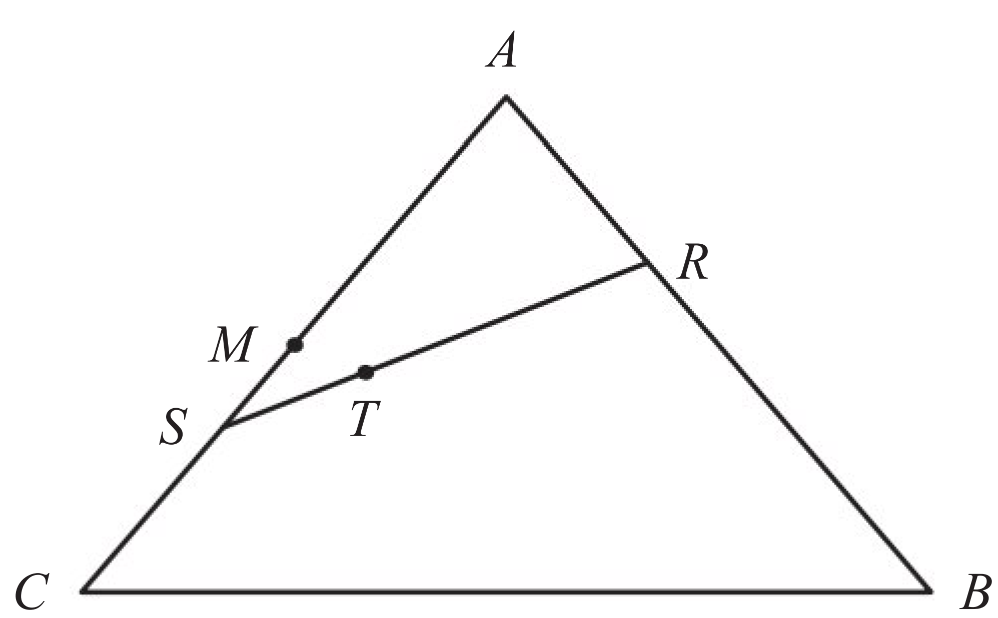


Diagram not to scale

M is the mid-point of $[AC]$.

S is a point on $[AC]$ such that $\overrightarrow{AS} = \frac{2}{3}\overrightarrow{AC}$.

R is a point on $[AB]$ such that $\overrightarrow{AR} = \frac{1}{3}\overrightarrow{AB}$.

T is a point on $[RS]$ such that $\overrightarrow{RT} = \frac{2}{3}\overrightarrow{RS}$.

a **i** Express $\overrightarrow{AM}$ in terms of $\mathbf{a}$ and $\mathbf{c}$.

ii Hence show that $\overrightarrow{BM} = \frac{1}{2}\mathbf{a} - \mathbf{b} + \frac{1}{2}\mathbf{c}$.

b **i** Express $\overrightarrow{RA}$ in terms of $\mathbf{a}$ and $\mathbf{b}$.

ii Show that $\overrightarrow{RT} = -\frac{2}{9}\mathbf{a} - \frac{2}{9}\mathbf{b} + \frac{4}{9}\mathbf{c}$.

c Prove that T lies on $[BM]$.

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17 Three forces, $\mathbf{F}_1 = (5\mathbf{i} + a\mathbf{k})\,\text{N}$, $\mathbf{F}_2 = (b\mathbf{i} - 7\mathbf{j} - 10\mathbf{k})\,\text{N}$ and $\mathbf{F}_3 = (8\mathbf{i} + c\mathbf{j} - 2\mathbf{k})\,\text{N}$ are applied to a particle. The resultant force is three times the sum of $\mathbf{F}_1$ and $\mathbf{F}_2$.

Find the values of a, b and c.

18 A crow flies in the direction $2\mathbf{i} - \mathbf{j} + \mathbf{k}$ with a speed of 30 miles per hour.

a Find its velocity vector.

b It takes off from point P. Find its position vector relative to P after 5 minutes.

19 Four points have coordinates $A(2, 4, 1)$, $B(k, 4, 2k)$, $C(k + 4, 2k + 4, 2k + 2)$ and $D(6, 2k + 4, 3)$.

a Show that $ABCD$ is a parallelogram for all values of k.

b When $k = 1$ find the angles of the parallelogram.

c Find the value of k for which $ABCD$ is a rectangle.

20 **a** Find the parametric equation of the line with vector equation $\mathbf{r} = \begin{pmatrix} 1/2 \\ -2 \\ 4/3 \end{pmatrix} + \lambda\begin{pmatrix} 2 \\ 3 \\ -2 \end{pmatrix}$.

b Determine whether the line intersects the x-axis.

c Find the angle the line makes with the x-axis.

21 Two lines have equations $l_1: \mathbf{r} = \begin{pmatrix} 5 \\ -3 \\ 1 \end{pmatrix} + \lambda\begin{pmatrix} 2 \\ -2 \\ 1 \end{pmatrix}$ and $l_2: \mathbf{r} = \begin{pmatrix} 0 \\ -4 \\ 2 \end{pmatrix} + \mu\begin{pmatrix} 1 \\ 5 \\ -3 \end{pmatrix}$

a Find the acute angle between the lines.

b The two lines intersect at the point X. Find the coordinates of X.

c Show that the point $Y(9, -7, 3)$ lies on l_1.

d Point Z lies on l_2 such that XY is perpendicular to YZ. Find the area of the triangle XYZ.

22 Two particles move with constant velocity. Particle A starts from the point $(7, -3, 2)$ and moves with velocity $\begin{pmatrix} -1 \\ 0 \\ 2 \end{pmatrix}\text{m s}^{-1}$. Particle B starts from the point $(1, 1, 26)$ and moves with velocity $\begin{pmatrix} 0 \\ 1 \\ 3 \end{pmatrix}\text{m s}^{-1}$.

a Write down expression for the position vector of A and B after time t.

b **i** Show that the paths taken by the two particles intersect.

ii Find the coordinates of this point of intersection.

c Show further that the two particles do not collide.

23 In this question the base vectors $\mathbf{i}$ and $\mathbf{j}$ point east and north, respectively.

A rowing boat, R, out at sea is first seen at 12:00 at the point with position vector $(-\mathbf{i}-3\mathbf{j})$ km. At 12:40 it is seen at the point with position vector $(5\mathbf{i}-5\mathbf{j})$ km.

a Calculate the bearing on which R is moving.

b Find an expression for the position vector of R at t hours after 12:00.

At 14:00 a speed boat leaves the origin travelling with speed $(p\mathbf{i}+q\mathbf{j})\,\text{km}\,\text{h}^{-1}$.

c Given that the speed boat intercepts R at 14:30, find the values of p and q.

24 The position vector of a particle at time t seconds is given by $\mathbf{r}=(4+3t)\mathbf{i}+(6-t)\mathbf{j}+(2t-7)\mathbf{k}$. The distance is measured in metres.

a Find the displacement of the particle from the starting point after 5 seconds.

b Find the speed of the particle.

c Determine whether the particle's path crosses the line connecting the points (3, 0, 1) and (1, 1, 5).

25 At time $t=0$, two aircraft have position vectors $5\mathbf{j}$ and $7\mathbf{k}$. The first moves with velocity $3\mathbf{i}-4\mathbf{j}+\mathbf{k}$ and the second with velocity $5\mathbf{i}+2\mathbf{j}-\mathbf{k}$.

a Write down the position vector of the first aircraft at time t.

b Show that at, time t, the distance, d, between the two aircraft is given by $d^2=44t^2-88t+74$.

c Show that the two aircraft will not collide.

d Find the minimum distance between the two aircraft.

26 Let $\mathbf{a}$ and $\mathbf{b}$ be unit vectors and let α be the angle between them.

a Express $|\mathbf{a}-\mathbf{b}|$ and $|\mathbf{a}+\mathbf{b}|$ in terms of $\cos\alpha$.

b Hence find the value of α such that $|\mathbf{a}+\mathbf{b}|=4|\mathbf{a}-\mathbf{b}|$.

27 Line l has equation $\mathbf{r}=\begin{pmatrix}4\\2\\-1\end{pmatrix}+\lambda\begin{pmatrix}2\\-1\\2\end{pmatrix}$ and point P has coordinates (7, 2, 3). Point C lies on l and PC is perpendicular to l. Find the coordinates of C.

28 Line l_1 has parametric equation $x=2+4\lambda$, $y=-1-3\lambda$, $z=3\lambda$.

Line l_2 is parallel to l_1 and passes through point $A(0,-1,2)$.

a Write a vector equation of l_2.

b Find the coordinates of the point B on l_1 such that AB is perpendicular to l_1.

c Hence find, to three significant figures, the shortest distance between the two lines.

29 Points A, B and C have position vectors $\mathbf{a} = \mathbf{i} - 19\mathbf{j} + 5\mathbf{k}$, $\mathbf{b} = 2\lambda\mathbf{i} + (\lambda + 2)\mathbf{j} + 2\mathbf{k}$ and $\mathbf{c} = -6\mathbf{i} - 15\mathbf{j} + 7\mathbf{k}$.

a Find the value of λ for which BC is perpendicular to AC.

For the value of λ found above

b find the angles of the triangle ABC

c find the area of the triangle ABC.

30 Let $\mathbf{a} = \begin{pmatrix} 3 \\ 0 \\ 1 \end{pmatrix}$, $\mathbf{b} = \begin{pmatrix} -1 \\ 5 \\ p \end{pmatrix}$ and $\mathbf{c} = \begin{pmatrix} 1 \\ 4 \\ -3 \end{pmatrix}$.

Find the value of p, given that $\mathbf{a} \times \mathbf{b}$ is parallel to $\mathbf{c}$.

31 Find the area of the triangle with vertices (2, 1, 2), (−1, 2, 2) and (0, 1, 5).

32 Given that $\mathbf{a} = \mathbf{i} - \mathbf{j} + 3\mathbf{k}$ and $\mathbf{b} = 2q\mathbf{i} + \mathbf{j} + q\mathbf{k}$, find the values of scalars p and q such that $p\mathbf{a} + \mathbf{b}$ is parallel to vector $\mathbf{i} + \mathbf{j} + 2\mathbf{k}$.

33 A kite has position vector at time t given by $\mathbf{r} = \begin{pmatrix} 2 \\ 1 \\ 1 \end{pmatrix} + t\begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix}\text{m s}^{-1}$.

The top of a nearby tree has position vector $\mathbf{r} = \begin{pmatrix} 1 \\ 1 \\ 3 \end{pmatrix}$ m.

Find the minimum distance between the kite and the treetop.

34 Two lines are given by $l_1: \mathbf{r} = \begin{pmatrix} -5 \\ 1 \\ 10 \end{pmatrix} + \lambda\begin{pmatrix} -3 \\ 0 \\ 4 \end{pmatrix}$ and $l_2: \mathbf{r} = \begin{pmatrix} 3 \\ 0 \\ -9 \end{pmatrix} + \mu\begin{pmatrix} 1 \\ 1 \\ 7 \end{pmatrix}$.

a l_1 and l_2 intersect at P. Find the coordinates of P.

b Show that the point $Q(5, 2, 5)$ lies on l_2.

c Find the coordinates of point M on l_1 such that QM is perpendicular to l_1.

d Find the area of the triangle PQM.

35 Two lines have equations $l_1: \mathbf{r} = \begin{pmatrix} 0 \\ -1 \\ 2 \end{pmatrix} + \lambda\begin{pmatrix} 1 \\ 5 \\ 3 \end{pmatrix}$ and $l_2: \mathbf{r} = \begin{pmatrix} 2 \\ 2 \\ 1 \end{pmatrix} + t\begin{pmatrix} -1 \\ 1 \\ 3 \end{pmatrix}$.

a Show that the point $P\left(\frac{5}{6}, \frac{19}{6}, \frac{9}{2}\right)$ lies on both lines.

b Find, in degrees, the acute angle between the two lines.

Point Q has coordinates $(-1, 5, 10)$.

c Show that Q lies on l_2.

d Find the distance PQ.

e Hence find the shortest distance from Q to the line l_1.

36 Two lines with equations $l_1: \mathbf{r} = \begin{pmatrix} 2 \\ -1 \\ 0 \end{pmatrix} + \lambda \begin{pmatrix} 1 \\ -2 \\ 2 \end{pmatrix}$ and $l_2: \mathbf{r} = \begin{pmatrix} 2 \\ -1 \\ 0 \end{pmatrix} + \mu \begin{pmatrix} 2 \\ 1 \\ 2 \end{pmatrix}$ intersect at point P.

a Show that $Q(8, 2, 6)$ lies on l_2.

b R is a point on l_1 such that $|PR| = |PQ|$. Find the possible coordinates of R.

c Find a vector equation of a line through P which bisects the angle QPR.

37 **a** Line l has equation $\mathbf{r} = \begin{pmatrix} 6 \\ 1 \\ 1 \end{pmatrix} + \lambda \begin{pmatrix} 2 \\ -1 \\ 2 \end{pmatrix}$ and point P has coordinates $(7, 2, 3)$.

Point C lies on l and PC is perpendicular to l. Find the coordinates of C.

b Hence find the shortest distance from P to l.

c Q is a reflection of P in line l. Find the coordinates of Q.

38 Lines l_1 and l_2 have equations:

$l_1: \mathbf{r} = (\mathbf{i} - 10\mathbf{j} + 12\mathbf{k}) + \lambda(\mathbf{i} + 3\mathbf{j} - 4\mathbf{k})$

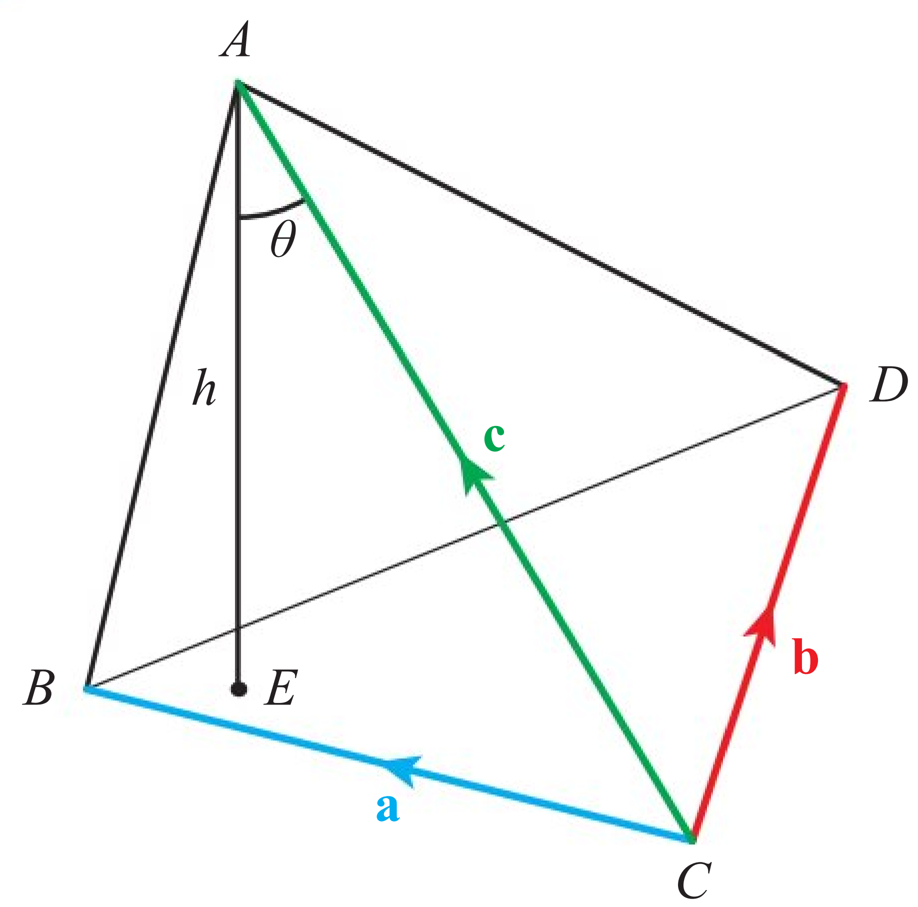
$l_2: \mathbf{r} = (4\mathbf{i} + \mathbf{j} + 5\mathbf{k}) + \mu(\mathbf{i} + 2\mathbf{k})$

P is a point on l_1 and Q is a point on l_2 such that $\overrightarrow{PQ}$ is perpendicular to both lines.

a Show that $26\lambda + 7\mu = 64$ and find another equation for λ and μ.

b Hence find the shortest distance between the lines l_1 and l_2.

39 Consider the tetrahedron shown in the diagram and define vectors $\mathbf{a} = \overrightarrow{CB}$, $\mathbf{b} = \overrightarrow{CD}$ and $\mathbf{c} = \overrightarrow{CA}$.



a Write down an expression for the area of the base in terms of vectors **a** and **b** only.

b AE is the height of the tetrahedron, $|AE| = h$ and $\angle CAE = \theta$. Express h in terms of **c** and θ.

c Use the results of **a** and **b** to prove that the volume of the tetrahedron is given by $\left|\frac{1}{6}(\mathbf{a}\times\mathbf{b})\cdot\mathbf{c}\right|$.

d Find the volume of the tetrahedron with vertices $A(0, 4, 0)$, $B(0, 6, 0)$, $C(1, 6, 1)$ and $D(3, -1, 2)$.

e Find the distance of the vertex A from the face BCD.

f Determine which of the vertices A and B is closer to its opposite face.

40 Port A is defined to be the origin of a set of coordinate axes and port B is located at the point (70, 30), where distances are measured in kilometres. A ship S_1 sails from port A at 10:00 in a straight line such that its position t hours after 10:00 is given by $\mathbf{r} = t\begin{pmatrix}10\\20\end{pmatrix}$.

A speedboat S_2 is capable of three times the speed of S_1 and is to meet S_1 by travelling the shortest possible distance. What is the latest time that S_2 can leave port B?

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3 Matrices

ESSENTIAL UNDERSTANDINGS

- Number and algebra allow us to represent patterns, show equivalences and make generalizations which enable us to model real-world situations.
- Algebra is an abstraction of numerical concepts and employs variables to solve mathematical problems.

In this chapter you will learn…

- how to determine the order of a matrix
- how to add, subtract and multiply matrices
- some properties of matrix multiplication
- about zero and identity matrices
- how to calculate the determinant and inverse of $n \times n$ matrices with technology and 2×2 matrices by hand
- how to solve systems of linear equations with matrices
- how to find the eigenvalues and eigenvectors of 2×2 matrices
- how to diagonalize 2×2 matrices
- how to use diagonalization to find powers of 2×2 matrices.

CONCEPTS

The following concepts will be addressed in this chapter:

- Matrices allow us to organize data so that it can be manipulated and **relationships** can be determined.

LEARNER PROFILE – Communicators

Is a proof the same as an explanation?

■ **Figure 3.1** Information can be displayed in different ways, such as a football league table, pixels on a screen or allergen information on a menu

PRIOR KNOWLEDGE

Before starting this chapter, you should already be able to complete the following:

1 Calculate

a $\begin{pmatrix}2\\5\end{pmatrix}+\begin{pmatrix}-1\\4\end{pmatrix}$ **b** $\begin{pmatrix}2\\5\end{pmatrix}-\begin{pmatrix}-1\\4\end{pmatrix}$ **c** $4\begin{pmatrix}2\\-3\end{pmatrix}$

2 Calculate $\left|\begin{pmatrix}3\\-5\end{pmatrix}\right|$

The idea of a column vector can be extended to a matrix by allowing two or more column vectors to be placed side by side in an array. Matrices provide an efficient way of storing and analyzing large amounts of data and are used widely in many branches of mathematics from calculus to probability.

They form the basis of coding and encryption, and computer programming, from graphics in a video game to a weather forecasting simulation, and are used extensively in physics, for example in quantum mechanics.

Just as with numbers, it is important to be able to perform operations such as addition, subtraction or multiplication with matrices and to be able to solve equations involving matrices.

Starter Activity

Look at the images in Figure 3.1. Discuss different ways to store and present the required information in each case.

Now look at this problem:

After 10 games of a league season, the results of four teams, Albion, City, Rovers and United are as follows.

	Won	Drawn	Lost
Albion	5	0	5
City	8	1	1
Rovers	2	3	5
United	2	7	1

A new points system is operating in the league this season where a win is now worth 1 more point than before.

	New points	Old points
Win	3	2
Draw	1	1
Loss	0	0

a Find the number of points each team has under the new system.

b Find the number of points each team would have had under the old system.

c Has the change in points system affected the position of any of the teams?

3A Definition and arithmetic of matrices

Definition of a matrix

A **matrix** (plural **matrices**) is a rectangular array of **elements**, which may be numerical or algebraic. For example,

$$\mathbf{A} = \begin{pmatrix} 2 & x & -3 \\ 0 & 4.5 & \pi \\ -1.8 & 1 & x^2 \end{pmatrix}$$

The number of rows and columns that a matrix has is called its **order**.

Tip

A matrix of order $n \times n$ (i.e. with the same number of rows as columns) is called a **square matrix**.

KEY POINT 3.1

A matrix with m rows and n columns has order $m \times n$.

WORKED EXAMPLE 3.1

State the order of the matrix $\begin{pmatrix} 2 & -3 & 0 \\ 5 & 1 & -4 \end{pmatrix}$.

3 columns

2 rows $\begin{pmatrix} 2 & -3 & 0 \\ 5 & 1 & -4 \end{pmatrix}$

Order 2×3

Tip

Note that a column vector such as $\begin{pmatrix} 3 \\ 5 \end{pmatrix}$ can also be thought of as a matrix of order 2×1.

Algebra of matrices

Two matrices are equal when each of their corresponding elements is equal. Of course, for this to be possible, the matrices must first have the same order.

WORKED EXAMPLE 3.2

$\mathbf{A} = \begin{pmatrix} 3 & 2 \\ -1 & x^2 \end{pmatrix}$ and $\mathbf{B} = \begin{pmatrix} x+1 & 2 \\ -1 & y \end{pmatrix}$

Given that $\mathbf{A} = \mathbf{B}$, find the values of x and y.

$$\mathbf{A} = \mathbf{B}$$

$$\begin{pmatrix} 3 & 2 \\ -1 & x^2 \end{pmatrix} = \begin{pmatrix} x+1 & 2 \\ -1 & y \end{pmatrix}$$

Equate the elements in the top left

$$3 = x + 1$$
$$x = 2$$

Equate the elements in the bottom right

$$x^2 = y$$
$$y = 4$$

If two matrices have the same order, then one can be added to or subtracted from the other.

KEY POINT 3.2

- Two matrices can be added/subtracted if they have the same order.
- To add/subtract matrices, add/subtract the corresponding elements.

WORKED EXAMPLE 3.3

$$\mathbf{A} = \begin{pmatrix} 6 & -1 \\ 2 & 0 \\ -3 & 5 \end{pmatrix} \text{ and } \mathbf{B} = \begin{pmatrix} 1 & 4 \\ 3 & -7 \\ -2 & 8 \end{pmatrix}$$

Find

a $\mathbf{A} + \mathbf{B}$ **b** $\mathbf{A} - \mathbf{B}$

$$\textbf{a}\ \mathbf{A} + \mathbf{B} = \begin{pmatrix} 6 & -1 \\ 2 & 0 \\ -3 & 5 \end{pmatrix} + \begin{pmatrix} 1 & 4 \\ 3 & -7 \\ -2 & 8 \end{pmatrix}$$

Add the corresponding elements of each matrix

$$= \begin{pmatrix} 6+1 & -1+4 \\ 2+3 & 0+(-7) \\ -3+(-2) & 5+8 \end{pmatrix}$$

$$= \begin{pmatrix} 7 & 3 \\ 5 & -7 \\ -5 & 13 \end{pmatrix}$$

$$\textbf{b}\ \mathbf{A} - \mathbf{B} = \begin{pmatrix} 6 & -1 \\ 2 & 0 \\ -3 & 5 \end{pmatrix} - \begin{pmatrix} 1 & 4 \\ 3 & -7 \\ -2 & 8 \end{pmatrix}$$

Subtract the corresponding element of matrix **B** from that of **A**

$$= \begin{pmatrix} 6-1 & -1-4 \\ 2-3 & 0-(-7) \\ -3-(-2) & 5-8 \end{pmatrix}$$

$$= \begin{pmatrix} 5 & -5 \\ -1 & 7 \\ -1 & -3 \end{pmatrix}$$

Multiplication of a matrix by a scalar works in the same way as for multiplication of a vector by a scalar.

Make sure you also know how to use your GDC to add, subtract and multiply matrices.

KEY POINT 3.3

To multiply a matrix by a scalar, multiply every element of the matrix by the scalar.

WORKED EXAMPLE 3.4

$$\mathbf{M} = \begin{pmatrix} -1 & 0.5 & 5 & 2.3 \\ 3 & 2 & 0 & -4 \end{pmatrix}$$

Find the matrix **A**, where **A** = 3**M**.

$$\mathbf{A} = 3\mathbf{M}$$

$$= 3\begin{pmatrix} -1 & 0.5 & 5 & 2.3 \\ 3 & 2 & 0 & -4 \end{pmatrix}$$

Multiply each element by 3

$$= \begin{pmatrix} 3(-1) & 3(0.5) & 3(5) & 3(2.3) \\ 3(3) & 3(2) & 3(0) & 3(-4) \end{pmatrix}$$

$$= \begin{pmatrix} -3 & 1.5 & 15 & 6.9 \\ 9 & 6 & 0 & -12 \end{pmatrix}$$

Matrix multiplication

The product of two matrices **AB** is another matrix **C**. The process involves multiplying the elements of each row of **A** by the corresponding elements of each column of **B** and summing each time to find the entry of **C**.

WORKED EXAMPLE 3.5

Find the matrix **M** where

$$\mathbf{M} = \begin{pmatrix} 3 & -1 \\ 2 & 4 \\ -5 & 1 \end{pmatrix}\begin{pmatrix} 2 & 1 \\ 3 & -4 \end{pmatrix}$$

The top left element is formed from the top row and the left column

$$\mathbf{M} = \begin{pmatrix} 3 & -1 \\ 2 & 4 \\ -5 & 1 \end{pmatrix}\begin{pmatrix} 2 & 1 \\ 7 & -4 \end{pmatrix}$$

The middle right row is formed from the middle row and the right column

$$= \begin{pmatrix} 3\times 2+(-1)\times 7 & 3\times 1+(-1)(-4) \\ 2\times 2+4\times 7 & 2\times 1+4\times(-4) \\ (-5)\times 2+1\times 7 & (-5)\times 1+1\times(-4) \end{pmatrix}$$

And so on

$$= \begin{pmatrix} -1 & 7 \\ 32 & -14 \\ -3 & -9 \end{pmatrix}$$

Notice that the product of two matrices can only exist if the number of columns of the left-hand matrix is the same as the number of rows of the right-hand matrix.

KEY POINT 3.4

If the matrix **A** has order $m \times n$ and the matrix **B** has order $n \times p$, then the product **AB** has order $m \times p$.

WORKED EXAMPLE 3.6

Matrix **A** has order 3×5. Matrix **B** has order 4×3.

Find the order of the following products or explain why the product does not exist.

a **AB** b **BA**

A has 5 columns (3×5) and **B** has 4 rows (4×3)

a **AB** does not exist because the number of columns of **A** does not equal the number of rows of **B**.

B has 3 columns (4×3) and **A** has 3 rows (3×5)

b Order of **BA** is 4×5

Properties of matrix multiplication

It is clear from Worked Example 3.6 that, in general, the product **AB** is not the same as the product **BA** – in this case the product **AB** does not even exist!

This property of matrix arithmetic is different from usual arithmetic but other properties are the same.

KEY POINT 3.5

For the matrices **A**, **B** and **C**:

- $\mathbf{AB} \neq \mathbf{BA}$ (non-commutative)
- $\mathbf{A}(\mathbf{BC}) = (\mathbf{AB})\mathbf{C}$ (associative)
- $\mathbf{A}(\mathbf{B}+\mathbf{C}) = \mathbf{AB} + \mathbf{AC}$ (distributive)

Be the Examiner

Expand $(\mathbf{A}+\mathbf{B})^2$, where **A** and **B** are both $n \times n$ matrices.

Which is the correct solution? Identify the errors in the incorrect solutions.

Solution 1	Solution 2	Solution 3
$(\mathbf{A}+\mathbf{B})^2 = \mathbf{A}^2 + \mathbf{B}^2$	$(\mathbf{A}+\mathbf{B})^2 = (\mathbf{A}+\mathbf{B})(\mathbf{A}+\mathbf{B})$ $= \mathbf{A}^2 + \mathbf{B}^2 + \mathbf{AB} + \mathbf{BA}$	$(\mathbf{A}+\mathbf{B})^2 = (\mathbf{A}+\mathbf{B})(\mathbf{A}+\mathbf{B})$ $= \mathbf{A}^2 + \mathbf{B}^2 + 2\mathbf{AB}$

Matrix multiplication is often useful in practical applications.

WORKED EXAMPLE 3.7

Two businesses, Any Tech and Bright Solutions, need to order varying quantities of three products P_1, P_2 and P_3 as given in the demand matrix, **D**:

$$\mathbf{D} = \begin{array}{c} \begin{array}{ccc} P_1 & P_2 & P_3 \end{array} \\ \begin{pmatrix} 80 & 55 & 75 \\ 70 & 60 & 60 \end{pmatrix} \end{array} \begin{array}{c} \\ A \\ B \end{array}$$

The price per unit of these products from suppliers Supply All and Transit It are given in the cost matrix, **C**:

$$\mathbf{C} = \begin{array}{c} \begin{array}{ccc} P_1 & P_2 & P_3 \end{array} \\ \begin{pmatrix} £30 & £45 & £48 \\ £33 & £50 & £40 \end{pmatrix} \end{array} \begin{array}{c} \\ S \\ T \end{array}$$

a Find the total cost matrix, **M**, that gives the overall cost to each business of ordering with each supplier.

b State which supplier each business should use to minimize cost.

You need to make sure that the quantities of P_1, P_2 and P_3 are multiplied by the price of P_1, P_2 and P_3 and added. This can be achieved by matrix multiplication if the rows of **C** are written as columns. We then have a 2×3 multiplied by a 3×2 matrix

a $$\mathbf{M} = \begin{array}{c} \\ A \\ B \end{array}\begin{array}{c} \begin{array}{ccc} P_1 & P_2 & P_3 \end{array} \\ \begin{pmatrix} 80 & 55 & 75 \\ 70 & 60 & 60 \end{pmatrix} \end{array} \begin{array}{c} \begin{array}{cc} S & T \end{array} \\ \begin{pmatrix} 30 & 33 \\ 45 & 50 \\ 48 & 40 \end{pmatrix} \end{array} \begin{array}{c} \\ P_1 \\ P_2 \\ P_3 \end{array}$$

Use your GDC to calculate the resulting 2×2 matrix

The top left entry, for example, will be the total cost to business A of using supplier S

$$= \begin{array}{c} \begin{array}{cc} S & T \end{array} \\ \begin{pmatrix} 8475 & 8390 \\ 7680 & 7710 \end{pmatrix} \end{array} \begin{array}{c} \\ A \\ B \end{array}$$

The top row gives the two potential costs to business A and the second row the two potential costs to business B

b Supplier T is cheaper for A (£8390 < £8475)
Supplier S is cheaper for B (£7680 < £7710)

Identity and zero matrices

The unique real number that has no effect when multiplied by another number is 1, that is, $1x = x$. The equivalent for matrices is the **identity matrix**.

KEY POINT 3.6

The identity matrix, **I**, is a square matrix with 1 as each element of the diagonal from top left to bottom right and 0 as every other element.

For example, the 2×2 identity is $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$, and the 3×3 identity is $\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$.

The unique real number that has no effect when added to another number is 0, that is, $x + 0 = x$. The equivalent for matrices is the **zero matrix**.

KEY POINT 3.7

The zero matrix, **0**, is a matrix whose elements are all 0.

WORKED EXAMPLE 3.8

Given that $\mathbf{A} = \begin{pmatrix} 3 & 0 \\ -1 & 2 \end{pmatrix}$, find the matrix $\mathbf{B}$ where

$$\mathbf{B} = \mathbf{A}^2 + 2\mathbf{I}$$

$\mathbf{A}$ is a 2×2 matrix so $\mathbf{I} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$

$$\mathbf{B} = \mathbf{A}^2 + 2\mathbf{I}$$

$$= \begin{pmatrix} 3 & 0 \\ -1 & 2 \end{pmatrix}^2 + 2\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

$\mathbf{A}^2$ means $\mathbf{A}$ multiplied by $\mathbf{A}$

$$= \begin{pmatrix} 3 & 0 \\ -1 & 2 \end{pmatrix}\begin{pmatrix} 3 & 0 \\ -1 & 2 \end{pmatrix} + \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}$$

Finally, add the matrices

$$= \begin{pmatrix} 9 & 0 \\ -5 & 4 \end{pmatrix} + \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}$$

$$= \begin{pmatrix} 11 & 0 \\ -5 & 6 \end{pmatrix}$$

Exercise 3A

For questions 1 to 3, use the method demonstrated in Worked Example 3.1 to find the order of the matrix.

1 **a** $\begin{pmatrix} 2 & 1 \\ -2 & 4 \end{pmatrix}$

b $\begin{pmatrix} 1 & 0 & 5 \\ 0 & -6 & 3 \\ 2 & 4 & -1 \end{pmatrix}$

2 **a** $\begin{pmatrix} 7 & 0 \\ 0 & -1 \\ 9 & 3 \end{pmatrix}$

b $\begin{pmatrix} 3 & 8 & -5 & 0 \\ -4 & 2 & 1 & 3 \end{pmatrix}$

3 **a** $\begin{pmatrix} -1 \\ 5 \\ 4 \end{pmatrix}$

b $\begin{pmatrix} -1 & 5 & 4 \end{pmatrix}$

For questions 4 and 5, use the method demonstrated in Worked Example 3.2 to find the values of x and y.

4 **a** $\begin{pmatrix} 1 & x \\ x-2 & 0 \end{pmatrix} = \begin{pmatrix} 1 & -y \\ 3 & 0 \end{pmatrix}$

b $\begin{pmatrix} x & -4 \\ y+3 & 2 \end{pmatrix} = \begin{pmatrix} 3-y & -4 \\ -1 & 2 \end{pmatrix}$

5 **a** $\begin{pmatrix} 2 & x+5 \\ x & y^2 \end{pmatrix} = \begin{pmatrix} 2 & 3 \\ x & x+3 \end{pmatrix}$

b $\begin{pmatrix} x^2 & x \\ y & 1 \end{pmatrix} = \begin{pmatrix} 9 & -y \\ -x & 1 \end{pmatrix}$

For questions 6 to 9, use the method demonstrated in Worked Example 3.3 to add/subtract the given matrices or state that this is not possible. Check your answers using your GDC.

6 **a** $\begin{pmatrix} 5 & -3 \\ 1 & 9 \end{pmatrix} + \begin{pmatrix} -4 & 10 \\ -2 & 7 \end{pmatrix}$

b $\begin{pmatrix} 5 & -3 \\ 1 & 9 \end{pmatrix} - \begin{pmatrix} -4 & 10 \\ -2 & 7 \end{pmatrix}$

7 **a** $\begin{pmatrix} 4 & 9 & 0 & -7 \\ -1 & 5 & 3 & 2 \end{pmatrix} - \begin{pmatrix} 3 & 9 & -2 & -1 \\ 1 & 8 & -6 & 2 \end{pmatrix}$

b $\begin{pmatrix} 4 & 9 & 0 & -7 \\ -1 & 5 & 3 & 2 \end{pmatrix} + \begin{pmatrix} 3 & 9 & -2 & -1 \\ 1 & 8 & -6 & 2 \end{pmatrix}$

8 **a** $\begin{pmatrix} 1 & 0 & 4 \\ 0 & -7 & 2 \end{pmatrix} + \begin{pmatrix} 2 & 8 \\ -6 & 1 \\ 3 & -5 \end{pmatrix}$

b $\begin{pmatrix} 1 & 0 & 4 \\ 0 & -7 & 2 \end{pmatrix} - \begin{pmatrix} 2 & 8 \\ -6 & 1 \\ 3 & -5 \end{pmatrix}$

9 **a** $\begin{pmatrix} 11 & -5 \end{pmatrix} - \begin{pmatrix} 3 & -2 \end{pmatrix}$

b $\begin{pmatrix} 11 & -5 \end{pmatrix} + \begin{pmatrix} 3 & -2 \end{pmatrix}$

For questions 10 to 12, use the method demonstrated in Worked Example 3.4 to multiply the scalar into the matrix. Check your answers using your GDC.

10 **a** $2\begin{pmatrix} 3 & 0 \\ 1 & -3 \\ -2 & 4 \end{pmatrix}$

b $5\begin{pmatrix} 3 & 0 \\ 1 & -3 \\ -2 & 4 \end{pmatrix}$

11 **a** $-4\begin{pmatrix} 0.5 & -1 & 3 \\ 2 & 0 & -1.2 \end{pmatrix}$

b $-3\begin{pmatrix} 0.5 & -1 & 3 \\ 2 & 0 & -1.2 \end{pmatrix}$

12 **a** $\frac{1}{5}\begin{pmatrix} -2 & 20 & 3 \\ 1 & -4 & -6.5 \\ 0.5 & 6 & 0 \end{pmatrix}$

b $\frac{1}{2}\begin{pmatrix} -2 & 20 & 3 \\ 1 & -4 & -6.5 \\ 0.5 & 6 & 0 \end{pmatrix}$

For questions 13 to 17, use the method demonstrated in Worked Example 3.5 to find the matrix product or state that it does not exist. Check your answers using your GDC.

13 **a** $\begin{pmatrix} 2 & -1 \\ 3 & 5 \end{pmatrix}\begin{pmatrix} 0 & 4 \\ 1 & -2 \end{pmatrix}$

b $\begin{pmatrix} 0 & 4 \\ 1 & -2 \end{pmatrix}\begin{pmatrix} 2 & -1 \\ 3 & 5 \end{pmatrix}$

14 **a** $\begin{pmatrix} 6 & 2 & 3 \\ 4 & -3 & 5 \\ 1 & -2 & 0 \end{pmatrix}\begin{pmatrix} 1 & 0 & 5 \\ -2 & 2 & -1 \\ -3 & 4 & 0 \end{pmatrix}$

b $\begin{pmatrix} 1 & 0 & 5 \\ -2 & 2 & -1 \\ -3 & 4 & 0 \end{pmatrix}\begin{pmatrix} 6 & 2 & 3 \\ 4 & -3 & 5 \\ 1 & -2 & 0 \end{pmatrix}$

15 **a** $\begin{pmatrix} 4 & 1 \\ -2 & 5 \\ 3 & -1 \\ 0 & -3 \end{pmatrix}\begin{pmatrix} 1 & -1 & 2 \\ 0 & 3 & 5 \end{pmatrix}$

b $\begin{pmatrix} 1 & -1 & 2 \\ 0 & 3 & 5 \end{pmatrix}\begin{pmatrix} 4 & 1 \\ -2 & 5 \\ 3 & -1 \\ 0 & -3 \end{pmatrix}$

16 **a** $\begin{pmatrix} 2 & 5 \\ -1 & -9 \end{pmatrix}\begin{pmatrix} 3 & 7 \end{pmatrix}$

b $\begin{pmatrix} 3 & 7 \end{pmatrix}\begin{pmatrix} 2 & 5 \\ -1 & -9 \end{pmatrix}$

17 **a** $\begin{pmatrix} 2 & -3 & 4 \end{pmatrix}\begin{pmatrix} 5 \\ 1 \\ -2 \end{pmatrix}$

b $\begin{pmatrix} 5 \\ 1 \\ -2 \end{pmatrix}\begin{pmatrix} 2 & -3 & 4 \end{pmatrix}$

For questions 18 to 20, use the method demonstrated in Worked Example 3.6 to find the order of the matrix product or state that it does not exist. In each case, matrix **A** is 2×3, matrix **B** is 3×3 and matrix **C** is 4×2.

18 **a** **AB**
b **BA**

19 **a** **AC**
b **CA**

20 **a** **BC**
b **CB**

For questions 21 and 22, use the method demonstrated in Worked Example 3.8 to find the matrix **M**.

In each case, $\mathbf{A} = \begin{pmatrix} 4 & 0 & 1 \\ 0 & -1 & 2 \\ 3 & -2 & 5 \end{pmatrix}$.

21 **a** $\mathbf{M} = 2\mathbf{A} - 3\mathbf{I}$
b $\mathbf{M} = \mathbf{I} - \mathbf{A}$

22 **a** $\mathbf{A} - 4\mathbf{M} = 0$
b $\mathbf{M} + 2\mathbf{A} = 0$

23 Three teams, the Meteors, the Novas and the Orbits achieve results at home and away (win (W), draw (D) and lose (L)) given by the matrices **H** and **A**.

$$\mathbf{H} = \begin{array}{c c} & \begin{array}{ccc} W & D & L \end{array} \\ & \begin{pmatrix} 5 & 2 & 3 \\ 4 & 2 & 4 \\ 6 & 3 & 1 \end{pmatrix} \begin{array}{c} M \\ N \\ O \end{array} \end{array} \qquad \mathbf{A} = \begin{array}{c c} & \begin{array}{ccc} W & D & L \end{array} \\ & \begin{pmatrix} 4 & 1 & 5 \\ 4 & 0 & 6 \\ 5 & 2 & 3 \end{pmatrix} \begin{array}{c} M \\ N \\ O \end{array} \end{array}$$

Find the matrix that gives their overall results.

24 Last week's takings (in \$) in a book shop for IB Biology (B), IB Chemistry (C) and IB Physics (P) books at Standard Level (SL) and Higher Level (HL) are given in the following matrix.

$$\begin{array}{c c} & \begin{array}{cc} SL & HL \end{array} \\ \begin{array}{c} B \\ C \\ P \end{array} & \begin{pmatrix} 800 & 560 \\ 640 & 320 \\ 680 & 300 \end{pmatrix} \end{array}$$

The manager projects an 8% increase in sales across all these books next week.

Find the matrix that gives next week's projected takings.

25 Find the values of x and y such that

$$\begin{pmatrix} 2x & 4 \\ 3 & y \end{pmatrix} + \begin{pmatrix} -y & -1 \\ 4 & 3x \end{pmatrix} = \begin{pmatrix} 12 & 3 \\ 7 & 13 \end{pmatrix}$$

26 Find the values of p and q such that

$$\begin{pmatrix} 1 & 2p & 0 \\ p & -2 & 3q \end{pmatrix} = \begin{pmatrix} 1 & 5-q & 0 \\ p & -2 & p-13 \end{pmatrix}$$

27 $\mathbf{A} = \begin{pmatrix} -3 & 0 \\ 5 & 4 \end{pmatrix}$ and $\mathbf{B} = \begin{pmatrix} -1 & 5 \\ k & -k \end{pmatrix}$

Find, in terms of k

a $\mathbf{A} + 3\mathbf{B}$

b $2\mathbf{A} - \mathbf{B} + 4\mathbf{I}$

28 $\mathbf{A} = \begin{pmatrix} 1 & k \\ 2 & 3 \end{pmatrix}$ and $\mathbf{B} = \begin{pmatrix} -k & 2k \\ 1 & 3 \end{pmatrix}$

Find, in terms of k

a $\mathbf{A} - \mathbf{B}$

b $\mathbf{AB}$

29 $\mathbf{P} = \begin{pmatrix} 1 & k & -3 \\ 2 & 4 & k \\ 1 & 0 & 5 \end{pmatrix}$ and $\mathbf{Q} = \begin{pmatrix} 6 & 1 & 4 \\ -1 & k & 3 \\ 2 & 0 & k \end{pmatrix}$

Find, in terms of k

a $\mathbf{P} + \mathbf{Q}$

b $\mathbf{PQ}$

30 The following matrix shows the cost (in euros) of four products P_1, P_2, P_3 and P_4 in France and Germany.

$$\begin{array}{c c} & \begin{array}{cccc} P_1 & P_2 & P_3 & P_4 \end{array} \\ \begin{array}{c} F \\ G \end{array} & \begin{pmatrix} 50 & 38 & 24 & 80 \\ 55 & 30 & 26 & 75 \end{pmatrix} \end{array}$$

A business wants to buy 100 units of P_1, 50 units of P_2, 125 units of P_3 and 30 units of P_4.

a Write a matrix equation for the cost matrix, **C**.

b Hence find the cost of ordering from each country.

31 Find the values of x and y such that

$$\begin{pmatrix} x & y \\ 2 & -1 \end{pmatrix} \begin{pmatrix} 3 & 1 \\ 1 & 2 \end{pmatrix} = \begin{pmatrix} 7 & 9 \\ 5 & 0 \end{pmatrix}$$

32 Find the values of p and q such that

$$\begin{pmatrix} 2 & 1 \\ -1 & 4 \end{pmatrix}\begin{pmatrix} 1 & p \\ 3 & q \end{pmatrix} = \begin{pmatrix} 5 & -4 \\ 11 & 11 \end{pmatrix}$$

33 Two manufacturers, Engineer Right and Forge Well, need to produce varying quantities of four products P_1, P_2, P_3 and P_4 as given in the matrix **D**.

$$\mathbf{D} = \begin{array}{cc} & \begin{array}{cccc} P_1 & P_2 & P_3 & P_4 \end{array} \\ & \begin{pmatrix} 210 & 180 & 320 & 400 \\ 250 & 150 & 200 & 450 \end{pmatrix} \begin{array}{c} E \\ F \end{array} \end{array}$$

The price per unit (in \$) of producing these products at factories G and H is given in the matrix **C**.

$$\mathbf{C} = \begin{array}{cc} & \begin{array}{cccc} P_1 & P_2 & P_3 & P_4 \end{array} \\ & \begin{pmatrix} 55 & 80 & 48 & 20 \\ 50 & 85 & 40 & 22 \end{pmatrix} \begin{array}{c} G \\ H \end{array} \end{array}$$

a Find the total cost matrix **M** that gives the overall cost to each manufacturer of using each factory.

b State which factory each manufacturer should use to minimize the cost.

34 The percentage of votes cast by males and females for the Red, Orange and Yellow parties at the last election is given by the following matrix.

$$\begin{array}{cc} & \begin{array}{cc} M & F \end{array} \\ \begin{array}{c} R \\ O \\ Y \end{array} & \begin{pmatrix} 28 & 34 \\ 42 & 31 \\ 30 & 35 \end{pmatrix} \end{array}$$

The number of eligible voters in districts C_1 and C_2 at the next election is given by the following matrix.

$$\begin{array}{cc} & \begin{array}{cc} M & F \end{array} \\ \begin{array}{c} C_1 \\ C_2 \end{array} & \begin{pmatrix} 26124 & 30125 \\ 28987 & 29846 \end{pmatrix} \end{array}$$

Assuming that the percentage of votes remains the same,

a calculate the 3×2 matrix that gives the projected number of votes for each party in these two districts at the next election

b find the projected total number of votes for the Yellow party in these two districts.

35 $\mathbf{A} = \begin{pmatrix} a & 3 \\ -2 & 1 \end{pmatrix}$ and $\mathbf{B} = \begin{pmatrix} 2 & b \\ 1 & 3 \end{pmatrix}$

Given that $\mathbf{A} + s\mathbf{B} = t\mathbf{I}$, find the values a, b, s and t.

36 $\boldsymbol{A} = \begin{pmatrix} 1 & a \\ 2 & 2 \end{pmatrix}$, $\boldsymbol{B} = \begin{pmatrix} b & 3 \\ -4 & -5 \end{pmatrix}$ and $\boldsymbol{C} = \begin{pmatrix} 2 & 2 \\ 4 & 0 \end{pmatrix}$

Given that $\frac{1}{2}\boldsymbol{C} = p\boldsymbol{A} + q\boldsymbol{B}$, find the values of a, b, p and q.

37 $\boldsymbol{M} = \begin{pmatrix} 1 & 2a \\ -a & 3 \end{pmatrix}$ and $\boldsymbol{N} = \begin{pmatrix} 4 & b+1 \\ 3b & 1 \end{pmatrix}$

Given that $c\mathbf{M} + d\mathbf{N} = \mathbf{I}$, find the values of a, b, c and d.

38 Find the value of k so that the matrices $\begin{pmatrix} 1 & 3 \\ 2 & 4 \end{pmatrix}$ and $\begin{pmatrix} k & 6 \\ 4 & -1 \end{pmatrix}$ commute.

39 Given that the matrices $\begin{pmatrix} 2 & 5 \\ 5 & -2 \end{pmatrix}$ and $\begin{pmatrix} c & d \\ d & -c \end{pmatrix}$ commute, find an expression for d in terms of c.

40 $\boldsymbol{A} = \begin{pmatrix} a_1 & a_2 \\ a_3 & a_4 \end{pmatrix}$, $\boldsymbol{B} = \begin{pmatrix} b_1 & b_2 \\ b_3 & b_4 \end{pmatrix}$ and $\boldsymbol{C} = \begin{pmatrix} c_1 & c_2 \\ c_3 & c_4 \end{pmatrix}$

a Calculate **AB** and **BC**.

b Show that $(\mathbf{AB})\mathbf{C} = \mathbf{A}(\mathbf{BC})$.

3B Determinants and inverses

In Section 4C, you will see that the determinant of a matrix gives the scale factor of the enlargement determined by the matrix.

$n \times n$ matrices with technology

The **determinant** of a square matrix is a numerical value calculated from the elements of the matrix. It is similar to the magnitude of a vector, except that a determinant can be positive or negative.

You do not need to know how the determinant is calculated in general; you just need to be able to use your GDC to find the determinant.

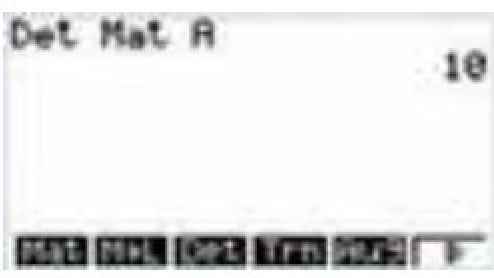
Tip

The notation **det M** is usually used for the determinant of the matrix **M** but in keeping with the idea of a determinant being like the magnitude of a vector, the notation $|\mathbf{M}|$ is also used.

You do need to be able to find the determinant of a 2×2 matrix without the GDC. You will see how to do this below.

WORKED EXAMPLE 3.9

Find the determinant of the matrix $\mathbf{A} = \begin{pmatrix} 2 & -1 & 3 \\ 0 & 4 & -2 \\ 1 & 5 & 0 \end{pmatrix}$.



$\det \mathbf{A} = 10$

The reciprocal of the real number x (where $x \neq 0$) is the number x^{-1}, giving $xx^{-1} = 1$. An **inverse matrix** is the equivalent idea in matrices to that of a reciprocal in real numbers.

Tip

Notice that a matrix must be square to have an inverse as otherwise the product $\mathbf{MM}^{-1}$ would not exist.

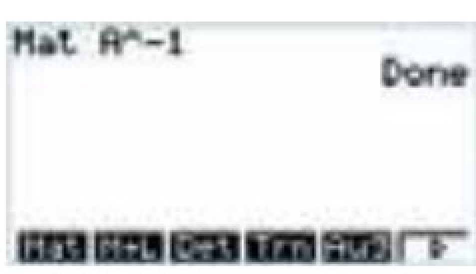
KEY POINT 3.8

The inverse of a square matrix **M** is the matrix $\mathbf{M}^{-1}$ such that

$$\mathbf{MM}^{-1} = \mathbf{M}^{-1}\mathbf{M} = \mathbf{I}$$

WORKED EXAMPLE 3.10

Find the inverse $\mathbf{A}^{-1}$ of the matrix $\mathbf{A} = \begin{pmatrix} 2 & -1 & 3 \\ 0 & 4 & -2 \\ 1 & 5 & 0 \end{pmatrix}$.



$$\mathbf{A}^{-1} = \begin{pmatrix} 1 & 1.5 & -1 \\ -0.2 & -0.3 & 0.4 \\ -0.4 & -1.1 & 0.8 \end{pmatrix}$$

Finding the inverse of a matrix allows simple matrix equations to be solved. However, because matrix multiplication is not commutative, it is important to multiply the inverse on the correct side of the equation.

KEY POINT 3.9

- If $\mathbf{AB} = \mathbf{C}$, then $\mathbf{B} = \mathbf{A}^{-1}\mathbf{C}$
- If $\mathbf{BA} = \mathbf{C}$, then $\mathbf{B} = \mathbf{CA}^{-1}$

The first result is proved here. The second result is proved similarly by multiplying $\mathbf{A}^{-1}$ on the right.

Proof 3.1

Prove that if $\mathbf{AB} = \mathbf{C}$, then $\mathbf{B} = \mathbf{A}^{-1}\mathbf{C}$.

$$\mathbf{AB} = \mathbf{C}$$

Multiply through the equation on the left by $\mathbf{A}^{-1}$

$$\mathbf{A}^{-1}\mathbf{AB} = \mathbf{A}^{-1}\mathbf{C}$$

Matrix multiplication is associative

$$(\mathbf{A}^{-1}\mathbf{A})\mathbf{B} = \mathbf{A}^{-1}\mathbf{C}$$

$\mathbf{A}^{-1}\mathbf{A} = \mathbf{I}$...

$$\mathbf{IB} = \mathbf{A}^{-1}\mathbf{C}$$

... and $\mathbf{IB} = \mathbf{B}$

$$\mathbf{B} = \mathbf{A}^{-1}\mathbf{C}$$

WORKED EXAMPLE 3.11

Matrix $\mathbf{A} = \begin{pmatrix} 2 & -1 & 3 \\ 0 & 4 & -2 \\ 1 & 5 & 0 \end{pmatrix}$ and matrix $\mathbf{C} = \begin{pmatrix} 5 & -2 & 0 \\ -8 & 9 & 1 \end{pmatrix}$.

Given that $\mathbf{BA} = \mathbf{C}$, find the matrix $\mathbf{B}$.

Multiplying by $\mathbf{A}^{-1}$ on the right gives $\mathbf{B} = \mathbf{CA}^{-1}$

$$\mathbf{BA} = \mathbf{C}$$
$$\mathbf{B} = \mathbf{CA}^{-1}$$

We found $\mathbf{A}^{-1}$ in Worked Example 3.10

$$= \begin{pmatrix} 5 & -2 & 0 \\ -8 & 9 & 1 \end{pmatrix} \begin{pmatrix} 1 & 1.5 & -1 \\ -0.2 & -0.3 & 0.4 \\ -0.4 & -1.1 & 0.8 \end{pmatrix}$$

$$= \begin{pmatrix} 5.4 & 8.1 & -5.8 \\ -10.2 & -15.8 & 12.4 \end{pmatrix}$$

CONCEPTS – RELATIONSHIPS

You can think of matrix **B** in Worked Example 3.10 as giving a **relationship** between matrices **A** and **C** – the two matrices are related through multiplication by **B**. This is a generalization of the idea of proportionality (when one quantity is a multiple of another). Another way to think of this type of a relationship is as a transformation: matrix **B** transforms **A** into **C**. We will apply this idea to geometrical transformations in the next chapter.

Tip

In the exam, you can always find numerical determinants using a GDC. However, it is useful to practise on numerical examples by hand first, so that you can then do algebraic examples.

Determinants and inverses of 2 × 2 matrices by hand

You need to be able to find the determinant and inverse of a 2 × 2 matrix by hand.

KEY POINT 3.10

The determinant of the 2 × 2 matrix $\mathbf{M} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ is

$$\det \mathbf{M} = ad - bc$$

WORKED EXAMPLE 3.12

Find the determinant of the matrix $\begin{pmatrix} -3 & -5 \\ 2 & 4 \end{pmatrix}$.

$$\det\begin{pmatrix} -3 & -5 \\ 2 & 4 \end{pmatrix} = (-3) \times 4 - (-5) \times 2$$

Use $\det \mathbf{M} = ad - bc$

$$= -12 - (-10)$$
$$= -2$$

If a matrix has a determinant of zero then the matrix is said to be **singular**.

WORKED EXAMPLE 3.13

Find the values of k for which the matrix $\begin{pmatrix} k & k-3 \\ 4 & k \end{pmatrix}$ is singular.

A singular matrix has $\det \mathbf{M} = 0$

$$\det\begin{pmatrix} k & 4 \\ k+3 & k \end{pmatrix} = 0$$

Use $\det \mathbf{M} = ad - bc$

$$k^2 - 4(k+3) = 0$$

Expand and solve with the GDC

$$k^2 - 4k - 12 = 0$$
$$k = 6 \text{ or } -2$$

Tip

You can see from the definition in Key Point 3.11 that a singular matrix does not have an inverse, because we cannot divide by zero.

You also need to be able to find the inverse of a 2 × 2 matrix by hand.

KEY POINT 3.11

The inverse of the 2 × 2 matrix $\mathbf{M} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, where $\det \mathbf{M} \neq 0$ is

$$\mathbf{M}^{-1} = \frac{1}{\det \mathbf{M}}\begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$$

You are the Researcher

What are other ways to define the determinant?

You can prove this by thinking about solving the equations $ax + by = p$ and $cx + dy = q$ for x and y. This also justifies the expression for the determinant in Key Point 3.10. One way of defining the determinant is the quantity which equals zero if there is no unique solution to the system of equations.

WORKED EXAMPLE 3.14

Find the inverse of the matrix $\begin{pmatrix} -3 & -5 \\ 2 & 4 \end{pmatrix}$ from Worked Example 3.12.

Use $\mathbf{M}^{-1} = \frac{1}{\det \mathbf{M}}\begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$

You already found that $\det \mathbf{M} = -2$

$$\begin{pmatrix} -3 & -5 \\ 2 & 4 \end{pmatrix}^{-1} = \frac{1}{-2}\begin{pmatrix} 4 & -(-5) \\ -2 & -3 \end{pmatrix}$$

$$= -\frac{1}{2}\begin{pmatrix} 4 & 5 \\ -2 & -3 \end{pmatrix}$$

You could either leave the factor of $-\frac{1}{2}$ out the front or multiply it into the matrix

$$= \begin{pmatrix} -2 & -2.5 \\ 1 & 1.5 \end{pmatrix}$$

Exercise 3B

For questions 1 to 6, use the method demonstrated in Worked Example 3.9 to find the determinant of each matrix with your GDC.

1 a $\begin{pmatrix} -2 & 1 & 3 \\ 6 & 4 & 1 \\ -3 & 2 & 5 \end{pmatrix}$

b $\begin{pmatrix} -2 & 3 & 1 \\ 4 & 1 & -1 \\ 5 & 0 & -2 \end{pmatrix}$

2 a $\begin{pmatrix} 3 & -18 & 2 \\ -1 & 13 & -3 \\ 1 & 8 & -4 \end{pmatrix}$

b $\begin{pmatrix} 2 & 1 & 8 \\ 3 & 5 & 19 \\ 4 & 7 & 26 \end{pmatrix}$

3 a $\begin{pmatrix} 1 & 0 & -2 \\ 2 & 4 & 0 \\ -3 & 1 & 5 \end{pmatrix}$

b $\begin{pmatrix} 4 & 5 & 0 \\ 0 & 1 & 3 \\ 2 & 3 & -2 \end{pmatrix}$

4 a $\begin{pmatrix} 1 & -1 & -1 & 1 \\ 2 & 0 & 2 & 0 \\ 0 & -1 & -2 & 0 \\ 3 & -3 & -2 & 4 \end{pmatrix}$

b $\begin{pmatrix} 1 & 3 & 1 & 4 \\ 0 & 2 & 1 & 1 \\ 3 & 9 & 5 & 15 \\ 0 & 4 & 2 & 3 \end{pmatrix}$

5 a $\begin{pmatrix} 1 & 1 & 1 & 0 \\ 0 & 3 & 1 & 2 \\ 2 & 3 & 1 & 0 \\ 1 & 0 & 2 & 1 \end{pmatrix}$

b $\begin{pmatrix} 3 & -2 & 2 & 3 \\ 1 & 7 & -1 & 4 \\ 2 & 9 & -2 & 14 \\ 2 & 8 & -2 & 15 \end{pmatrix}$

6 a $\begin{pmatrix} 4 & -2 & 1 & 8 \\ 7 & 2 & 8 & 15 \\ 9 & -3 & 3 & 17 \\ 18 & -8 & -2 & 26 \end{pmatrix}$

b $\begin{pmatrix} -5 & 3 & 2 & 4 \\ 1 & -7 & 2 & -4 \\ 2 & 0 & 5 & -1 \\ 4 & 6 & 10 & 1 \end{pmatrix}$

For questions 7 to 12, use the method demonstrated in Worked Example 3.10 to find the inverse of each matrix with your GDC or state that it does not exist.

7 a $\begin{pmatrix} -2 & 1 & 3 \\ 6 & 4 & 1 \\ -3 & 2 & 5 \end{pmatrix}$

b $\begin{pmatrix} -2 & 3 & 1 \\ 4 & 1 & -1 \\ 5 & 0 & -2 \end{pmatrix}$

8 a $\begin{pmatrix} 3 & -18 & 2 \\ -1 & 13 & -3 \\ 1 & 8 & -4 \end{pmatrix}$

b $\begin{pmatrix} 2 & 1 & 8 \\ 3 & 5 & 19 \\ 4 & 7 & 26 \end{pmatrix}$

9 a $\begin{pmatrix} 1 & 0 & -2 \\ 2 & 4 & 0 \\ -3 & 1 & 5 \end{pmatrix}$

b $\begin{pmatrix} 4 & 5 & 0 \\ 0 & 1 & 3 \\ 2 & 3 & -2 \end{pmatrix}$

10 **a** $\begin{pmatrix} 1 & -1 & -1 & 1 \\ 2 & 0 & 2 & 0 \\ 0 & -1 & -2 & 0 \\ 3 & -3 & -2 & 4 \end{pmatrix}$

b $\begin{pmatrix} 1 & 3 & 1 & 4 \\ 0 & 2 & 1 & 1 \\ 3 & 9 & 5 & 15 \\ 0 & 4 & 2 & 3 \end{pmatrix}$

11 **a** $\begin{pmatrix} 1 & 1 & 1 & 0 \\ 0 & 3 & 1 & 2 \\ 2 & 3 & 1 & 0 \\ 1 & 0 & 2 & 1 \end{pmatrix}$

b $\begin{pmatrix} 3 & -2 & 2 & 3 \\ 1 & 7 & -1 & 4 \\ 2 & 9 & -2 & 14 \\ 2 & 8 & -2 & 15 \end{pmatrix}$

12 **a** $\begin{pmatrix} 4 & -2 & 1 & 8 \\ 7 & 2 & 8 & 15 \\ 9 & -3 & 3 & 17 \\ 18 & -8 & -2 & 26 \end{pmatrix}$

b $\begin{pmatrix} -5 & 3 & 2 & 4 \\ 1 & -7 & 2 & -4 \\ 2 & 0 & 5 & -1 \\ 4 & 6 & 10 & 1 \end{pmatrix}$

For questions 13 to 16, use the method demonstrated in Worked Example 3.11 to find **B**. In these questions,

$$\mathbf{A} = \begin{pmatrix} -3 & 2 \\ 7 & -6 \end{pmatrix} \text{ and } \mathbf{C} = \begin{pmatrix} 0 & 5 & -6 \\ 7 & -3 & 0 \\ 4 & 0 & -2 \end{pmatrix}.$$

13 **a** $\mathbf{AB} = \begin{pmatrix} 7 & 9 & -7 \\ 19 & -21 & 11 \end{pmatrix}$

b $\mathbf{BA} = \begin{pmatrix} 6 & -8 \\ 5 & -2 \\ 1 & -2 \end{pmatrix}$

14 **a** $\mathbf{BA} = \begin{pmatrix} 25 & -22 \end{pmatrix}$

b $\mathbf{AB} = \begin{pmatrix} 1 \\ -9 \end{pmatrix}$

15 **a** $\mathbf{BC} = \begin{pmatrix} 13 & -4 & -2 \\ 19 & -15 & 8 \\ 6 & -24 & 26 \end{pmatrix}$

b $\mathbf{CB} = \begin{pmatrix} -12 & -15 & 5 \\ 14 & -2 & -10 \\ 4 & -6 & -4 \end{pmatrix}$

16 **a** $\mathbf{CB} = \begin{pmatrix} 9 \\ 5 \\ 6 \end{pmatrix}$

b $\mathbf{BC} = \begin{pmatrix} 5 & 1 & -4 \end{pmatrix}$

For questions 17 to 19, use the method demonstrated in Work Example 3.12 to find the determinant of each matrix.

17 **a** $\begin{pmatrix} 1 & -4 \\ 3 & 2 \end{pmatrix}$ **b** $\begin{pmatrix} 3 & 4 \\ -2 & -1 \end{pmatrix}$

18 **a** $\begin{pmatrix} -6 & 2 \\ -5 & 3 \end{pmatrix}$ **b** $\begin{pmatrix} 0 & -2 \\ -1 & 8 \end{pmatrix}$

19 **a** $\begin{pmatrix} 2 & 6 \\ 1 & 3 \end{pmatrix}$ **b** $\begin{pmatrix} -5 & 10 \\ -2 & 4 \end{pmatrix}$

For questions 20 to 22, use the method demonstrated in Worked Example 3.13 to find the value(s) of k for which each matrix is singular.

20 **a** $\begin{pmatrix} k & -3 \\ 1 & 2 \end{pmatrix}$ **b** $\begin{pmatrix} 1 & 5 \\ k & 2 \end{pmatrix}$

21 **a** $\begin{pmatrix} 6k & 3 \\ 2 & 4k \end{pmatrix}$ **b** $\begin{pmatrix} 9 & 2k \\ 2k & 4 \end{pmatrix}$

22 **a** $\begin{pmatrix} k-1 & k+5 \\ 2 & k \end{pmatrix}$ **b** $\begin{pmatrix} 2k & k-2 \\ k & 1 \end{pmatrix}$

For questions 23 to 25, use the method demonstrated in Worked Example 3.14 to find the inverse of each matrix (from questions 17 to 19) or to state that the inverse does not exist.

23 **a** $\begin{pmatrix} 1 & -4 \\ 3 & 2 \end{pmatrix}$ **b** $\begin{pmatrix} 3 & 4 \\ -2 & -1 \end{pmatrix}$

24 **a** $\begin{pmatrix} -6 & 2 \\ -5 & 3 \end{pmatrix}$ **b** $\begin{pmatrix} 0 & -2 \\ -1 & 8 \end{pmatrix}$

25 **a** $\begin{pmatrix} 2 & 6 \\ 1 & 3 \end{pmatrix}$ **b** $\begin{pmatrix} -5 & 10 \\ -2 & 4 \end{pmatrix}$

26 **a** Find the values of k for which the matrix $\mathbf{A} = \begin{pmatrix} 2k & 1 \\ 4 & k \end{pmatrix}$ is singular.

b For all values of k apart from those in part **a**, find $\mathbf{A}^{-1}$ in terms of k.

27 a Show that the matrix $\mathbf{A} = \begin{pmatrix} 1 & 3c \\ -c & 2 \end{pmatrix}$ has an inverse for all values of c.

b Find $\mathbf{A}^{-1}$ in terms of c.

28 $\mathbf{A} = \begin{pmatrix} 8 & 4 \\ -3 & -2 \end{pmatrix}$ and $\mathbf{C} = \begin{pmatrix} 28 & 20 & 40 \\ -13 & -7 & -18 \end{pmatrix}$

Given that $\mathbf{AB} = \mathbf{C}$,

a find $\mathbf{A}^{-1}$

b hence find $\mathbf{B}$.

29 $\mathbf{B} = \begin{pmatrix} 2 & -1 & 6 \\ 1 & 3 & -2 \\ 7 & -4 & 5 \end{pmatrix}$ and $\mathbf{C} = \begin{pmatrix} -9 & 19 & -31 \\ 25 & -10 & 38 \end{pmatrix}$

Given that $\mathbf{AB} = \mathbf{C}$,

a find $\mathbf{B}^{-1}$

b hence find $\mathbf{A}$.

30 A 6-digit number is written in a 3×2 matrix and encoded by multiplying by the matrix $\begin{pmatrix} 1 & 0 & -2 \\ 3 & -1 & 2 \\ 4 & 1 & 1 \end{pmatrix}$.

The result is $\begin{pmatrix} 1 & -16 \\ 23 & 20 \\ 35 & 21 \end{pmatrix}$.

Find the original 3×2 matrix.

31 Given that $\mathbf{P} = \begin{pmatrix} 1 & -3 \\ 1 & 5 \end{pmatrix}$, find the matrix $\mathbf{Q}$ such that $\mathbf{P}^{-1}\mathbf{QP} = \begin{pmatrix} 4 & 0 \\ 0 & -2 \end{pmatrix}$.

32 Given that $\mathbf{A} = \begin{pmatrix} 5 & 1 & -4 \\ 3 & 0 & 2 \\ -1 & -1 & 6 \end{pmatrix}$, $\mathbf{B} = \begin{pmatrix} 6 & 2 & -2 \\ -3 & -1 & 2 \\ -5 & -1 & 2 \end{pmatrix}$ and that $\mathbf{ABC} = \mathbf{I}$, find the matrix $\mathbf{C}$.

33 For the matrix 2×2 matrix $\mathbf{M} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, show that $\det(k\mathbf{M}) = k^2 \det \mathbf{M}$.

34 $\mathbf{M} = \begin{pmatrix} 1 & 1 & 2 \\ 0 & 2 & -1 \\ 0 & 0 & 4 \end{pmatrix}$

a Show that $\mathbf{M}^3 = 7\mathbf{M}^2 - 14\mathbf{M} + 8\mathbf{I}$.

b Hence deduce that $\mathbf{M}^{-1} = \frac{1}{8}(\mathbf{M}^2 - 7\mathbf{M} + 14\mathbf{I})$.

35 Prove that if $\mathbf{M} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, then $\mathbf{M}^{-1} = \frac{1}{ad - bc}\begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$.

36 Prove that $(\mathbf{AB})^{-1} = \mathbf{B}^{-1}\mathbf{A}^{-1}$.

Hence simplify $(\mathbf{A}^{-1}\mathbf{B})^{-1}\mathbf{A}^{-1}$.

3C Solutions of systems of equations using the inverse matrix

You already know how to use the GDC to solve systems of two or three equations. You now also need to be able to set up these systems as a matrix equation and use your knowledge of inverse matrices to solve that equation.

Tip

Note that the system of equations will not have a unique solution if the matrix $\mathbf{A}$ is singular (since, in this case, $\mathbf{A}^{-1}$ does not exist).

KEY POINT 3.12

- A system of simultaneous equations can be written as a matrix equation
  $$\mathbf{Ax} = \mathbf{b}$$
  where $\mathbf{A}$ contains the coefficients and $\mathbf{x}$ is a column vector of the variables.
- If a solution exists, it is given by
  $$\mathbf{x} = \mathbf{A}^{-1}\mathbf{b}$$

WORKED EXAMPLE 3.15

By first forming a matrix equation, solve the system of equations

$$\begin{cases} 2x - 7y = 5 \\ 3x - 8y = 4 \end{cases}$$

Set up the matrix equation $\mathbf{Ax} = \mathbf{b}$, where the entries of $\mathbf{A}$ are the coefficients of x and y

$$\begin{cases} 2x - 7y = 5 \\ 3x - 8y = 4 \end{cases}$$

$$\begin{pmatrix} 2 & -7 \\ 3 & -8 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 5 \\ 4 \end{pmatrix}$$

The solution is given by $\mathbf{x} = \mathbf{A}^{-1}\mathbf{b}$

$$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 2 & -7 \\ 3 & -8 \end{pmatrix}^{-1}\begin{pmatrix} 5 \\ 4 \end{pmatrix}$$

Find $\mathbf{A}^{-1}$

$$= \frac{1}{5}\begin{pmatrix} -8 & 7 \\ -3 & 2 \end{pmatrix}\begin{pmatrix} 5 \\ 4 \end{pmatrix}$$

$$= \frac{1}{5}\begin{pmatrix} -12 \\ -7 \end{pmatrix}$$

$$= \begin{pmatrix} -2.4 \\ -1.4 \end{pmatrix}$$

WORKED EXAMPLE 3.16

Hotel rooms in Alpha Hotel, Bravo Hotel and Charlie Hotel all offer nightly rates on Double (D), Superior (S) and Luxury (L) rooms as shown:

$$\begin{array}{cc} & \begin{array}{ccc} D & S & L \end{array} \\ \begin{array}{c} A \\ B \\ C \end{array} & \begin{pmatrix} £120 & £175 & £225 \\ £125 & £200 & £275 \\ £100 & £150 & £250 \end{pmatrix} \end{array}$$

A company wants to book several rooms for one night. It would cost £3230 at Alpha, £3575 at Bravo and £2900 at Charlie.

Find the number of Double, Superior and Luxury rooms the company wants to book.

Let d be the number of double rooms, s be the number of superior rooms and l be the number of luxury rooms.
Then,

Set up a matrix equation of the form $\mathbf{Ax} = \mathbf{b}$

$$\begin{pmatrix} 120 & 175 & 225 \\ 125 & 200 & 275 \\ 100 & 150 & 250 \end{pmatrix}\begin{pmatrix} d \\ s \\ l \end{pmatrix} = \begin{pmatrix} 3230 \\ 3575 \\ 2900 \end{pmatrix}$$

Solve the equation using $\mathbf{x} = \mathbf{A}^{-1}\mathbf{b}$

$$\begin{pmatrix} d \\ s \\ l \end{pmatrix} = \begin{pmatrix} 120 & 175 & 225 \\ 125 & 200 & 275 \\ 100 & 150 & 250 \end{pmatrix}^{-1}\begin{pmatrix} 3230 \\ 3575 \\ 2900 \end{pmatrix}$$

Use the GDC to find the inverse of the 3×3 matrix

$$= -\frac{1}{900}\begin{pmatrix} -70 & 80 & -25 \\ 30 & -60 & 39 \\ 10 & 4 & -17 \end{pmatrix}\begin{pmatrix} 3230 \\ 3575 \\ 2900 \end{pmatrix}$$

$$= \begin{pmatrix} 14 \\ 5 \\ 3 \end{pmatrix}$$

So, $d = 14$, $s = 5$, $l = 3$

Exercise 3C

As well as doing the questions in this exercise, revisit Exercise 12A of Mathematics: applications and interpretation SL and make sure you can set up and solve those systems of equations using matrices.

For questions 1 to 6, use the method demonstrated in Worked Example 3.15 to solve each system of equations by first forming a matrix equation.

1 **a** $\begin{cases} x - 4y = 11 \\ 3x + 5y = -1 \end{cases}$

b $\begin{cases} 2x + 3y = -5 \\ x + 5y = 1 \end{cases}$

2 **a** $\begin{cases} 6x - 4y = 7 \\ 9x - 6y = 2 \end{cases}$

b $\begin{cases} 3x + 9y = -4 \\ 4x + 12y = -1 \end{cases}$

3 **a** $\begin{cases} 7x - 3y = -1 \\ 4x - 2y = 3 \end{cases}$

b $\begin{cases} 3x + y = -2 \\ 8x + 4y = 5 \end{cases}$

4 **a** $\begin{cases} 2x + 4y + z = 8 \\ 3x + y - 2z = 7 \\ 5x - y + 3z = 9 \end{cases}$

b $\begin{cases} x + y + z = 3 \\ x + 2y + 3z = -5 \\ 3x - 2y + 2z = 4 \end{cases}$

5 **a** $\begin{cases} x + 2y - z = 2 \\ 2x + y = 5 \\ x + z = 4 \end{cases}$

b $\begin{cases} x - y = 4 \\ y + z = 1 \\ x - z = 3 \end{cases}$

6 **a** $\begin{cases} 3x - y + z = 17 \\ x + 2y - z = 8 \\ 2x - 3y + 2z = 3 \end{cases}$

b $\begin{cases} x - 2y - z = -2 \\ 2x + y - 3z = 9 \\ 5x + 5y - 8z = 15 \end{cases}$

7 **a** Write the system of equations

$\begin{cases} 5x - 4y = 7 \\ 3x - 2y = -1 \end{cases}$ in the form $\mathbf{Ax} = \mathbf{b}$, where $\mathbf{A}$ is a matrix and $\mathbf{x}$ and $\mathbf{b}$ are column vectors.

b Find $\mathbf{A}^{-1}$.

c Hence solve the system of equations.

8 **a** Write the system of equations

$\begin{cases} x - y + z = 1 \\ 5x + 3y + 2z = 1 \\ 4y - 3z = 4 \end{cases}$ in the form $\mathbf{Ax} = \mathbf{b}$, where $\mathbf{A}$ is a matrix and $\mathbf{x}$ and $\mathbf{b}$ are column vectors.

b Find $\mathbf{A}^{-1}$.

c Hence solve the system of equations.

9 A restaurant owner wants to buy a supply of raspberries (R) and strawberries (S). The cost matrix per kilogram (in euros) from suppliers X and Y is

$$\begin{array}{cc} & \begin{array}{cc} R & S \end{array} \\ \begin{array}{c} X \\ Y \end{array} & \begin{pmatrix} 8.5 & 5.5 \\ 8 & 6 \end{pmatrix} \end{array}$$

€26.50 is spent with supplier X and €27 is spent with supplier Y.

Find the number of kilograms of raspberries and the number of kilograms of strawberries the restaurant owner buys.

10 Gill invested a total of \$50 000 in three different share funds, Allshare, Baserate and Chartwise. She invested \$10 000 more in Allshare than Chartwise.

After one year, these funds had returned growth rates of 5.2%, 3.1% and 6.5%, respectively. The total value of Gill's investment after one year was \$52 447.50.

a Set up a matrix equation of the form $\mathbf{Mx} = \mathbf{b}$ to represent this information.

b Hence find the amount Gill invested in each of the three share funds.

11 **a** Write the system of equations

$\begin{cases} kx + 3y = -1 \\ (k-2)x + 4y = 1 \end{cases}$ in the form $\mathbf{Ax} = \mathbf{b}$, where $\mathbf{A}$ is a matrix and $\mathbf{x}$ and $\mathbf{b}$ are column vectors.

b Hence find the value of k for which the equations do not have a solution.

c Assuming that the value of k is not equal to that found in part **b**, solve the system of equations in terms of k.

12 **a** Write the system of equations

$\begin{cases} kx + 5y = 2 \\ 2x + (k-3)y = -1 \end{cases}$ in the form $\mathbf{Ax} = \mathbf{b}$, where $\mathbf{A}$ is a matrix and $\mathbf{x}$ and $\mathbf{b}$ are column vectors.

b Hence find the values of k for which the equations do not have a solution.

c Assuming that the value of k is not equal to either of those found in part **b**, solve the system of equations in terms of k.

3D Eigenvalues and eigenvectors

Characteristic polynomial of 2 × 2 matrices

It is particularly useful to be able to solve equations of the form $\mathbf{Mv} = \lambda\mathbf{v}$, where $\mathbf{M}$ is a matrix, $\mathbf{v}$ is a column vector and λ is a scalar.

You will see geometrical applications of solutions to equations of this form in Section 4C and applications to probability in Section 8D.

KEY POINT 3.13

If $\mathbf{Mv} = \lambda\mathbf{v}$, then λ is called an **eigenvalue** of the matrix $\mathbf{M}$ and the non-zero vector $\mathbf{v}$ is its associated **eigenvector**.

To find the eigenvalues of a matrix you need to solve the **characteristic equation** of the matrix.

Tip

In this course, the characteristic equation will always be a quadratic.

KEY POINT 3.14

The eigenvalues of the matrix $\mathbf{M}$ satisfy the characteristic equation $\det(\mathbf{M} - \lambda\mathbf{I}) = 0$.

Proof 3.2

Prove that the eigenvalues of the matrix $\mathbf{M}$ satisfy the characteristic equation $\det(\mathbf{M} - \lambda\mathbf{I}) = 0$.

Rearrange the eigenvalue equation so that you have the zero matrix on the right

$$\mathbf{Mv} = \lambda\mathbf{v}$$

$$\mathbf{Mv} - \lambda\mathbf{v} = \mathbf{0}$$

Multiply $\lambda\mathbf{v}$ by the identity matrix so that $\mathbf{M} - \lambda\mathbf{I}$ is a square matrix

$$\mathbf{Mv} - \lambda\mathbf{Iv} = \mathbf{0}$$

$$(\mathbf{M} - \lambda\mathbf{I})\mathbf{v} = \mathbf{0}$$

If the matrix $\mathbf{M} - \lambda\mathbf{I}$ had an inverse then the equation $(\mathbf{M} - \lambda\mathbf{I})\mathbf{v} = \mathbf{0}$ could be solved to give $\mathbf{v} = \mathbf{0}$, which is not permissible

Since $\mathbf{v} \neq \mathbf{0}$, the matrix $\mathbf{M} - \lambda\mathbf{I}$ is singular, so

$$\det(\mathbf{M} - \lambda\mathbf{I}) = 0$$

WORKED EXAMPLE 3.17

Find the eigenvalues and associated eigenvectors of the matrix $\mathbf{M} = \begin{pmatrix} 1 & 4 \\ 2 & 3 \end{pmatrix}$.

Solve the characteristic equation to find eigenvalues

$$\det(\mathbf{M} - \lambda\mathbf{I}) = 0$$

$$\det\begin{pmatrix} 1 & 4 \\ 2 & 3 \end{pmatrix} - \lambda\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = 0$$

$$\det\left(\begin{pmatrix} 1 & 4 \\ 2 & 3 \end{pmatrix} - \begin{pmatrix} \lambda & 0 \\ 0 & \lambda \end{pmatrix}\right) = 0$$

You will always end up with a matrix of this form so you can start at this point in future

$$\det\begin{pmatrix} 1-\lambda & 4 \\ 2 & 3-\lambda \end{pmatrix} = 0$$

Expand the determinant and solve for λ (by factorising or with your GDC)

$$(1-\lambda)(3-\lambda)-8=0$$
$$3-4\lambda+\lambda^2-8=0$$
$$\lambda^2-4\lambda-5=0$$
$$\lambda=5,\ -1$$

You now need to find the eigenvectors associated with these eigenvalues. To do this return to the definition of an eigenvector: $\mathbf{Mv}=\lambda\mathbf{v}$

When $\lambda = 5$,

$$\begin{pmatrix}1 & 4\\ 2 & 3\end{pmatrix}\begin{pmatrix}x\\ y\end{pmatrix}=5\begin{pmatrix}x\\ y\end{pmatrix}$$

Write this as two simultaneous equations

$$\begin{cases}x+4y=5x\\ 2x+3y=5y\end{cases}$$

Simplifying each, we see that they just give the same equation twice

$$\begin{cases}4x=4y\\ 2x=2y\end{cases}$$

So, you have a free choice of x or y. Any multiple of the vector $\begin{pmatrix}1\\ 1\end{pmatrix}$ would also be correct but this is the simplest version

$x=y$, so choosing $x=1$ gives the eigenvector $\mathbf{v}=\begin{pmatrix}1\\ 1\end{pmatrix}$

When $\lambda = -1$,

Now repeat for the other eigenvalue

$$\begin{pmatrix}1 & 4\\ 2 & 3\end{pmatrix}\begin{pmatrix}x\\ y\end{pmatrix}=-\begin{pmatrix}x\\ y\end{pmatrix}$$

$$\begin{cases}x+4y=-x\\ 2x+3y=-y\end{cases}$$

Again you get the same equation twice, so you can choose x or y

$$\begin{cases}2x=-4y\\ 2x=-4y\end{cases}$$

The simplest choice this time is to let $y=1$ so that $x=-1$

$x=-2y$, so choosing $y=1$ gives the eigenvector $\mathbf{v}=\begin{pmatrix}-2\\ 1\end{pmatrix}$.

(If you chose $x=1$ again you would get $y=-\frac{1}{2}$, which is still correct but not in such a simple form.)

Tip

The sum of the values on the diagonal running from top left of a matrix to bottom right will always be the same as the sum of the eigenvalues. This provides a useful way of checking whether the eigenvalues you get are likely to be correct.

Diagonalization of 2 × 2 matrices

Once you know the eigenvalues and eigenvectors of a matrix **M** you can find matrices **P** and **D** such that $\mathbf{M}=\mathbf{PDP}^{-1}$, where **D** is a diagonal matrix, that is a matrix of the form $\begin{pmatrix}a & 0\\ 0 & b\end{pmatrix}$.

This process is called **diagonalization**.

Tip

In this course, you only need to diagonalize matrices with two real distinct eigenvalues.

KEY POINT 3.15

Any matrix **M** with two real, distinct eigenvalues, λ_1 and λ_2, can be expressed as $\mathbf{M} = \mathbf{PDP}^{-1}$, where **P** is a matrix whose columns are the eigenvectors of **M** and $\mathbf{D} = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix}$.

Proof 3.3

Prove the result in Key Point 3.15.

Let $\begin{pmatrix} x_1 \\ y_1 \end{pmatrix}$ be the eigenvector corresponding to λ_1

and $\begin{pmatrix} x_2 \\ y_2 \end{pmatrix}$ be the eigenvector corresponding to λ_2

Use the definition of eigenvectors of **M**

Then,

$$\mathbf{M}\begin{pmatrix} x_1 \\ y_1 \end{pmatrix} = \lambda_1 \begin{pmatrix} x_1 \\ y_1 \end{pmatrix}$$

and

$$\mathbf{M}\begin{pmatrix} x_2 \\ y_2 \end{pmatrix} = \lambda_2 \begin{pmatrix} x_2 \\ y_2 \end{pmatrix}$$

Using the relationships above, the first column is $\lambda_1 \begin{pmatrix} x_1 \\ y_1 \end{pmatrix}$ and the second $\lambda_2 \begin{pmatrix} x_2 \\ y_2 \end{pmatrix}$

$$\mathbf{MP} = \mathbf{M}\begin{pmatrix} x_1 & x_2 \\ y_1 & y_2 \end{pmatrix}$$

$$= \begin{pmatrix} \lambda_1 x_1 & \lambda_2 x_2 \\ \lambda_1 y_1 & \lambda_2 y_2 \end{pmatrix}$$

Separate into the product of two matrices

$$= \begin{pmatrix} x_1 & x_2 \\ y_1 & y_2 \end{pmatrix}\begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix}$$

$$= \mathbf{PD}$$

Multiply on the right by $\mathbf{P}^{-1}$

$$\mathbf{MPP}^{-1} = \mathbf{PDP}^{-1}$$

$$\mathbf{M} = \mathbf{PDP}^{-1}$$

Tip

It is a good idea to use technology to verify that this multiplication does indeed give **M**.

WORKED EXAMPLE 3.18

Express the matrix $\mathbf{M} = \begin{pmatrix} 1 & 4 \\ 2 & 3 \end{pmatrix}$ from Worked Example 3.17 in the form $\mathbf{PDP}^{-1}$, where **D** is a diagonal matrix.

M has eigenvalues 5 and −1 with corresponding eigenvectors $\begin{pmatrix} 1 \\ 1 \end{pmatrix}$ and $\begin{pmatrix} -2 \\ 1 \end{pmatrix}$.

Use $\mathbf{M} = \mathbf{PDP}^{-1}$, where the columns of **P** are the eigenvectors of **D** and $\mathbf{D} = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix}$

So,

$$\mathbf{M} = \begin{pmatrix} 1 & -2 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} 5 & 0 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} 1 & -2 \\ 1 & 1 \end{pmatrix}^{-1}$$

Applications to powers of 2 × 2 matrices

One of the main uses of diagonalization is in finding powers of a matrix. The key idea to note is that finding powers of a diagonal matrix is straightforward.

KEY POINT 3.16

$$\begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix}^n = \begin{pmatrix} a^n & 0 \\ 0 & b^n \end{pmatrix}$$

This then leads to a result for the power of a matrix.

KEY POINT 3.17

If $\mathbf{M} = \mathbf{PDP}^{-1}$ for a diagonal matrix **D**, then

$$\mathbf{M}^n = \mathbf{PD}^n\mathbf{P}^{-1}$$

Proof 3.4

Prove the result in Key Point 3.17.

$(\mathbf{PDP}^{-1})^n$ means $\mathbf{PDP}^{-1}$ multiplied together n times

$$\mathbf{M}^n = (\mathbf{PDP}^{-1})^n$$
$$= (\mathbf{PDP}^{-1})(\mathbf{PDP}^{-1})(\mathbf{PDP}^{-1})\ldots(\mathbf{PDP}^{-1})$$
$$= \mathbf{PD}(\mathbf{P}^{-1}\mathbf{P})\mathbf{D}(\mathbf{P}^{-1}\mathbf{P})\mathbf{D}(\mathbf{P}^{-1}\mathbf{P})\ldots\mathbf{DP}^{-1}$$

Since matrix multiplication is associative we can group together each adjacent $\mathbf{PP}^{-1}$ to give **I**

$$= \mathbf{PDIDIDI}\ldots\mathbf{DP}^{-1}$$
$$= \mathbf{PDDD}\ldots\mathbf{DP}^{-1}$$
$$= \mathbf{PD}^n\mathbf{P}^{-1}$$

WORKED EXAMPLE 3.19

For the matrix $\mathbf{M} = \begin{pmatrix} 1 & 4 \\ 2 & 3 \end{pmatrix}$ from Worked Example 3.17,

a find $\mathbf{M}^n$ in terms of n

b hence find $\mathbf{M}^4$.

From Worked Example 3.17, $\mathbf{M}$ can be written in the form $\mathbf{M} = \mathbf{PDP}^{-1}$

a $\mathbf{M} = \begin{pmatrix} 1 & -2 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} 5 & 0 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} 1 & -2 \\ 1 & 1 \end{pmatrix}^{-1}$

Use $\mathbf{M}^n = \mathbf{PD}^n\mathbf{P}^{-1}$

$\mathbf{M}^n = \begin{pmatrix} 1 & -2 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} 5 & 0 \\ 0 & -1 \end{pmatrix}^n\begin{pmatrix} 1 & -2 \\ 1 & 1 \end{pmatrix}^{-1}$

Now use the fact that $\begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix}^n = \begin{pmatrix} a^n & 0 \\ 0 & b^n \end{pmatrix}$

$= \begin{pmatrix} 1 & -2 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} 5^n & 0 \\ 0 & (-1)^n \end{pmatrix}\begin{pmatrix} 1 & -2 \\ 1 & 1 \end{pmatrix}^{-1}$

Find the inverse of $\begin{pmatrix} 1 & -2 \\ 1 & 1 \end{pmatrix}$ using Key Point 3.11 or with a GDC

$= \begin{pmatrix} 1 & -2 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} 5^n & 0 \\ 0 & (-1)^n \end{pmatrix}\begin{pmatrix} \frac{1}{3} & \frac{2}{3} \\ -\frac{1}{3} & \frac{1}{3} \end{pmatrix}$

Multiply together the first two matrices ...

$= \begin{pmatrix} 5^n & -2(-1)^n \\ 5^n & (-1)^n \end{pmatrix}\begin{pmatrix} \frac{1}{3} & \frac{2}{3} \\ -\frac{1}{3} & \frac{1}{3} \end{pmatrix}$

... and then multiply the result by the third matrix

$= \begin{pmatrix} \dfrac{5^n + 2(-1)^n}{3} & \dfrac{2(5^n) - 2(-1)^n}{3} \\ \dfrac{5^n - (-1)^n}{3} & \dfrac{2(5^n) + (-1)^n}{3} \end{pmatrix}$

Let $n = 4$ in the matrix from part **a**

b $\mathbf{M}^4 = \begin{pmatrix} \dfrac{5^4 + 2(-1)^4}{3} & \dfrac{2(5^4) - 2(-1)^4}{3} \\ \dfrac{5^4 - (-1)^4}{3} & \dfrac{2(5^4) + (-1)^4}{3} \end{pmatrix}$

$= \begin{pmatrix} 209 & 416 \\ 208 & 417 \end{pmatrix}$

Exercise 3D

For questions 1 to 3, use the method demonstrated in Worked Example 3.17 to find the eigenvalues and eigenvectors of each matrix.

1 **a** $\begin{pmatrix} 4 & -5 \\ 1 & -2 \end{pmatrix}$ **b** $\begin{pmatrix} 4 & -2 \\ 1 & 1 \end{pmatrix}$

2 **a** $\begin{pmatrix} 7 & 3 \\ -5 & -1 \end{pmatrix}$ **b** $\begin{pmatrix} 5 & 15 \\ -2 & -8 \end{pmatrix}$

3 **a** $\begin{pmatrix} 0.6 & 0.3 \\ 0.4 & 0.7 \end{pmatrix}$ **b** $\begin{pmatrix} 0.85 & 0.25 \\ 0.15 & 0.75 \end{pmatrix}$

For questions 4 to 6, use the method demonstrated in Worked Example 3.18 to express each matrix (from questions 1 to 3) in the form $\mathbf{PDP}^{-1}$, where $\mathbf{D}$ is a diagonal matrix.

4 a $\begin{pmatrix} 4 & -5 \\ 1 & -2 \end{pmatrix}$
b $\begin{pmatrix} 4 & -2 \\ 1 & 1 \end{pmatrix}$

5 a $\begin{pmatrix} 7 & 3 \\ -5 & -1 \end{pmatrix}$
b $\begin{pmatrix} 5 & 15 \\ -2 & -8 \end{pmatrix}$

6 a $\begin{pmatrix} 0.6 & 0.3 \\ 0.4 & 0.7 \end{pmatrix}$
b $\begin{pmatrix} 0.85 & 0.25 \\ 0.15 & 0.75 \end{pmatrix}$

For questions 7 to 9, use the method demonstrated in Worked Example 3.19 to find $\mathbf{M}^n$ for each matrix (from questions 4 to 6).

7 a $\begin{pmatrix} 4 & -5 \\ 1 & -2 \end{pmatrix}$
b $\begin{pmatrix} 4 & -2 \\ 1 & 1 \end{pmatrix}$

8 a $\begin{pmatrix} 7 & 3 \\ -5 & -1 \end{pmatrix}$
b $\begin{pmatrix} 5 & 15 \\ -2 & -8 \end{pmatrix}$

9 a $\begin{pmatrix} 0.6 & 0.3 \\ 0.4 & 0.7 \end{pmatrix}$
b $\begin{pmatrix} 0.85 & 0.25 \\ 0.15 & 0.75 \end{pmatrix}$

10 $\mathbf{M} = \begin{pmatrix} 7 & 2 \\ 2 & 4 \end{pmatrix}$

a i Find the characteristic polynomial of $\mathbf{M}$.
ii Hence find the eigenvalues of $\mathbf{M}$.

b Find the eigenvectors of $\mathbf{M}$.

11 Find the eigenvalues and corresponding eigenvectors of the matrix $\begin{pmatrix} -2 & 2 \\ 2 & 1 \end{pmatrix}$.

12 Find the eigenvalues and corresponding eigenvectors of the matrix $\begin{pmatrix} 4 & -5 \\ 6 & -9 \end{pmatrix}$.

13 $\begin{pmatrix} 1 \\ 2 \end{pmatrix}$ is an eigenvector of matrix $\begin{pmatrix} 3 & 1 \\ 4 & a \end{pmatrix}$.

a Find the value of a.

b Find
i the eigenvalue associated with $\begin{pmatrix} 1 \\ 2 \end{pmatrix}$
ii the remaining eigenvalue and associated eigenvector.

14 a Find the eigenvalues and corresponding eigenvectors of the matrix $\mathbf{A} = \begin{pmatrix} -5 & 8 \\ 3 & -7 \end{pmatrix}$.

b Hence state a matrix $\mathbf{C}$ and a diagonal matrix $\mathbf{D}$ such that $\mathbf{A} = \mathbf{CDC}^{-1}$.

15 $\mathbf{M} = \begin{pmatrix} 8 & 2 \\ -1 & 5 \end{pmatrix}$

a Find matrices $\mathbf{P}$ and $\mathbf{D}$ such that $\mathbf{M} = \mathbf{PDP}^{-1}$, where $\mathbf{D}$ is a diagonal matrix.

b Hence find $\mathbf{M}^3$, clearly showing your working.

16 $\mathbf{M} = \begin{pmatrix} 7 & 4 \\ -2 & 1 \end{pmatrix}$

a Find matrices $\mathbf{P}$ and $\mathbf{D}$ such that $\mathbf{M} = \mathbf{PDP}^{-1}$, where $\mathbf{D}$ is a diagonal matrix.

b Hence find $\mathbf{M}^n$.

17 $\mathbf{A} = \begin{pmatrix} 4 & -4 \\ 3 & -4 \end{pmatrix}$

a Find a matrix $\mathbf{P}$ and a diagonal matrix $\mathbf{D}$ such that $\mathbf{A} = \mathbf{PDP}^{-1}$.

b Find $\mathbf{A}^n$ for integer n when
i n is odd
ii n is even.

18 Matrix $\mathbf{M}$ has eigenvectors $\begin{pmatrix}1\\3\end{pmatrix}$ and $\begin{pmatrix}1\\1\end{pmatrix}$ and eigenvalues 1 and –2.

a Find two possible matrices $\mathbf{M}_1$ and $\mathbf{M}_2$.

b Find the product $\mathbf{M}_1\mathbf{M}_2$ and comment on your answer.

19 $\mathbf{M} = \begin{pmatrix}0.25 & 0.5\\0.75 & 0.5\end{pmatrix}$

a Find the eigenvalues and corresponding eigenvectors of $\mathbf{M}$.

b State a matrix $\mathbf{P}$ and a matrix $\mathbf{D}$ such that $\mathbf{M} = \mathbf{PDP}^{-1}$, where $\mathbf{D}$ is a diagonal matrix.

c Show that as n becomes large, $\mathbf{M}^n$ approaches $\begin{pmatrix}0.4 & 0.4\\0.6 & 0.6\end{pmatrix}$.

20 $\mathbf{M} = \begin{pmatrix}a & b\\b & a\end{pmatrix}$, where a and b are constants and $b \neq 0$.

a Find the eigenvalues of $\mathbf{M}$ in terms of a and b.

b Find corresponding eigenvectors of $\mathbf{M}$.

c Express $\mathbf{M}$ in the form $\mathbf{M} = \mathbf{PDP}^{-1}$.

d Hence find $\mathbf{M}^n$.

e Given that $a = 0.6$ and $b = 0.4$, use your answer to part **d** to find the matrix that $\mathbf{M}^n$ approaches as n becomes very large.

Checklist

- You should be able to determine the order of a matrix.
  - A matrix with m rows and n columns has order $m \times n$.
- You should be able to add, subtract and multiply matrices.
  - Two matrices can be added/subtracted if they have the same order. To add/subtract matrices, add/subtract the corresponding elements.
  - To multiply a matrix by a scalar, multiply every element of the matrix by the scalar.
  - If the matrix $\mathbf{A}$ has order $m \times n$ and the matrix $\mathbf{B}$ has order $n \times p$, then the product $\mathbf{AB}$ has order $m \times p$.
- You should know some properties of matrix multiplication.
  - For the matrices $\mathbf{A}$, $\mathbf{B}$ and $\mathbf{C}$:
    - — $\mathbf{AB} \neq \mathbf{BA}$ (non-commutative)
    - — $\mathbf{A}(\mathbf{BC}) = (\mathbf{AB})\mathbf{C}$ (associative)
    - — $\mathbf{A}(\mathbf{B} + \mathbf{C}) = \mathbf{AB} + \mathbf{AC}$ (distributive)
- You should know about zero and identity matrices.
  - The identity matrix, $\mathbf{I}$, is a square matrix with 1 as each element of the diagonal from top left to bottom right and 0 as every other element.
  - The zero matrix, $\mathbf{0}$, is a matrix whose elements are all 0.
- You should be able to calculate the determinant and inverse of $n \times n$ matrices with technology and 2×2 matrices by hand.
  - The inverse of a square matrix $\mathbf{M}$ is the matrix $\mathbf{M}^{-1}$ such that
    $$\mathbf{MM}^{-1} = \mathbf{M}^{-1}\mathbf{M} = \mathbf{I}$$
  - If $\mathbf{AB} = \mathbf{C}$, then $\mathbf{B} = \mathbf{A}^{-1}\mathbf{C}$
  - If $\mathbf{BA} = \mathbf{C}$, then $\mathbf{B} = \mathbf{CA}^{-1}$
  - The determinant of the 2×2 matrix $\mathbf{M} = \begin{pmatrix}a & b\\c & d\end{pmatrix}$ is
    $$\det \mathbf{M} = ad - bc$$

- The inverse of the 2×2 matrix $\mathbf{M} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, where $\det \mathbf{M} \neq 0$ is

  $$\mathbf{M}^{-1} = \frac{1}{\det \mathbf{M}} \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$$

- You should be able to solve systems of linear equations with matrices.
  - A system of simultaneous equations can be written as a matrix equation

    $$\mathbf{Ax} = \mathbf{b}$$

    — where $\mathbf{A}$ contains the coefficients and $\mathbf{x}$ is a column vector of the variables.

    — If a solution exists, it is given by

    $$\mathbf{x} = \mathbf{A}^{-1}\mathbf{b}$$

- You should be able to find the eigenvalues and eigenvectors of 2×2 matrices.
- If $\mathbf{Mv} = \lambda\mathbf{v}$, then λ is called an eigenvalue of the matrix $\mathbf{M}$ and the non-zero vector $\mathbf{v}$ is its associated eigenvector.
- The eigenvalues of the matrix $\mathbf{M}$ satisfy the characteristic equation

  $$\det(\mathbf{M} - \lambda\mathbf{I}) = 0$$

- You should be able to diagonalize 2×2 matrices.
  - Any matrix $\mathbf{M}$ with two real, distinct eigenvalues λ_1 and λ_2 can be expressed as

    $$\mathbf{M} = \mathbf{PDP}^{-1}$$

    where $\mathbf{P}$ is a matrix whose columns are the eigenvectors of $\mathbf{M}$ and $\mathbf{D} = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix}$.
- You should be able to use diagonalization to find powers of 2×2 matrices.
  - $\begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix}^n = \begin{pmatrix} a^n & 0 \\ 0 & b^n \end{pmatrix}$
  - If $\mathbf{M} = \mathbf{PDP}^{-1}$ for a diagonal matrix $\mathbf{D}$, then $\mathbf{M}^n = \mathbf{PD}^n\mathbf{P}^{-1}$

Mixed Practice

1 Matrix $\mathbf{R}$ gives the weekly revenue (in \$) from the sales of toy cats (C), dogs (D) and elephants (E) in shops X and Y of *Soft Toys 'r' Us*.

Matrix $\mathbf{F}$ gives the fixed costs of producing that number of toys and transporting them to the two stores.

$$\mathbf{R} = \begin{array}{cc} \begin{array}{cc} X & Y \end{array} & \\ \begin{pmatrix} 110 & 85 \\ 154 & 102 \\ 130 & 175 \end{pmatrix} & \begin{array}{c} C \\ D \\ E \end{array} \end{array} \qquad \mathbf{F} = \begin{array}{cc} \begin{array}{cc} X & Y \end{array} & \\ \begin{pmatrix} 42 & 30 \\ 65 & 54 \\ 88 & 106 \end{pmatrix} & \begin{array}{c} C \\ D \\ E \end{array} \end{array}$$

a Find the matrix $\mathbf{P}$, which gives the weekly profit for each product at each store.

b **i** If the revenue of each product at each store increases by 5% and the costs increase by 2%, write an equation relating the new profit matrix $\mathbf{Q}$ to $\mathbf{R}$ and $\mathbf{F}$.

ii Hence find matrix $\mathbf{Q}$.

2 $\mathbf{A} = \begin{pmatrix} 2 & -3 \\ 0 & 4 \end{pmatrix}$ and $\mathbf{B} = \begin{pmatrix} 2k & -k \\ k & 1 \end{pmatrix}$

Find, in terms of k, the matrix $\mathbf{AB}$.

3 $\mathbf{A} = \begin{pmatrix} 2k & 0 \\ k-1 & -1 \end{pmatrix}$ and $\mathbf{B} = \begin{pmatrix} 2 & 0 \\ 3 & -1 \end{pmatrix}$, where k is a constant.

a If **A** and **B** are commutative, find k.

b Show, by choosing matrices **C** and **D**, that matrix multiplication is not always commutative.

4 $\mathbf{A} = \begin{pmatrix} 3 & -1 \\ 5 & -2 \end{pmatrix}$

Find $\mathbf{A}^{-1}$.

5 $\mathbf{A} = \begin{pmatrix} -5 & 3 \\ -2 & 2 \end{pmatrix}$ and $\mathbf{B} = \begin{pmatrix} 1 & 3 \\ k & 4 \end{pmatrix}$

Given that $\mathbf{AC} = \mathbf{B}$, find **C**.

6 The matrix **A** is given by $\mathbf{A} = \begin{pmatrix} 3 & k \\ 0 & 1 \end{pmatrix}$, where k is a constant.

a Find $\mathbf{A}^{-1}$.

The matrix **B** is given by $\mathbf{B} = \begin{pmatrix} 3 & k \\ 6 & 1 \end{pmatrix}$.

b Given that $\mathbf{CA} = \mathbf{B}$, find the matrix **C**.

7 Two 4-digit ID numbers are written in a 2×4 matrix and encoded by multiplying by the matrix

$$\begin{pmatrix} 9 & 18 & -13 & 3 \\ -3 & -6 & 4 & -1 \\ 7 & 12 & -15 & 2 \\ 1 & 1 & -13 & 1 \end{pmatrix}$$

The result is $\begin{pmatrix} 68 & 127 & -130 & 22 \\ 119 & 218 & -240 & 38 \end{pmatrix}$

Find the original ID numbers.

8 $\mathbf{M} = \begin{pmatrix} 2 & -1 \\ 7 & -3 \end{pmatrix}$

a Show that $\mathbf{M}^2 = \mathbf{M}^{-1}$.

b Hence state the matrix $\mathbf{M}^3$.

c Use a matrix method to solve the equations

$$\begin{cases} 2x - y = 7 \\ 7x - 3y = 23 \end{cases}$$

9 Three sweet shops, *XS Sweets*, *Your Candy* and *Zorn's Confectionery* sell aniseed balls (A), bonbons (B) and caramels (C).

The prices per kilogram (£) of each type of sweet at the three shops are:

$$\begin{array}{c} \\ X \\ Y \\ Z \end{array} \begin{array}{c} \begin{array}{ccc} A & B & C \end{array} \\ \begin{pmatrix} 1.00 & 1.80 & 2.50 \\ 1.10 & 2.00 & 2.10 \\ 1.20 & 1.70 & 2.40 \end{pmatrix} \end{array}$$

Jack wants to order a certain number of kilograms of each sweet. He works out that his order would cost him £24.90 at *XS Sweets*, £22.90 at *Your Candy* and £24.30 at *Zorn's Confectioner.*

How many kilograms of each sweet does he want to order?

10 $\mathbf{M} = \begin{pmatrix} 3 & -5 \\ -4 & 2 \end{pmatrix}$

a **i** Find the characteristic polynomial of **M**.

ii Hence find the eigenvalues of **M**.

b Find the eigenvectors of **M**.

11 Find the eigenvalues and corresponding eigenvectors of the matrix $\begin{pmatrix} 3 & -2 \\ 0 & 4 \end{pmatrix}$.

12 $\mathbf{A} = \begin{pmatrix} a & -1 \\ 2 & 1 \end{pmatrix}$ $\mathbf{B} = \begin{pmatrix} 3 & b \\ 0 & -4 \end{pmatrix}$ $\mathbf{C} = \begin{pmatrix} 1 & 4 \\ 2 & c \end{pmatrix}$

Given that $\mathbf{AB} + k\mathbf{C} = \mathbf{I}$, find a, b, c and k.

13 $\mathbf{M} = \begin{pmatrix} 3 & k \\ 5 & 8 \end{pmatrix}$

a **i** Find the value of k for which **M** is singular.

ii Given that **M** is non-singular, find $\mathbf{M}^{-1}$ in terms of k.

b When $k = 7$, use $\mathbf{M}^{-1}$ to solve the simultaneous equations

$$\begin{cases} 3x + 7y = -1 \\ 5x + 8y = 13 \end{cases}$$

14 **a** If $\mathbf{ABC} = \mathbf{I}$, prove that $\mathbf{B}^{-1} = \mathbf{CA}$.

b $\mathbf{A} = \begin{pmatrix} 4 & 5 \\ -2 & -3 \end{pmatrix}$ and $\mathbf{C} = \begin{pmatrix} 0 & -1 \\ -3 & 4 \end{pmatrix}$

Find **B**.

15 $\mathbf{M} = \begin{pmatrix} x & 2 \\ y & 0 \end{pmatrix}$

Given that $\mathbf{M}^{-1}$ exists and $\mathbf{M} + \mathbf{M}^{-1} = \mathbf{I}$, find x and y.

16 A matrix **M** has eigenvectors $\begin{pmatrix} 2 \\ 1 \end{pmatrix}$ and $\begin{pmatrix} -1 \\ 3 \end{pmatrix}$, with corresponding eigenvalues 2 and 3.

Find the matrix M.

17 $\mathbf{M} = \begin{pmatrix} 3 & 4 \\ -2 & 9 \end{pmatrix}$

a Find matrices **P** and **D** such that $\mathbf{M} = \mathbf{PDP}^{-1}$, where **D** is a diagonal matrix.

b Hence find $\mathbf{M}^n$.

18 $\mathbf{M} = \begin{pmatrix} a & 1-b \\ 1-a & b \end{pmatrix}$, where $0 < a < 1$ and $0 < b < 1$.

a **i** Show that one of the eigenvalues of **M** is $\lambda = 1$ and find the other eigenvalue.

ii Find the corresponding eigenvectors of **M**.

b State matrices **P** and **P**, where **D** is a diagonal matrix, such that $\mathbf{M} = \mathbf{PDP}^{-1}$.

c Hence find $\mathbf{M}^n$.

4 Geometry and trigonometry

ESSENTIAL UNDERSTANDINGS

- Trigonometry allows us to quantify the physical world.
- This topic provides us with the tools for analysis, measurement and transformation of quantities, movements and relationships.

In this chapter you will learn...

- about different units for measuring angles, called radians
- how to find the length of an arc of a circle
- how to find the area of a sector of a circle
- how to define the sine and cosine functions in terms of the unit circle
- how to define the tangent function
- about the ambiguous case of the sine rule
- about the Pythagorean identity $\cos^2\theta + \sin^2\theta \equiv 1$
- how to sketch the graphs of trigonometric functions
- how to solve trigonometric equations graphically
- how to use matrices to represent geometrical transformations
- how to combine transformations
- how to find the area of an image after a transformation.

CONCEPTS

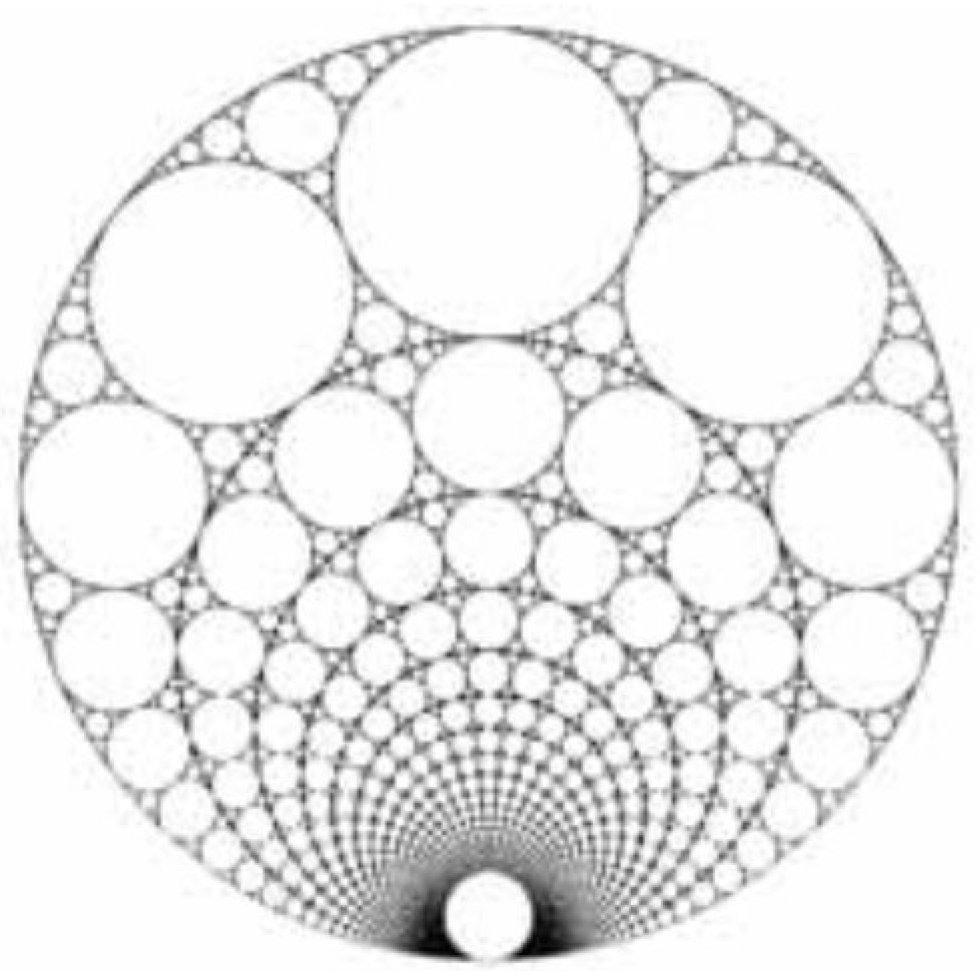
The following concepts will be addressed in this chapter:

- **Equivalent** measurement systems, such as degrees and radians, can be used for angles to facilitate our ease of calculation.
- Different **representations** of the values of trigonometric relationships, such as exact or **approximate**, may not be **equivalent** to one another.
- The trigonometric functions of angles may be defined on the unit circle, which can visually and algebraically **represent** the periodic or symmetric nature of their values.

LEARNER PROFILE – Thinkers

Is mathematics an art or a science? How much creativity do you use when tackling a mathematics problem? What would mathematics and art assignments look like if their teaching approaches were reversed?

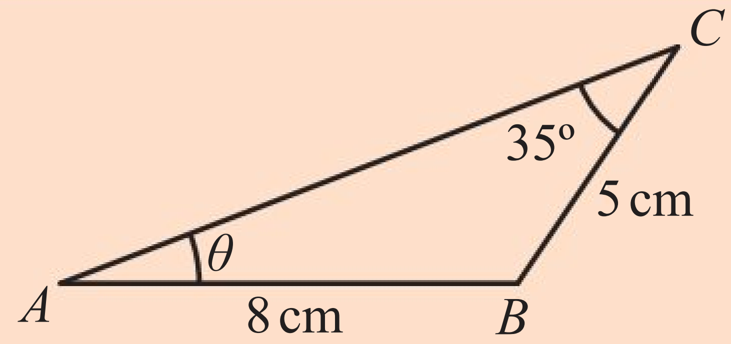
Figure 4.1 Trigonometric functions and the repeated application of transformation matrices can be combined to make detailed geometric designs.



PRIOR KNOWLEDGE

Before starting this chapter, you should already be able to complete the following:

1 Find the size of angle θ in triangle ABC.



2 Use technology to solve the equation $2^x = x^3 - 4x$.

3 Given that $\mathbf{A} = \begin{pmatrix} 2 & 1 \\ -3 & 3 \end{pmatrix}$, $\mathbf{B} = \begin{pmatrix} 1 & -1 \\ 1 & 2 \end{pmatrix}$ and $\mathbf{C} = \begin{pmatrix} 2 \\ -5 \end{pmatrix}$, find

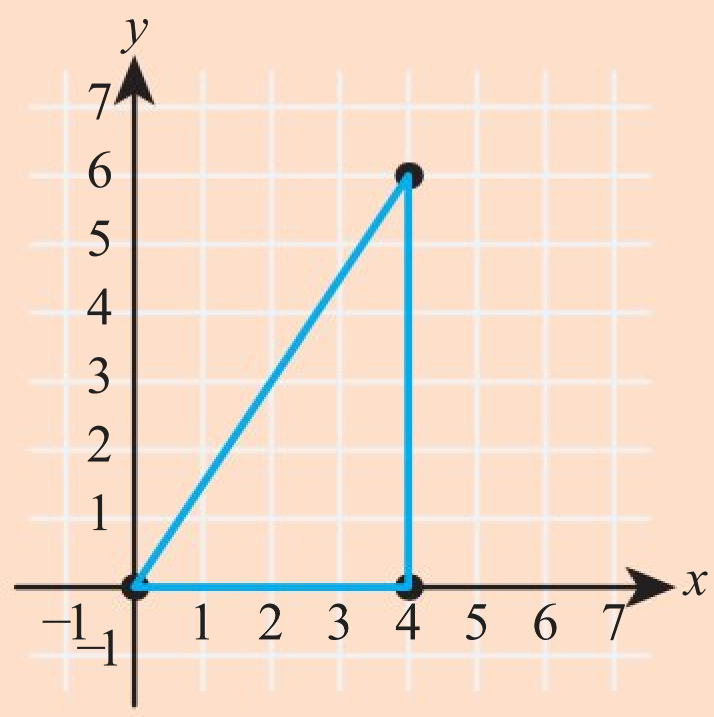
a **AC** b **BA** c $\mathbf{B}^{-1}$

4 Draw the image of the triangle in the diagram after the following transformations:

a reflection in the line $y = x$

b rotation 90° clockwise around the origin

c translation with vector $\begin{pmatrix} 4 \\ -3 \end{pmatrix}$



Up until now you have used trigonometric functions (sine, cosine and tangent) to find side lengths and angles in triangles, measuring angles in degrees. It turns out that a different measure of angle, the radian, is more useful when modelling other real-life situations with trigonometric functions.

Trigonometric functions can be combined with matrices to calculate coordinates of points after geometrical transformations such as rotations and reflections. This is used, for example, to generate patterns in graphic design.

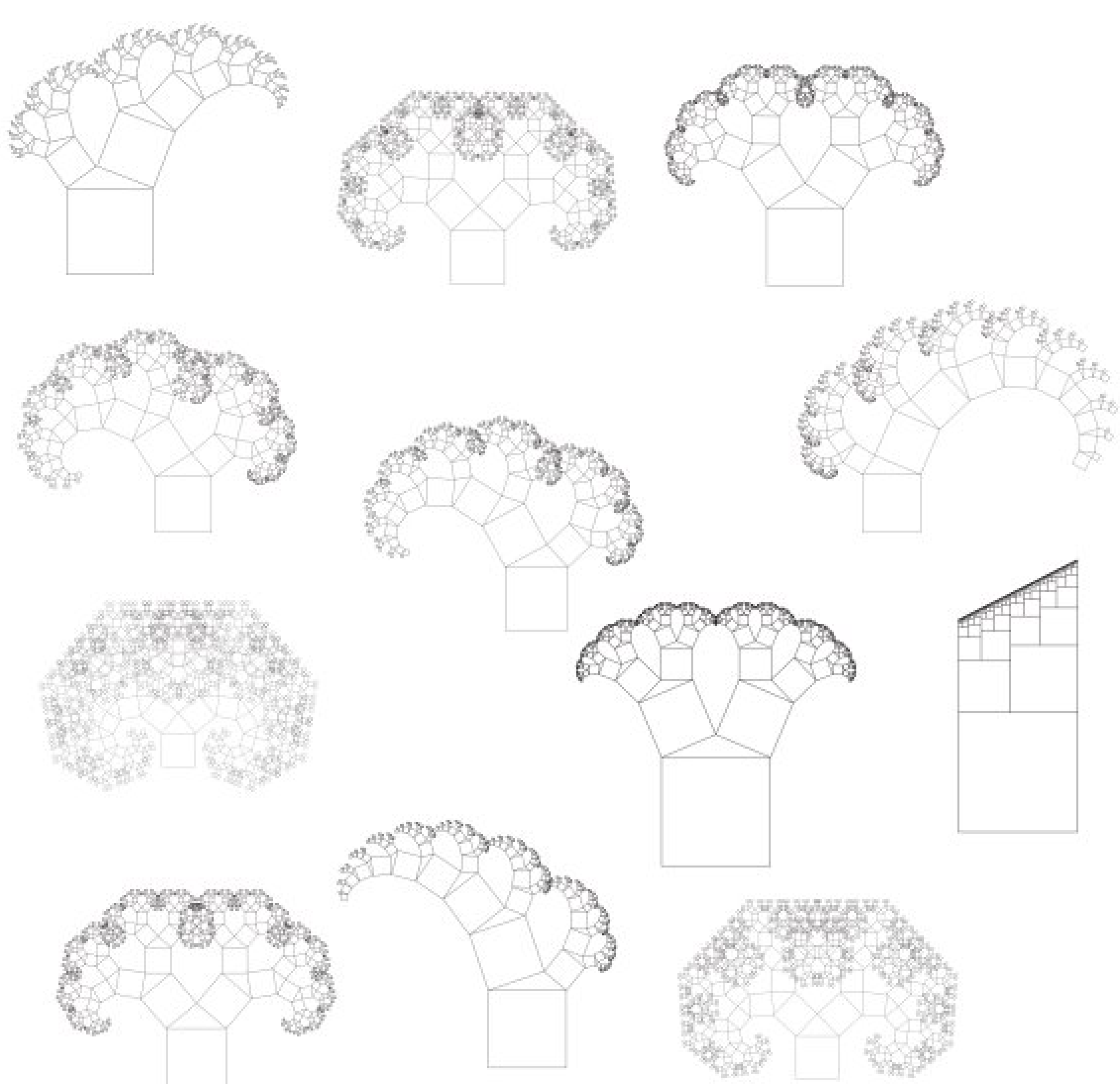
Starter Activity

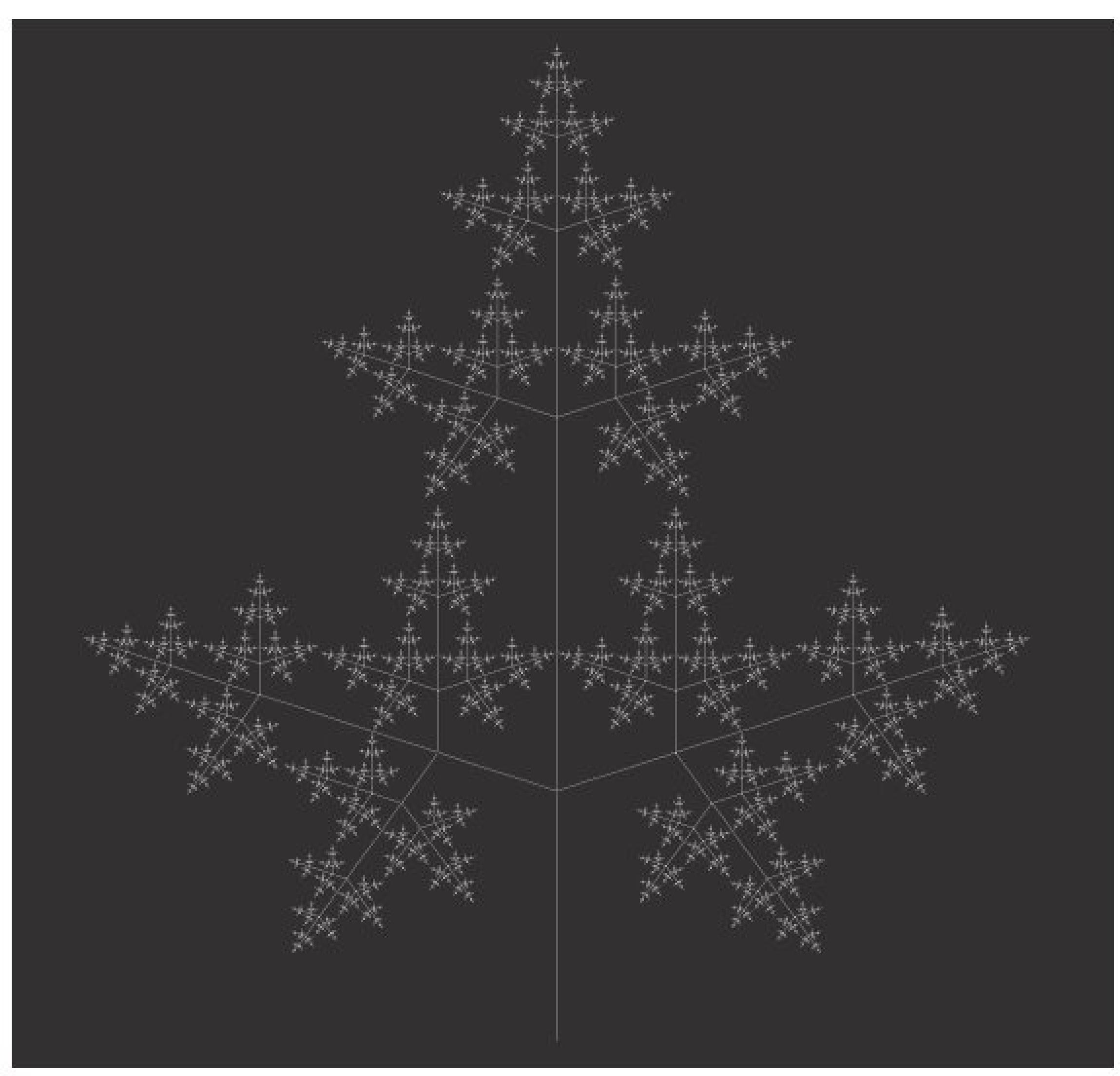
Look at the pictures in Figure 4.1. In small groups, identify which transformations were used to generate these pictures.

Now look at this problem:

1 How many triangles can you draw given the following information:
$AB = 15\text{cm}$, $BC = 10\text{cm}$, $A\hat{C}B = 40°$?

2 What if the information is as given above except that $BC = 18\text{cm}$?

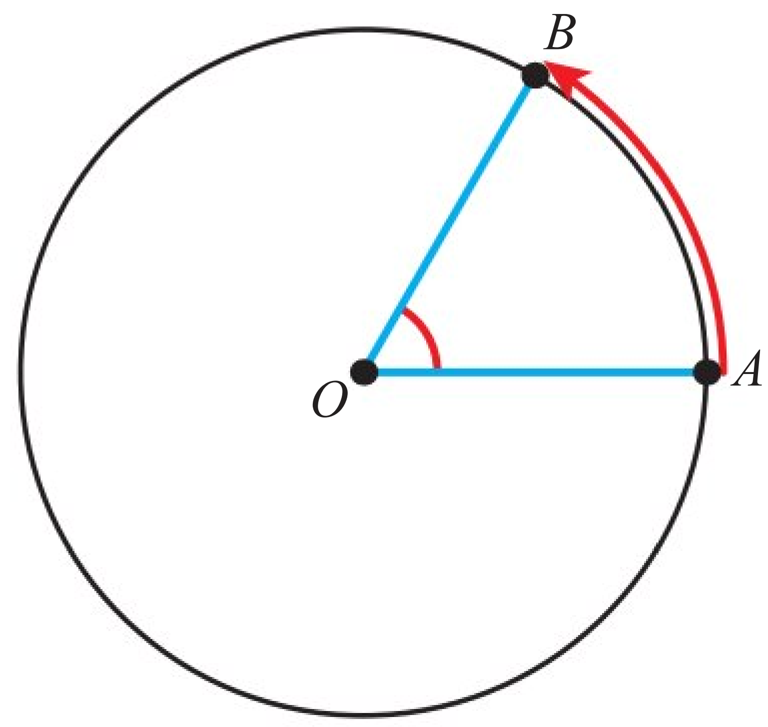




4A Radian measure

Although degrees are the units you are familiar with for measuring angles, they are not always the best unit to use. Instead, **radians** are far more useful in many branches of mathematics.

Radians relate the size of an angle at the centre of the unit circle (a circle with radius 1) to the distance a point moves round the circumference of that circle.



The length of the arc AB is equal to the size of angle AOB in radians.

Since the circumference of the unit circle is $2\pi \times 1 = 2\pi$, there are 2π radians in one full turn.

Tip

Radians are often given as multiples of π, but can also be given as decimals.

KEY POINT 4.1

$360° = 2\pi$ radians

WORKED EXAMPLE 4.1

a Convert 75° to radians.

b Convert 1.5 radians to degrees.

$360° = 2\pi$ radians, so $1° = \frac{2\pi}{360}$ radians

a $75° = \frac{2\pi}{360} \times 75$

$= \frac{75\pi}{180}$

$= \frac{5\pi}{12}$ radians

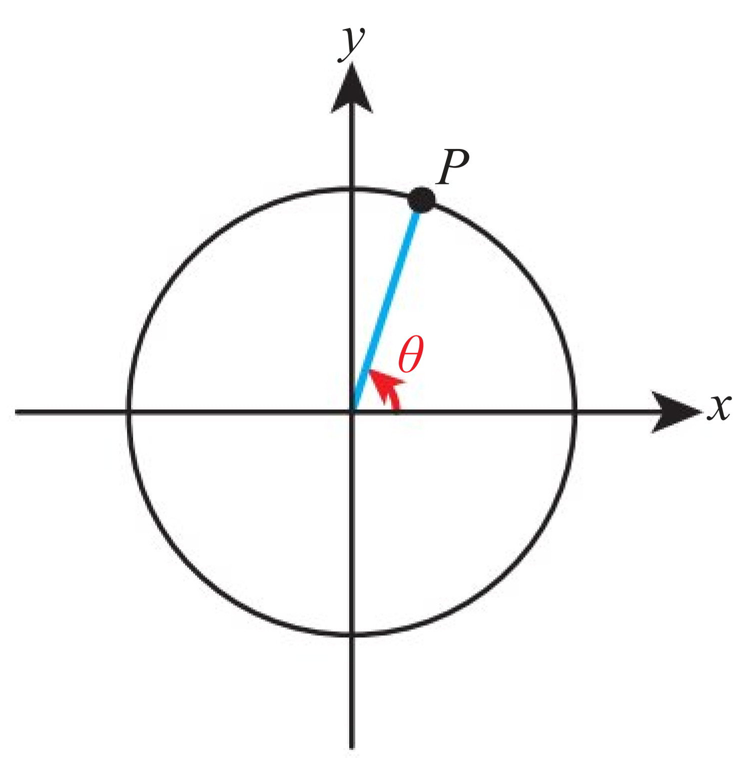
2π radians $= 360°$, so 1 radian $= \left(\frac{360}{2\pi}\right)°$

b $1.5 \text{ radians} = \frac{360}{2\pi} \times 1.5$

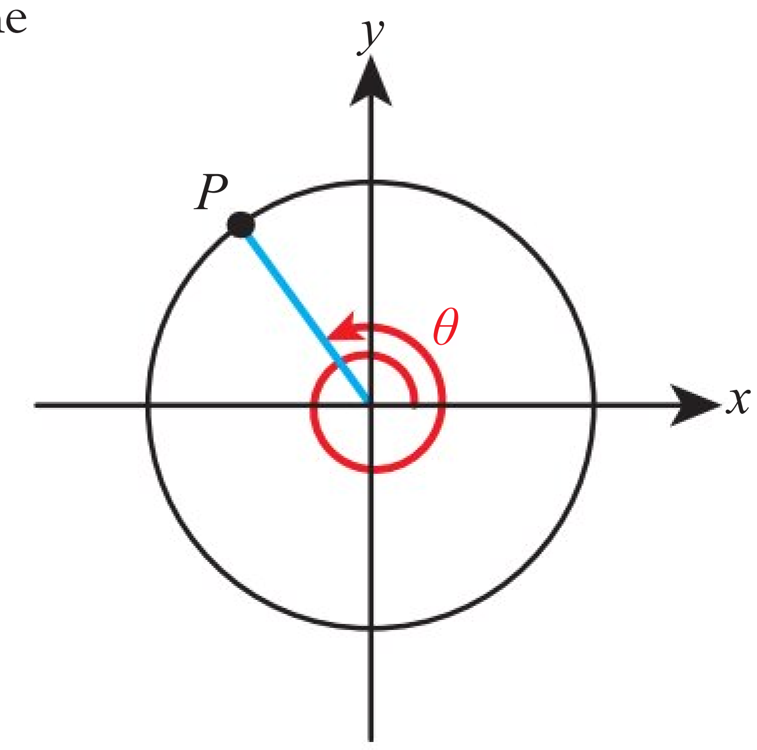
$= 85.9°$

During the French Revolution there was a move towards decimalization, including the introduction of a new unit called the gradian (often abbreviated to grad), which split right angles into 100 subdivisions. It is still used in some areas of surveying today.

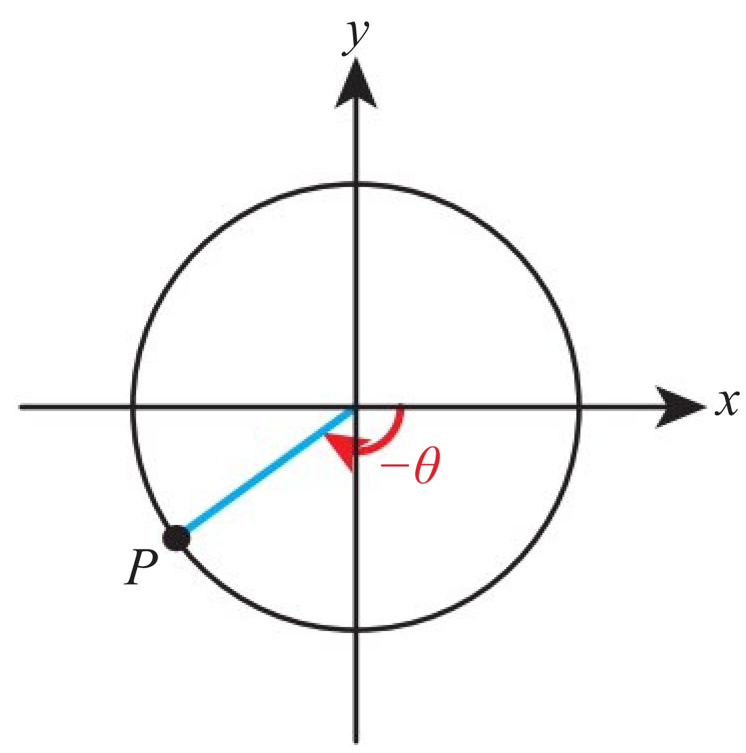
Using the unit circle, we can define any positive angle as being between the positive x-axis and the radius from a point P as P moves anti-clockwise around the circle:



If the angle is greater than 2π, P just goes around the circle again.



If the angle is negative, P moves clockwise from the positive x-axis.



WORKED EXAMPLE 4.2

Mark on the unit circle the points corresponding to these angles.

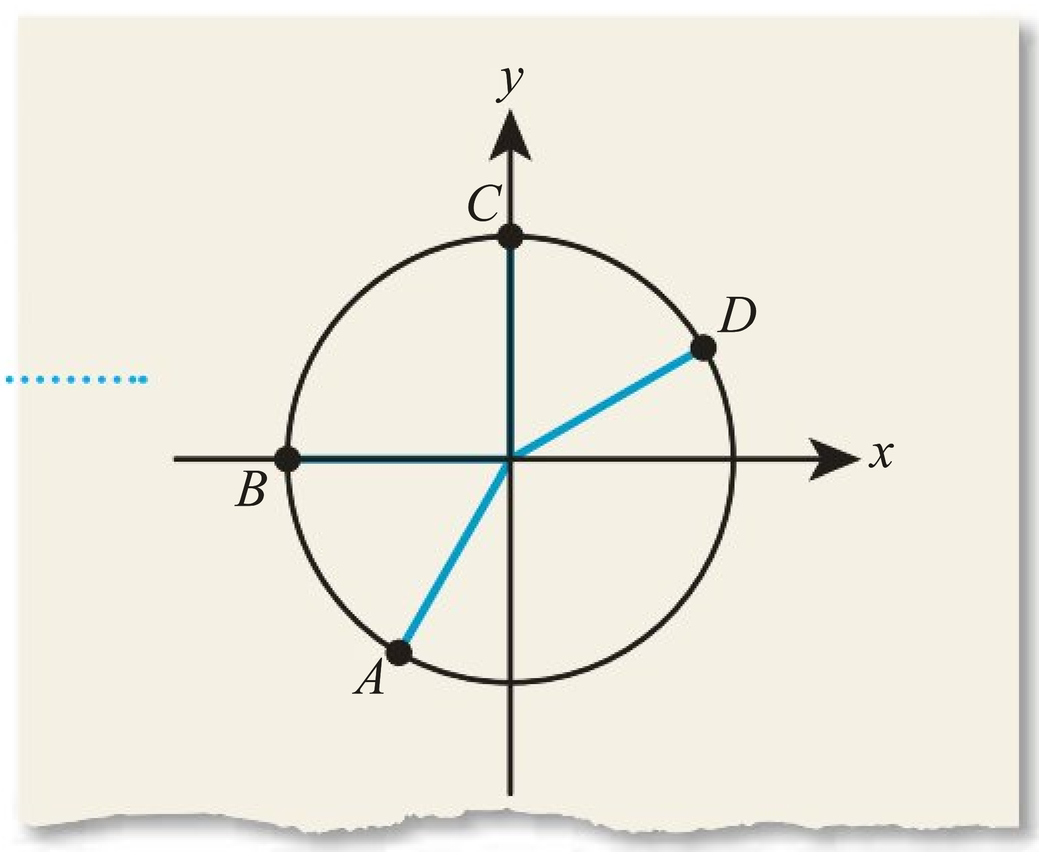
A $\frac{4\pi}{3}$ **B** -3π **C** $\frac{5\pi}{2}$ **D** $\frac{25\pi}{6}$

A: $\frac{4\pi}{3} = \frac{2}{3} \times 2\pi$ so it is $\frac{2}{3}$ of a whole turn

B: $-3\pi = -2\pi - \pi$ so it is a whole turn 'backwards' (clockwise) followed by half a turn in the same negative direction

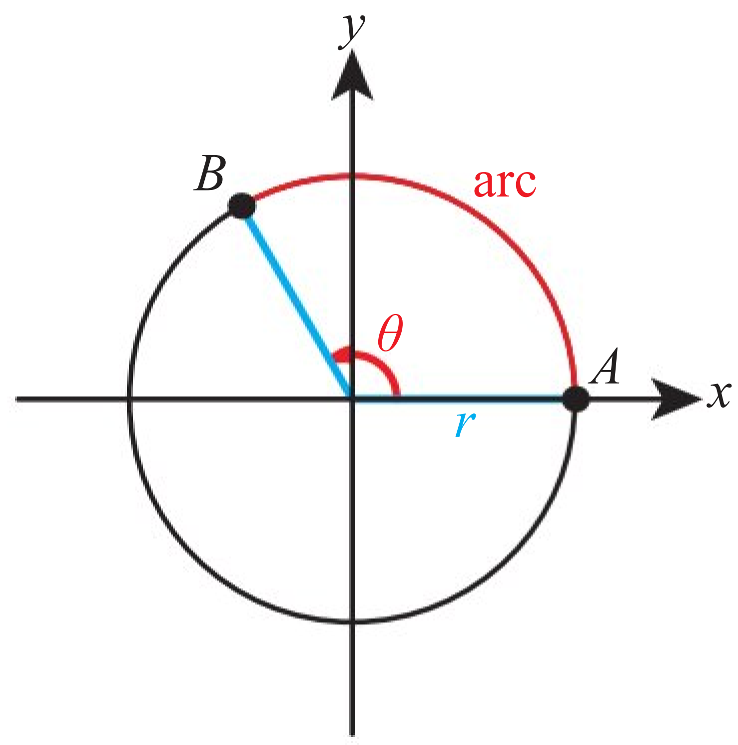
C: $\frac{5\pi}{2} = 2\pi + \frac{\pi}{2}$ so it is a whole turn followed by $\frac{1}{4}$ of a turn

D: $\frac{25\pi}{6} = 4\pi + \frac{\pi}{6}$ so it is two whole turns followed by $\frac{1}{12}$ of a turn



Length of an arc

The arc AB **subtends** an angle θ at the centre of the circle, where the angle at the centre is in radians.



Since the ratio of the arc length, s, to the circumference will be the same as the ratio of θ to 2π radians, this gives

$$\frac{s}{2\pi r} = \frac{\theta}{2\pi}$$

Rearranging gives the formula for arc length when θ is measured in radians.

KEY POINT 4.2

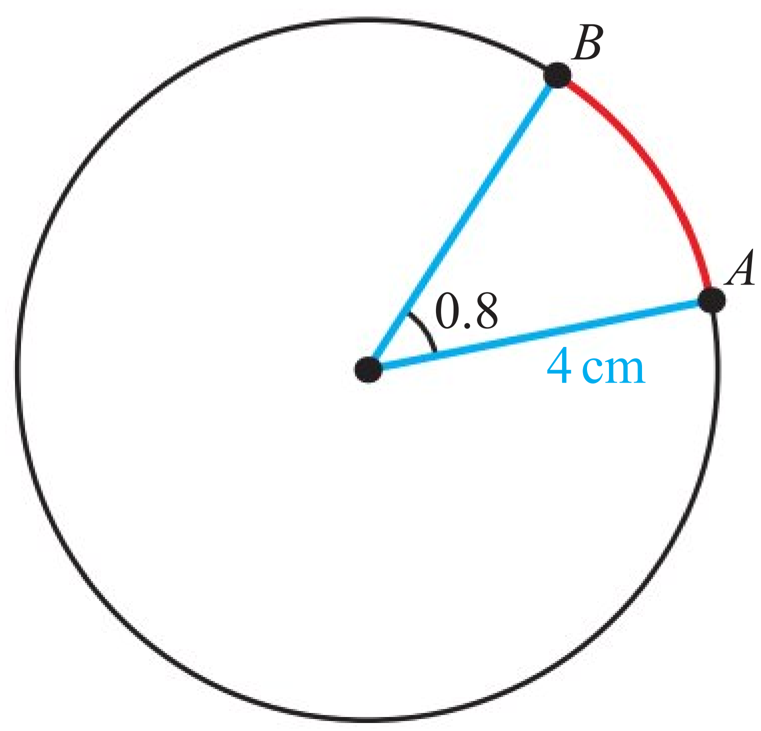
The length of an arc is

$$s = r\theta$$

where r is the radius of the circle and θ is the angle subtended at the centre measured in radians.

WORKED EXAMPLE 4.3

Find the length of the arc AB in the circle shown, where the angle at the centre is in radians.

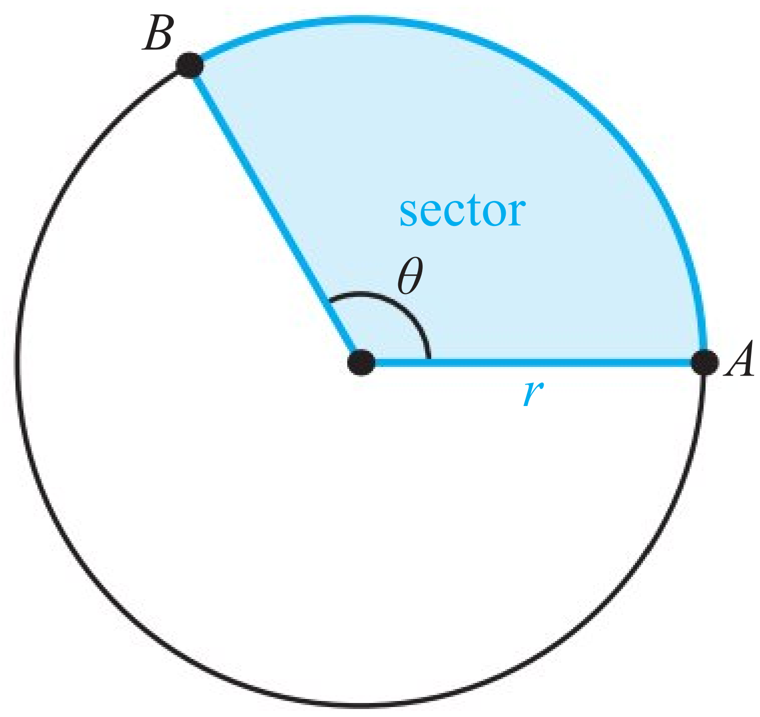


Use $s = r\theta$ with θ in radians

$$\begin{aligned} s &= r\theta \\ &= 4 \times 0.8 \\ &= 3.2\,\text{cm} \end{aligned}$$

Area of a sector

By a very similar argument to that above, we can obtain a formula for the area of a sector.



Since the ratio of the area of a sector, A, to the area of the circle will be the same as the ratio of θ to 2π radians, this gives

$$\frac{A}{\pi r^2} = \frac{\theta}{2\pi}$$

Rearranging gives the formula for the area of a sector when θ is measured in radians.

KEY POINT 4.3

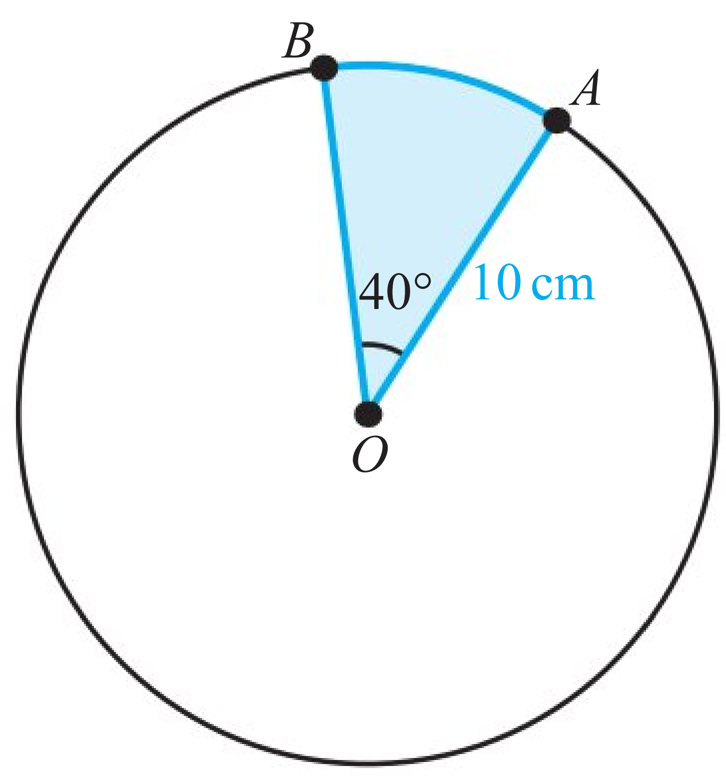
The area of a sector is

$$A = \frac{1}{2}r^2\theta$$

where r is the radius of the circle and θ is the angle subtended at the centre measured in radians.

WORKED EXAMPLE 4.4

Find the area of the sector AOB in the circle shown.



First convert 40° to radians $\quad 40° = \frac{2\pi}{360} \times 40$

$= \frac{2\pi}{9}$ radians

Then use $A = \frac{1}{2}r^2\theta$ $\quad A = \frac{1}{2} \times 10^2 \times \frac{2\pi}{9}$

$= 34.9\,\text{cm}^2$

KEY CONCEPTS – EQUIVALENCE

You have seen various quantities that can be measured in different units. For example, lengths can be measured in feet or metres and temperature in degrees Fahrenheit or Celsius. There is no mathematical reason to prefer one over the other because the units that we use for measurement do not change any formulae used; they are **equivalent**. You might have expected it to be the same for this new unit of angle measurement. However, as seen in Worked Example 4.4, in this case, the units used actually do change the formulae. This has some very important consequences when it comes to differentiating trigonometric functions, as you will see in Chapter 10.

You met the sine rule, cosine rule, and area of a triangle formula in Section 5B of the Mathematics: applications and interpretation SL book.

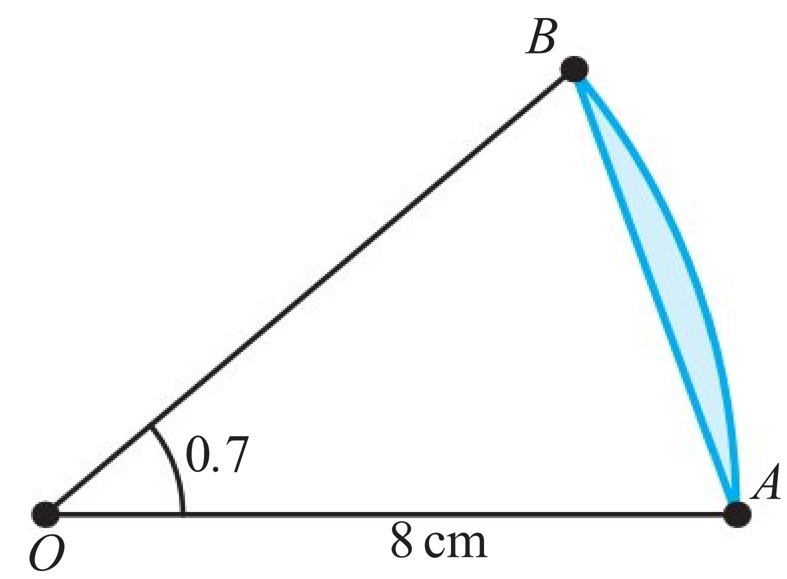
Tip

If you are working with $\sin\theta$, $\cos\theta$ or $\tan\theta$ with θ in radians, you must set your calculator to radian (not degree) mode.

You will often need to combine the results for arc length and area of sector with the results you know for triangles.

WORKED EXAMPLE 4.5

The diagram shows a sector of a circle with radius 8 cm and angle 0.7 radians.



For the shaded region, find

a the perimeter

b the area.

Use $s = r\theta$ to find the arc length

a $s = 8 \times 0.7$

$= 5.6$

By cosine rule,

Use the cosine rule to find the length of the chord AB. Remember to make sure your calculator is in radian mode

$AB^2 = 8^2 + 8^2 - 2 \times 8 \times 8 \cos 0.7$

$= 30.1002$

So, $AB = 5.4864$

The perimeter of the shaded region is the length of the arc plus the length of the chord

Hence, $p = 5.6 + 5.49$

$= 11.1\text{ cm}$ (3 s.f.)

Use $A = \frac{1}{2}r^2\theta$ to find the area of the sector

b Area of sector $= \frac{1}{2} \times 8^2 \times 0.7$

$= 22.4$

Use $A = \frac{1}{2}ab\sin C$ with $a = b = r$ and $C = \theta$ to find the area of the triangle

Area of triangle $= \frac{1}{2} \times 8^2 \sin 0.7$

$= 20.615$

The area of the shaded region is the area of the sector minus the area of the triangle

$A = 22.4 - 20.6$

$= 1.79\text{ cm}^2$ (3 s.f.)

Exercise 4A

For questions 1 to 12, use the method demonstrated in Worked Example 4.1 to convert between degrees and radians.

Convert to radians, giving your answers in terms of π:

1 **a** $60°$ **b** $45°$

2 **a** $150°$ **b** $120°$

3 **a** $90°$ **b** $270°$

Convert to radians, giving your answer to three significant figures:

4 **a** $28°$ **b** $36°$

5 **a** $67°$ **b** $78°$

6 **a** $196°$ **b** $236°$

Convert to degrees, giving your answers to one decimal place where appropriate:

7 **a** 0.62 radians **b** 0.83 radians

8 **a** 1.26 radians **b** 1.35 radians

9 **a** 4.61 radians **b** 5.24 radians

10 **a** $\frac{\pi}{5}$ radians **b** $\frac{\pi}{8}$ radians

11 **a** $\frac{7\pi}{12}$ radians **b** $\frac{4\pi}{15}$ radians

12 **a** $\frac{7\pi}{3}$ radians **b** $\frac{11\pi}{6}$ radians

For questions 13 to 18, use the method demonstrated in Worked Example 4.2 to mark on the unit circle the points corresponding to these angles.

13 **a** $\frac{2\pi}{3}$ **b** $\frac{3\pi}{4}$

14 **a** $\frac{5\pi}{6}$ **b** $\frac{7\pi}{4}$

15 **a** $-\frac{\pi}{2}$ **b** $-\frac{3\pi}{2}$

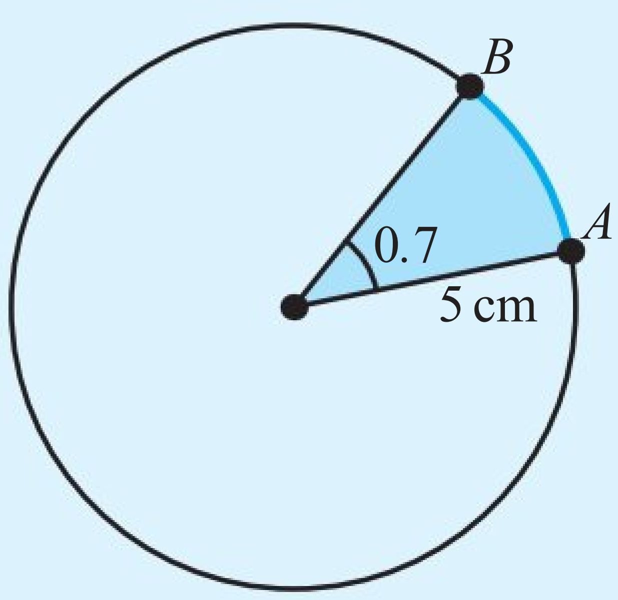
16 **a** $-\frac{\pi}{3}$ **b** $-\frac{\pi}{4}$

17 **a** $\frac{8\pi}{3}$ **b** $\frac{11\pi}{4}$

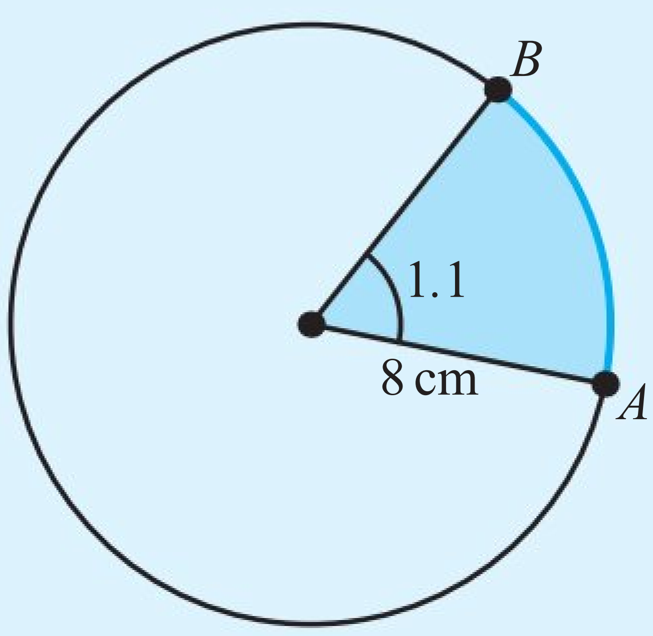
18 **a** $\frac{11\pi}{2}$ **b** $\frac{19\pi}{2}$

For questions 19 to 21, use the methods demonstrated in Worked Examples 4.3 and 4.4 to find the length of the arc AB that subtends the given angle (in radians), and the area of the corresponding sector.

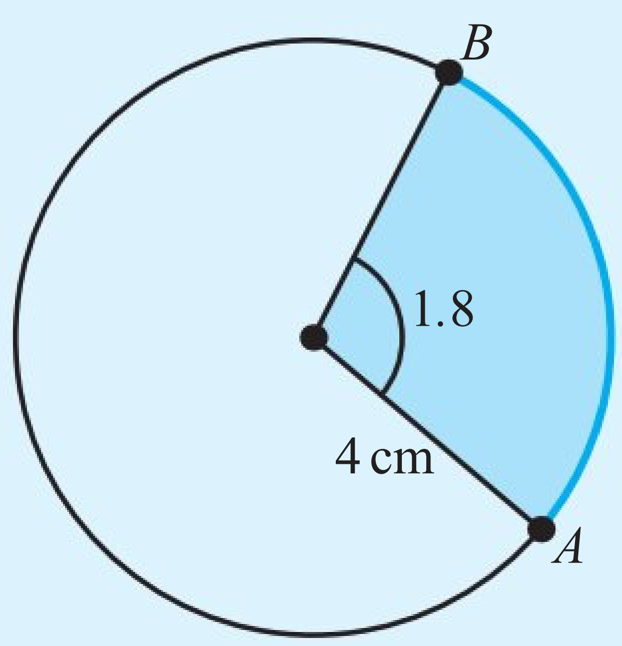
19 **a**



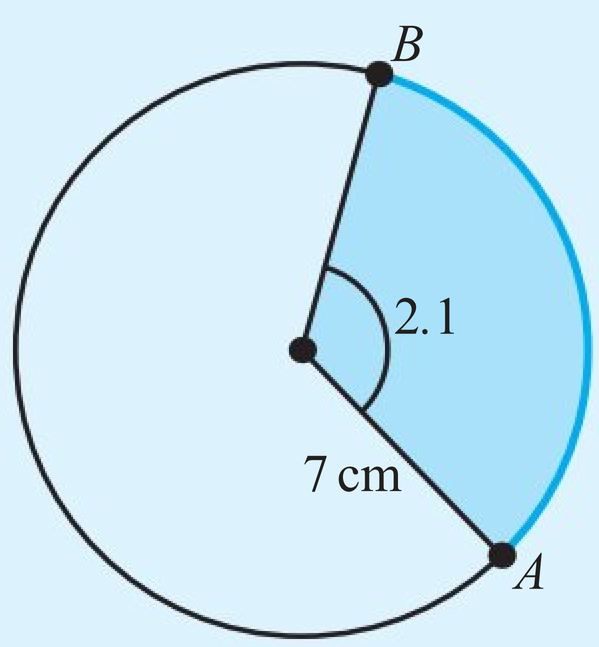
b



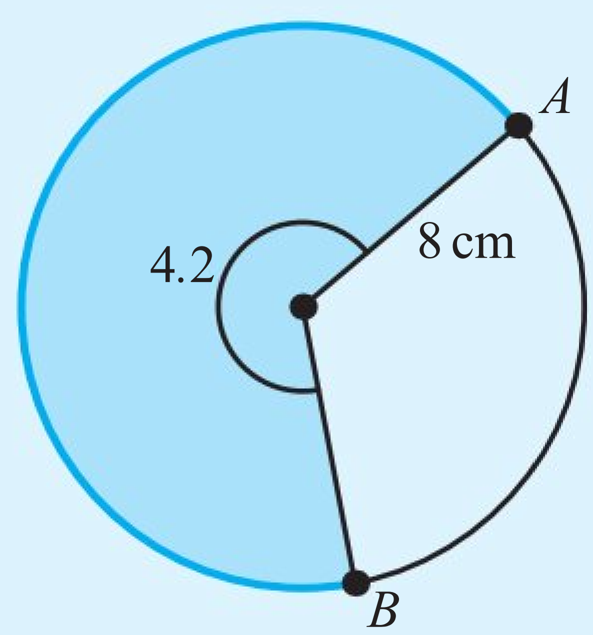
20 **a**



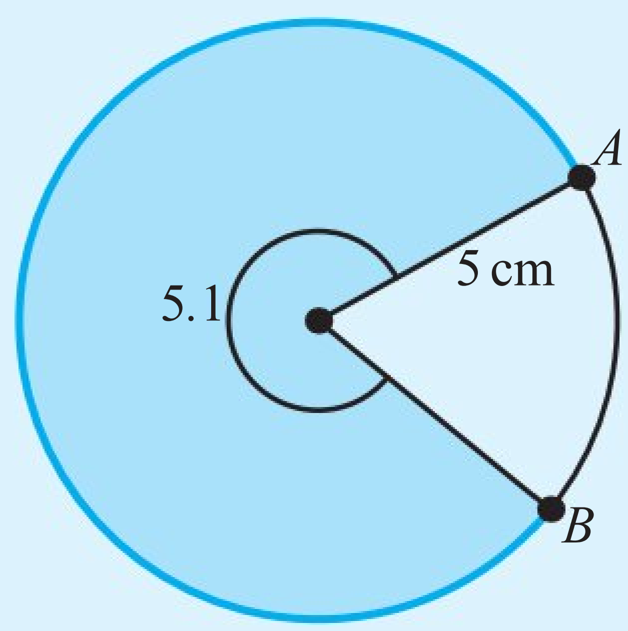
b



21 **a**

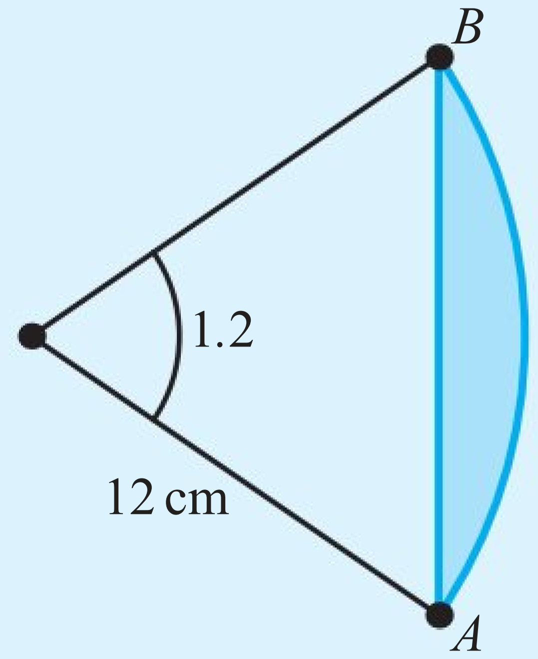


b

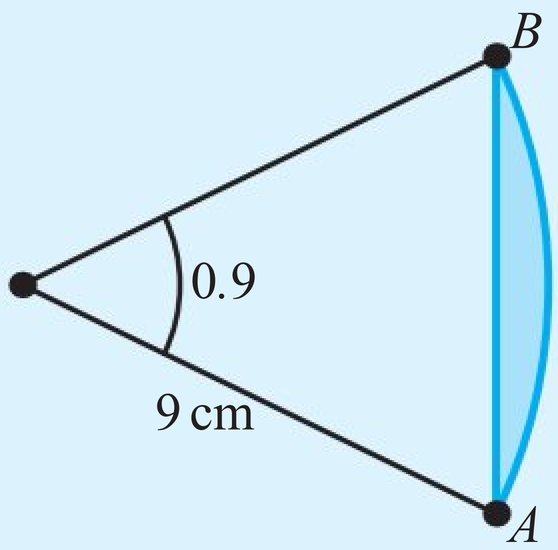


For questions 22 to 24, use the method demonstrated in Worked Example 4.5 to find the area and the perimeter of the shaded region.

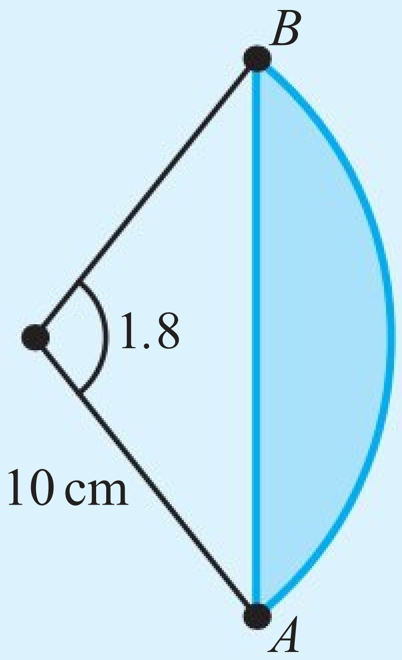
22 **a**



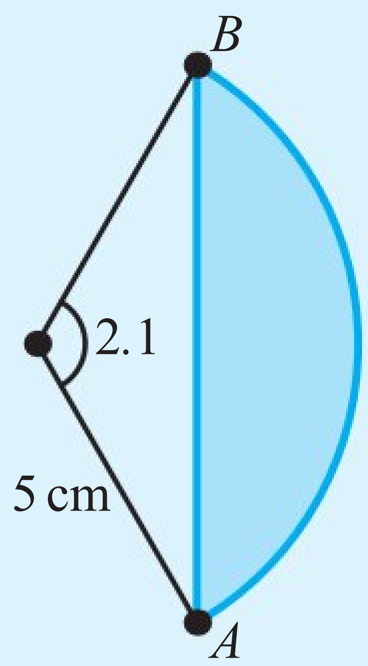
b



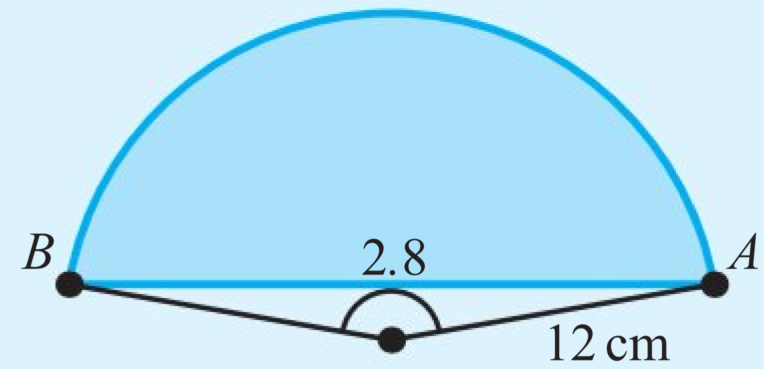
23 **a**



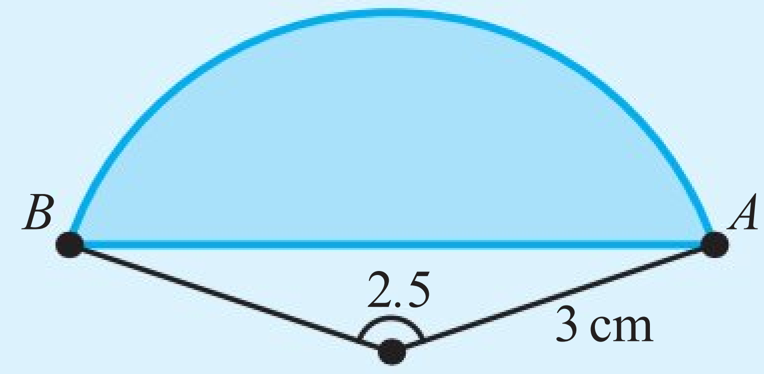
b



24 **a**

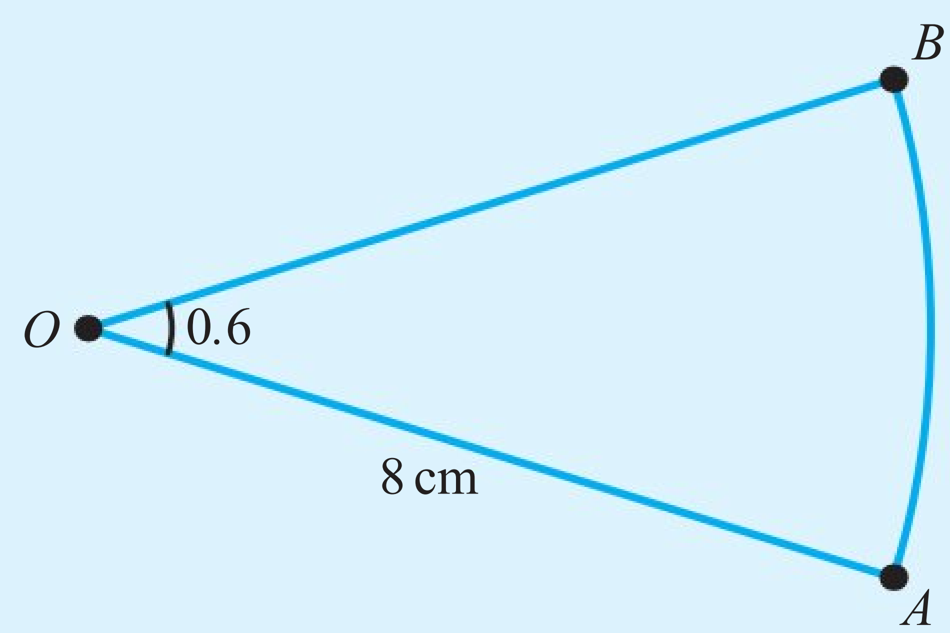


b



25 The diagram shows a sector AOB of a circle of radius 8 cm. The angle of the centre of the sector is 0.6 radians.

Find the perimeter and the area of the sector.



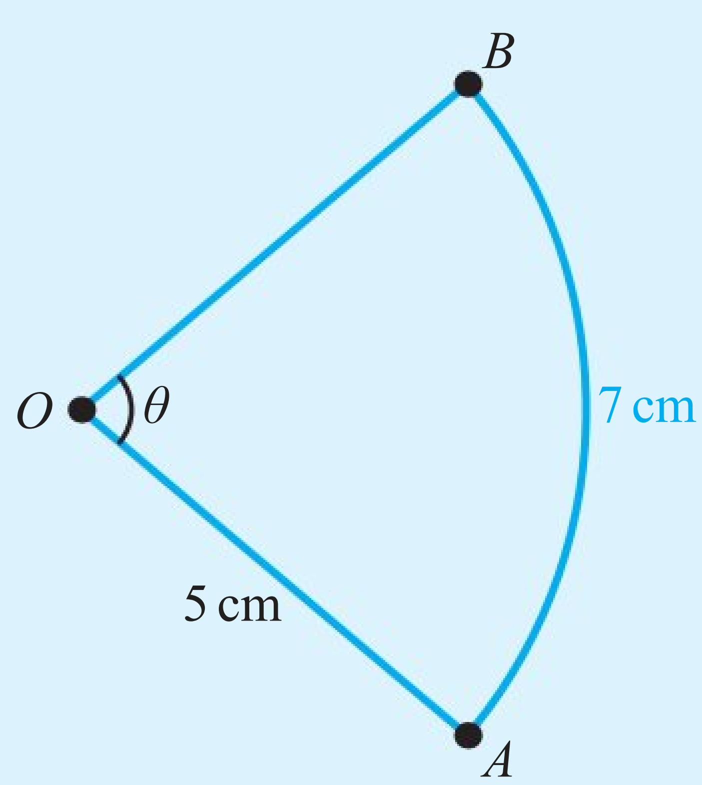
26 A circle has centre O and radius 6.2 cm. Points A and B lie on the circumference of the circle so that the arc AB subtends an angle of 2.5 radians at the centre of the circle. Find the perimeter and the area of the sector AOB.

27 An arc of a circle has length 12.3 cm and subtends an angle of 1.2 radians at the centre of the circle. Find the radius of the circle.

28 The diagram shows a sector of a circle of radius 5 cm. The length of the arc AB is 7 cm.

a Find the value of θ.

b Find the area of the sector.



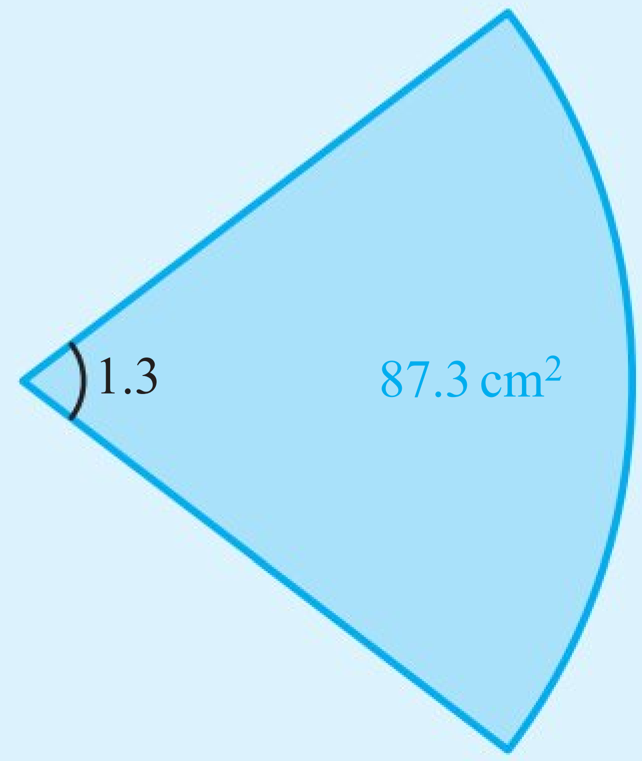
29 A circle with centre O has radius 23 cm. Arc AB subtends angle θ radians at the centre of the circle. Given that the area of the sector AOB is 185 cm^2, find the value of θ.

30 A sector of a circle has area 326 cm^2 and an angle at the centre of 2.7 radians. Find the radius of the circle.

31 The diagram shows a sector of a circle. The area of the sector is 87.3 cm^2.

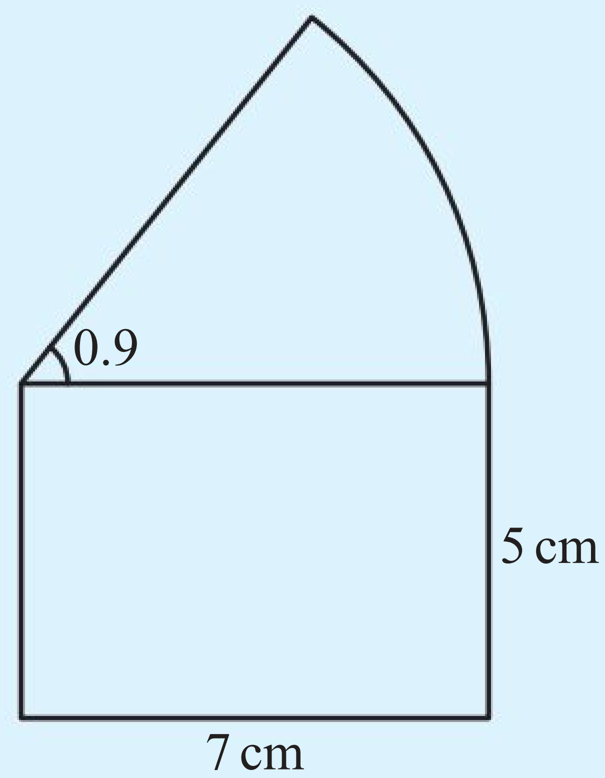
a Find the radius of the circle.

b Find the perimeter of the sector.



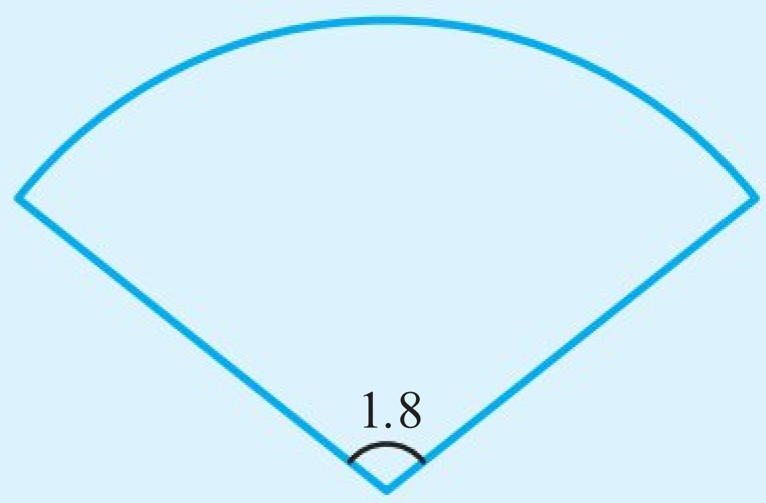
32 The figure shown in the diagram consists of a rectangle and a sector of a circle.

Calculate the area and the perimeter of the figure.



33 The diagram shows a sector of a circle.

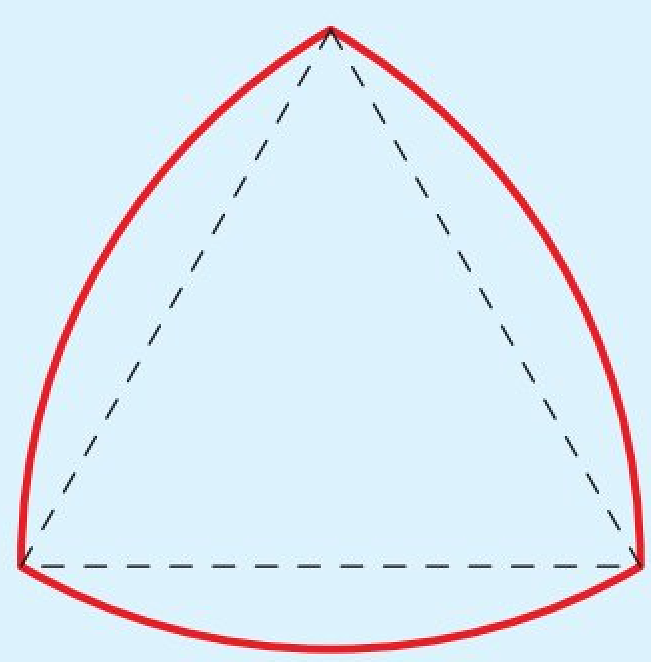
The perimeter of the sector is 26 cm. Find the radius of the circle.



34 A sector of a circle has area 18 cm^2 and perimeter 30 cm. Find the possible values of the radius of the circle.

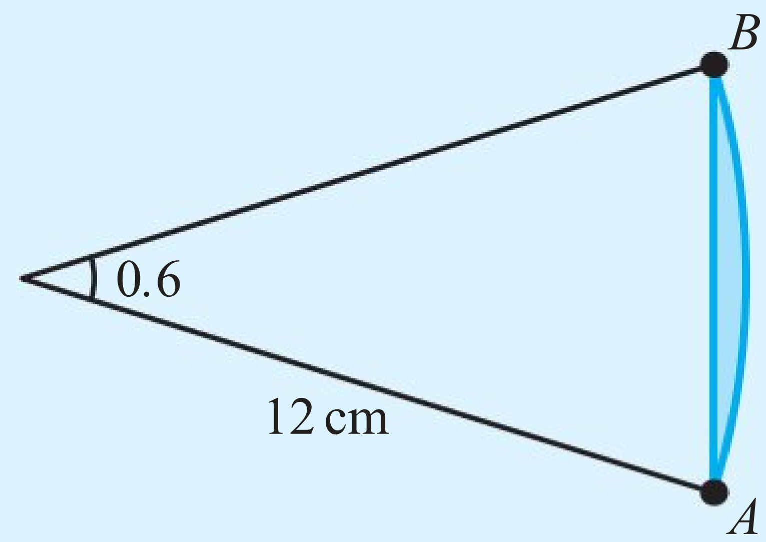
35 The diagram shows an equilateral triangle and three arcs of circles with centres that are the vertices of the triangle.

The length of the sides of the triangle are 12 cm. Find the perimeter of the figure.



36 A circle has centre O and radius 8 cm. Chord PQ subtends angle 0.9 radians at the centre. Find the difference between the length of the arc PQ and the length of the chord PQ.

37 The arc AB of a circle of radius 12 cm subtends an angle of 0.6 radians at the centre. Find the perimeter and the area of the shaded region.

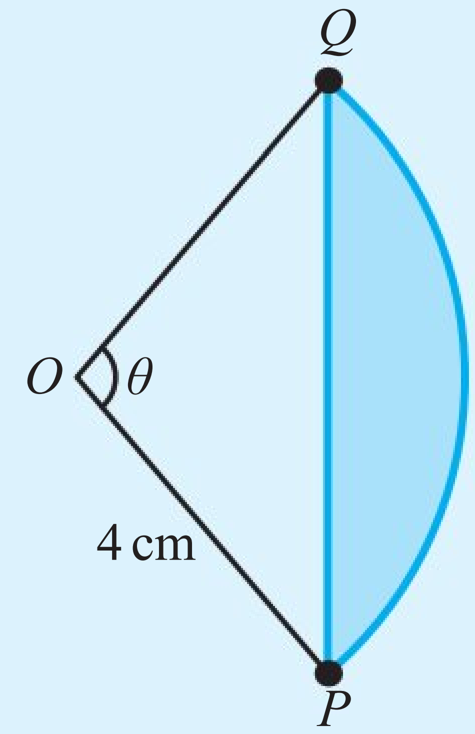


38 A circle has centre O and radius 4 cm. Chord PQ subtends angle θ radians at the centre. The area of the shaded region is 6 cm^2.

a Show that $\theta - \sin\theta = 0.75$.

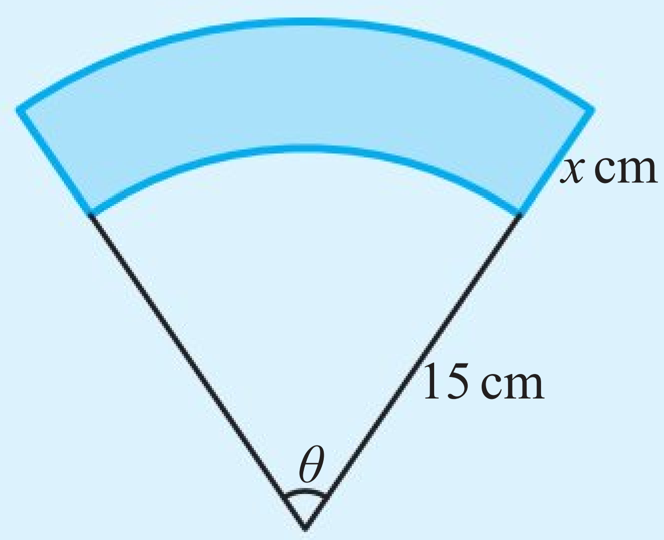
b Find the value of θ.

c Find the perimeter of the shaded region.



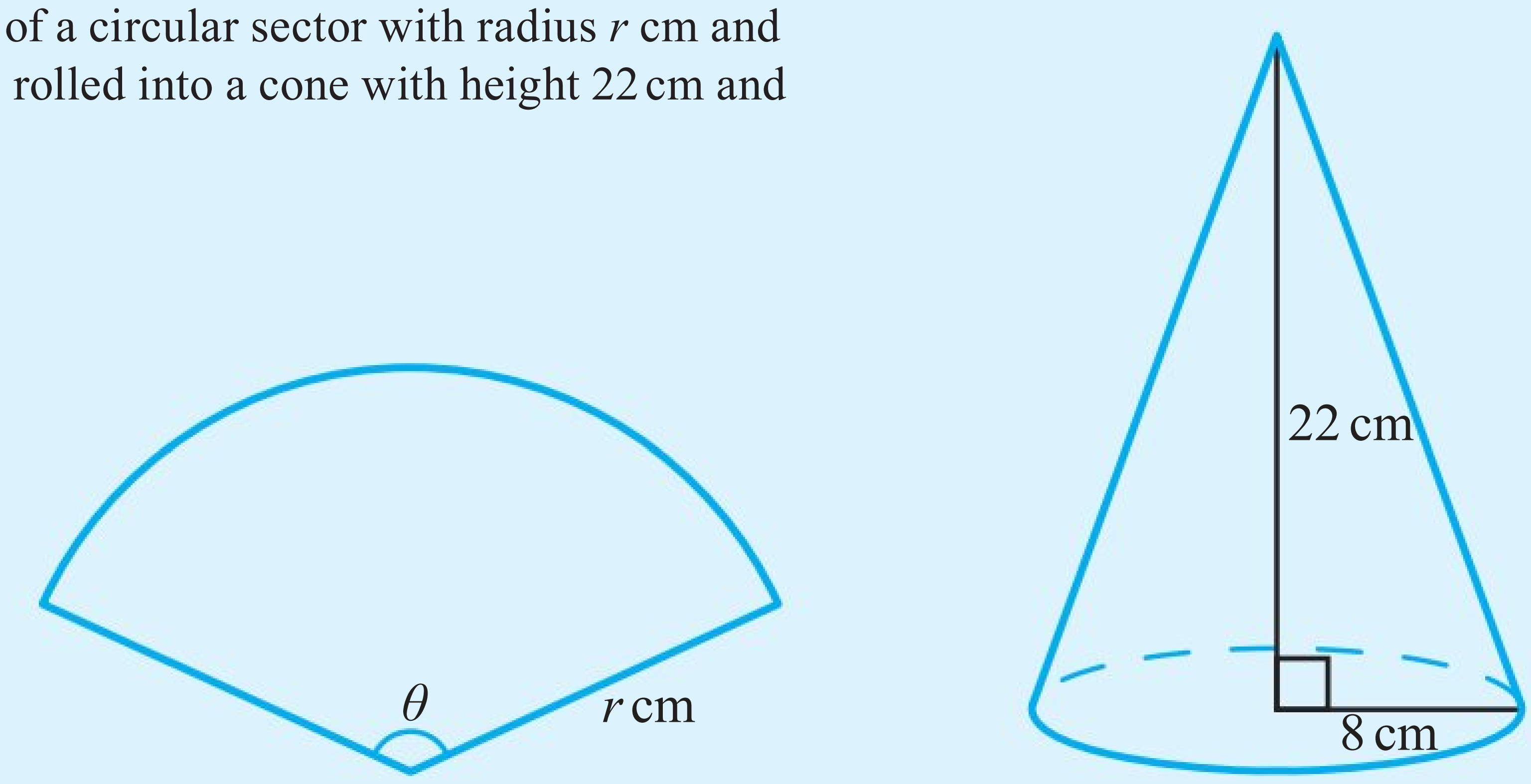
39 The diagram shows two circular sectors with angle 1.2 radians at the centre. The radius of the smaller circle is 15 cm and the radius of the larger circle is x cm larger.

Find the value of x so that the area of the shaded region is 59.4 cm^2.



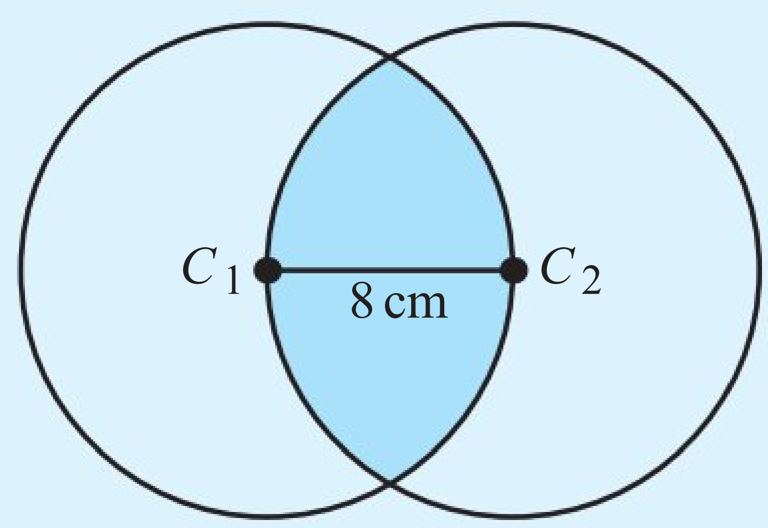
40 A piece of paper has a shape of a circular sector with radius r cm and angle θ radians. The paper is rolled into a cone with height 22 cm and base radius 8 cm.

Find the values of r and θ.



41 Two identical circles each have radius 8 cm. They overlap in such a way that the centre of each circle lies on the circumference of the other, as shown in the diagram.

Find the perimeter and the area of the shaded region.



4B Further trigonometry

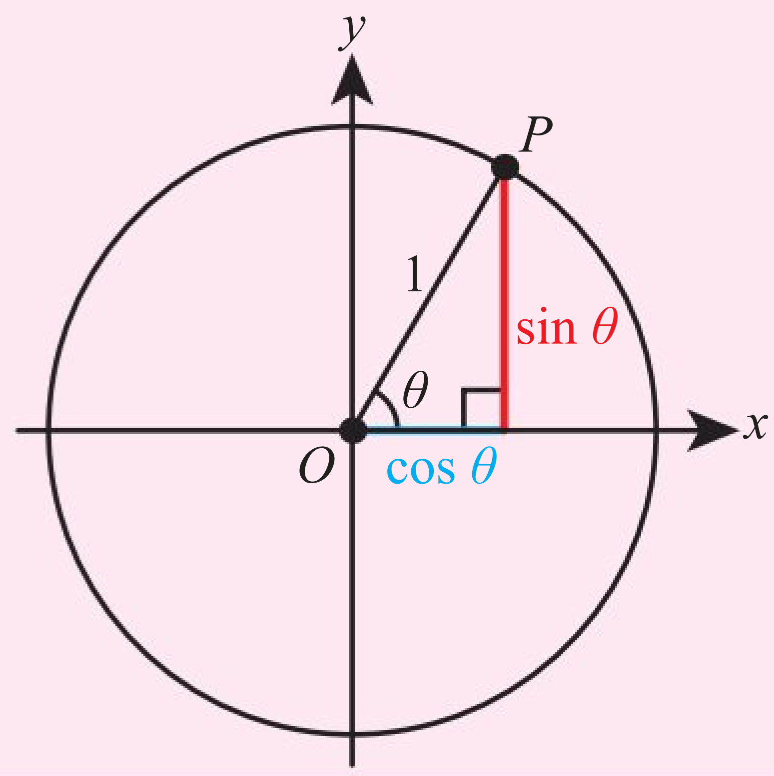
Definition of cos θ and sin θ in terms of the unit circle

Until now you have only used $\sin\theta$ and $\cos\theta$ in triangles, where θ had to be less than 180° (or π radians). However, using the unit circle we can define these functions so that θ can be any size, positive or negative.

KEY POINT 4.4

For a point, P, on the unit circle at an angle θ to the positive x-axis:

- $\sin\theta$ is the y-coordinate of the point P
- $\cos\theta$ is the x-coordinate of the point P.



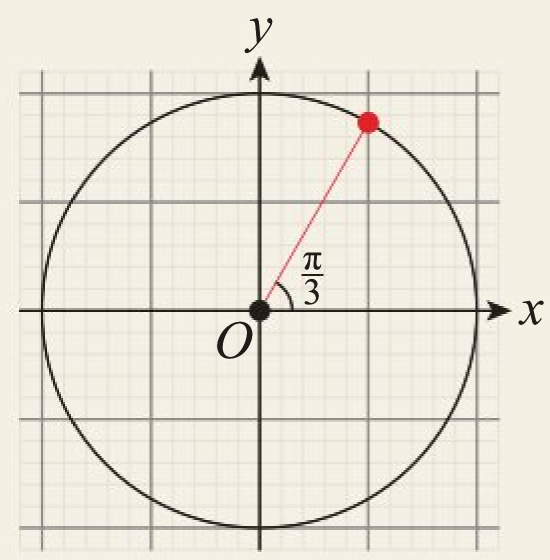
WORKED EXAMPLE 4.6

Mark on the unit circle the point corresponding to each angle θ. Hence find, or estimate, the values of $\sin\theta$ and $\cos\theta$.

a $\theta = \frac{\pi}{3}$ **b** $\theta = \frac{7\pi}{2}$ **c** $\theta = -\frac{5\pi}{4}$

$\frac{\pi}{3} = \frac{1}{6}$ of 2π, so rotate one-sixth around the circle going anticlockwise, starting from the positive x-axis. (In degrees, this corresponds to an angle of 60°.)

a



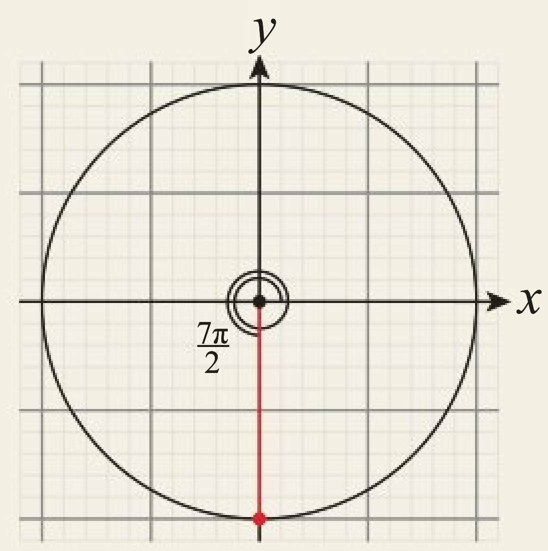
$\sin\left(\frac{\pi}{3}\right)$ is the y-coordinate and $\cos\left(\frac{\pi}{3}\right)$ is the x-coordinate From this diagram, you can only estimate the values to 1 d.p.

$\sin\left(\frac{\pi}{3}\right) \approx 0.9$

$\cos\left(\frac{\pi}{3}\right) \approx 0.5$

$\frac{7\pi}{2} = 2\pi + \frac{3\pi}{2}$, so rotate one full turn plus another three-quarters of a turn anticlockwise

b



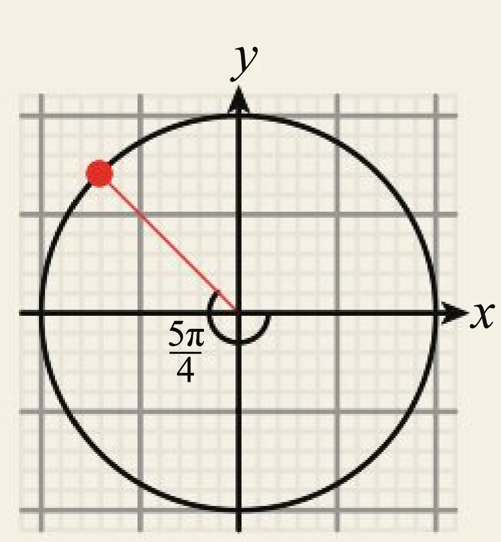
$$\sin\left(\frac{7\pi}{2}\right) = -1$$

The point is on the y-axis, so its x-coordinate is zero

$$\cos\left(\frac{7\pi}{2}\right) = 0$$

c

The negative sign means you need to rotate clockwise around the circle, in this case half a turn followed by another eighth. (Remember that an eighth of a turn corresponds to 45°)



$$\sin\left(-\frac{5\pi}{4}\right) \approx 0.7$$

You can see that the x-coordinate is negative but the y-coordinate is positive

$$\cos\left(-\frac{5\pi}{4}\right) \approx -0.7$$

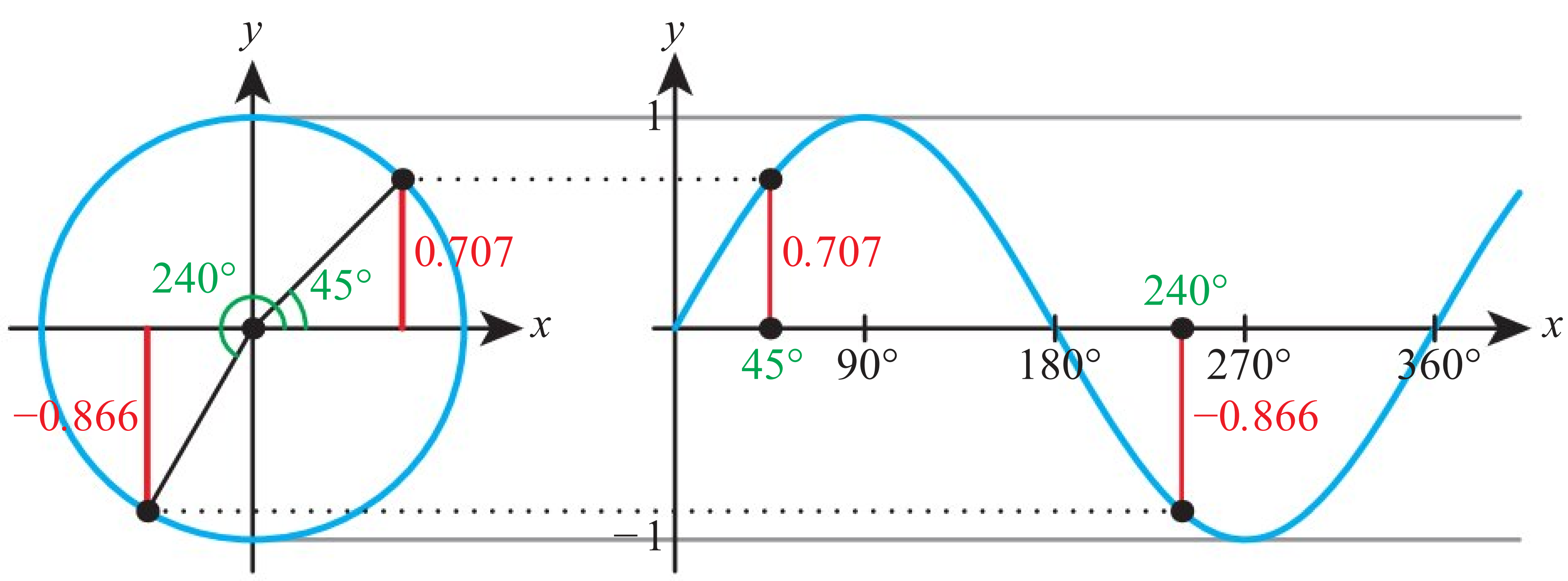
CONCEPTS – APPROXIMATION

For most angles, the values of sine and cosine found from the unit circle are only **approximations**. There are other methods for finding more accurate approximations. In a few special cases, you can use Pythagoras's theorem to find the exact values.

For an architect, is it more useful to know that $\sin\frac{\pi}{3} = \frac{\sqrt{3}}{2}$ or 0.866?

Graphs of f(x) = sin(x) and f(x) = cos(x)

Using the definition of sine from the unit circle, we can draw its graph:

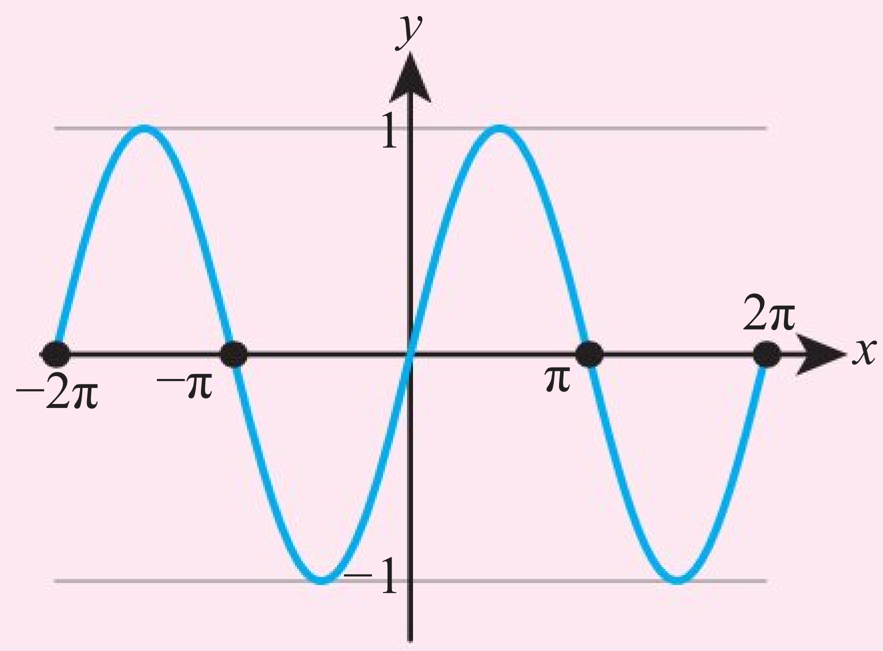


Sine repeats every 2π radians, so we say it is a periodic function with **period** 2π radians.

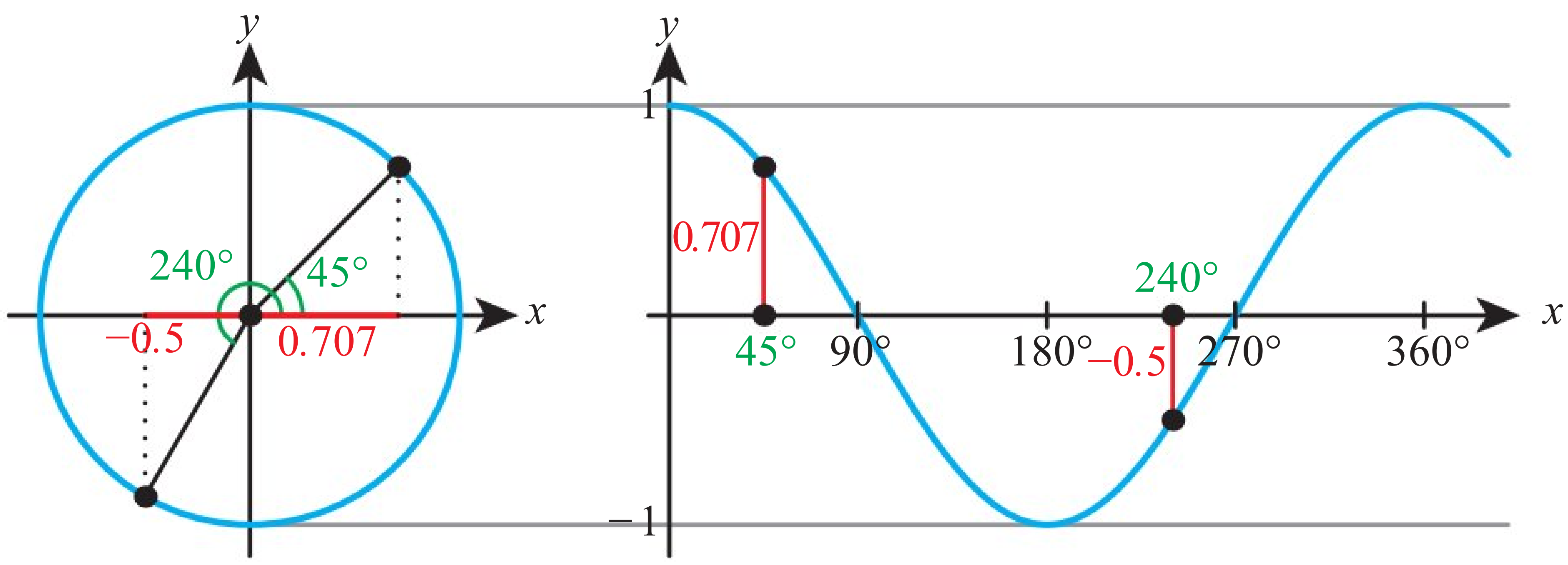
It has a minimum value of -1 and a maximum value of $+1$, so we say it has an **amplitude** of 1.

KEY POINT 4.5

The graph of $y = \sin x$:



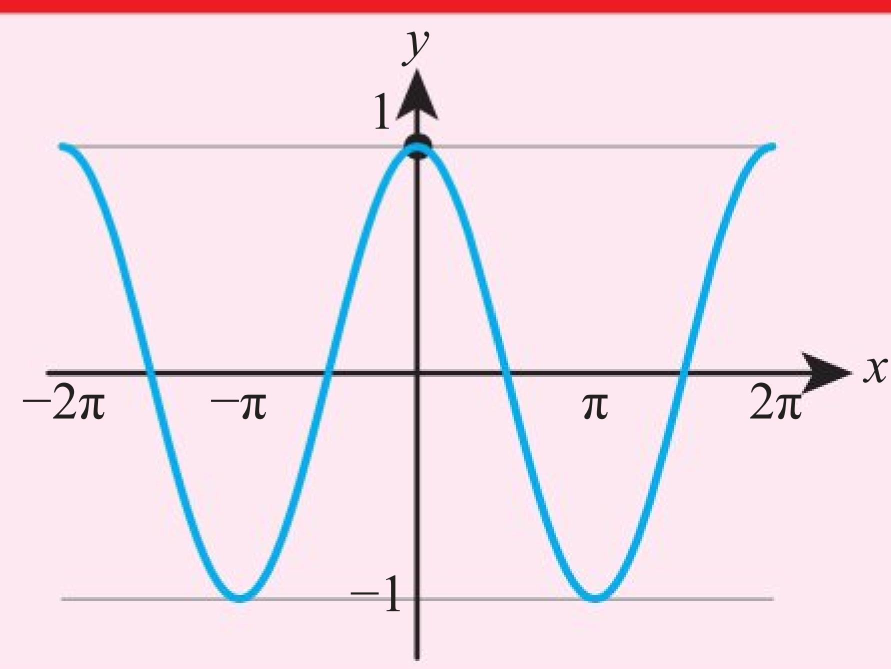
We can construct the cosine graph from the unit circle definition in the same way:



Like sine, cosine has a period of 2π radians and an amplitude of 1.

KEY POINT 4.6

The graph of $y = \cos x$:



You used the sine rule to find angles (and side lengths) of triangles in IB Diploma: applications and interpretation SL Section 5B.

Tip

Note that there is not a corresponding relationship for cos, so you do not get this second possibility when using the cosine rule.

Extension of the sine rule to the ambiguous case

You can see from the graph of $\sin(x)$ (or from the unit circle) that $\sin(\theta) = \sin(180° - \theta)$ (or $\sin(\pi - \theta)$ if working with radians). This means that, if you know the value of $\sin\theta$, there are two possible values between 0° and 180° that the angle θ could take.

This has an immediate implication for finding angles in triangles using the sine rule (where angles will usually be measured in degrees).

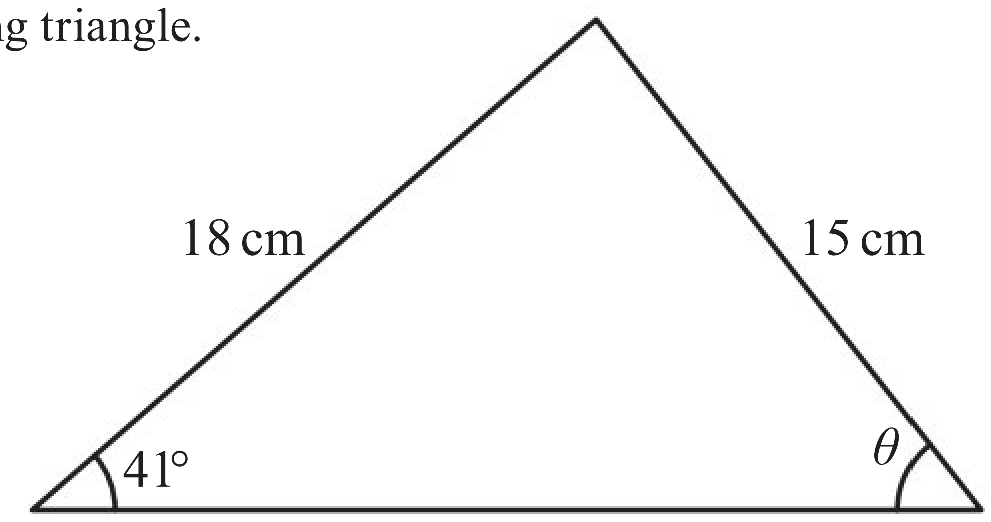
KEY POINT 4.7

When using the sine rule to find an angle, there may be two possible solutions: θ and $180 - \theta$.

Be aware that just because there is the possibility of a second value for an angle, this does not necessarily mean that a second triangle exists. You always need to check whether the angle sum is less than 180°.

WORKED EXAMPLE 4.7

Find the size of the angle θ in the following triangle.



Since we know a side and the angle opposite, the sine rule is useful

By sine rule, $\frac{\sin\theta}{18} = \frac{\sin 41}{15}$

This expression can be rearranged to find θ_1, the first possible value, which is the one given by the inverse sine function

$\theta_1 = \sin^{-1}\left(\frac{\sin 41}{15} \times 18\right)$

$180 - 51.9 = 128.1°$ is also a possible value for θ

$\theta = 51.9°$ or $128.1°$

Check each possible value of θ to see that the angle sum is less than 180°

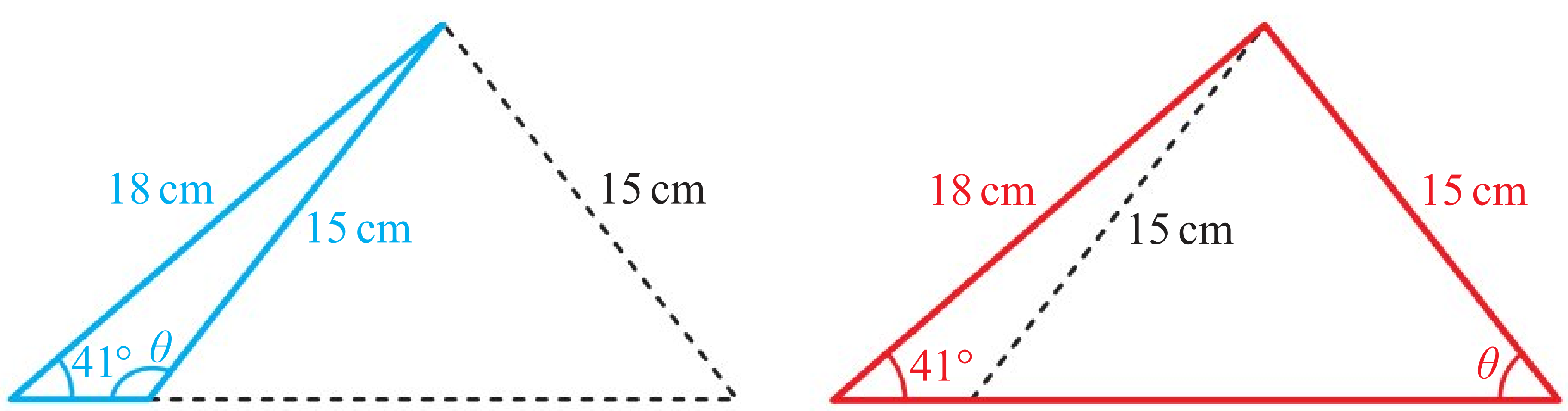
$51.9 + 41 = 92.9 < 180$

$128.1 + 41 = 169.1 < 180$

Both solutions are possible

So, $\theta = 51.9°$ or $128.1°$

The diagram below shows both possible triangles. Note that the 41° angle must remain opposite the 15 cm side, as given.



Be the Examiner 4.1

In triangle ABC, $AB = 10\,\text{cm}$, $AC = 12\,\text{cm}$ and $A\hat{B}C = 70°$.

Find the size of $A\hat{C}B$.

Which is the correct solution? Identify the errors made in the incorrect solutions.

Solution 1	Solution 2	Solution 3
$\frac{\sin\theta}{10} = \frac{\sin 70°}{12}$ $\sin\theta = \frac{10\sin 70°}{12}$ $\theta = \sin^{-1}\left(\frac{10\sin 70°}{12}\right)$ $= 51.5°$ So, $\theta = 51.5°$	$\frac{\sin\theta}{12} = \frac{\sin 70°}{10}$ $\sin\theta = \frac{12\sin 70°}{10}$ $= 1.13$ $1.13 > 1$ So, there are no solutions.	$\frac{\sin\theta}{10} = \frac{\sin 70°}{12}$ $\sin\theta = \frac{10\sin 70°}{12}$ $\theta = \sin^{-1}\left(\frac{10\sin 70°}{12}\right)$ $= 51.5°$ $180 - 51.5 = 128.5$ So, $\theta = 51.5°$ or $128.5°$

Definition of $\tan\theta$ as $\frac{\sin\theta}{\cos\theta}$

The tangent function is defined as the ratio of the sine function to the cosine function.

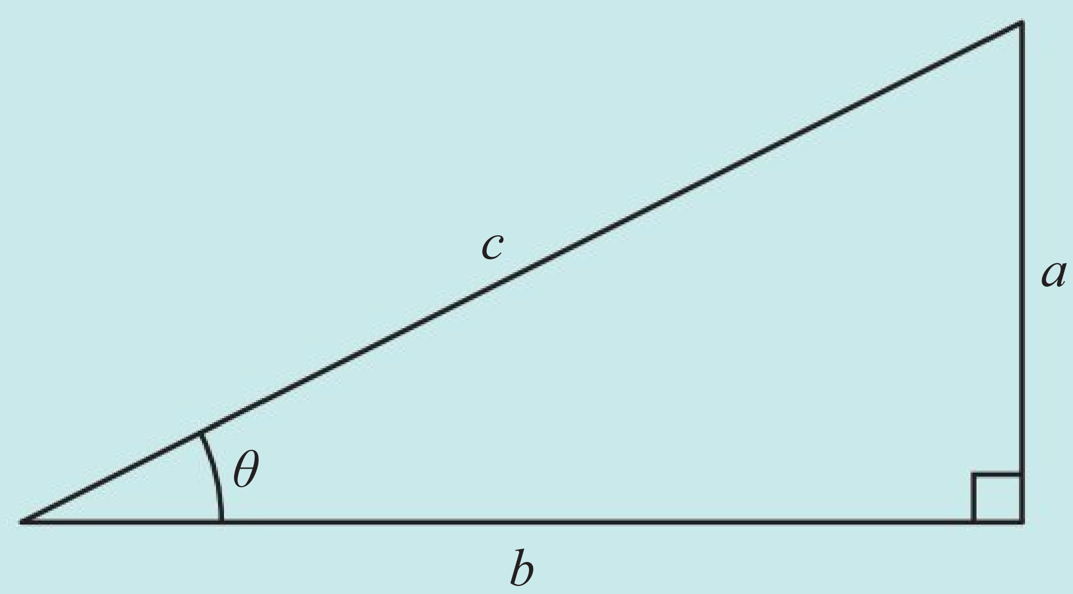
KEY POINT 4.8

$$\tan\theta = \frac{\sin\theta}{\cos\theta}$$

For acute angles of θ, this definition fits in with the definition you already know based on right-angled triangles:

$\sin\theta = \frac{a}{c}$, $\cos\theta = \frac{b}{c}$, so

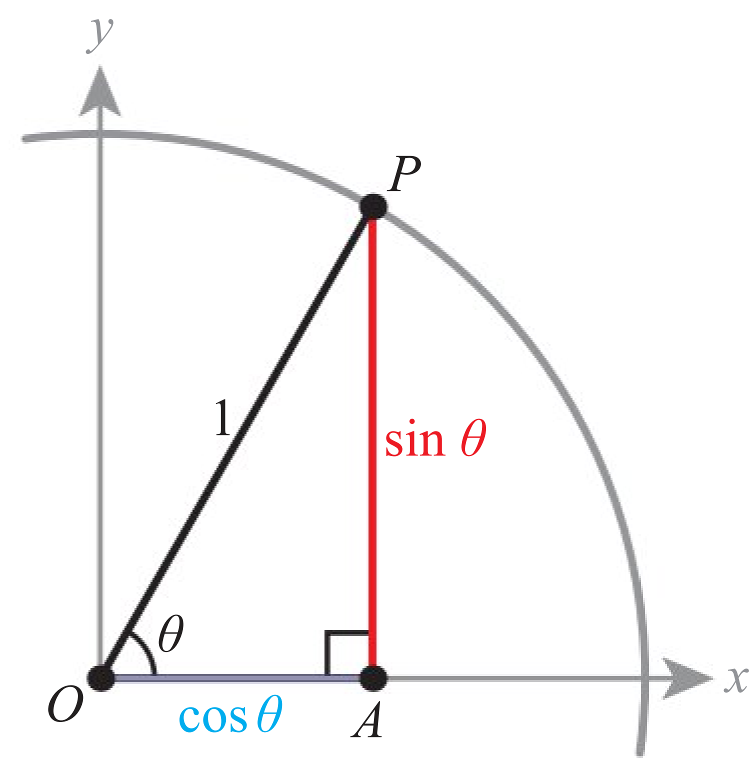
$$\tan\theta = \frac{\left(\frac{a}{c}\right)}{\left(\frac{b}{c}\right)} = \frac{a}{b}$$



The Pythagorean identity $\cos^2\theta + \sin^2\theta \equiv 1$

This identity, which relates cosine and sine, follows directly from their definitions on the unit circle.

Using Pythagoras' theorem in the right-angled triangle OAP gives the identity.



KEY POINT 4.9

$\cos^2\theta + \sin^2\theta \equiv 1$

WORKED EXAMPLE 4.8

Given that $\frac{3\pi}{2} < x < 2\pi$ and that $\cos x = \frac{1}{3}$, find the exact value of

a $\sin x$ **b** $\tan x$

Use $\cos^2 x + \sin^2 x \equiv 1$ to relate sin to cos

a $\sin^2 x \equiv 1 - \cos^2 x$

Substitute in the given value of $\cos x$

$$= 1 - \left(\frac{1}{3}\right)^2$$

$$= 1 - \frac{1}{9}$$

$$= \frac{8}{9}$$

Remember to take $\pm$ when square rooting

$$\sin x = \pm\sqrt{\frac{8}{9}}$$

Looking at the unit circle or the graph, you can see that $\sin x$ is negative for x between $\frac{3\pi}{2}$ and 2π

But $\sin x < 0$ for $\frac{3\pi}{2} < x < 2\pi$, so

$$\sin x = -\sqrt{\frac{8}{9}}$$

You may be able to use the calculator to simplify the surd

$$= -\frac{2\sqrt{2}}{3}$$

Use $\tan x = \frac{\sin x}{\cos x}$ to relate tan to sin and cos

b $\tan x = \frac{\sin x}{\cos x}$

Substitute in the values for $\sin x$ and $\cos x$ and simplify

$$= \frac{\left(-\frac{2\sqrt{2}}{3}\right)}{\left(\frac{1}{3}\right)}$$

$$= -2\sqrt{2}$$

Solving trigonometric equations in a finite interval graphically

You can solve a trigonometric equation graphically in the same way that you have solved other equations with your GDC. Because trigonometric functions are periodic, you will often find more than one solution.

See Chapter 3 of the Mathematics: applications and interpretation SL book for a reminder of how to use your GDC to solve equations.

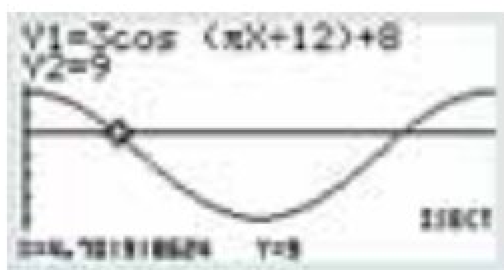
Tip

When using trigonometric models, you should work in radians unless told otherwise.

WORKED EXAMPLE 4.9

The depth of water (d m) in harbour, t hours after midnight, can be modelled by the equation $d = 3\cos\left(\frac{\pi}{12}t\right) + 8$. Find the times at which the depth of water in the harbour is 9 m, giving your answer to the nearest minute.

Use your GDC to find the intersection of $y = 3\cos\left(\frac{\pi}{12}x\right) + 8$ and $y = 9$



$t = 4.702, 19.298$

Convert 0.702 and 0.298 to minutes by multiplying by 60

So, at 04:42 and 19:18

You studied sinusoidal models in Chapter 13 of the Mathematics: applications and interpretation SL book and will revisit them in Chapter 5.

Exercise 4B

For questions 1 to 4, use the method demonstrated in Worked Example 4.6 to mark the point corresponding to angle θ on the unit circle, and hence estimate the values of $\sin\theta$ and $\cos\theta$.

1 **a** $\theta = \frac{2\pi}{3}$
b $\theta = \frac{\pi}{4}$

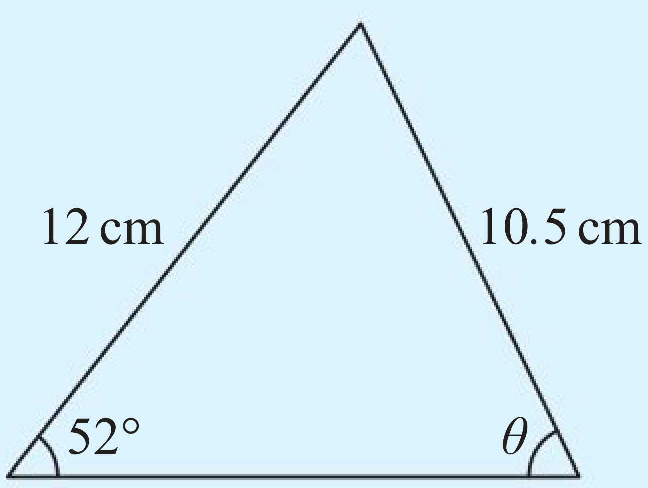
2 **a** $\theta = \pi$
b $\theta = 4\pi$

3 **a** $\theta = \frac{9\pi}{2}$
b $\theta = \frac{11\pi}{2}$

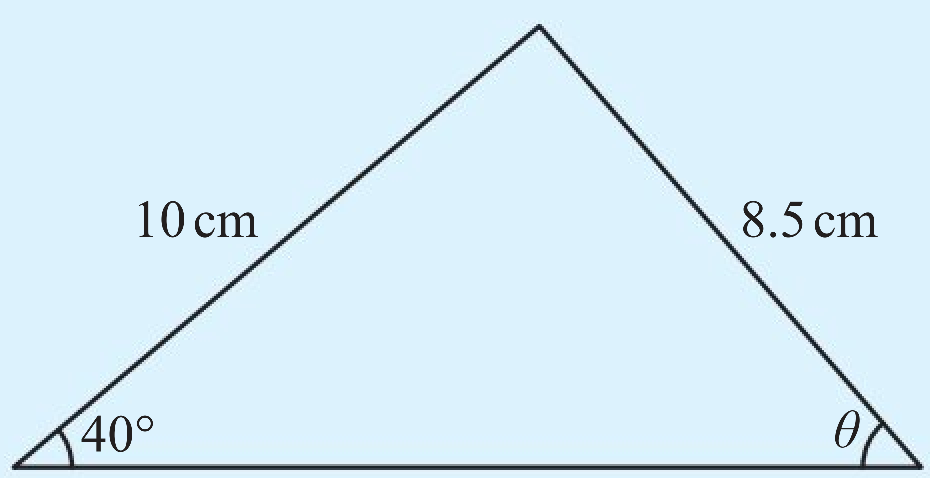
4 **a** $\theta = -\frac{5\pi}{6}$
b $\theta = -\frac{9\pi}{4}$

For questions 5 to 8, use the method demonstrated in Worked Example 4.7 to find the possible size(s) of the angle θ in each triangle. Give your answers correct to the nearest degree.

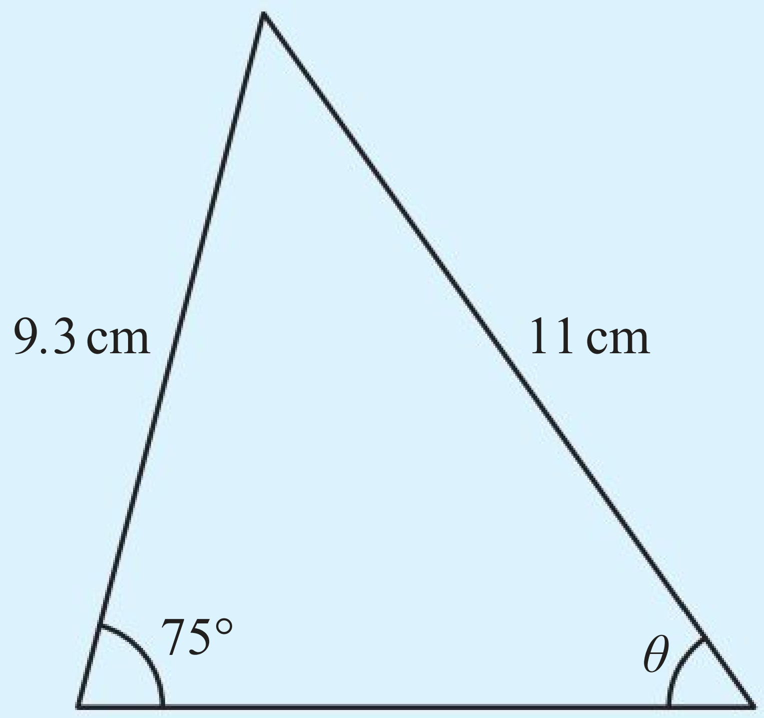
5 **a**



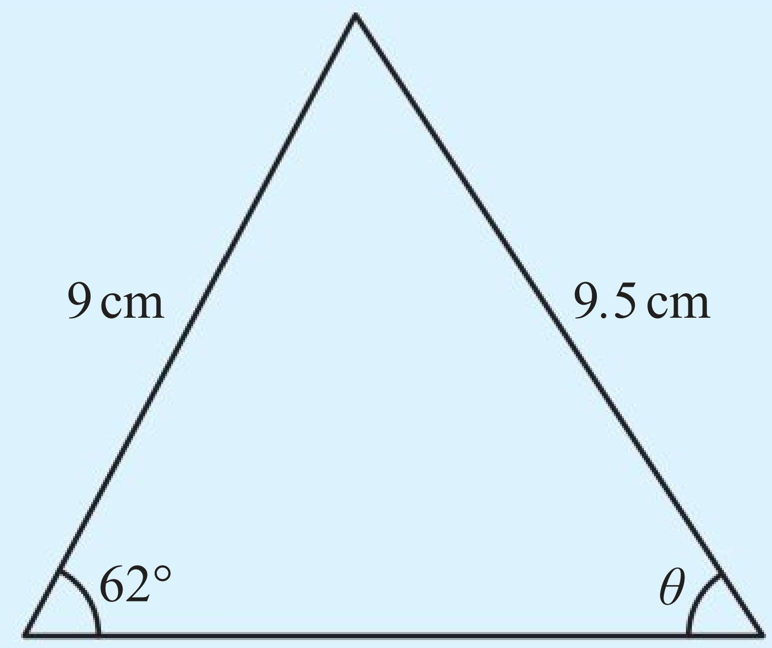
b



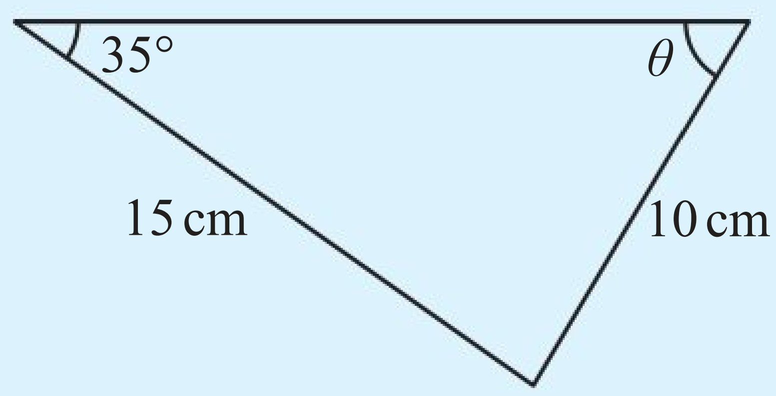
6 a



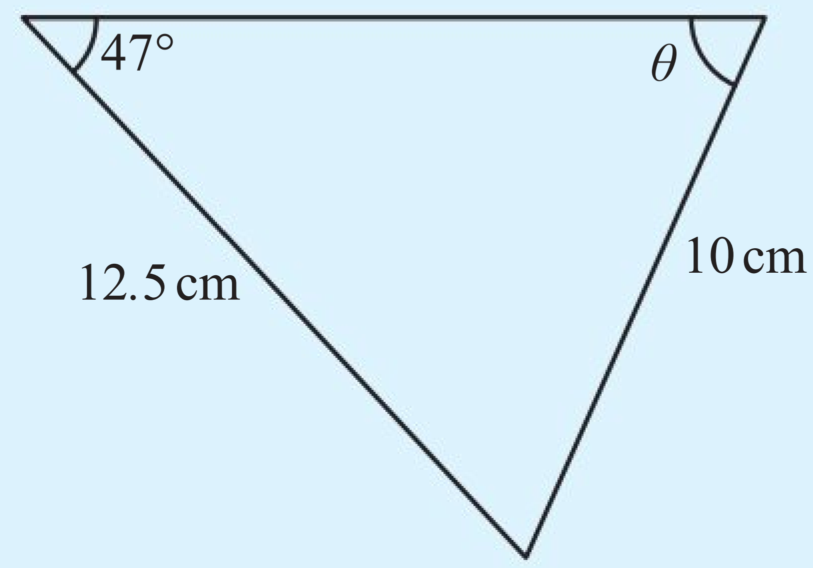
b



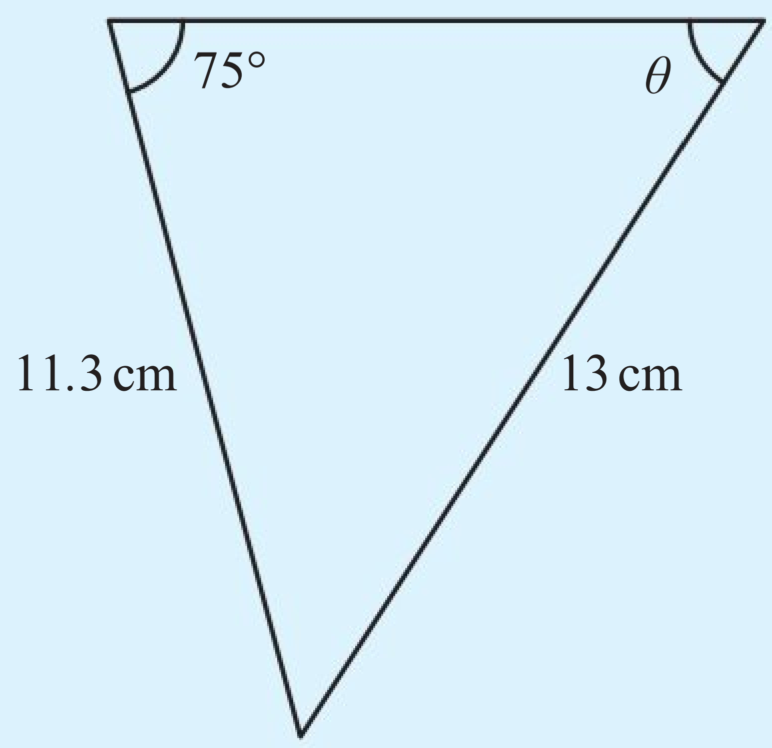
7 a



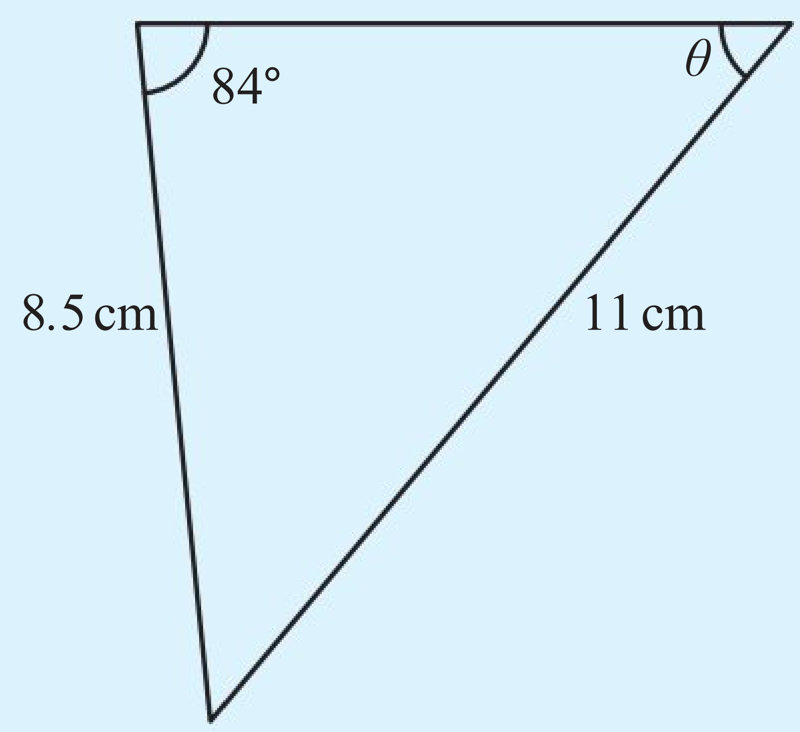
b



8 a



b



For questions 9 to 12, use the method demonstrated in Worked Example 4.8 to find the possible values of the required trigonometric functions.

9 a Find $\sin x$ and $\tan x$ given that $\cos x = \frac{1}{4}$ and $0 < x < \frac{\pi}{2}$.

b Find $\sin x$ and $\tan x$ given that $\cos x = \frac{2}{3}$ and $0 < x < \frac{\pi}{2}$.

10 a Find $\cos x$ and $\tan x$ given that $\sin x = 0.381$ and $\frac{\pi}{2} < x < \pi$.

b Find $\cos x$ and $\tan x$ given that $\sin x = 0.722$ and $0 < x < \frac{\pi}{2}$.

11 a Find $\sin x$ and $\tan x$ given that $\cos x = -0.062$ and $\pi < x < \frac{3\pi}{2}$.

b Find $\sin x$ and $\tan x$ given that $\cos x = -0.831$ and $\frac{\pi}{2} < x < \pi$.

12 a Find $\cos x$ and $\tan x$ given that $\sin x = -\frac{1}{3}$ and $\pi < x < \frac{3\pi}{2}$.

b Find $\cos x$ and $\tan x$ given that $\sin x = -\frac{3}{5}$ and $\frac{3\pi}{2} < x < 2\pi$.

For questions 13 to 16, use a graphical method, as illustrated in Worked Example 4.9, to find all solutions of the equation in the given interval.

13 a $3\sin x = 0.5$ for $0 \leqslant x < 3\pi$

b $2\cos x = -0.2$ for $0 \leqslant x \leqslant 4\pi$

14 a $2 + 5\cos(3x) = 4$ for $0 \leqslant x \leqslant \pi$

b $3 + 4\cos(2x) = 2$ for $0 \leqslant x \leqslant \pi$

15 a $4 - 2\sin(2x) = 1$ for $0 \leqslant x \leqslant 360°$

b $3\sin(4x) - 5 = -1$ for $0 \leqslant x \leqslant 180°$

16 a $4\tan(\pi(x-2)) = 3$ for $-1 < x < 1$

b $3\tan\left(\frac{x-\pi}{2}\right) = 5$ for $0 < x < 4\pi$

17 Given that $\sin\theta = -\frac{4}{9}$, find the possible values of $\cos\theta$.

18 Given that $\cos\theta = -\frac{2}{5}$ and $\pi < \theta < 2\pi$, find the exact value of

a $\sin\theta$

b $\tan\theta$.

19 Given that $\sin x = \frac{3}{7}$ and $\frac{\pi}{2} < x < \pi$, find the exact value of

a $\cos x$

b $\tan x$.

20 The depth of water in a harbour varies according to the equation $d = 5 + 1.6\sin\left(\frac{\pi}{12}t\right)$, where d is measured in metres and t is the time, in hours, after midnight.

a Find the depth of water at high tide.

b Find the first time after midnight when the high tide occurs.

21 The height of a seat on a Ferris wheel above ground is modelled by the equation $h = 6.2 - 4.8\cos\left(\frac{\pi}{4}t\right)$, where h is measured in metres and t is the time, in minutes, since the start of the ride.

a How long does the wheel take to complete one revolution?

b Find the maximum height of the seat above ground.

c Find the height of the seat above ground 2 minutes and 40 seconds after the start of the ride.

22 A ball is attached to one end of a spring and hangs vertically. The ball is then pulled down and released. In the subsequent motion, the height of the ball, h metres, above ground at time t seconds is given by $1.4 - 0.2\cos(15t)$.

a Find the greatest height of the ball above the ground.

b How many full oscillations does the ball perform during the first 3 seconds?

c Find the second time at which the ball is 1.5 m above ground.

23 In triangle ABC, $AB = 8$ cm, $AC = 11$ cm and angle $BAC = \theta°$. The area of the triangle is 35 cm^2. Find the possible values of θ.

24 In triangle ABC, $AB = 7.5$ cm, $CM = 5.3$ cm and angle $BAC = 44°$. Find the two possible values of the length AC.

25 Triangle KLM has $KL = 12$ cm, $KM = 15$ cm and angle $MLK = 55°$. Show that there is only one possible value for the length of the side LM and find this value.

26 Express $3\sin^2 x + 7\cos^2 x$ in terms of $\sin x$ only.

27 Express $4\cos^2 x - 5\sin^2 x$ in terms of $\cos x$ only.

28 Prove the identity $(\sin x + \cos x)^2 + (\sin x - \cos x)^2 \equiv 2$.

29 Prove that $1 + \tan^2\theta \equiv \frac{1}{\cos^2\theta}$.

30 Prove that $\frac{1}{\sin^2 A} - \frac{1}{\tan^2 A} \equiv 1$.

31 Prove the identity $\frac{1}{\cos\theta} - \cos\theta \equiv \sin\theta\tan\theta$.

32 It is given that $2\cos^2 x + \sin x = k$, where $1 \leqslant k \leqslant 2$. Find, in terms of k, the possible values of $\sin x$.

4C Matrices as transformations

Tip

Notice that the coordinates of a point are written as a position vector and go after the matrix.

You learnt more about position vectors in Chapter 2.

You already know how to draw the image of an object after a transformation such as rotation, reflection or enlargement. It can be more difficult to calculate its exact coordinates. For example, what are the coordinates of the image when the point (4, 1) is reflected in the line $y = 3x$? In this section you will learn how to use matrices to do this.

If a transformation is represented by a matrix $\mathbf{M}$, then the image of the point with coordinates (x, y) is given by $\mathbf{M}\begin{pmatrix} x \\ y \end{pmatrix}$. For example, if a transformation has matrix $\mathbf{M} = \begin{pmatrix} 3 & 1 \\ 1 & 2 \end{pmatrix}$, then the point with coordinates (x, y) is mapped to the point with coordinates $(3x + y, x + 2y)$, because

$$\begin{pmatrix} 3 & 1 \\ 1 & 2 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 3x + y \\ x + 2y \end{pmatrix}$$

WORKED EXAMPLE 4.10

A transformation is represented by the matrix $\mathbf{M} = \begin{pmatrix} 2 & -3 \\ 1 & 1 \end{pmatrix}$.

a Find the coordinates of the image of the point $(3, -2)$.

b Find the coordinates of the point whose image is $(1, 3)$.

The image of the point (x, y) is given by $\mathbf{M}\begin{pmatrix} x \\ y \end{pmatrix}$

a $\begin{pmatrix} 2 & -3 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} 3 \\ -2 \end{pmatrix} = \begin{pmatrix} 12 \\ 1 \end{pmatrix}$

The coordinates of the image point are (12, 1).

You can use the inverse matrix to find the point with the given image

b $\mathbf{M}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 1 \\ 3 \end{pmatrix} \Rightarrow \begin{pmatrix} x \\ y \end{pmatrix} = \mathbf{M}^{-1}\begin{pmatrix} 1 \\ 3 \end{pmatrix}$

You can find the inverse matrix using technology

$\mathbf{M}^{-1} = \begin{pmatrix} 0.4 & 0.6 \\ -0.2 & 0.2 \end{pmatrix}$

So,

$$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0.2 & 0.6 \\ -0.2 & 0.4 \end{pmatrix}\begin{pmatrix} 1 \\ 3 \end{pmatrix} = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$$

The coordinates of the original point are (2, 1).

How can we find matrices to represent common transformations, such as rotations or reflections? Conversely, if you are given a matrix, can you describe geometrically the transformation it represents?

A key observation is that the images of the points (1, 0) and (1, 0) correspond to the columns of the matrix. For example, for the transformation with the matrix $\mathbf{M} = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}$,

$$\mathbf{M}\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ 3 \end{pmatrix} \text{ and } \mathbf{M}\begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 2 \\ 4 \end{pmatrix}$$

Tip

The origin (0, 0) is always mapped to itself under a matrix transformation.

KEY POINT 4.10

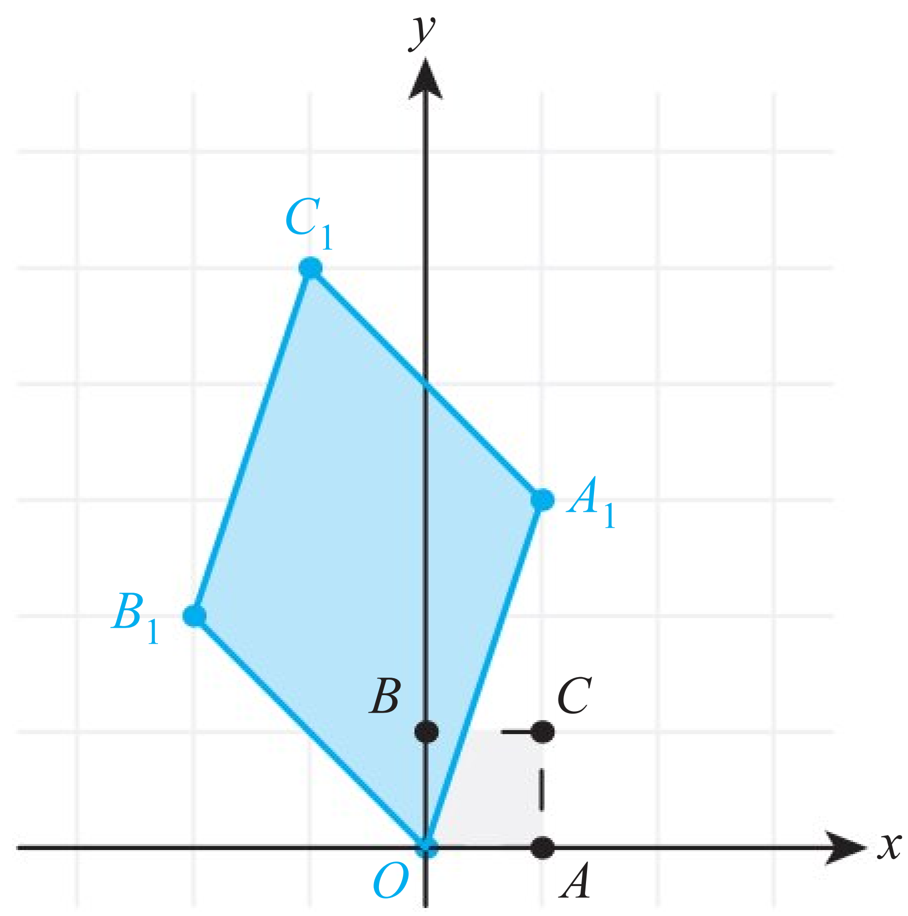
For a transformation represented by a matrix **M**, the image of the point (1, 0) is the first column of **M** and the image of the point (0, 1) is the second column of **M**.

To visualize the effect of the transformation it is often useful to look at the image of the **unit square**, the square with vertices (0, 0), (1, 0), (1, 1) and (0, 1).

WORKED EXAMPLE 4.11

a Draw the unit square and its image under the transformation with matrix $\mathbf{M} = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}$.

b The diagram shows the unit square and its image under a transformation. Find the matrix representing the transformation.

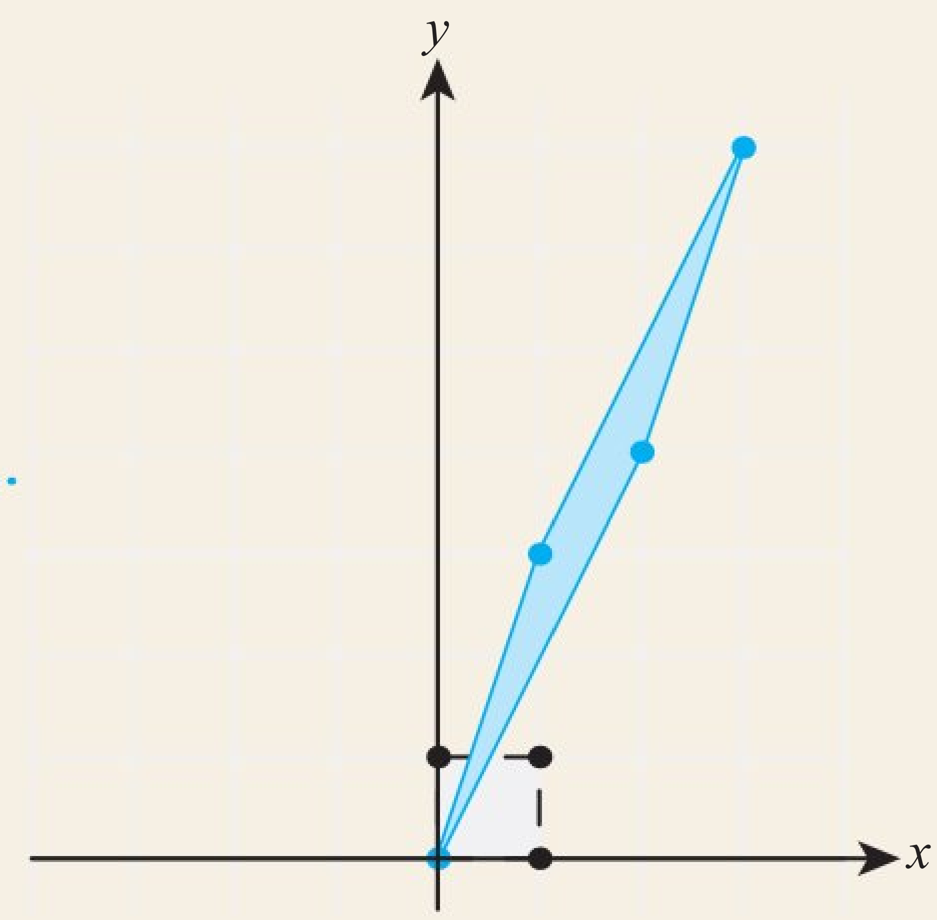


Find the coordinates of the images of the vertices (0, 0), (1, 0), (1, 1) and (0, 1)
You can use the fact that the images of (1, 0) and (0, 1) are the columns of the matrix

a $\begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}\begin{pmatrix} 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$, $\begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ 3 \end{pmatrix}$,

$\begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}\begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 3 \\ 7 \end{pmatrix}$, $\begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}\begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 2 \\ 4 \end{pmatrix}$

The unit square is drawn with dashed lines and its image with solid lines



The first column is the image of the point $A(1, 0)$, which is $A_1(1, 3)$

The second column is the image of the point $B(0, 1)$, which is $B_1(-2, 2)$

b $\mathbf{M} = \begin{pmatrix} 1 & -2 \\ 3 & 2 \end{pmatrix}$

Some common transformations

You can use the result of Key Point 4.10 to find matrices representing common transformations.

KEY POINT 4.11

Matrices representing common transformations:

reflection in the line $y = (\tan\theta)x$	$\begin{pmatrix} \cos 2\theta & \sin 2\theta \\ \sin 2\theta & -\cos 2\theta \end{pmatrix}$
horizontal stretch with scale factor k	$\begin{pmatrix} k & 0 \\ 0 & 1 \end{pmatrix}$
vertical stretch with scale factor k	$\begin{pmatrix} 1 & 0 \\ 0 & k \end{pmatrix}$
enlargement with scale factor k	$\begin{pmatrix} k & 0 \\ 0 & k \end{pmatrix}$
anticlockwise rotation of angle θ about the origin $(\theta > 0)$	$\begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$
clockwise rotation of angle θ about the origin $(\theta > 0)$	$\begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}$

There are several things you should note about using these matrices:

- The reflection line $y = (\tan\theta)x$ passes through the origin and makes angle θ with the positive x-axis. You will often need to use special cases such as the lines $y = x$ (where $\theta = \frac{\pi}{4}$) and $y = -x$ (where $\theta = \frac{3\pi}{4}$).
- Make sure that you distinguish between a stretch (which changes only one of the x- or y- coordinates) and an enlargement (which changes both).
- In the matrices for the rotations, the angle θ needs to be positive.

The transformation matrices in Key Point 4.11 and an explanation of their respective actions are all given in the Mathematics: applications and interpretation formula booklet.

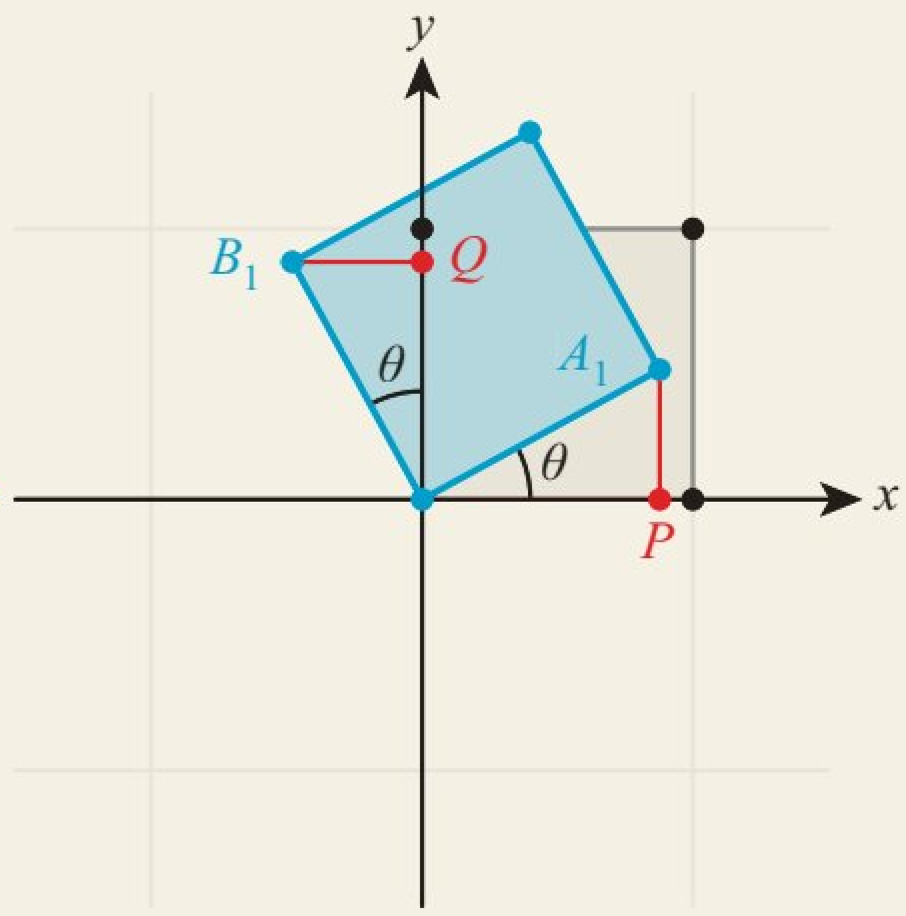
TOOLKIT: Proof

See if you can use similar methods to prove the other transformation matrices from Key Point 4.11.

Proof 4.1

Prove that the matrix representing an anticlockwise rotation of angle θ is $\begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$.

Draw a diagram showing the images of the points (1, 0) and (0, 1)



Triangle OPA_1 is right-angled with hypotenuse $OA_1 = 1$ (as this is the same as the length of OA). You can use trigonometry to find the lengths OP and PA_1

$$OP = \cos\theta,\ PA_1 = \sin\theta$$

So,

$$\begin{pmatrix} 1 \\ 0 \end{pmatrix} \mapsto \begin{pmatrix} \cos\theta \\ \sin\theta \end{pmatrix}$$

Triangle OQB_1 is also right-angled with hypotenuse $OB_1 = 1$, so you can find the lengths OQ and OB_1

You can see from the diagram that point B_1 has a negative x-coordinate and a positive y-coordinate

$$OQ = \sin\theta,\ QB_1 = \cos\theta$$

So,

$$\begin{pmatrix} 0 \\ 1 \end{pmatrix} \mapsto \begin{pmatrix} -\sin\theta \\ \cos\theta \end{pmatrix}$$

The images of (1, 0) and (0, 1) give the coordinates of the transformation matrix

Hence, the matrix is

$$\begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$$

WORKED EXAMPLE 4.12

Write down the matrix representing each transformation, and hence find the image of the point (–2, 3).

a Vertical stretch with scale factor 4.8.

b Clockwise rotation of angle 60° about the origin.

c Reflection in the line $y = 2x$.

You can use the table in Key Point 4.11, but you can also remember that a vertical translation only affects y-coordinates, so the point (0, 1) is mapped to (0, 4.8)

a $\begin{pmatrix} 1 & 0 \\ 0 & 4.8 \end{pmatrix}$

$\cos 60° = \frac{1}{2}$ and $\sin 60° = \frac{\sqrt{3}}{2}$. Note that the rotation is clockwise

b $\begin{pmatrix} \frac{1}{2} & \frac{\sqrt{3}}{2} \\ -\frac{\sqrt{3}}{2} & \frac{1}{2} \end{pmatrix}$

The gradient of the line is $\tan\theta$. Find θ (given here in radians).

c $\tan\theta = 2 \Rightarrow \theta = 1.107...$

Then find $\cos 2\theta$ and $\sin 2\theta$:

$\mathbf{M} = \begin{pmatrix} \cos 2\theta & \sin 2\theta \\ \sin 2\theta & -\cos 2\theta \end{pmatrix}$

$\mathbf{M} = \begin{pmatrix} -\frac{3}{5} & \frac{4}{5} \\ \frac{4}{5} & \frac{3}{5} \end{pmatrix}$

The table in Key Point 4.11 contains all the familiar transformations except for translations. A translation cannot be achieved by matrix multiplication, because it moves the origin. Instead, a translation is determined by its translation vector. To translate a point, you simply add the translation vector to the point's position vector.

WORKED EXAMPLE 4.13

Transformation **T** is a reflection in the x-axis and transformation **S** is a translation with vector $\begin{pmatrix} -1 \\ 3 \end{pmatrix}$. Find the image of the point (2, –3) after the following sequence of transformations:

a **T** followed by **S**

b **S** followed by **T**.

The reflection in the x-axis transforms (1, 0) to (1, 0) and (0, 1) to (0, –1), so the matrix for **T** is

$\mathbf{T} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$

Apply the reflection matrix to the position vector of (2, –3) and then add the translation vector

a $\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} 2 \\ -3 \end{pmatrix} + \begin{pmatrix} -1 \\ 3 \end{pmatrix} = \begin{pmatrix} 2 \\ 3 \end{pmatrix} + \begin{pmatrix} -1 \\ 3 \end{pmatrix}$

$= \begin{pmatrix} 1 \\ 6 \end{pmatrix}$

The image of (2, –3) is (1, 6).

This time, add the translation vector first, then apply the reflection matrix

b $\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\left(\begin{pmatrix} 2 \\ -3 \end{pmatrix} + \begin{pmatrix} -1 \\ 3 \end{pmatrix}\right)$

$= \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$

The image of $(2, -3)$ is $(1, -1)$.

Composition of transformations

As you can see from Worked Example 4.13, the result of a composition of two transformations depends on the order in which they are applied. For two transformations represented by matrices, the combined transformation is represented by the product of the two matrices.

KEY POINT 4.12

If the transformation with matrix **A** is followed by the transformation with matrix **B**, the combined transformation has matrix **BA**.

Notice the order of the matrices in the product: if **A** is applied to a vector **x** first, the image is the vector **Ax**. When transformation **B** is applied to this image, the final image is **B**(**Ax**) = (**BA**)**x**.

WORKED EXAMPLE 4.14

Find the matrix representing a 45° anticlockwise rotation about the origin followed by a reflection in the line $y = x$.

Find the matrix for the first transformation. It is an anticlockwise rotation so use $\begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$

$\mathbf{A} = \begin{pmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{pmatrix}$

The second transformation is a reflection in the line $y = x$, so use $\begin{pmatrix} \cos 2\theta & \sin 2\theta \\ \sin 2\theta & -\cos 2\theta \end{pmatrix}$ where $\theta = 45°$

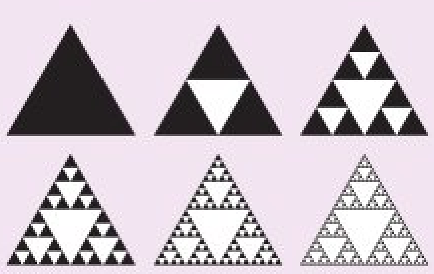
$\mathbf{B} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$

The combined transformation (**A** followed by **B**) has matrix **BA**

$\mathbf{BA} = \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \end{pmatrix}$

You are the Researcher

Two examples of fractals are given in questions 49 and 50 of Exercise 4C. Investigate other examples of fractals, such as the Koch Snowflake and the Sierpinski Triangle (shown).



An important example of a composition of transformations is repeatedly combining a transformation with itself. This is written as $\mathbf{A}^2$, $\mathbf{A}^3$, etc. Applying a transformation repeatedly can generate interesting shapes known as **fractals**.

Be the Examiner 4.2

Transformations **A** and **B** have matrices $\mathbf{A} = \begin{pmatrix} 3 & -1 \\ 1 & 2 \end{pmatrix}$ and $\mathbf{B} = \begin{pmatrix} 1 & 2 \\ -2 & 5 \end{pmatrix}$. Transformation **C** is **B** followed by **A**. Find the matrix of the transformation **C**.

Which is the correct solution? Identify the errors made in the incorrect solutions.

Solution 1	Solution 2	Solution 3
$\mathbf{C} = \begin{pmatrix} 3 & -1 \\ 1 & 2 \end{pmatrix} + \begin{pmatrix} 1 & 2 \\ -2 & 5 \end{pmatrix}$ $= \begin{pmatrix} 4 & 1 \\ -1 & 7 \end{pmatrix}$	$\mathbf{C} = \begin{pmatrix} 3 & -1 \\ 1 & 2 \end{pmatrix}\begin{pmatrix} 1 & 2 \\ -2 & 5 \end{pmatrix}$ $= \begin{pmatrix} 5 & 1 \\ -3 & 12 \end{pmatrix}$	$\mathbf{C} = \begin{pmatrix} 1 & 2 \\ -2 & 5 \end{pmatrix}\begin{pmatrix} 3 & -1 \\ 1 & 2 \end{pmatrix}$ $= \begin{pmatrix} 5 & 3 \\ -1 & 12 \end{pmatrix}$

The determinant of a transformation matrix

When a shape is transformed using a matrix transformation, there is a simple way to find the area of the image.

You learnt how to find the determinant of a 2×2 matrix in Section 3B.

KEY POINT 4.13

For a transformation represented by a 2×2 matrix **A**,

$$\text{area of image} = |\det \mathbf{A}| \times \text{area of object}$$

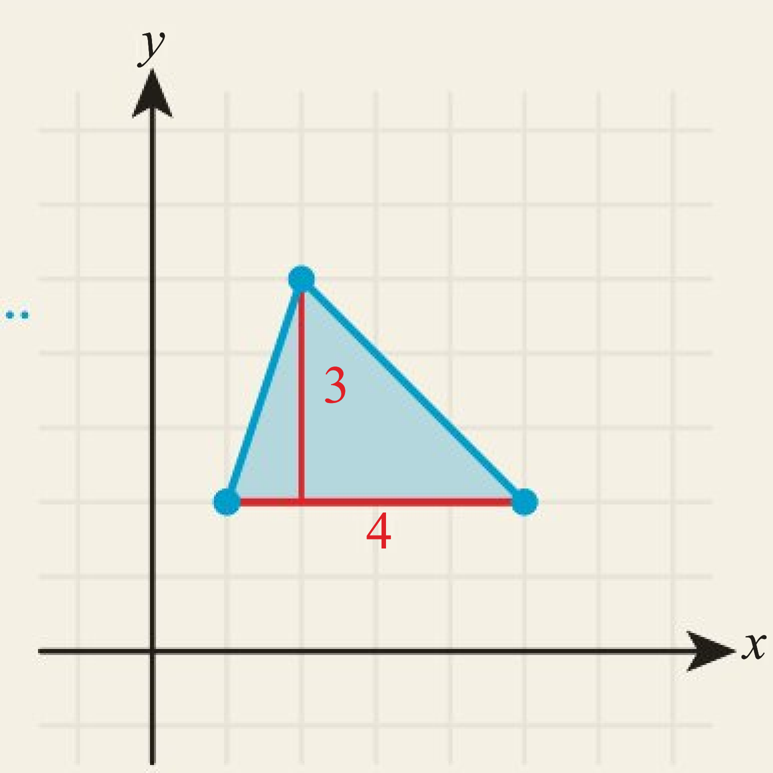
TOOLKIT: Proof

Find the image of the unit square under the transformation $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$. By splitting it up into suitable triangles, find the area of the resulting shape. What does this tell you about the area transformation of any shape?

WORKED EXAMPLE 4.15

The triangle with vertices (1, 2), (5, 2) and (2, 5) is transformed using the matrix $\mathbf{A} = \begin{pmatrix} 2 & 5 \\ 1 & -1 \end{pmatrix}$. Find the area of the image triangle.

The original triangle has a horizontal base, so it is easy to find its area



$$\text{Area} = \frac{1}{2}(4)(3) = 6$$

Find the determinant of the matrix

$$\det \mathbf{A} = -2 - 5 = -7$$

So,

Remember to take the modulus of the determinant

$$\text{image area} = 7 \times 6 = 42$$

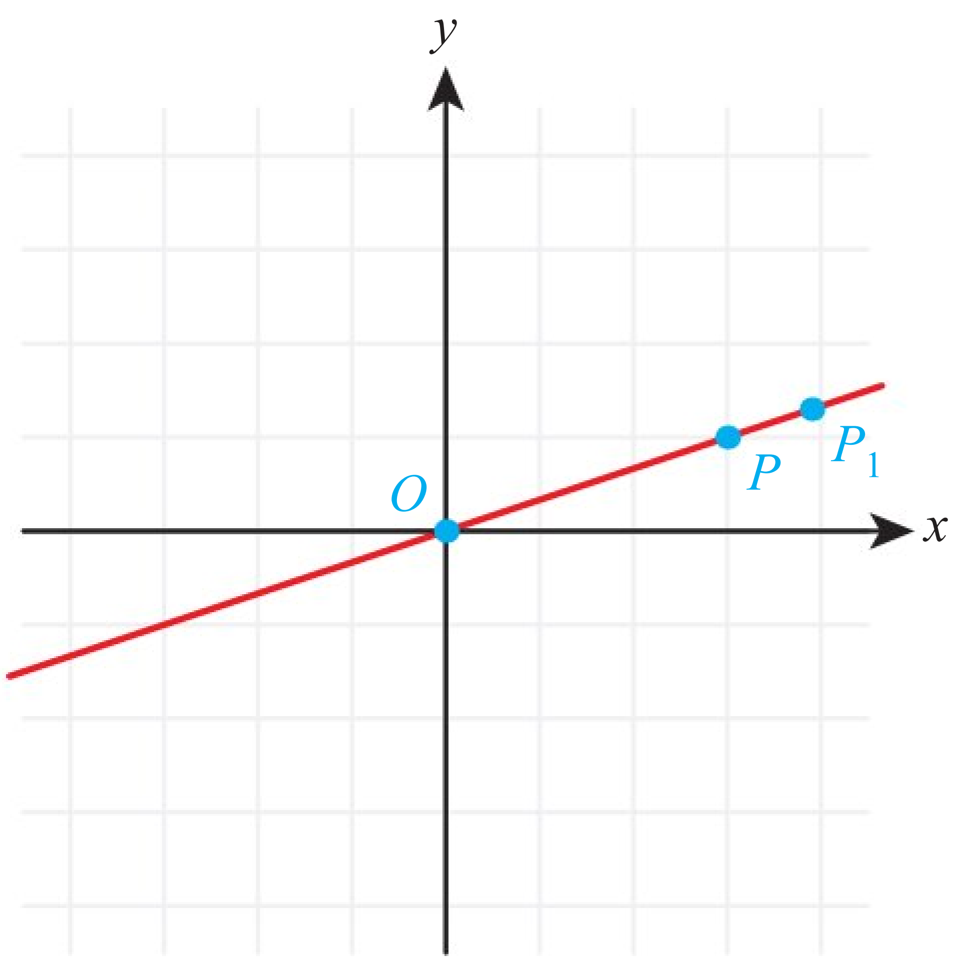
Looking at the table in Key Point 4.11, you can see that rotation matrices always have determinant 1 and reflection matrices always have determinant –1. This corresponds to the fact that they do not change the size of the object.

Tip

Transformations with a negative determinant reverse the orientation of the object.

Eigenvalues and eigenvectors

You know from Section 3D that, for a matrix **A**, an eigenvector **v** with eigenvalue λ satisfies $\mathbf{Av} = \lambda\mathbf{v}$. Geometrically, this means that, if point P has position vector $\overrightarrow{OP} = \mathbf{v}$, then its image P_1 (under the transformation with matrix **A**) has position vector $\overrightarrow{OP_1} = \lambda\mathbf{v} = \lambda\overrightarrow{OP}$. Hence any point on the line through the origin in the direction of vector **v** is mapped to another point on the same line (with the distance from the origin increasing by a factor λ).



We say that the line OP is **invariant** under the transformation. If the eigenvalue $\lambda = 1$, each point on the line is actually mapped to itself.

WORKED EXAMPLE 4.16

Using eigenvectors,

a find the invariant lines for the horizontal stretch with scale factor 3

b show that a rotation through 90° anticlockwise about the origin has no invariant lines.

a

$$\mathbf{A} = \begin{pmatrix} 3 & 0 \\ 0 & 1 \end{pmatrix}$$

$$\det(\mathbf{A} - \lambda\mathbf{I}) = 0$$

First find the eigenvalues of the transformation matrix

$$\det\begin{pmatrix} 3-\lambda & 0 \\ 0 & 1-\lambda \end{pmatrix} = 0$$

$$(3-\lambda)(1-\lambda) = 0$$

$$\lambda = 1, 3$$

When $\lambda = 1$,

Now find the eigenvector associated with each eigenvalue

$$\begin{pmatrix} 3 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x \\ y \end{pmatrix}$$

$$\begin{cases} 3x = x \Rightarrow x = 0 \\ y = y \end{cases}$$

You can choose any value for y; choose $y = 1$

The eigenvector is $\begin{pmatrix} 0 \\ 1 \end{pmatrix}$

When $\lambda = 3$,

$$\begin{pmatrix} 3 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = 3\begin{pmatrix} x \\ y \end{pmatrix}$$

$$\begin{cases} 3x = 3x \\ y = 3y \Rightarrow y = 0 \end{cases}$$

The eigenvector is $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$

The invariant lines are the lines through the origin in the direction of each eigenvector

The invariant lines are $y = 0$ and $x = 0$.

b

$$\mathbf{A} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$$

$$\det(\mathbf{A} - \lambda\mathbf{I}) = 0$$

Again, start by trying to find the eigenvalues of the transformation matrix

$$\det\begin{pmatrix} -\lambda & -1 \\ 1 & -\lambda \end{pmatrix} = 0$$

$$\lambda^2 + 1 = 0$$

The characteristic equation has no real roots

No real roots, so the transformation matrix has no real eigenvalues and hence no invariant lines.

You are the Researcher

In Worked Example 4.16, you found that the horizontal stretch with scale factor 3 has two eigenvectors: one in the direction of the x-axis with eigenvalue 3, and one in the direction of the y-axis with eigenvalue 1. Geometrically, this corresponds to the fact that the points on the x-axis are moved three times further away from the origin, while the points on the y-axis stay fixed. Use eigenvalues and eigenvectors to investigate invariant lines of other common transformations and their combinations. How might you develop this idea into a mathematical exploration that shows sophistication and rigour?

Exercise 4C

For questions 1 to 3, use the method demonstrated in Worked Example 4.10a to find the coordinates of the image of the given point under the given transformation.

1 **a** $\begin{pmatrix} 2 & 1 \\ 1 & 3 \end{pmatrix}$, point (2, 3)

b $\begin{pmatrix} 1 & 4 \\ 2 & 1 \end{pmatrix}$, point (4, 2)

2 **a** $\begin{pmatrix} 3 & -1 \\ 1 & -2 \end{pmatrix}$, point (2, 1)

b $\begin{pmatrix} -3 & 2 \\ -1 & 1 \end{pmatrix}$, point (1, 3)

3 **a** $\begin{pmatrix} 4 & 1 \\ -1 & -5 \end{pmatrix}$, point (2, −3)

b $\begin{pmatrix} 2 & 1 \\ -1 & -3 \end{pmatrix}$, point (−2, 5)

For questions 4 to 6, use the method demonstrated in Worked Example 4.10b to find the coordinates of the point whose image under the given transformation is given.

4 **a** $\begin{pmatrix} 2 & 1 \\ 1 & 3 \end{pmatrix}$, image point (6, 8)

b $\begin{pmatrix} 1 & 4 \\ 2 & 1 \end{pmatrix}$, image point (13, 5)

5 **a** $\begin{pmatrix} 3 & -1 \\ 1 & -2 \end{pmatrix}$, image point (−6, −7)

b $\begin{pmatrix} -3 & 2 \\ -1 & 1 \end{pmatrix}$, image point (−8, −3)

6 **a** $\begin{pmatrix} 4 & 1 \\ -1 & -5 \end{pmatrix}$, image point (−2, 10)

b $\begin{pmatrix} 2 & 1 \\ 1 & 3 \end{pmatrix}$, image point (6, 3)

For questions 7 to 10, use the method demonstrated in Worked Example 4.11a to draw the image of the unit square under the given transformation.

7 **a** $\begin{pmatrix} 1 & 2 \\ 1 & 0 \end{pmatrix}$

b $\begin{pmatrix} -1 & 1 \\ 0 & 1 \end{pmatrix}$

8 **a** $\begin{pmatrix} 3 & 0 \\ 0 & 3 \end{pmatrix}$

b $\begin{pmatrix} 0 & 2 \\ -3 & 0 \end{pmatrix}$

9 **a** $\begin{pmatrix} 2 & 1 \\ 0 & 2 \end{pmatrix}$

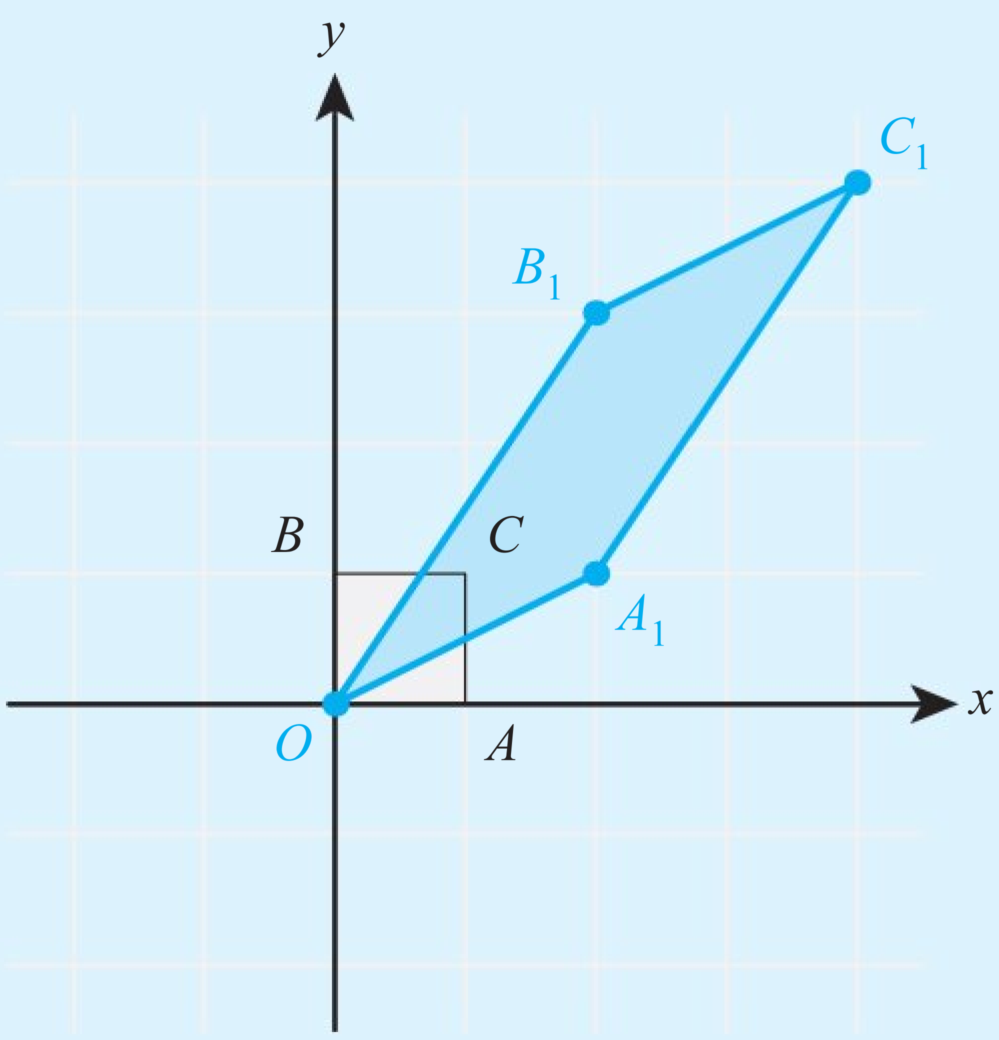
b $\begin{pmatrix} 2 & 3 \\ 1 & -1 \end{pmatrix}$

10 **a** $\begin{pmatrix} 1 & -2 \\ -4 & 1 \end{pmatrix}$

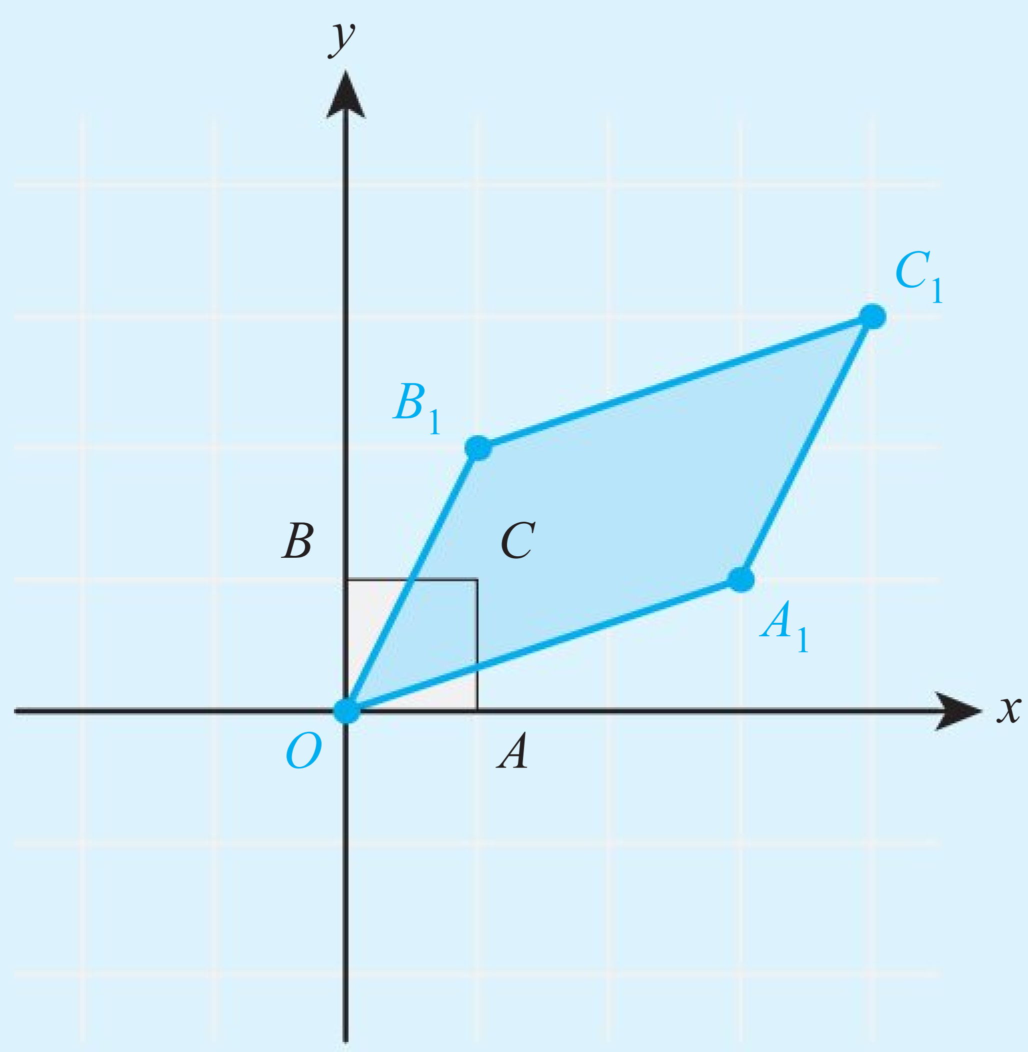
b $\begin{pmatrix} 2 & 1 \\ -1 & 1 \end{pmatrix}$

For questions 11 to 16, each diagram shows the unit square and its image under a transformation. Use the method demonstrated in Worked Example 4.11b to find the matrix representing the transformation.

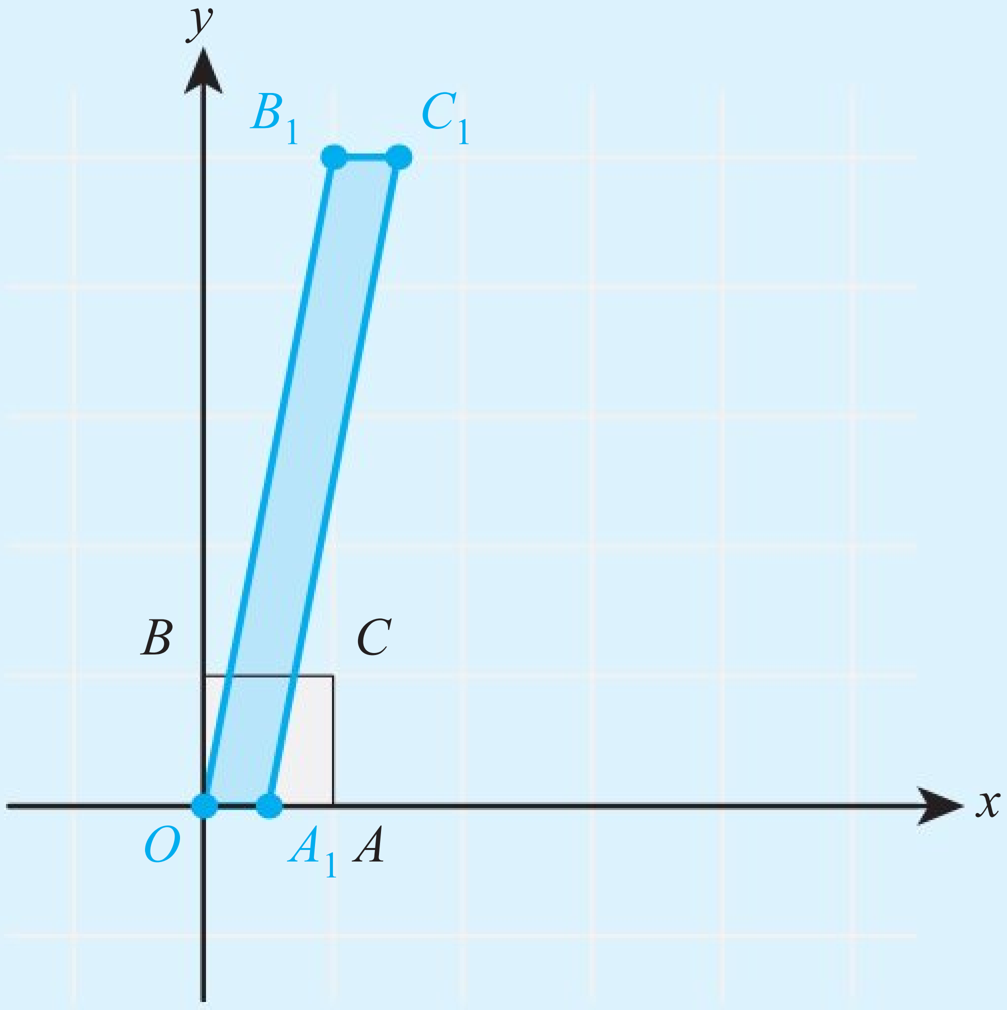
11 **a**



b



12 a



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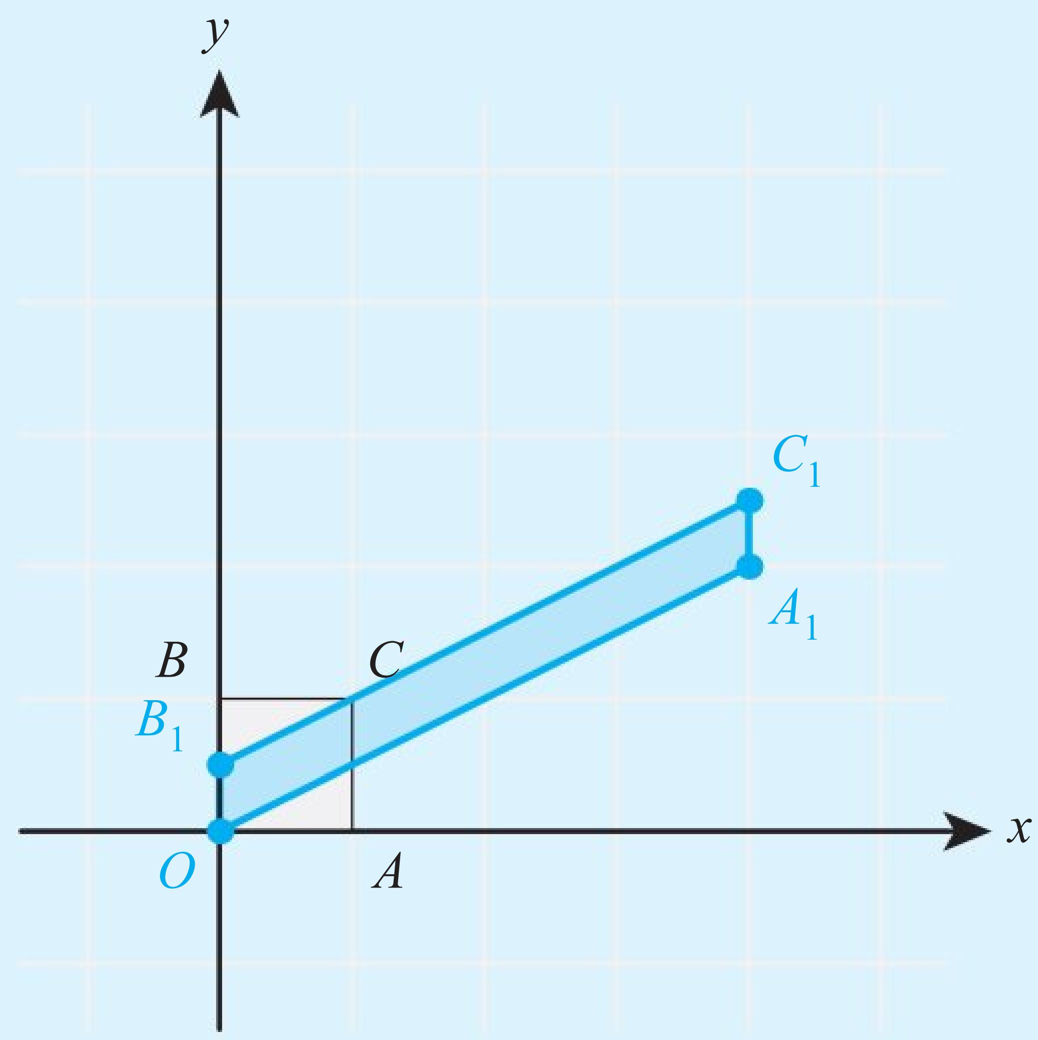
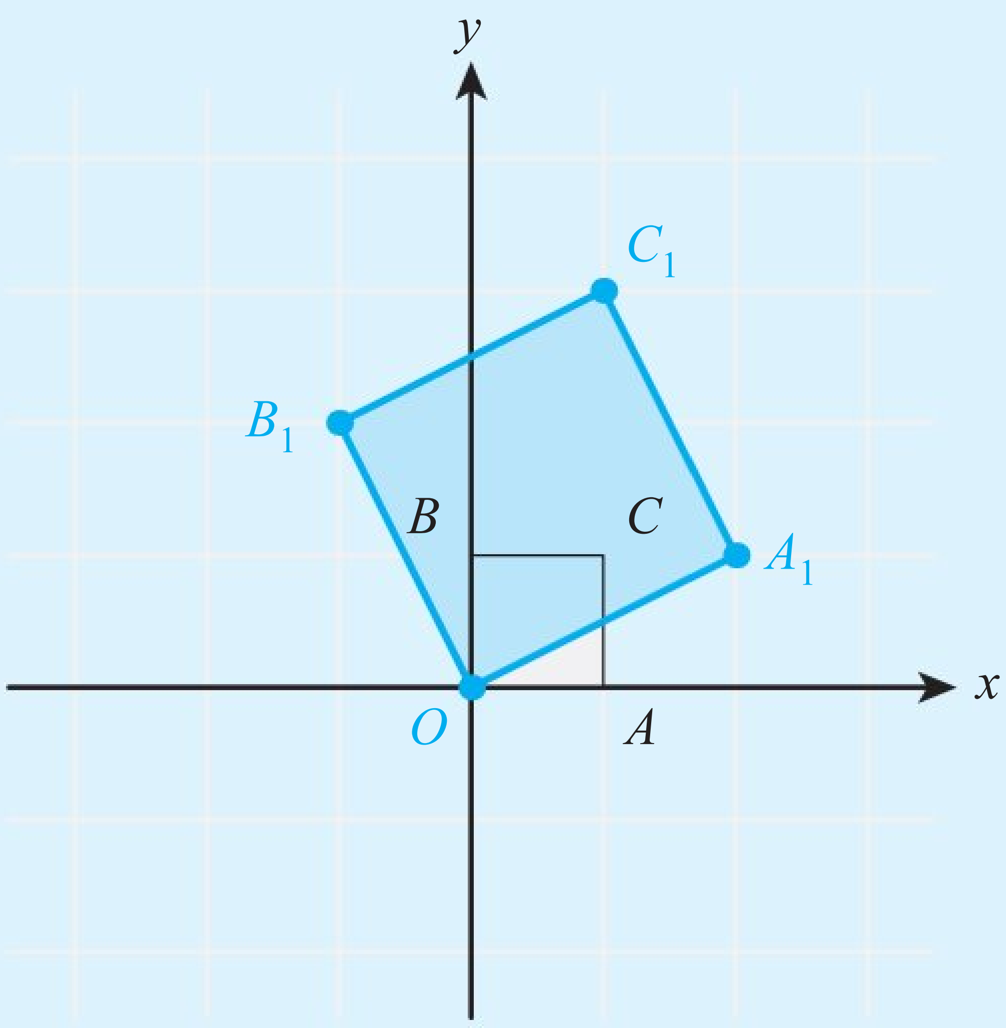
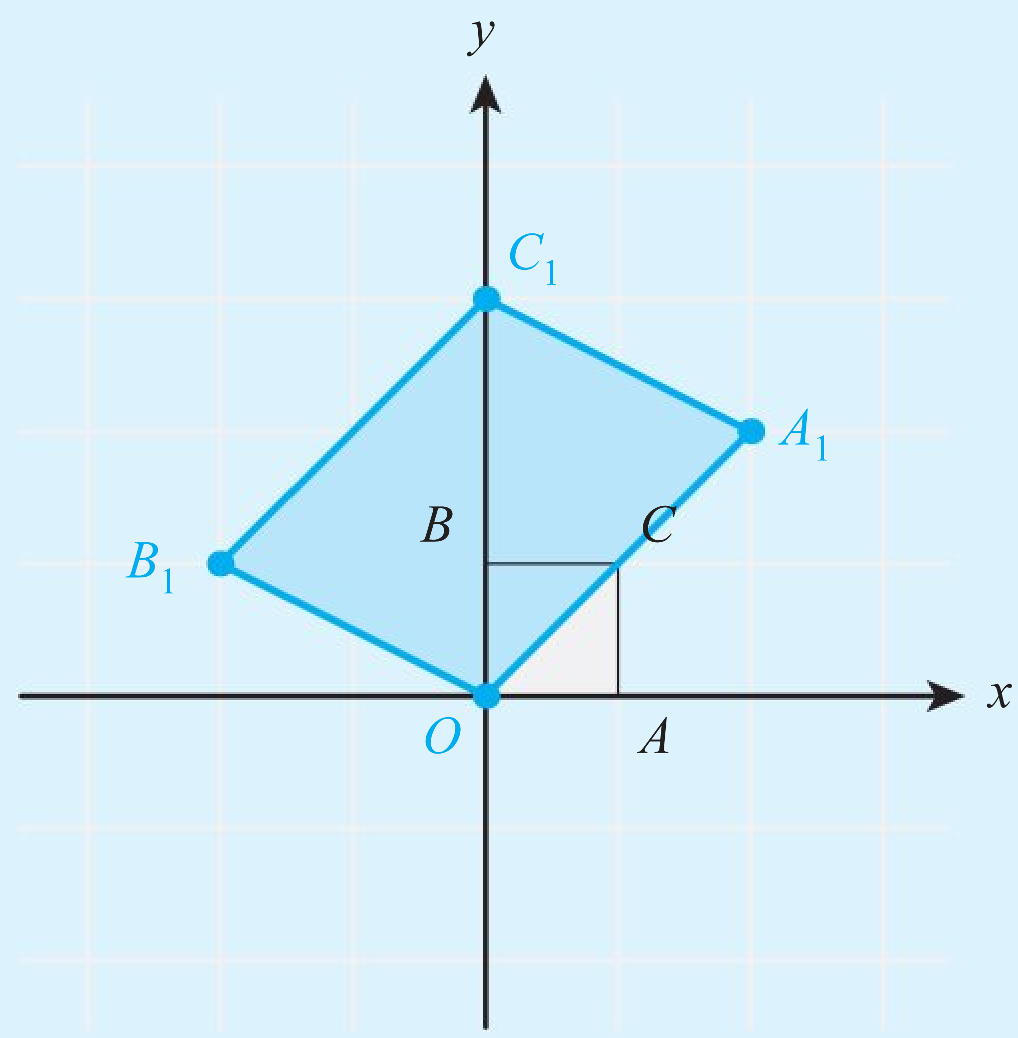


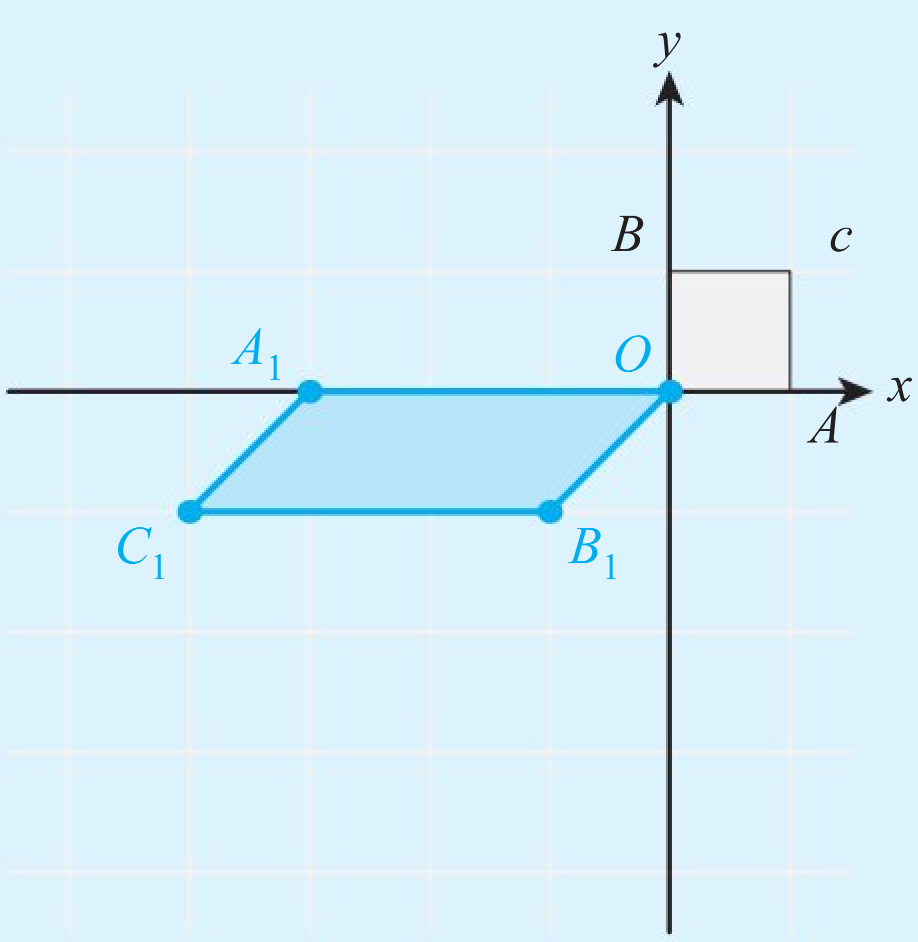
13 a



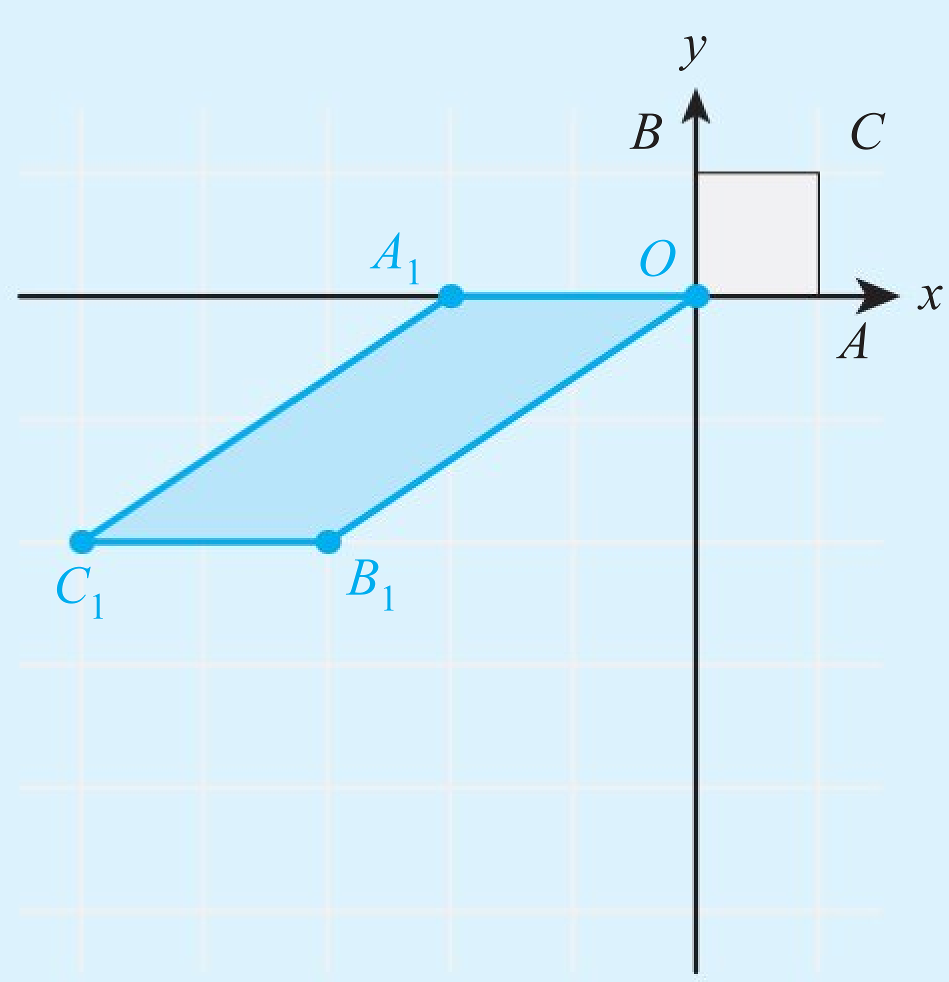
b



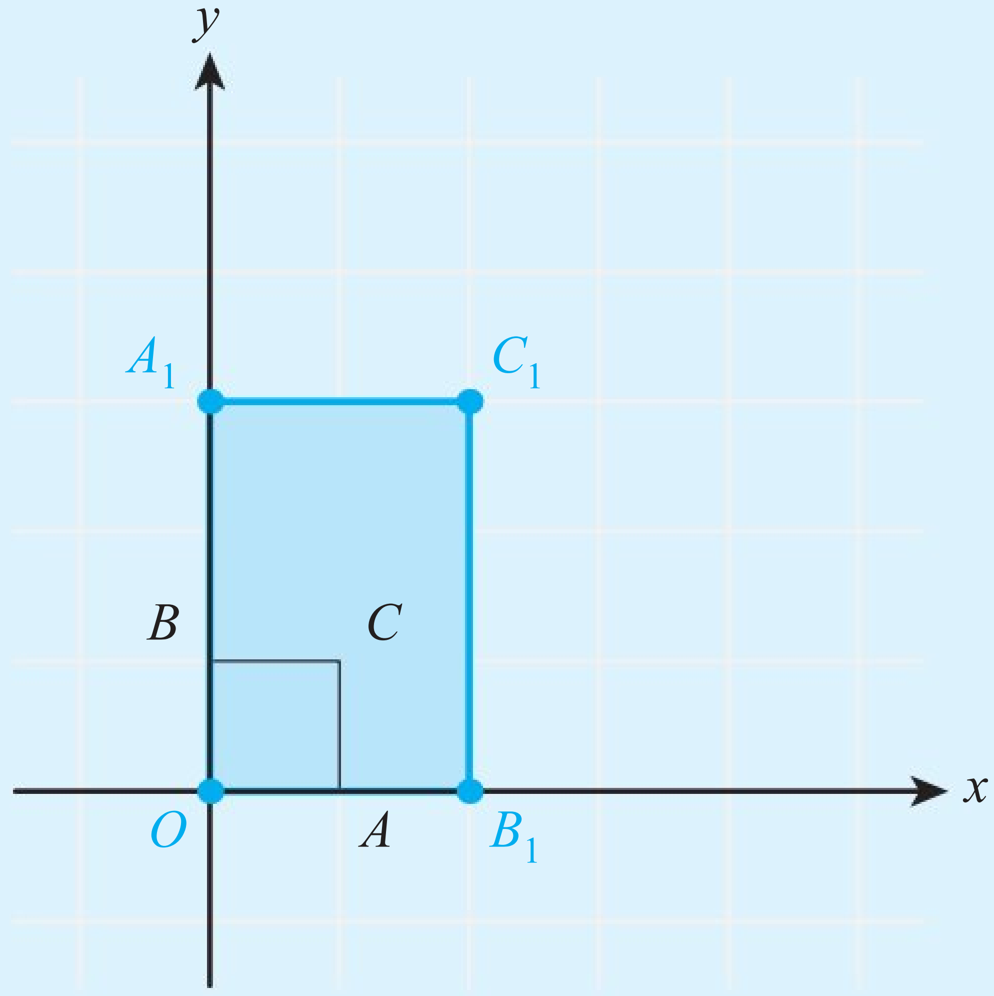
14 a



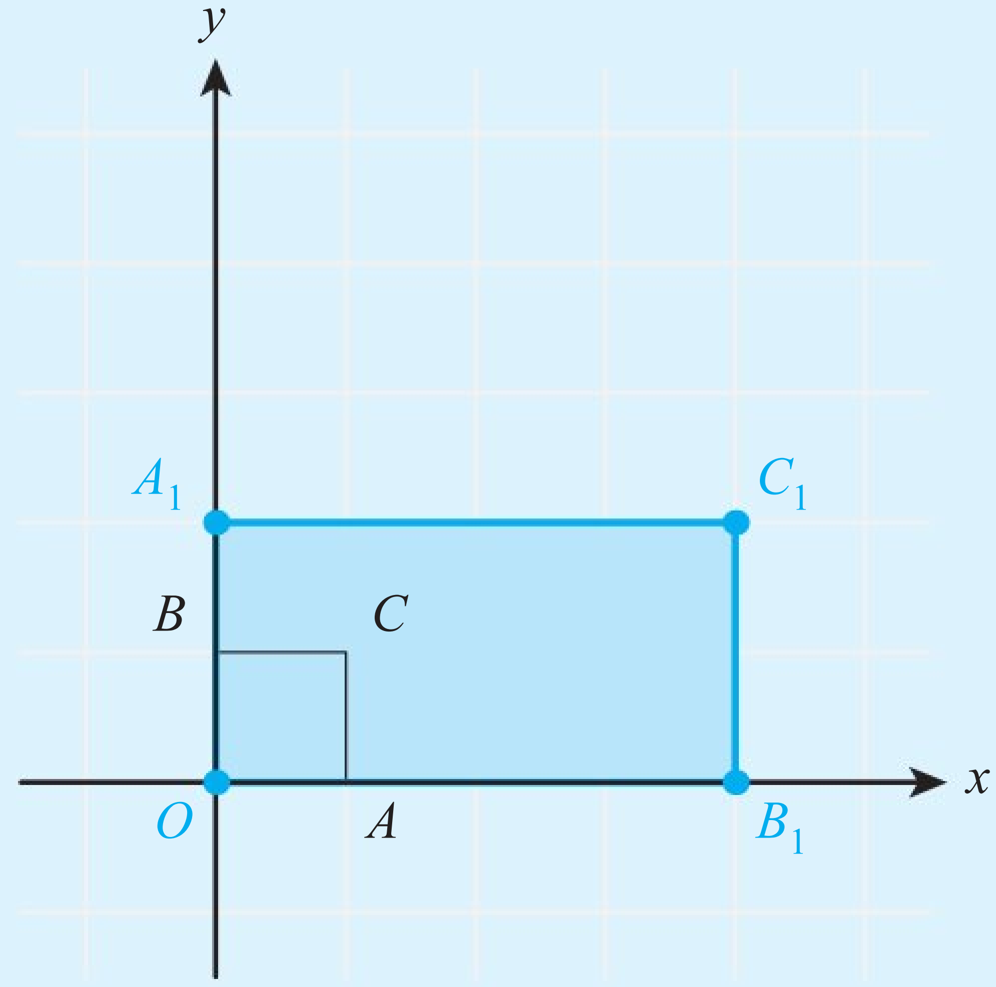
b



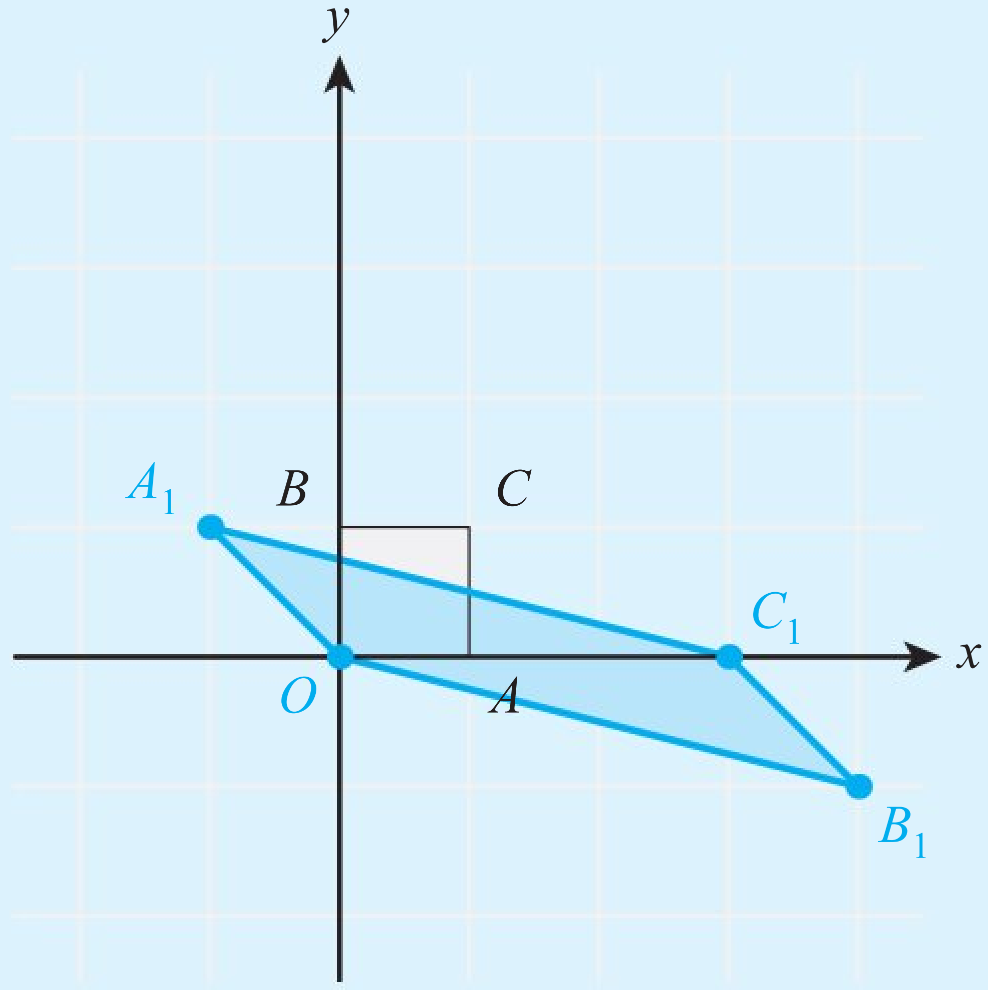
15 **a**



b



16 **a**



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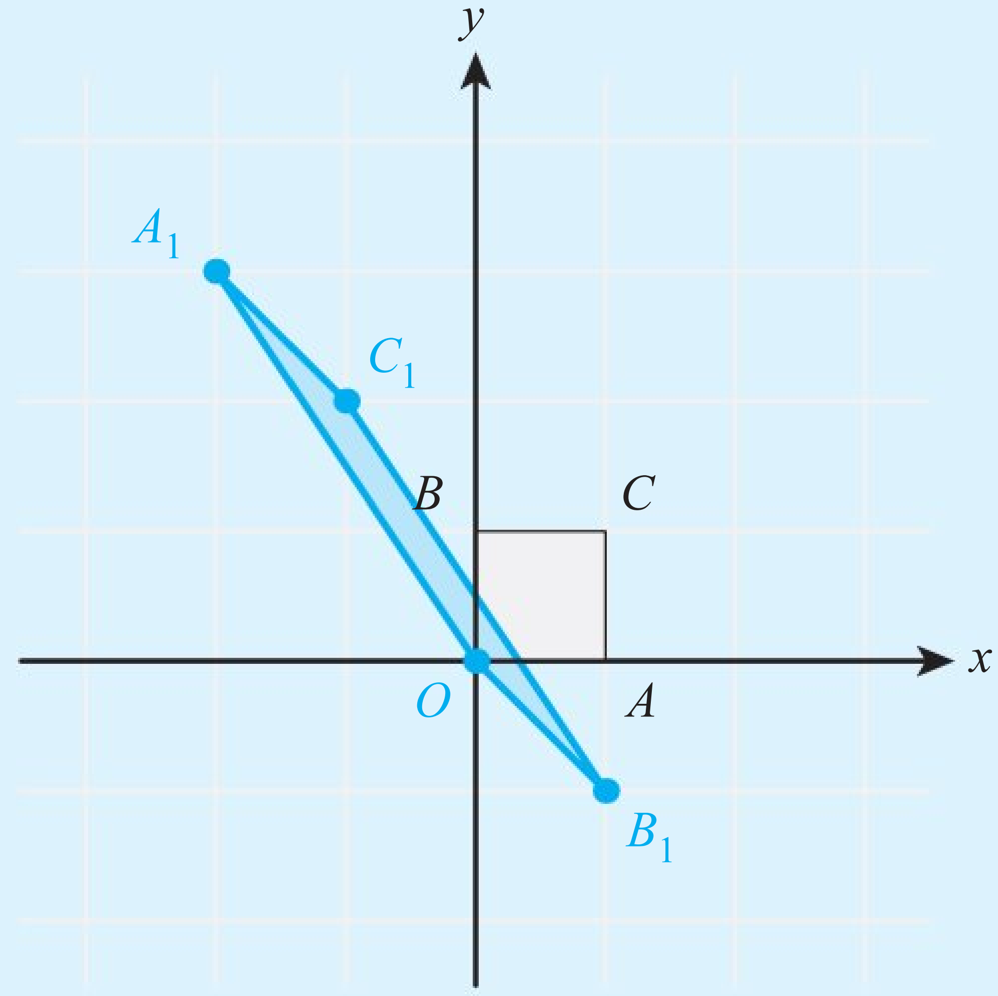


For questions 17 to 24, use the method demonstrated in Worked Example 4.12 to write down the matrix representing each transformation.

17 **a** Horizontal stretch with scale factor 3

b Vertical stretch with scale factor 2

18 **a** Vertical stretch with scale factor $\frac{1}{4}$

b Horizontal stretch with scale factor $\frac{1}{3}$

19 **a** Enlargement with scale factor 5

b Enlargement with scale factor 6

20 **a** Reflection in the line $y = x$

b Reflection in the line $y = -x$

21 **a** Reflection in the line $y = \sqrt{3}x$

b Reflection in the line $y = \frac{1}{\sqrt{3}}x$

22 **a** Rotation 30° anticlockwise about the origin

b Rotation 45° anticlockwise about the origin

23 **a** Rotation 90° clockwise about the origin

b Rotation 180° about the origin

24 **a** Rotation 120°clockwise about the origin

b Rotation 150° clockwise about the origin

For questions 25 to 27, use the method demonstrated in Worked Example 4.13 to find the image of the given point under the given sequence of transformations.

25 **a** Point $(2, -3)$; rotation 90° anticlockwise about the origin followed by translation with vector $\begin{pmatrix} -1 \\ 4 \end{pmatrix}$

b Point $(2, 4)$; reflection in the y-axis followed by translation with vector $\begin{pmatrix} 3 \\ -5 \end{pmatrix}$

26 **a** Point $(1, 1)$; translation with vector $\begin{pmatrix} 2 \\ -1 \end{pmatrix}$ followed by an enlargement with scale factor 3

b Point $(0, 6)$; translation with vector $\begin{pmatrix} -1 \\ 5 \end{pmatrix}$ followed by a horizontal stretch with scale factor 4

27 **a** Point $(1, -2)$; translation with vector $\begin{pmatrix} 2 \\ 3 \end{pmatrix}$ followed by reflection in the line $y = x$ followed by translation with vector $\begin{pmatrix} -2 \\ -3 \end{pmatrix}$

b Point $(3, -1)$; translation with vector $\begin{pmatrix} -2 \\ 5 \end{pmatrix}$ followed by rotation 90° clockwise about the origin followed by translation with vector $\begin{pmatrix} 2 \\ -5 \end{pmatrix}$

For questions 28 to 30, use the method demonstrated in Worked Example 4.14 to find the matrix representing the composite transformation.

28 **a** Transformation with matrix $\begin{pmatrix} 2 & -1 \\ 3 & 2 \end{pmatrix}$ followed by the transformation with matrix $\begin{pmatrix} 1 & 2 \\ -2 & 4 \end{pmatrix}$

b Transformation with matrix $\begin{pmatrix} 1 & 3 \\ 3 & 1 \end{pmatrix}$ followed by the transformation with matrix $\begin{pmatrix} 1 & -1 \\ 3 & 1 \end{pmatrix}$

29 **a** Rotation 90° anticlockwise about the origin followed by the transformation with matrix $\begin{pmatrix} 2 & 0 \\ -1 & 3 \end{pmatrix}$

b Transformation with matrix $\begin{pmatrix} -3 & 1 \\ 1 & 3 \end{pmatrix}$ followed by the reflection in the line $y = x$

30 **a** Reflection in the x-axis followed by the reflection in the line $y = -x$

b Rotation 180° followed by reflection in the y-axis

For questions 31 to 34, the triangle with vertices $(1, 0)$, $(4, 0)$ and $(2, 4)$ is transformed by the given transformation. Use the method demonstrated in Worked Example 4.15 to find the area of the image.

31 **a** Vertical stretch with scale factor 3

b Horizontal stretch with scale factor 0.2

32 **a** Enlargement with scale factor $\frac{1}{2}$

b Enlargement with scale factor –2

33 **a** Rotation 90° clockwise about the origin

b Reflection in the x-axis

34 **a** Transformation with matrix $\begin{pmatrix} 3 & -2 \\ -4 & 2 \end{pmatrix}$

b Transformation with matrix $\begin{pmatrix} 1 & 2 \\ -1 & 3 \end{pmatrix}$

35 A transformation is represented by the matrix $\mathbf{M} = \begin{pmatrix} 2 & 5 \\ 1 & -1 \end{pmatrix}$.

a Find the coordinates of the image of the point $(-1, 2)$ under this transformation.

b Draw the image of the unit square.

c Find the area of the image of the unit square.

36 The transformation with matrix $\mathbf{A} = \begin{pmatrix} -1 & 1 \\ 3 & -2 \end{pmatrix}$ maps the point with coordinates $(3,\ p)$ to the point with coordinates $(q,\ -1)$.

- **a** Find the values of p and q.
- **b** Find the area of the image of the triangle with vertices $(0,\ 0)$, $(6,\ 0)$ and $(2,\ 5)$.
- **c** Find the image of the point $(2,\ 3)$ under the transformation with matrix $\mathbf{A}^3$.

37 Transformation **R** is a rotation 90° clockwise around the origin. Transformation **T** is a translation with vector $\begin{pmatrix} -3 \\ 1 \end{pmatrix}$.

- **a** Transformation **R** maps point $(p,\ q)$ to the point $(3,\ -2)$. Find the values of p and q.
- **b** Point $(a,\ b)$ is mapped using transformation **R** followed by transformation **T**. Find the coordinates of the image in terms of a and b.
- **c** Point $(a,\ b)$ is now mapped using transformation **T** followed by transformation **R**. Show that the final image is different from the one in part **b**.

38
- **a** Find the image of the point (2, –3) under a rotation through 135° anticlockwise about the origin.
- **b** Find the image of the point (2, –3) under a rotation through 135° anticlockwise about the origin, followed by an enlargement with scale factor 4.
- **c** Find the area of the unit square after the sequence of transformations from part **b**.

39
- **a** Find a 2×2 matrix representing reflection in the line $y = 3x$.
- **b** Find the coordinates of the point whose image under the reflection is (2, 2).

40 Transformation **P** has matrix $\begin{pmatrix} 2 & 1 \\ 1 & 3 \end{pmatrix}$ and transformation **Q** has matrix $\begin{pmatrix} 1 & 0 \\ 2 & 1 \end{pmatrix}$.

- **a** Find the image of the point (3, 1) under the transformation **P** followed by transformation **Q**.
- **b** Point C is transformed using transformation **P** followed by transformation **Q**. The image has coordinates (–2, 2). Find the coordinates of C.

41 Let **N** be an enlargement with scale factor 2 centred at the origin, and let **R** be the rotation through 30° anticlockwise about the origin.

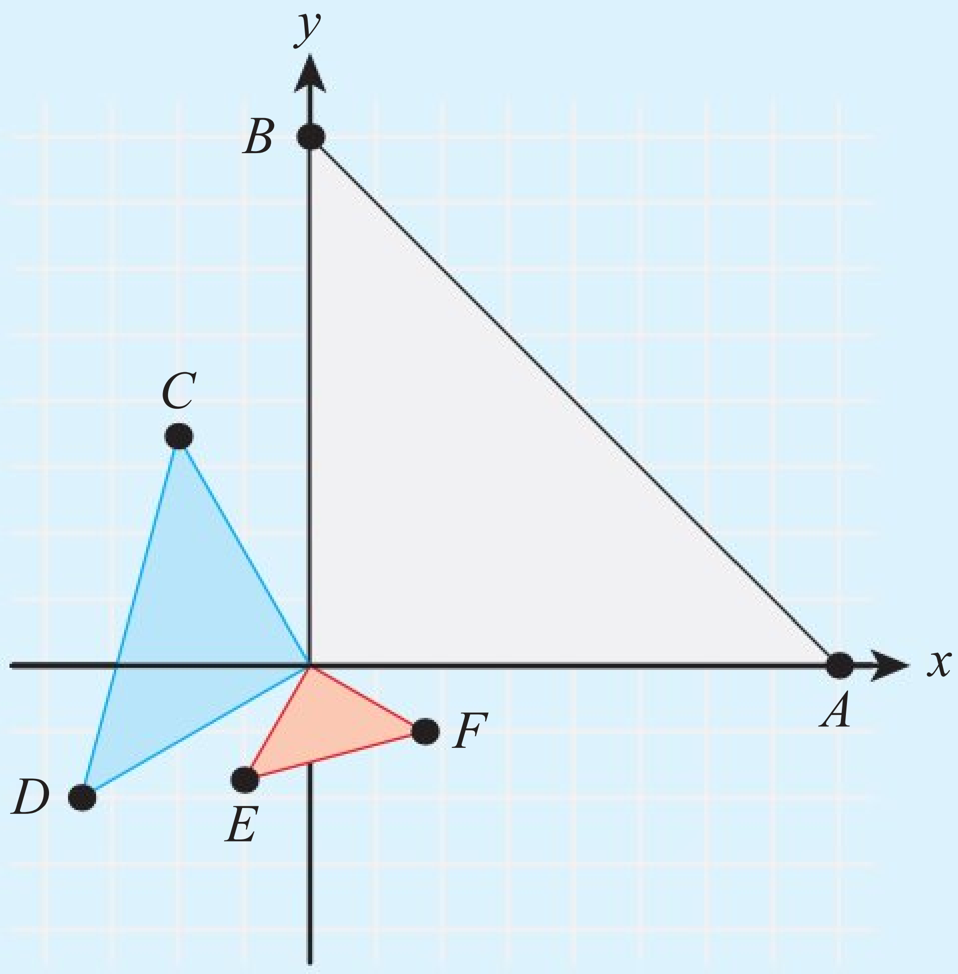
- **a** The point (2, 1) is transformed using the rotation followed by the enlargement. Find the coordinates of the image.
- **b** Find the matrix representing the transformation resulting from the enlargement followed by the rotation.
- **c** Show that **N** followed by **R** always gives the same result as **R** followed by **N**.

42 Use matrices to prove that rotation through 30° about the origin followed by rotation through 60° about the origin (in the same direction) results in rotation through 90° about the origin.

43 Let **S** be the matrix representing reflection in the x-axis and **R** be the matrix representing the rotation 90° anticlockwise about the origin.

- **a** Show that **S** followed by **R** results in a reflection and find the equation of its mirror line.
- **b** Show that **R** followed by **S** results in a different reflection and find the equation of its mirror line.

44 A logo is designed by rotating and shrinking triangle OAB, as shown in the diagram, where the coordinates of the vertices are $A(8, 0)$ and $B(0, 8)$.



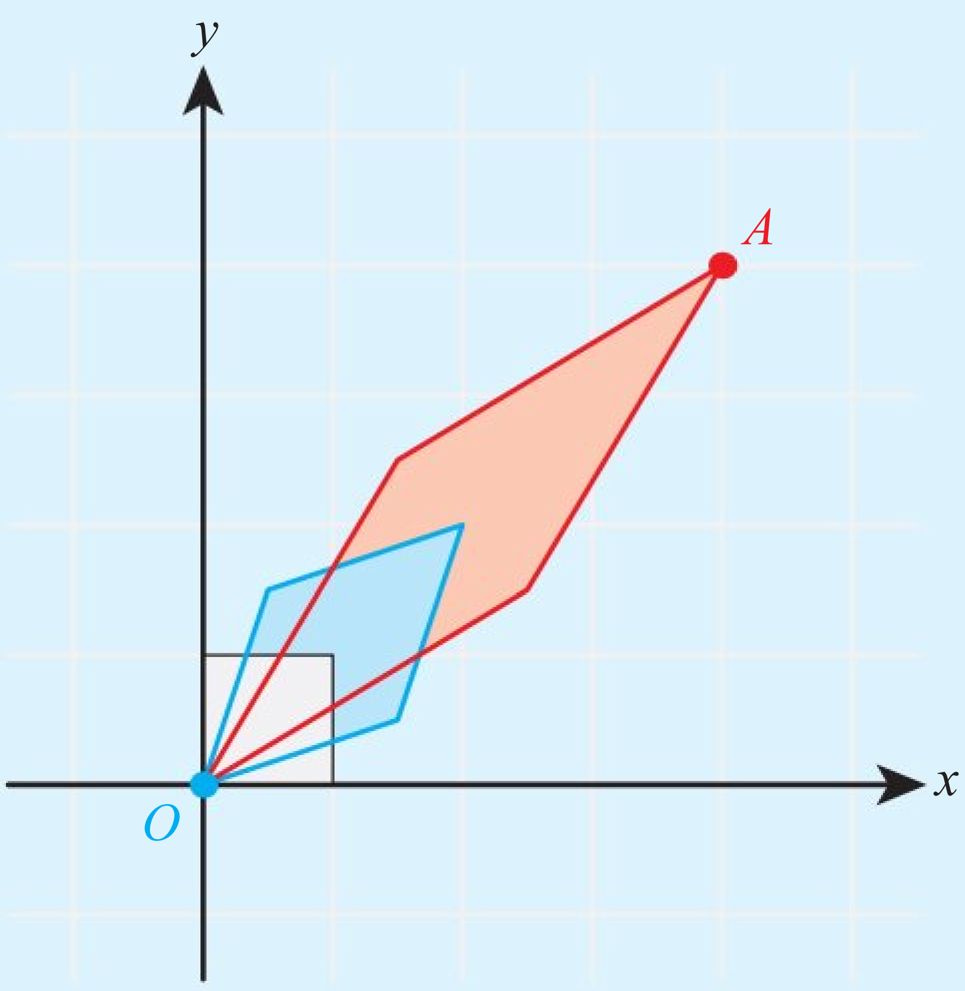
Let **R** denote rotation 120° anticlockwise around the origin, and **S** an enlargement with scale factor $\frac{1}{2}$. Let **T** be the composite transformation **R** followed by **S**.

a Find the matrices representing **T** and $\mathbf{T}^2$.

b Find the coordinates of D and F.

c Find the area of the whole logo.

45 Transformation **M** has matrix $\mathbf{M} = \begin{pmatrix} \frac{3}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{3}{2} \end{pmatrix}$. The diagram shows the unit square and its images under **M** and $\mathbf{M}^2$.



a Find the coordinates of the point **A**.

b Find the area of the parallelogram which is the image of the unit square under $\mathbf{M}^6$.

46 Let **S** be the matrix representing a reflection in the line $y = -x$.

a Write down the matrix **S**.

b Find the eigenvectors and corresponding eigenvalues of **S**.

c Interpret geometrically the meaning of the two eigenvectors.

47 **a** Find the eigenvalues and eigenvectors of the matrix $\mathbf{B}_1 = \begin{pmatrix} -4 & 1 \\ 3 & -2 \end{pmatrix}$.

b Hence determine the equations of the invariant lines of the form $y = mx$ of the transformation represented by $\mathbf{B}_1$.

c Show that the transformation represented by the matrix $\mathbf{B}_2 = \begin{pmatrix} -5 & -3 \\ 4 & 1 \end{pmatrix}$ has no invariant lines passing through the origin.

48 **R** is a reflection in the line $y = 2x$.

S is a 90° rotation anticlockwise about the origin.

T the transformation **R** followed by **S**.

a Find the matrix representing **T**.

b Hence find the equations of invariant lines of **T** which pass through the origin.

c Show that **T** is a reflection and find the equation of the mirror line.

49 Transformation **T** is an enlargement with scale factor $\frac{1}{2}$ followed by a translation with vector $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$.

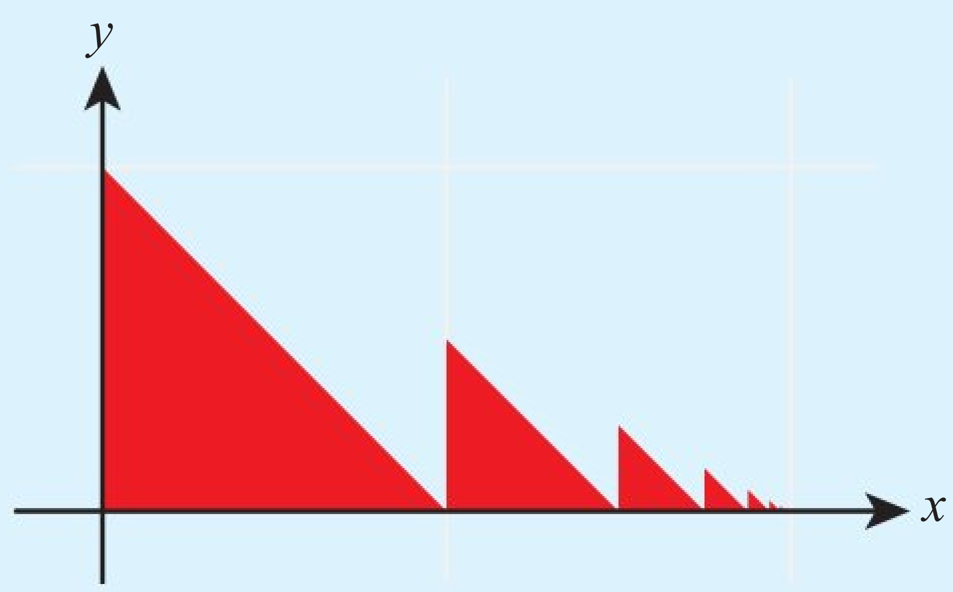
a Write down the coordinates of the image of the point $P_0(x, y)$.

Let P_n be the image of P_0 under the transformation $\mathbf{T}^n$.

b Find the coordinates of P_2, P_3 and P_4.

c Hence conjecture an expression for the coordinates of P_n.

The diagram shows a design formed by repeatedly applying transformation **T** to the triangle with vertices $(0, 0)$, $(1, 0)$ and $(0, 1)$.



In questions 49 and 50, you will need the formula for the sum of an infinite geometric series, which you met in Section 1C.

d Find the total area of the design.

50 The fractal shown in the diagram is formed as follows:

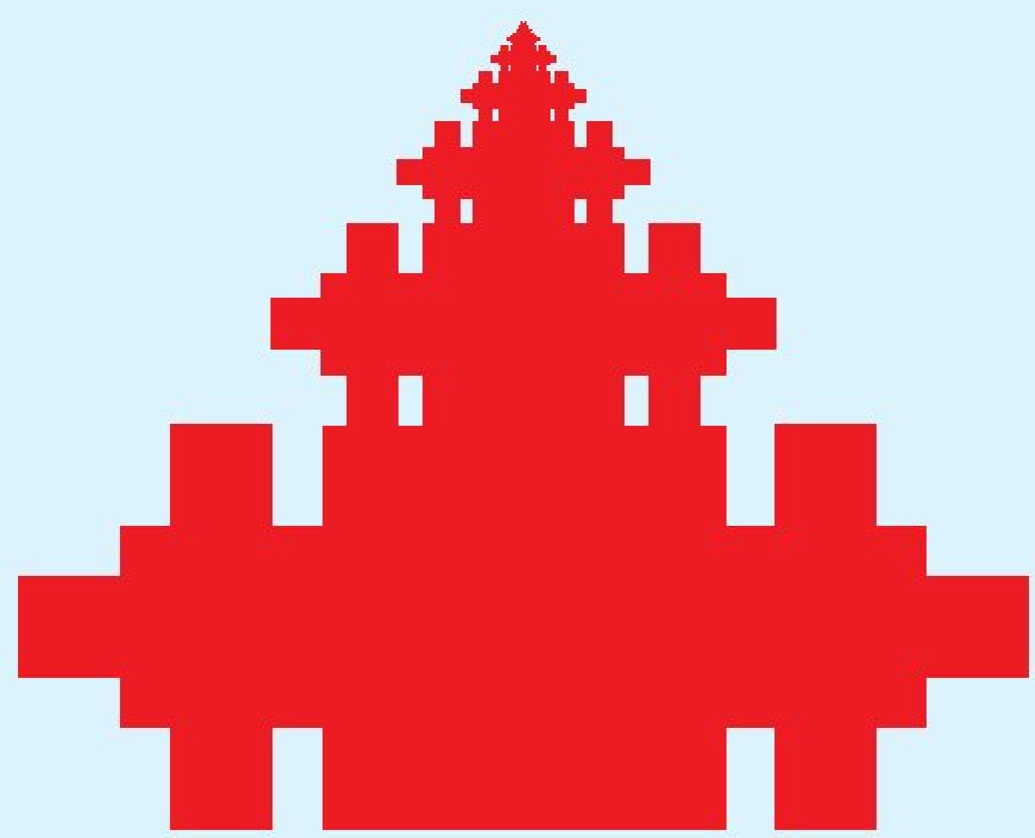
1 Start with a square of side 1.

2 At step 1, construct three squares of side $\frac{1}{2}$ on three of the sides of the original square.

3 At each subsequent step, construct nine new squares, on the middle half of the sides of three of the squares from the previous step.

a Write down the length of the side of each square constructed at step **4**.

b Find the total area of the fractal.



Checklist

- You should be able to convert between degrees and radians:

  $360° = 2\pi$ radians

- You should be able to find the length of an arc of a circle:

  $s = r\theta$

  where r is the radius of the circle and θ is the angle subtended at the centre measured in radians.

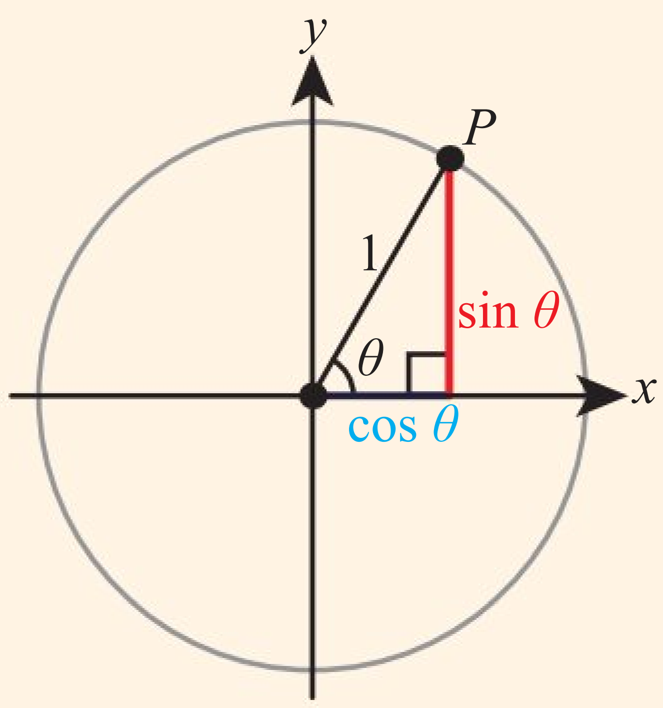
- You should be able to find the area of a sector of a circle:

  $A = \frac{1}{2}r^2\theta$

  where r is the radius of the circle and θ is the angle subtended at the centre measured in radians.

- You should be able to define the sine and cosine functions in terms of the unit circle.

  For a point P on the unit circle:
  - $\sin\theta$ is the y-coordinate of the point P
  - $\cos\theta$ is the x-coordinate of the point P

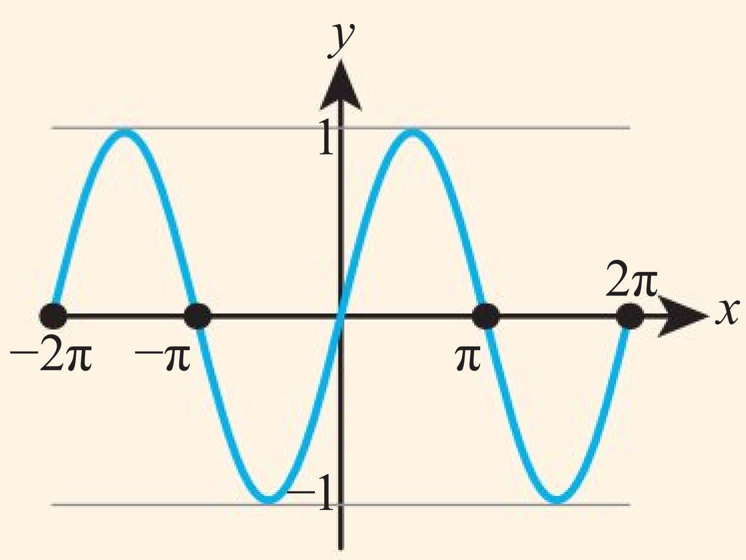


- You should be able to define the tangent function:

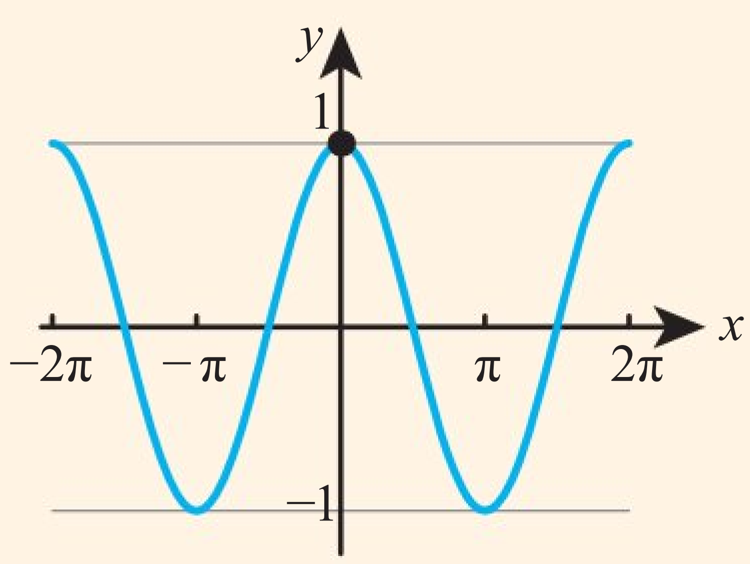
  $\tan\theta = \frac{\sin\theta}{\cos\theta}$

- You should be able to sketch the graphs of trigonometric functions:

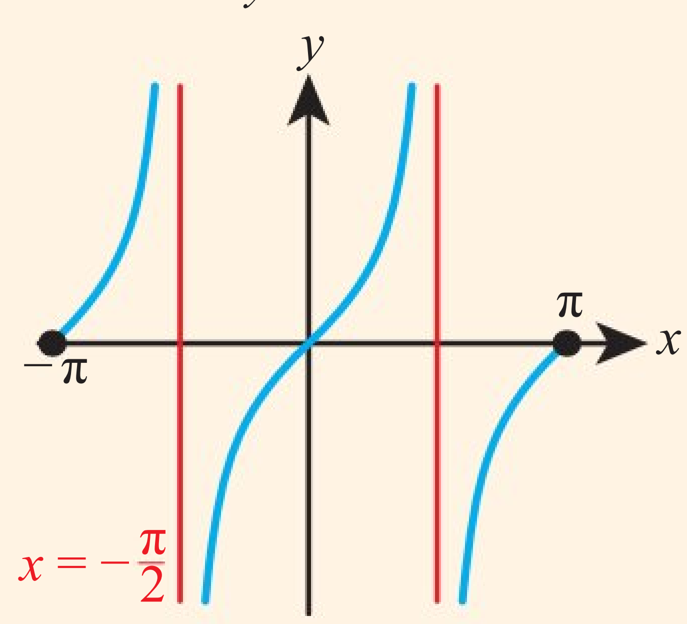
$y = \sin\theta$



$y = \cos\theta$



$y = \tan\theta$



- You should know about the ambiguous case of the sine rule:
  - When using the sine rule to find an angle, there may be two possible solutions: θ and $180 - \theta$.
- You should know the Pythagorean identity $\cos^2\theta + \sin^2\theta \equiv 1$
- You should be able to solve trigonometric equations graphically.
- You should be able to use matrices to represent transformations of the form $\mathbf{T}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} e \\ f \end{pmatrix}$.
- You should be able to find a matrix representing a linear transformation by considering the image of the unit square:
  - For a transformation represented by a matrix **M**, the image of the point $(1, 0)$ is the first column of **M** and the image of the point $(0, 1)$ is the second column of **M**.
- You should be able to use the matrices representing the following common transformations:

reflection in the line $y = (\tan\theta)x$	$\begin{pmatrix} \cos 2\theta & \sin 2\theta \\ \sin 2\theta & -\cos 2\theta \end{pmatrix}$
horizontal stretch with scale factor k	$\begin{pmatrix} k & 0 \\ 0 & 1 \end{pmatrix}$
vertical stretch with scale factor k	$\begin{pmatrix} 1 & 0 \\ 0 & k \end{pmatrix}$
enlargement with scale factor k	$\begin{pmatrix} k & 0 \\ 0 & k \end{pmatrix}$
anticlockwise rotation of angle θ about the origin $(\theta > 0)$	$\begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$
clockwise rotation of angle θ about the origin $(\theta > 0)$	$\begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}$

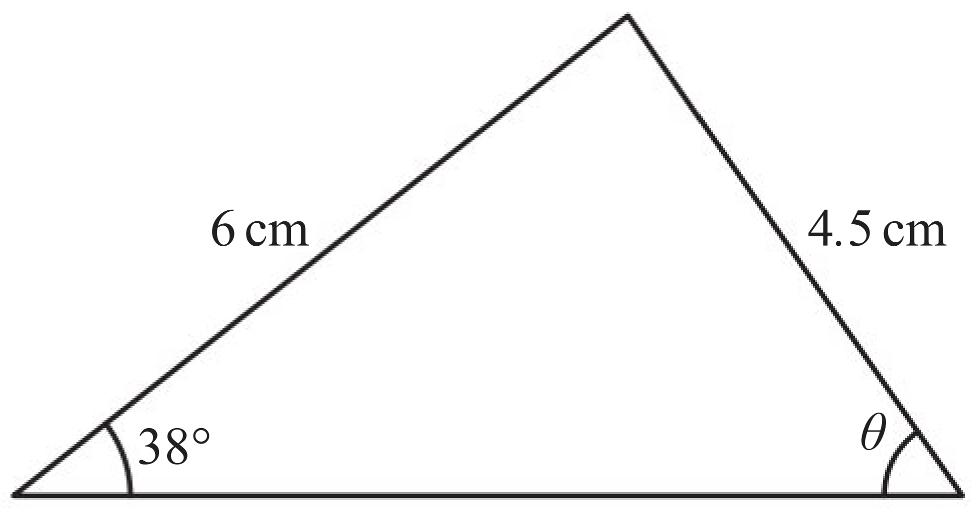
- You should know that the image of a point (x, y) under a translation with vector $\begin{pmatrix} e \\ f \end{pmatrix}$ is $(x + e, y + f)$.
- You should be able to find a matrix representing a composition of two transformations:
  - If the transformation with matrix **A** is followed by the transformation with matrix **B**, the combined transformation has matrix **BA**.
- You should know that $\mathbf{A}^n$ represents transformation **A** repeated n times.
- You should know that, for a transformation represented by matrix **A**, area of image $= |\det \mathbf{A}| \times$ area of object.

Mixed Practice

1 The height of a wave (h m) at a distance (x m) from the shore is modelled by the equation $h = 1.3\sin(2.5x)$.

a Write down the amplitude of the wave.
b Find the distance between consecutive peaks of the wave.

2 For the triangle shown in the diagram, find the two possible values of θ.



3 The obtuse angle A has $\sin A = \frac{5}{13}$.

Find the exact value of

a $\cos A$
b $\tan A$.

4 Transformation **M** is represented by the matrix $\begin{pmatrix} 2 & 1 \\ 3 & 1 \end{pmatrix}$.

a Find the coordinates of the image of the point (3, –2) under the transformation **M**.
b Find the coordinates of the point whose image is (2, 1).
c Find the area of the image of the unit square under the transformation $\mathbf{M}^2$.

5 **a** Write down the matrix representing the rotation 90° clockwise about the origin.
b Find the matrix representing transformation **T**, which is the rotation from part **a** followed by an enlargement with scale factor 3.
c Draw the image of the unit square under transformation **T**.

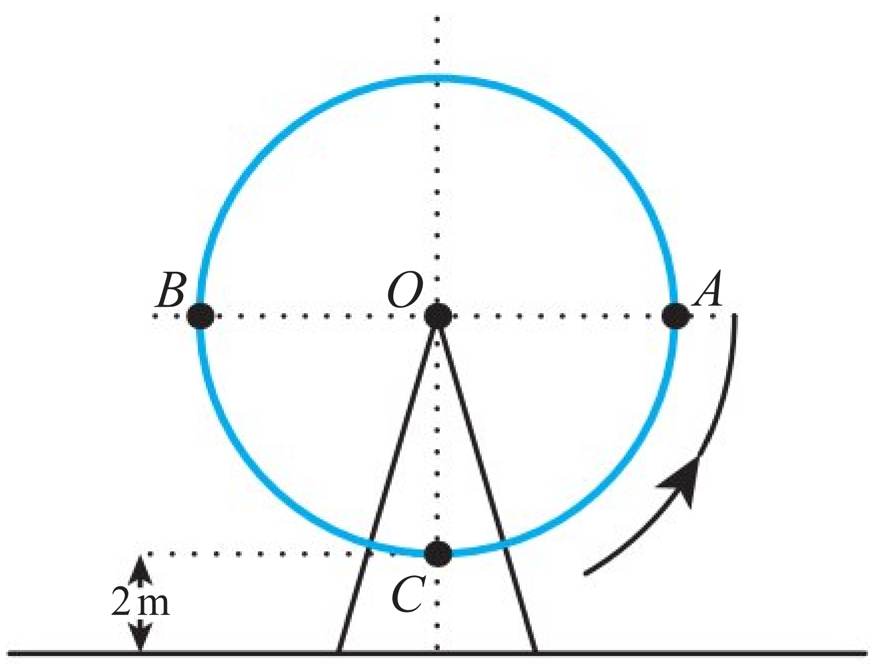
6 Let **R** be the matrix representing a reflection in the line $y = x$ and let **S** be the matrix representing a horizontal stretch with scale factor 2.

a Find the matrix representing the composite transformation when **R** is followed by **S**.
b A second composite transformation is **S** followed by **R**. Determine whether this is the same transformation as in part **a**.

7 Find the period of the function $f(x) = 3\sin\left(\frac{x}{2}\right) - 4\cos\left(\frac{x}{5}\right)$, where x is in radians.

8 Find the range of the function $f(x) = (4\cos(x - \pi) - 1)^2$.

9 The diagram shows a Ferris wheel that moves with constant speed and completes a rotation every 40 seconds. The wheel has a radius of 12 m and its lowest point is 2 m above the ground.



Initially, a seat C is vertically below the centre of the wheel, O. It then rotates in an anticlockwise (counterclockwise) direction.

a Write down

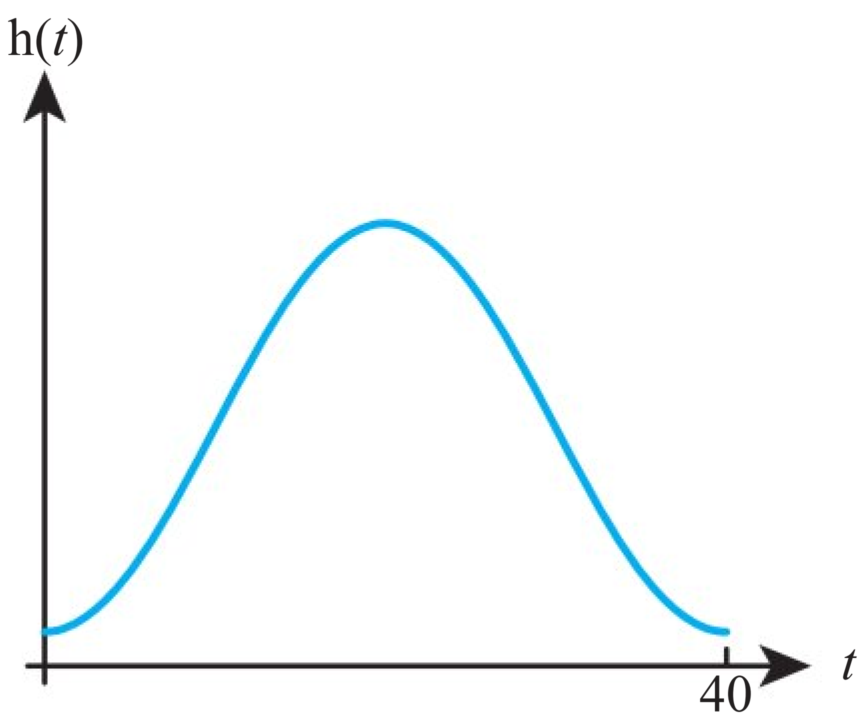
i the height of O above the ground

ii the maximum height above the ground reached by C.

In a revolution, C reaches points A and B, which are at the same height above the ground as the centre of the wheel.

b Write down the number of seconds taken for C to first reach A and then B.

The sketch below shows the graph of function, $h(t)$, for the height above ground of C, where h is measured in metres and t is the time in seconds, $0 \leqslant t \leqslant 40$.



c Copy the sketch and show the results of part **a** and part **b** on your diagram. Label the points clearly with their coordinates.

The height of C above ground can be modelled by the function, $h(t) = a\cos(bt) + c$, where bt is measured in degrees and t is the time in seconds.

d Find the value of

i a **ii** b **iii** c.

C **first** reaches a height of 20 m above the ground after T seconds.

e **i** Sketch a clearly labelled diagram of the wheel to show the position of C.

ii Find the angle that C has rotated through to reach this position.

iii Find the value of T.

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10 Line l makes a 30° angle with the positive x-axis. Transformation **M** (in two dimensions) is the result of the reflection in the x-axis followed by the reflection in l.

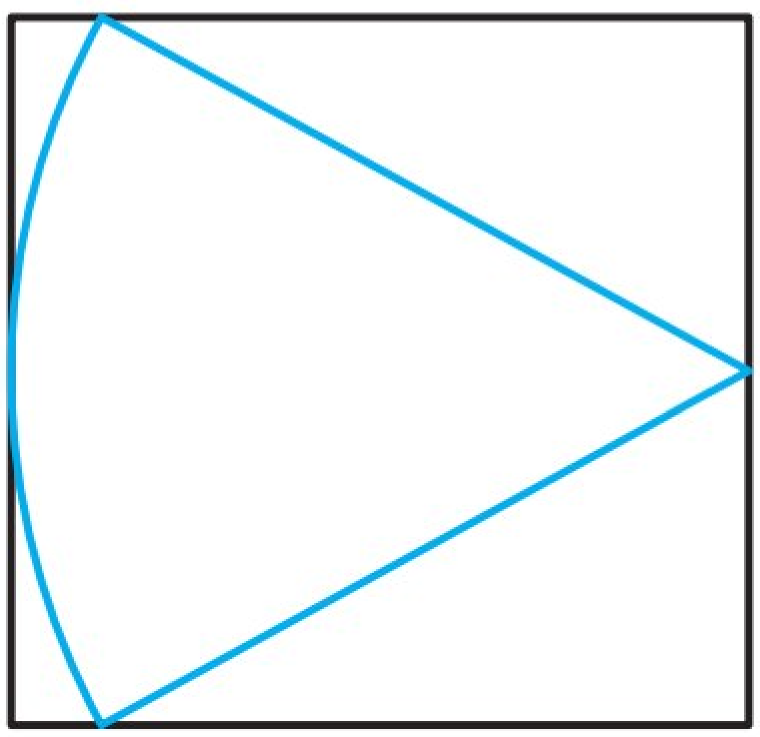
a Find the matrix representing **M**.
b Describe the transformation **M**.
c Describe the transformation resulting from the reflection in l followed by the reflection in the x-axis.

11 Transformation **A** is a 90° clockwise rotation around the origin and transformation **B** is a stretch, scale factor 3, parallel to the y-axis.

a Write down the 2×2 matrices for **A** and **B**.
b Transformation **C** is **A** followed by **B**. Find the matrix for **C**.
c Find the coordinates of the point whose image under **C** is (6, 12).

12 In triangle ABC, $AB = 10\,\text{cm}$, $AC = 7\,\text{cm}$ and angle $ABC = 40°$. Find the difference in areas between the two possible triangles ABC.

13 A rectangle is drawn around a sector of a circle as shown. If the angle of the sector is 1 radian and the area of the sector is $7\,\text{cm}^2$, find the dimensions of the rectangle, giving your answers to the nearest millimetre.



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14 Prove the identity $\dfrac{1}{1+\cos x} + \dfrac{1}{1-\cos x} \equiv \dfrac{2}{\sin^2 x}$.

15 If $0 < k < 1$, find the sum of the solutions to $\sin x = k$ for $-\pi < x < 3\pi$.

16 **a** Sketch $y = x^2 - x$.
b Hence find the values of k for which $\sin^2 x - \sin x - k = 0$ has solutions.

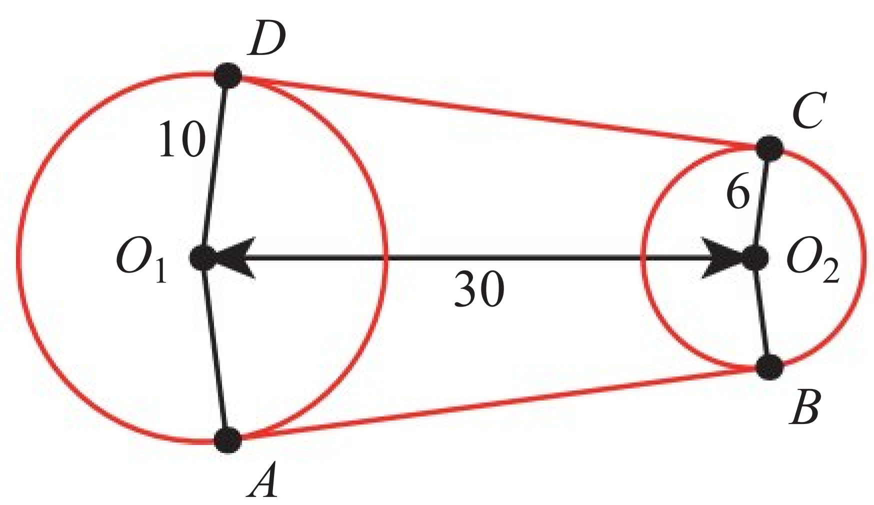
17 Let **S** be the matrix representing reflection in the y-axis and **R** be the matrix representing rotation through 30° anticlockwise about the origin.

a Find the matrix $\mathbf{T} = \mathbf{R}^{-1}\mathbf{SR}$.
b Describe fully the transformation represented by **T**.

18 Let **M** be the matrix representing reflection in the line $y = x$ and **N** be the matrix representing rotation through 45° anticlockwise about the origin. Describe fully the transformation represented by the matrix $\mathbf{M}^{-1}\mathbf{NM}$.

19 A bicycle chain is modelled by the arcs of 2 circles connected by 2 straight lines which are tangents to both circles.

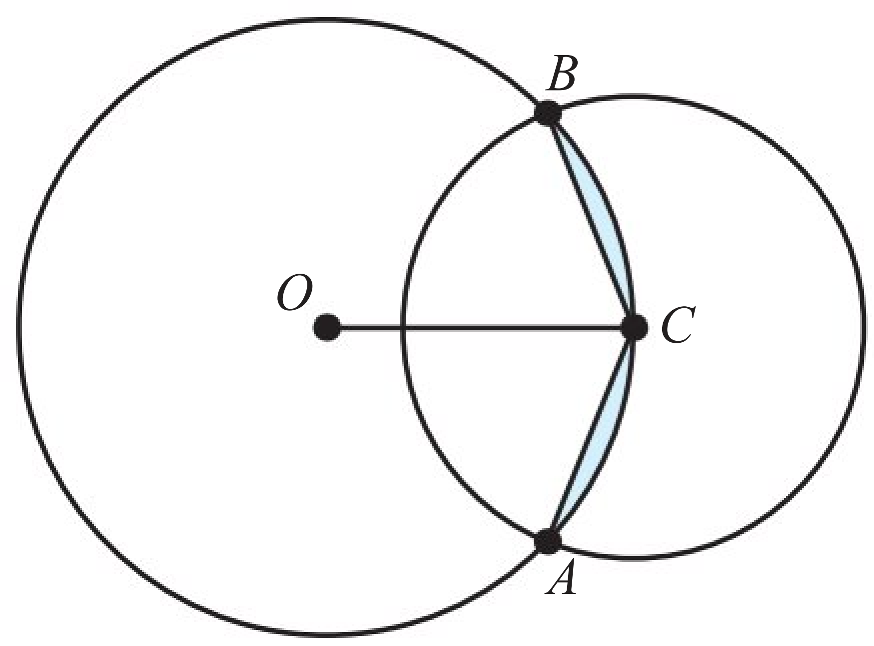
The radius of the larger circle is 10 cm and the radius of the smaller circle is 6 cm. The distance between the centre of the circles is 30 cm.



a Find angle AO_1O_2.

b Hence find the length of the bicycle chain, giving your answer to the nearest cm.

20 The following diagram shows two intersecting circles of radii 4 cm and 3 cm. The centre C of the smaller circle lies on the circumference of the bigger circle. O is the centre of the bigger circle and the two circles intersect at points A and B.



Find

a $B\hat{O}C$

b the area of the shaded region.

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5 Functions and modelling

ESSENTIAL UNDERSTANDINGS

- Creating different representations of functions to model relationships between variables, visually and symbolically, as graphs, equations and tables represents different ways to communicate mathematical ideas.

In this chapter you will learn...

- how to form composite functions
- when composite functions exist
- how to find inverse functions
- when inverse functions exist
- the effect of transforming functions on their graphs
- about more functions which can be used to model real-world situations.

CONCEPTS

The following concepts will be addressed in this chapter:

- Functions **represent** mappings that assign to each value of the independent variable (input) one, and only one, dependent variable (output).
- The parameters in a function or equation may correspond to notable geometrical features of a graph and can represent physical quantities in **spatial** dimensions.
- Changing the parameters of a trigonometric function **changes** the position, orientation and shape of the corresponding graph.
- **Generalization** provides an insight into variation and allows us to access ideas such as half-life and scaling logarithmically to adapt theoretical **models** to solve complex real-life problems.

LEARNER PROFILE – Inquirers

Is mathematics just about answering other people's questions? Before you can do this, you need to get used to questioning other people's mathematics – asking questions such as 'When does this work?', 'What assumptions are being made here?' or 'How does this link to what I already know?' are all second nature to mathematicians.

■ **Figure 5.1** Does the order in which we do things always matter?

PRIOR KNOWLEDGE

Before starting this chapter, you should already be able to complete the following:

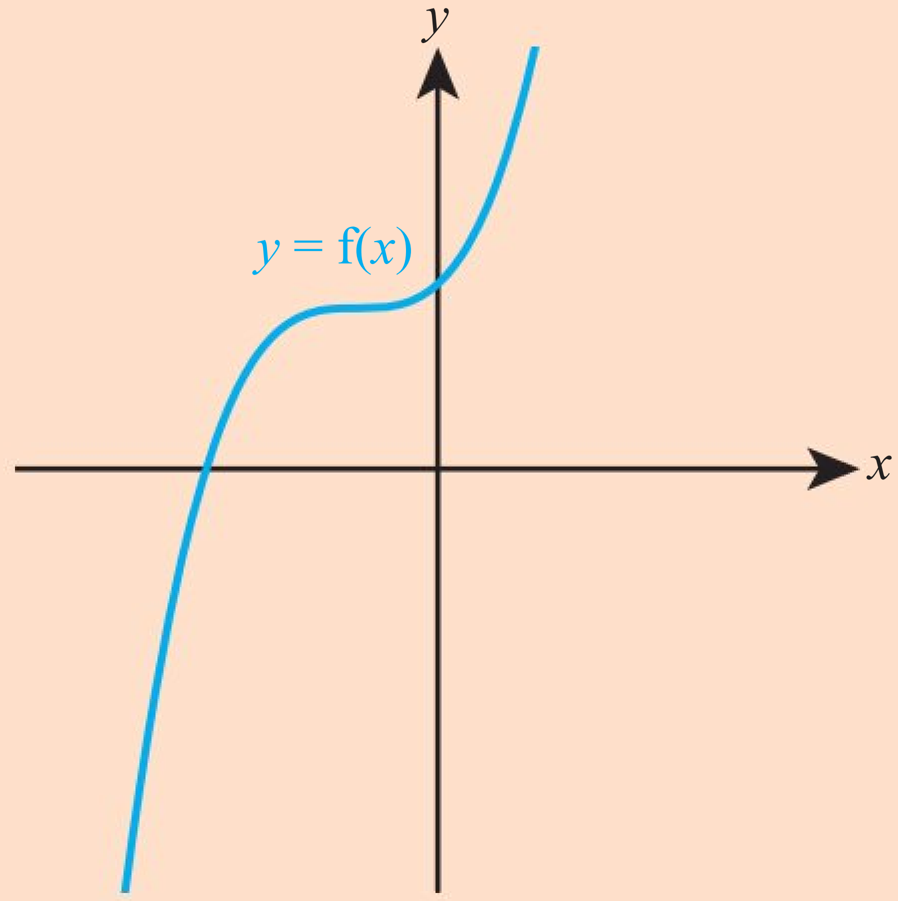
1 If $f(x) = 3 - 4x$, calculate $f(-2)$.

2 Find the largest possible domain of $f(x) = \frac{1}{x+3}$.

3 a Find the range of $f(x) = x^2 + 4x + 1,\ x \in \mathbb{R}$.

b State whether $f(x)$ is a one-to-one or many-to-one function.

4 Sketch the inverse function of the graph shown alongside.



5 Make x the subject of the following:

a $y = \frac{3x+1}{x-1}$

b $y = e^x - 2$

Functions can be combined in many different ways: they can be added, subtracted, multiplied, divided or we can take one function and apply it to another function. Sometimes the order in which functions are combined doesn't matter, but at other times we get different results depending on the order in which the functions have been combined.

Starter Activity

Look at the pictures in Figure 5.1. In small groups, label each part of the process of baking a cake shown. Does the order in which they occur matter? Can each process be reversed?

Now look at this problem:

Ann thinks of a number and squares it. She then adds five to the result.

Bill thinks of a number and adds five to it. He then squares the result.

Both Ann and Bill end up with the same answer.

a What was the answer?

b Will Ann and Bill always end up with the same answer?

5A Composite functions

If, after applying one function, g, to a number you then apply a second function, f, to the result, you have a composite function.

This is written as $f(g(x))$ or $(f \circ g)(x)$.

Tip

You must apply the function nearest to x first. So, $f(g(x))$ or $(f \circ g)(x)$ means you apply g first and then f.

WORKED EXAMPLE 5.1

$f(x) = 2x - 3$ and $g(x) = x^2$

Find

a $f(g(x))$ **b** $g(f(x))$.

Replace x in $f(x)$ with $g(x)$ **a** $f(g(x)) = f(x^2)$

$= 2x^2 - 3$

Replace x in $g(x)$ with $f(x)$ **b** $g(f(x)) = g(2x - 3)$

$= (2x - 3)^2$

Be the Examiner 5.1

$f(x) = x + 4$ and $g(x) = 5x$

Find $(f \circ g)(3)$.

Which is the correct solution? Identify the errors made in the incorrect solutions.

Solution 1	Solution 2	Solution 3
$f(3) = 3 + 4 = 7$ $g(3) = 5 \times 3 = 15$ So, $(f \circ g)(3) = 7 \times 15 = 105$	$(f \circ g)(3) = 5(3 + 4)$ $= 5 \times 7$ $= 35$	$(f \circ g)(3) = 5 \times 3 + 4$ $= 15 + 4$ $= 19$

TOOLKIT: Problem Solving

If $f(g(x)) \equiv g(f(x))$ then the two functions are said to be commutative.

Find examples of two functions $f(x)$ and $g(x)$ which commute. Can you make and prove any general conjectures about some types of functions which always commute with each other?

Can you find a function $f(x)$ which commutes with any other function?

For a composite function f(g(x)) to exist, all possible outputs from g must be allowed as inputs to f, i.e. the range of g must lie entirely within the domain of f.

You met domain and range in Section 3A of Mathematics: applications and interpretation SL.

WORKED EXAMPLE 5.2

$f(x) = \ln(x+2),\ x > -2$

$g(x) = x + 1$

Find the largest possible domain of $(f \circ g)(x)$.

The only thing restricting the domain is that the range of $g(x)$ must fit into the domain of $f(x)$

Largest possible domain of $(f \circ g)$ occurs when $g(x) > -2$

The domain of f is $x > -2$. So, the largest the range of g can be is $g(x) > -2$

Solve the inequality for x

$$x + 1 > -2$$
$$x > -3$$

This is the largest possible domain.

Exercise 5A

For questions 1 to 8, use the method demonstrated in Worked Example 5.1 to find an expression for the composite function $(f \circ g)(x)$.

1 a $f(x) = 3x - 1$, $g(x) = x^2$
  b $f(x) = 4x + 2$, $g(x) = x^2$

2 a $f(x) = x^2$, $g(x) = 2x + 1$
  b $f(x) = x^2$, $g(x) = 3x - 2$

3 a $f(x) = 3x^2 + 2x$, $g(x) = 2x$
  b $f(x) = 5x^2 - 3x$, $g(x) = 2x$

4 a $f(x) = 3e^x$, $g(x) = 2x + 5$
  b $f(x) = 4e^x$, $g(x) = 3x + 1$

5 a $f(x) = 3x + 1$, $g(x) = 4e^x$
  b $f(x) = 2x + 5$, $g(x) = 3e^x$

6 a $f(x) = x^3 - 2x$, $g(x) = e^x$
  b $f(x) = x^3 + 4x$, $g(x) = e^x$

7 a $f(x) = \dfrac{1}{x+1}$, $g(x) = 3x + 2$
  b $f(x) = \dfrac{1}{x-2}$, $g(x) = 2x + 5$

8 a $f(x) = 3x - 2x^2$, $g(x) = \dfrac{1}{x}$
  b $f(x) = 4x + 3x^2$, $g(x) = \dfrac{1}{x}$

For questions 9 to 14, use the method demonstrated in Worked Example 5.2 to find the largest possible domain for the function $(f \circ g)(x)$. Where the domain of f is not given, assume the largest possible real domain.

9 a $f(x) = \ln(x+3)$, $x > -3$ and $g(x) = x + 5$
  b $f(x) = \ln(x+8)$, $x > -8$ and $g(x) = x + 1$

10 a $f(x) = \ln(4 - x)$, $x < 4$ and $g(x) = x + 3$
  b $f(x) = \ln(1 - x)$, $x < 1$ and $g(x) = x + 1$

11 a $f(x) = \sqrt{2x+1}$, $x \geqslant -\dfrac{1}{2}$ and $g(x) = 3x - 2$
  b $f(x) = \sqrt{3x+1}$, $x \geqslant \dfrac{1}{3}$ and $g(x) = 2x - 3$

12 a $f(x) = \sqrt{2x-1}$ and $g(x) = 4 - x$
  b $f(x) = \sqrt{3x-1}$ and $g(x) = 2 - x$

13 a $f(x) = \dfrac{1}{x-2}$, $x \neq 2$ and $g(x) = 4x + 1$
  b $f(x) = \dfrac{1}{x+5}$, $x \neq -5$ and $g(x) = 2x - 7$

14 a $f(x) = \dfrac{1}{x-3}$ and $g(x) = e^x$
  b $f(x) = \dfrac{1}{x-7}$ and $g(x) = e^x$

15 Let $f(x) = 3x - 1$ and $g(x) = 4 - 3x$.
  a Find and simplify and expression for $(f \circ g)(x)$.
  b Solve the equation $(g \circ f)(x) = 4$.

16 Let $f(x) = x^2 + 1$ and $g(x) = x - 1$.
  a Find and simplify an expression for $f(f(x))$.
  b Solve the equation $f(g(x)) = g(f(x))$.

17 Given that $f(x) = 2x^3$, find a simplified expression for $(f \circ f)(x)$.

18 Given that $f(x) = \sqrt{x-4}$, $x \geqslant 4$ and $g(x) = 3x + 10$,

a find the largest possible domain for the function $(f \circ g)$

b solve the equation $(f \circ g)(x) = 5$.

19 Let $f(x) = \ln x (x > 0)$ and $g(x) = x - 5$.

a Write down the exact values of $(f \circ g)(8)$ and $(g \circ f)(8)$.

b Solve the equation $fg(x) = 8$.

20 Functions f and g both have domain {1, 2, 3, 4, 5} and their values are given in the following table.

x	1	2	3	4	5
$f(x)$	5	4	3	2	1
$g(x)$	3	1	4	5	2

a Find:

i $(f \circ g)(3)$

ii $(g \circ f)(4)$

b Solve the equation $(f \circ g)(x) = 1$.

21 Some of the values of the functions f and g are given in the following table.

x	1	3	5	7	9
$f(x)$	3	7	5	1	9
$g(x)$	5	7	9	3	1

a Find:

i $(f \circ g)(3)$

ii $(g \circ f)(9)$

b Solve the equation $(f \circ g)(x) = 5$.

22 Let $f(x) = \ln x$, $x > 0$ and $g(x) = x - 3$.

a Find the largest possible domain for the function $(f \circ g)$.

b Solve the equation $(f \circ g)(x) = 1$.

c Solve the equation $(g \circ f)(x) = 1$.

d Solve the equation $(f \circ g)(x) = (g \circ f)(x)$.

23 Given that $f(x) = \dfrac{1}{x+2}$ and $g(x) = \dfrac{1}{x-3}$,

a find and simplify an expression for $(f \circ g)(x)$

b find the largest possible domain for the function $(f \circ g)(x)$

c solve the equation $(f \circ g)(x) = 2$.

24 Let $f(x) = \dfrac{4}{x}$, $x \neq 0$ and $g(x) = 2x^2$, $x \in \mathbb{R}$. Prove that $gf(x) \equiv k\,fg(x)$ for some constant k, which you should find.

25 Let $f(x) = \dfrac{1}{3x+2}$ for $x \neq -\dfrac{2}{3}$.

a Find the largest possible domain for $(f \circ f)(x)$.

b Use technology to find the range of $(f \circ f)(x)$ for the domain from part a.

c Solve the equation $(f \circ f)(x) = 1$.

You met the idea of inverse function in Section 3A of Mathematics: applications and interpretation SL.

5B Inverse functions

Identity function

You have already met the idea that the inverse, f^{-1}, of a function f reverses the effect of f. This can be expressed more formally in terms of composite functions.

KEY POINT 5.1

$(f \circ f^{-1})(x) = (f^{-1} \circ f)(x) = x$

Finding the inverse function

In order to find an expression for f^{-1} from f you need to rearrange f to find the input (x) in terms of the output (y).

KEY POINT 5.2

To find an expression for $f^{-1}(x)$:

1. let $y = f(x)$
2. rearrange to make x the subject
3. state $f^{-1}(x)$ by replacing any ys with xs.

WORKED EXAMPLE 5.3

$f(x) = \dfrac{3x-5}{2}$

Find the inverse function, f^{-1}.

Let $y = f(x)$

$$y = \frac{3x-5}{2}$$

Rearrange to make x the subject

$$2y = 3x - 5$$

$$2y + 5 = 3x$$

$$x = \frac{2y+5}{3}$$

Write the resulting function in terms of x

$$\text{So, } f^{-1}(x) = \frac{2x+5}{3}$$

See Section 3A of Mathematics: applications and interpretation SL for a reminder of one-to-one and many-to-one functions.

The existence of an inverse for one-to-one functions

For $f^{-1}(x)$ to be a function, it must map each input value to a single output value. But since the graph $y = f^{-1}(x)$ is a reflection in $y = x$ of the graph $y = f(x)$, it follows that for $f(x)$ each output must come from a single input, i.e. $f(x)$ must be one-to-one.

If $f(x)$ is many-to-one, $f^{-1}(x)$ would not be a function as there would be input values that map to more than one output value.

KEY POINT 5.3

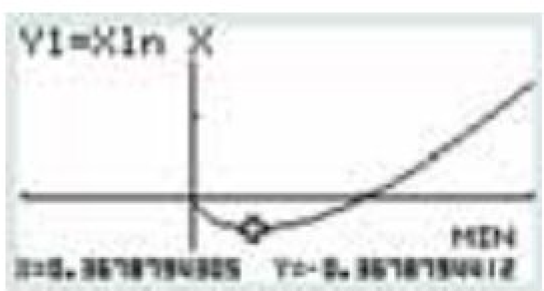
A function has to be one-to-one to have an inverse.

Any function can be made one-to-one (and therefore to have an inverse) by restricting its domain.

WORKED EXAMPLE 5.4

Use technology to find the largest possible domain of the function $f(x) = x\ln x$ of the form $x \geqslant k$ for which the inverse f^{-1} exists.

f^{-1} will only exist if f is one-to-one. Use the GDC to sketch the graph and find the minimum point. Eliminating values of x to the left of the minimum will leave f being one-to-one



Largest possible domain of f for which f^{-1} exists:

$x \geqslant 0.368$

Be the Examiner 5.2

Find the inverse function of $f(x) = x^2 - 4$, $x < 0$.

Which is the correct solution? Identify the errors made in the incorrect solutions.

Solution 1	Solution 2	Solution 3
$y = x^2 - 4$ $x^2 = y + 4$ $x = \sqrt{y+4}$ So, $f^{-1}(x) = \sqrt{x+4}$	$y = x^2 - 4$ $x^2 = y + 4$ $x = -\sqrt{y+4}$ So, $f^{-1}(x) = -\sqrt{x+4}$	f isn't one-to-one, so the inverse function doesn't exist.

Exercise 5B

For questions 1 to 10, use the method demonstrated in Worked Example 5.3 to find an expression for the inverse function, $f^{-1}(x)$.

1 a $f(x) = \frac{4x+1}{2}$
b $f(x) = \frac{3x+4}{5}$

2 a $f(x) = 4x - 3$
b $f(x) = 5x + 1$

3 a $f(x) = e^{4x}$
b $f(x) = e^{3x}$

4 a $f(x) = 3e^{x-2}$
b $f(x) = 2e^{x+3}$

5 a $f(x) = \log_2(3x+1)$
b $f(x) = \log_3(4x-1)$

6 a $f(x) = \sqrt{x-2}$
b $f(x) = \sqrt{x+3}$

7 a $f(x) = (x+2)^3$
b $f(x) = (x-3)^3$

8 a $f(x) = x^3 - 2$
b $f(x) = x^3 + 5$

9 a $f(x) = \frac{x+3}{x-2}$
b $f(x) = \frac{x+1}{x-3}$

10 a $f(x) = \frac{2x+1}{3x-2}$
b $f(x) = \frac{3x-1}{2x-3}$

For questions 11 to 17, use the method demonstrated in Worked Example 5.4. For each function f find the largest possible domain of the given form such that f has an inverse function.

11 a $f(x) = (x-2)^2,\ x \geqslant a$
b $f(x) = (x+5)^2,\ x \geqslant a$

12 a $f(x) = (x+1)^2,\ x \leqslant b$
b $f(x) = (x-3)^2,\ x \leqslant b$

13 a $f(x) = x^3 - 3x^2,\ x \geqslant c$
b $f(x) = x^3 + 2x^2,\ x \geqslant c$

14 a $f(x) = x^3 - 3x,\ c \leqslant x \leqslant d$
b $f(x) = x^3 - 12x,\ c \leqslant x \leqslant d$

15 a $f(x) = xe^x,\ x \leqslant a$
b $f(x) = xe^{-x},\ x \leqslant a$

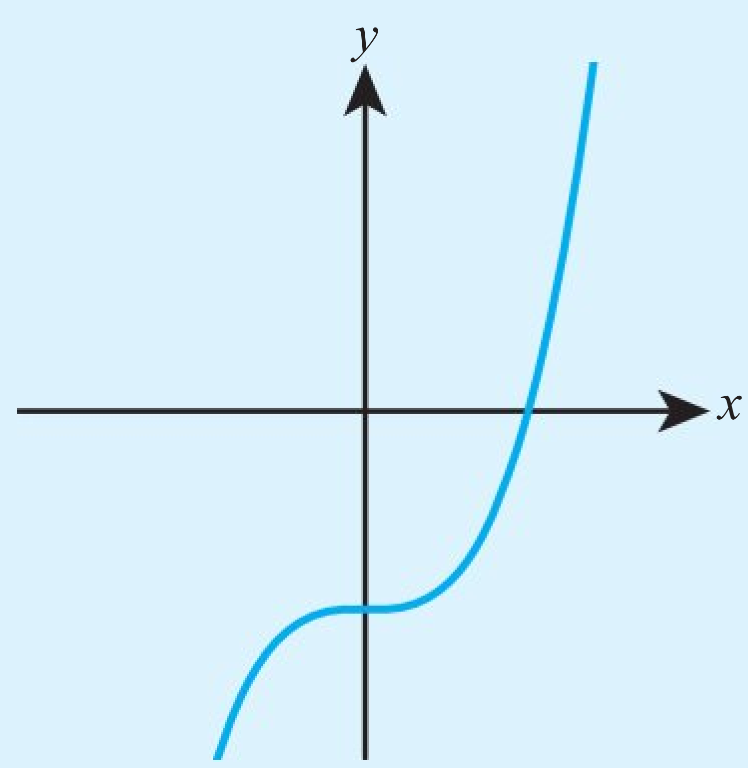
16 a $f(x) = xe^{\frac{x}{2}},\ x \geqslant a$
b $f(x) = xe^{-\frac{x}{3}},\ x \geqslant a$

17 a $f(x) = \frac{x^2+1}{x^2-4},\ x < b$
b $f(x) = \frac{x^2-2}{x^2-1},\ x < b$

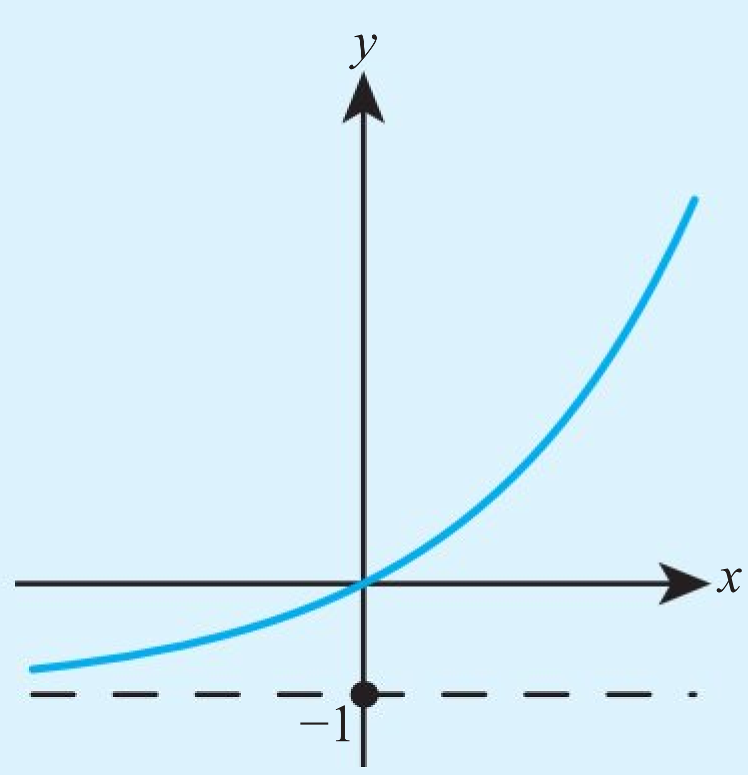
18 For the function $f(x) = \frac{4}{3-x}\ (x \neq 3)$,
a find $f^{-1}(3)$
b find an expression for $f^{-1}(x)$.

19 Given that $f(x) = 3e^{5x}$, find an expression for $f^{-1}(x)$.

20 The diagram shows the graph of $y = f(x)$, where $f(x) = \frac{1}{5}x^3 - 3$.
a Copy the graph and, on the same axes, sketch $y = f^{-1}(x)$.
b Find an expression for $y = f^{-1}(x)$.



21 Let $f(x) = e^{\frac{x}{2}} - 1$ for all $x \in \mathbb{R}$. The diagram shows the graph of $y = f(x)$.
a Sketch the graph of $y = f^{-1}(x)$.
b Find an expression for $y = f^{-1}(x)$ and state its domain.



22 Given that $f(x) = \dfrac{x^2+1}{x^2-4}$ for $x > 2$,

a find an expression for $f^{-1}(x)$

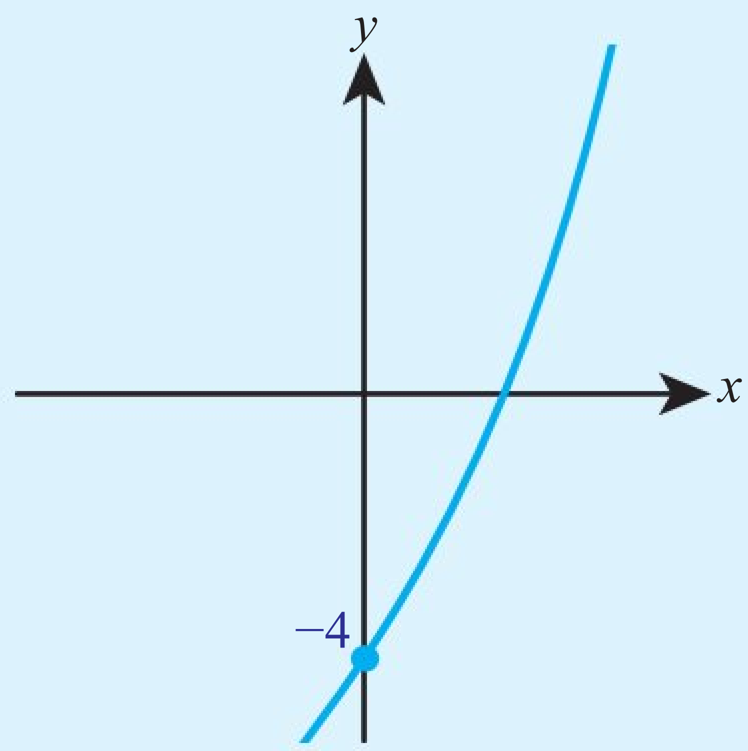
b state the range of f^{-1}.

23 If $f(x) = \dfrac{1}{4+\sqrt{x}}$, $x > 0$ and $g(x) = 2x + 1$, solve $(g^{-1} \circ f^{-1})(x) = 4$.

24 Let $f(x) = e^{\frac{x}{2}} + x - 5$ for $x \in \mathbb{R}$. The diagram shows a part of the graph of $y = f(x)$.

a Copy the graph and sketch $y = f^{-1}(x)$ on the same axes.

b Find the exact solution of the equation $f(x) = f^{-1}(x)$.



25 Let $f(x) = x^2 + 3$ for $x \leqslant a$.

a Find the largest possible value of a such that f has an inverse function.

b For this domain, find an expression for $f^{-1}(x)$.

26 **a** For the function $g(x) = 9(x-5)^2$, find the largest possible domain of the form $x \leqslant k$ such that g has an inverse function.

b For this domain, find an expression for $g^{-1}(x)$.

27 A function f is called *self-inverse* if $f(x) \equiv f^{-1}(x)$. Find the value of a such that $f(x) = \dfrac{ax+3}{x-4}$ is a self-inverse function.

TOOLKIT: Problem Solving

Can you find any other functions which are self inverse?

5C Transformations of graphs

It is useful to be able to take a familiar graph such as $y = \dfrac{1}{x}$ and transform it into a more complicated graph. You need to be familiar with three types of transformation and how they affect the equation of a graph: **reflections** in the x-axis and y-axis, **translations** and **stretches**.

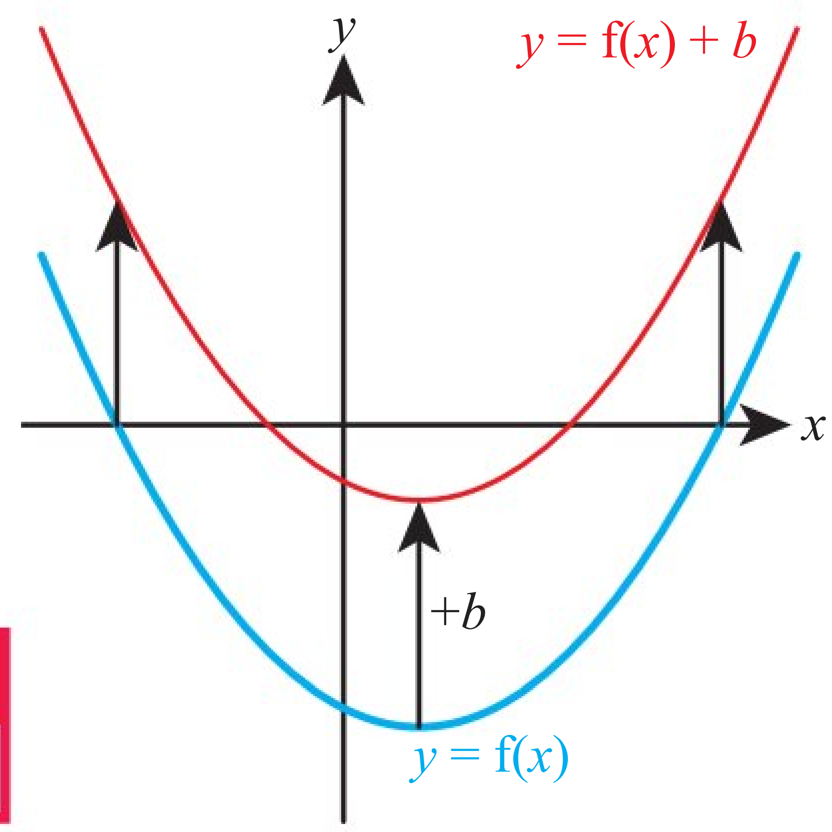
The standard Cartesian coordinate system of x- and y-axes meeting at an origin is usually attributed to the French mathematician René Descartes. Modern philosophers have suggested that this way of describing graphs is actually a reflection of Western European language and philosophy, where position tends to be described relative to the observer (for example, ‘that stool is 5 metres in front of me’). It has been suggested that in other cultures this is not the convention and positions tend to be described relative to both the observer and the audience. For example, in Maori culture it would be usual to say ‘that stool is 5 metres away from me and 2 metres away from you’. For this reason, it has been suggested that the natural coordinate system used in these cultures would require two origins.

Translations

Tip

Remember that b can be negative. This results in a downward translation.

Adding a constant to a function translates its graph vertically:



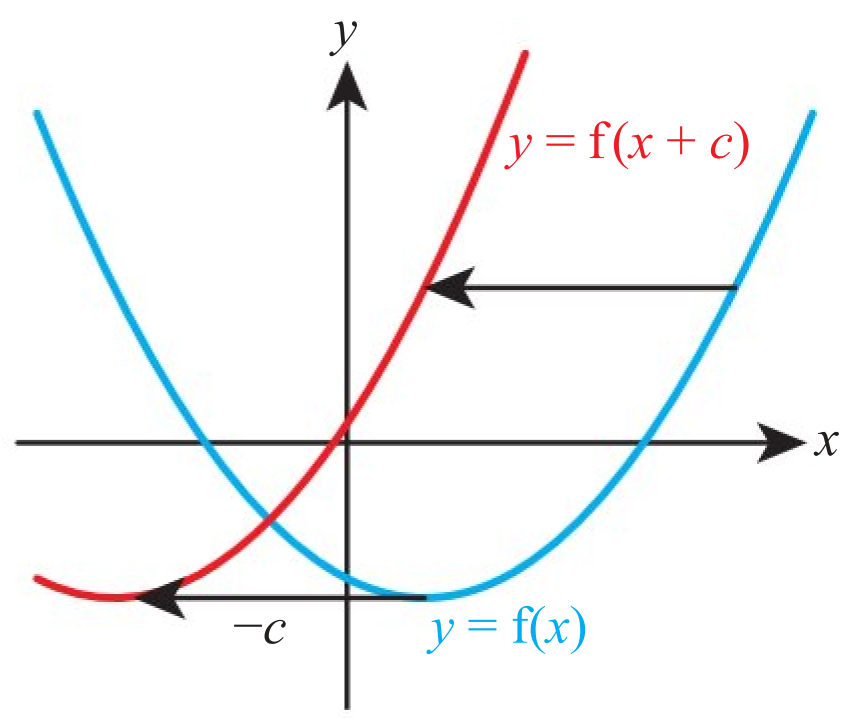
KEY POINT 5.4

$y = \mathrm{f}(x) + b$ is a vertical translation by b of $y = \mathrm{f}(x)$.

Tip

Notice that $y = \mathrm{f}(x + c)$ moves **to the left** by c, and not to the right by c.

Replacing x with $x + c$ in a function translates its graph horizontally:



KEY POINT 5.5

$y = \mathrm{f}(x + c)$ is a horizontal translation by $-c$ of $y = \mathrm{f}(x)$.

Proof 5.1

Prove that $y = \mathrm{f}(x + c)$ is a horizontal translation by $-c$ of $y = \mathrm{f}(x)$.

Define a point (x_1, y_1) on the original curve and a point $(x_2 + c, y_2)$ on the transformed curve

Let
$y_1 = \mathrm{f}(x_1)$
$y_2 = \mathrm{f}(x_2 + c)$

One way to make the two points have the same height (y) is to set $x_1 = x_2 + c$

If $x_1 = x_2 + c$, then $y_1 = y_2$.

Rearrange to make x_2 the subject

Equivalently, if $x_2 = x_1 - c$, then $y_1 = y_2$.

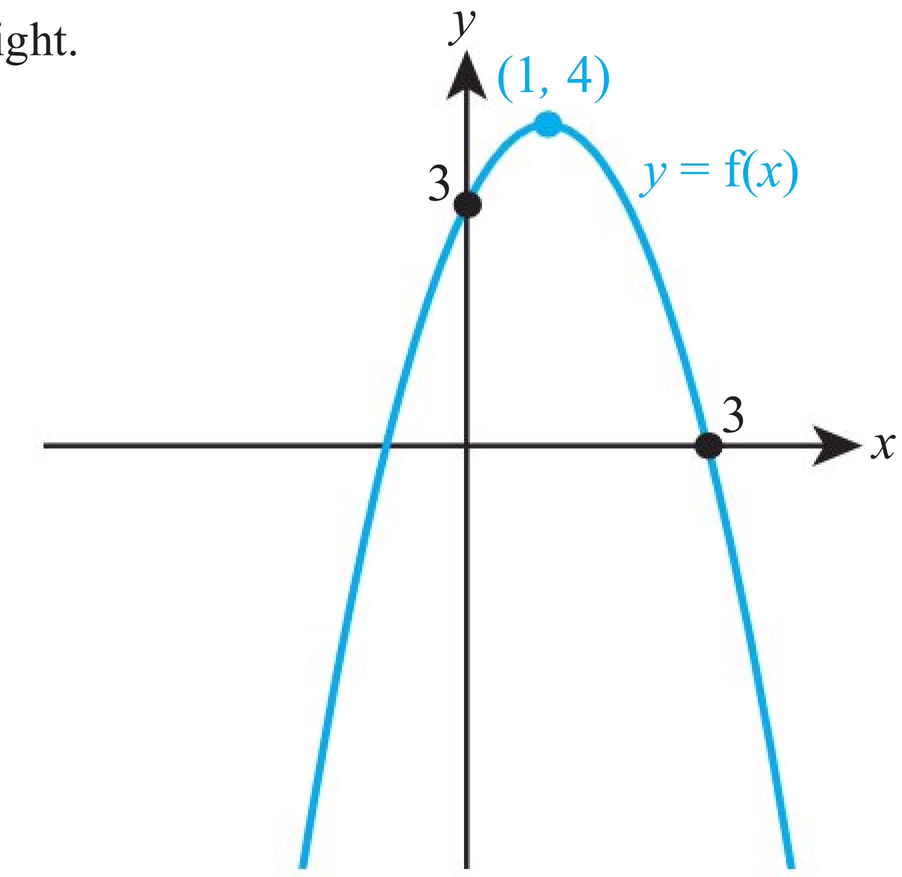
Interpret this graphically

So, the equivalent point to (x_1, y) on $y = \mathrm{f}(x)$ is $(x_1 - c, y)$ on $y = \mathrm{f}(x + c)$, which means that the points on $y = \mathrm{f}(x)$ have been translated horizontally by $-c$ to form $y = \mathrm{f}(x + c)$.

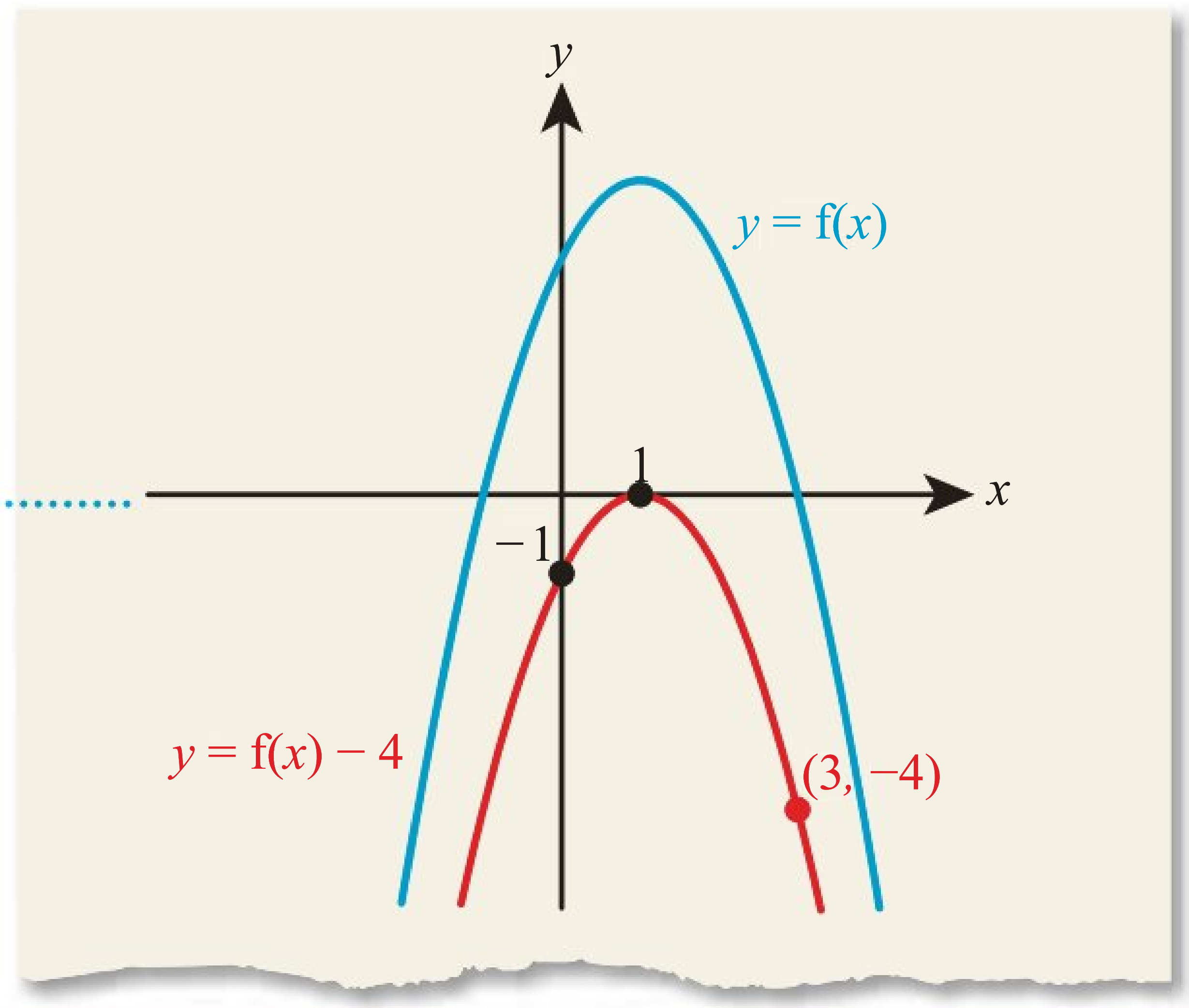
WORKED EXAMPLE 5.5

The graph of $y = f(x)$ is shown on the right.

Sketch the graph of $y = f(x) - 4$.



$y = f(x) - 4$ is a vertical translation by -4



WORKED EXAMPLE 5.6

The graph of $y = x^2 - 3x + 5$ is translated to the left by 2 units.

Find the equation of the resulting graph in the form $y = ax^2 + bx + c$.

Let $f(x) = x^2 - 3x + 5$

Then the new graph is

A translation to the left (or horizontally by -2) means that $y = f(x + 2)$

$$y = f(x + 2)$$
$$= (x + 2)^2 - 3(x + 2) + 5$$

Expand and simplify

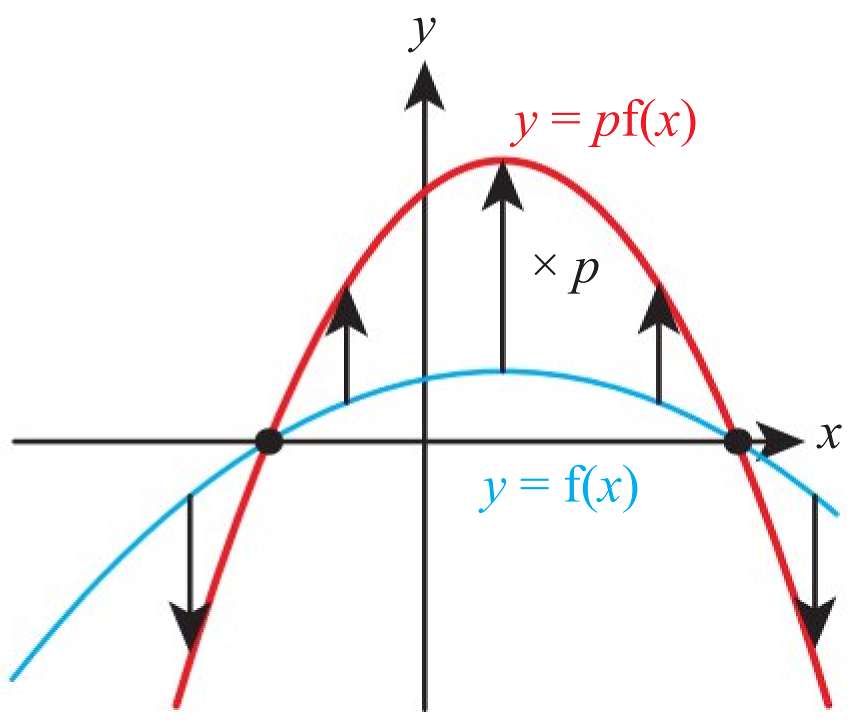
$$= x^2 + 4x + 4 - 3x - 6 + 5$$
$$= x^2 + x + 3$$

Tip

You should try checking this by sketching both graphs on the GDC.

Stretches

Multiplying a function by a constant stretches its graph vertically:



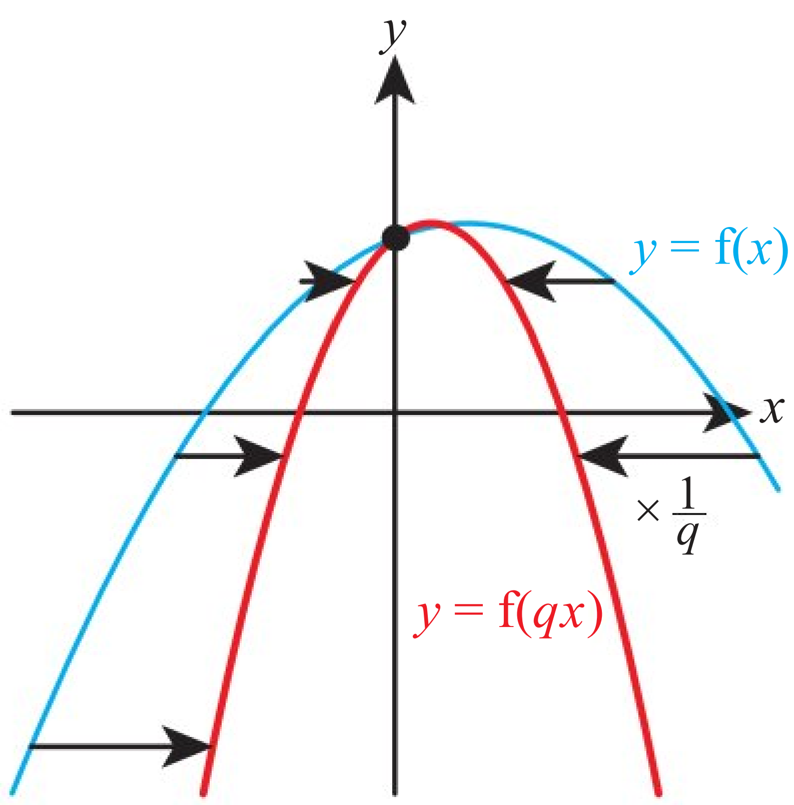
Tip

Remember that p can be a fraction less than one.

KEY POINT 5.6

$y = p\mathrm{f}(x)$ is a vertical stretch with scale factor p of $y = \mathrm{f}(x)$.

Replacing x with qx in a function stretches its graph horizontally:



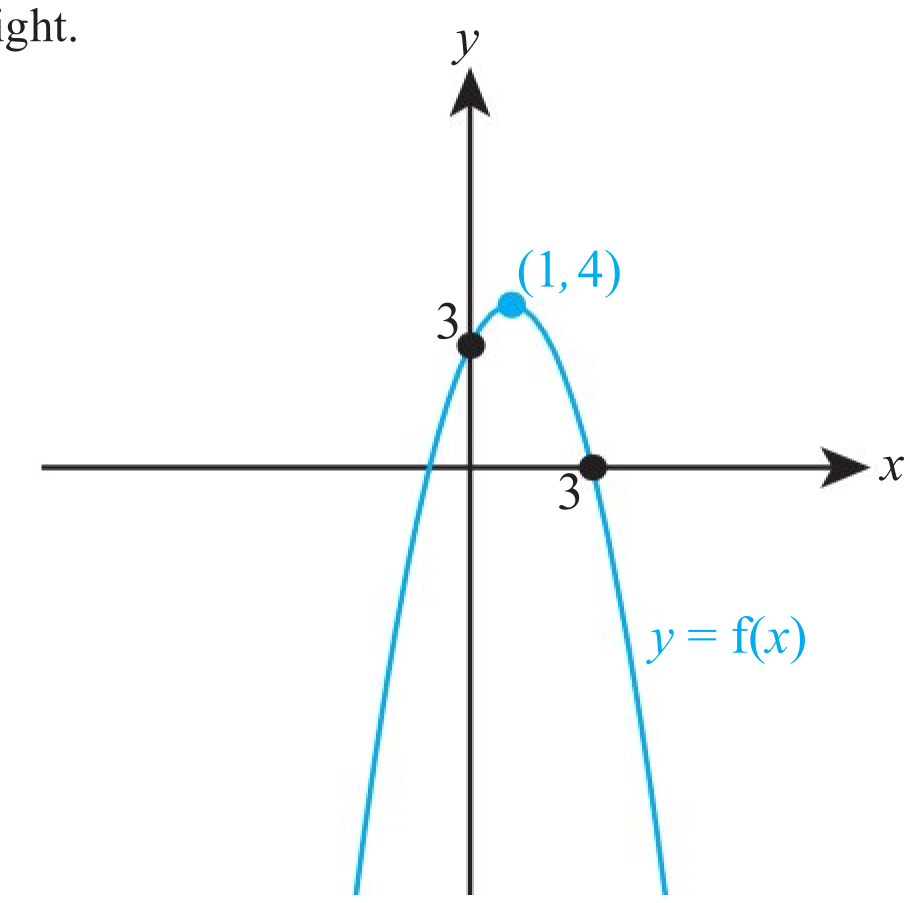
KEY POINT 5.7

$y = \mathrm{f}(qx)$ is a horizontal stretch with scale factor $\frac{1}{q}$ of $y = \mathrm{f}(x)$.

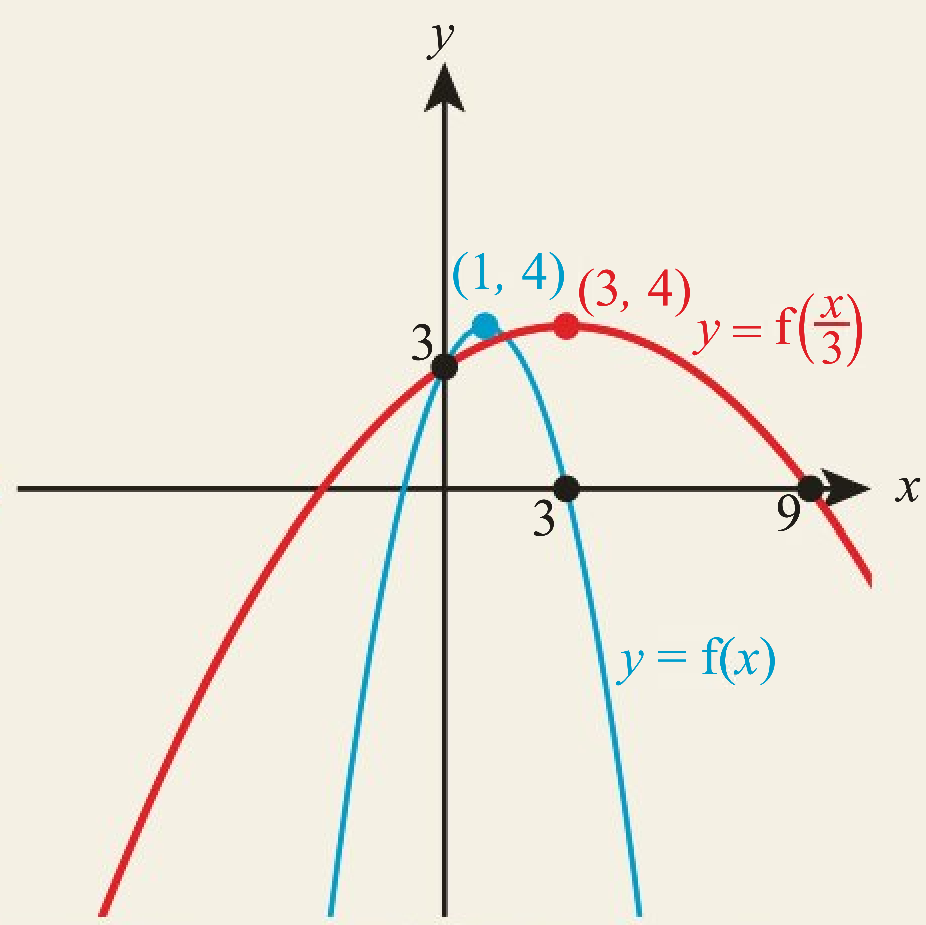
WORKED EXAMPLE 5.7

The graph of $y = f(x)$ is shown on the right.

Sketch the graph $y = f\left(\frac{1}{3}x\right)$.



$y = f\left(\frac{1}{3}x\right)$ is a horizontal stretch with scale factor $\frac{1}{\left(\frac{1}{3}\right)} = 3$



WORKED EXAMPLE 5.8

Describe the transformation that maps the graph of $y = 2x^2 - 8x - 5$ to the graph of $y = x^2 - 4x - 2.5$.

Let $f(x) = 2x^2 - 8x - 5$

Relate the second equation to the first: Then,

$$y = x^2 - 4x - 2.5$$
$$= \frac{1}{2}(2x^2 - 8x - 5)$$

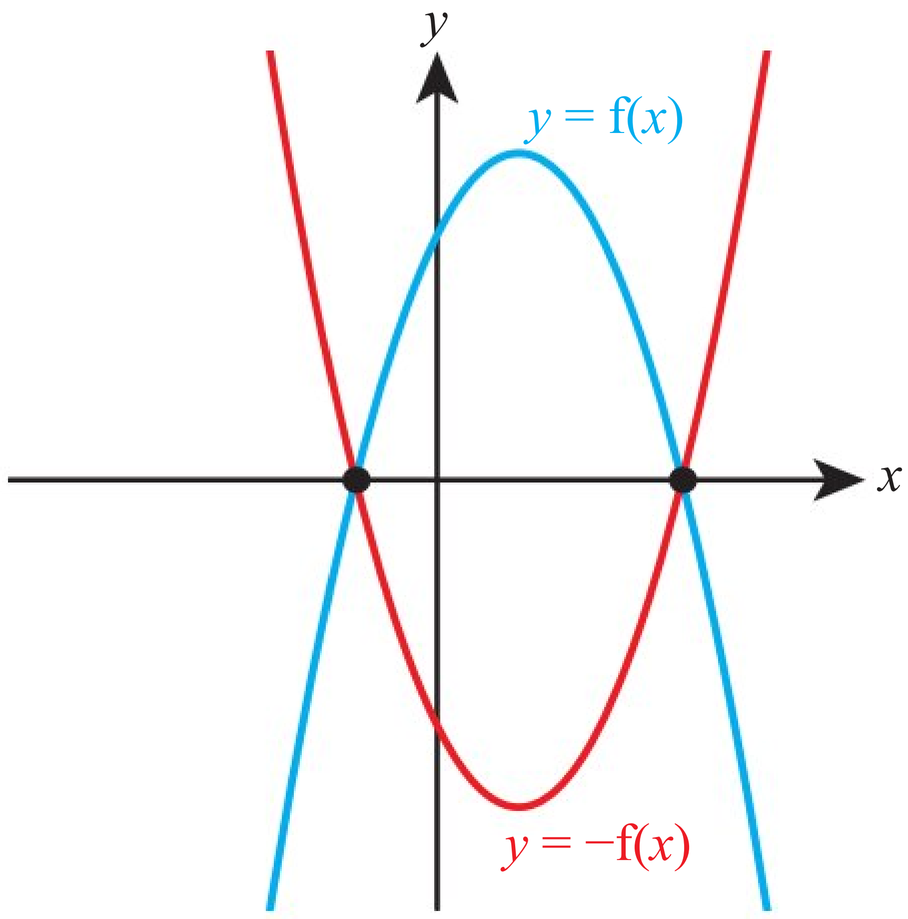
Express this in function notation:

$$= \frac{1}{2}f(x)$$

State the transformation: Vertical stretch with scale factor $\frac{1}{2}$

Reflections

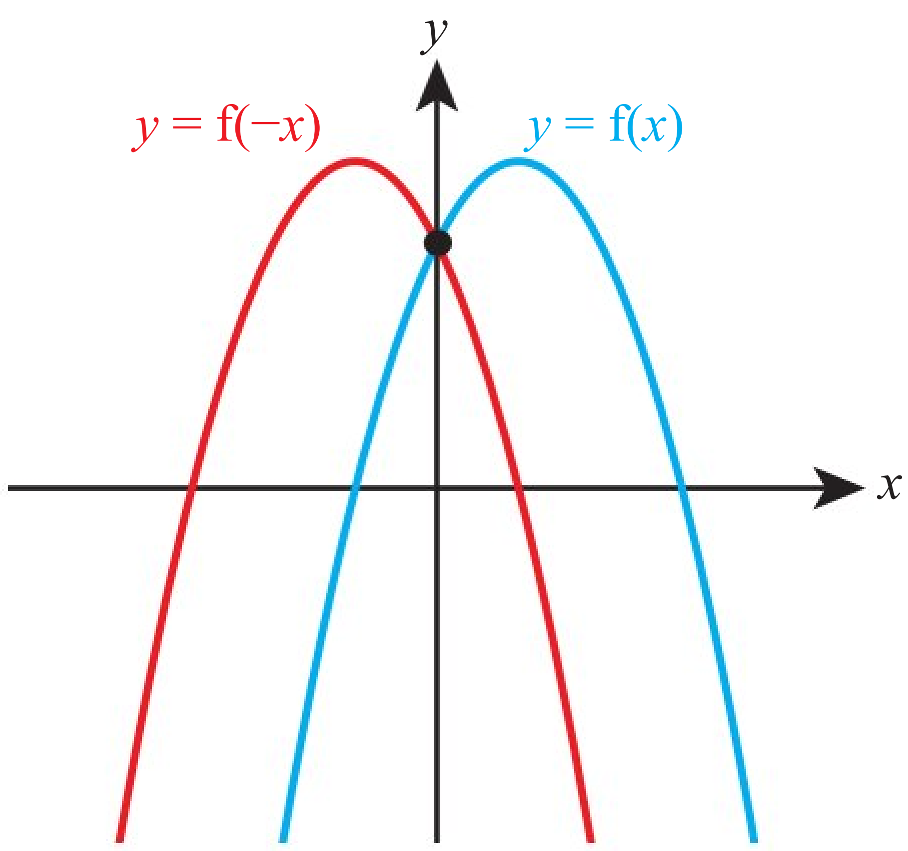
Multiplying a function by –1 reflects its graph in the x-axis:



KEY POINT 5.8

$y = -f(x)$ is a reflection in the x-axis of $y = f(x)$.

Replacing x with $-x$ in a function reflects its graph in the y-axis:



Tip

Notice that the negative sign inside the brackets affects the x-coordinates. This is consistent with translations and stretches where any alterations inside the brackets result in horizontal movements of the graph.

KEY POINT 5.9

$y = f(-x)$ is a reflection in the y-axis of $y = f(x)$.

WORKED EXAMPLE 5.9

The graph of $y = f(x)$ has a single vertex at $(5, -2)$.

Find the coordinates of the vertex on the graph of $y = f(-x)$.

$y = f(-x)$ is a reflection in the y-axis. This will just multiply the x-coordinate by -1

The vertex of $y = f(-x)$ is at $(-5, -2)$.

Composite transformations

You saw how the order in which the functions in a composite function were applied could affect the result; the same is true for the order in which transformations are applied.

Tip

If there is both a vertical translation by b and a horizontal translation by c, then this can be described using a vector as $\begin{pmatrix} -c \\ b \end{pmatrix}$.

KEY POINT 5.10

- When two vertical transformations are applied, the order matters: $y = p\mathrm{f}(x) + b$ is a vertical stretch with scale factor p followed by a vertical translation by b.
- When two horizontal transformations are applied, the order matters: $y = \mathrm{f}(qx + c)$ is a horizontal translation of $-c$ followed by a horizontal stretch factor $\frac{1}{q}$.
- When one vertical and one horizontal transformation are applied, the order does not matter.

TOOLKIT: Problem Solving

Try using a flowchart to explain why two composite horizontal transformations work in the way described in Key Point 5.10. What function is produced if the order of the transformations is reversed?

WORKED EXAMPLE 5.10

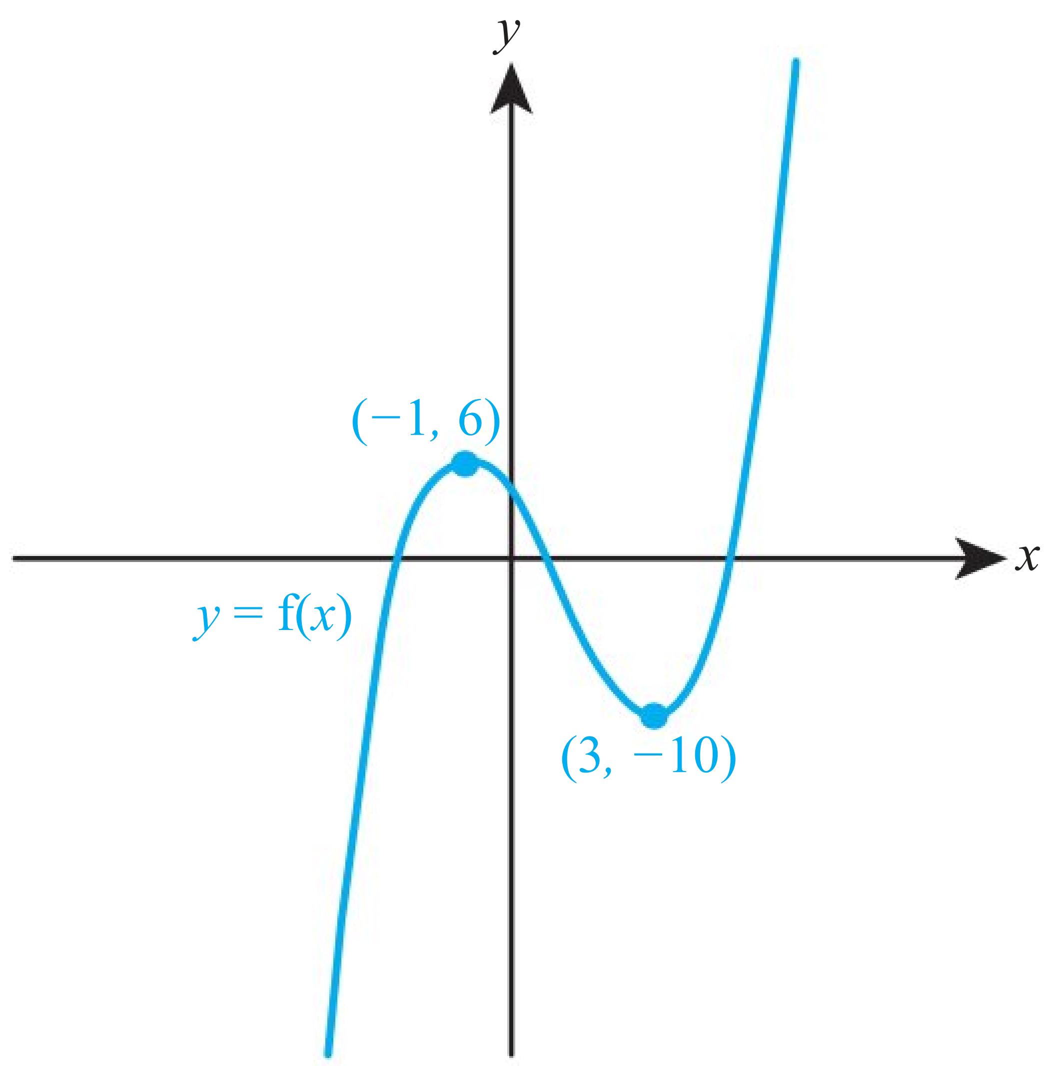
The diagram shows the graph of $y = \mathrm{f}(x)$.

On separate sets of axes, sketch the graphs of

a $y = 3\mathrm{f}(x - 2)$

b $y = -\mathrm{f}(x) + 4$

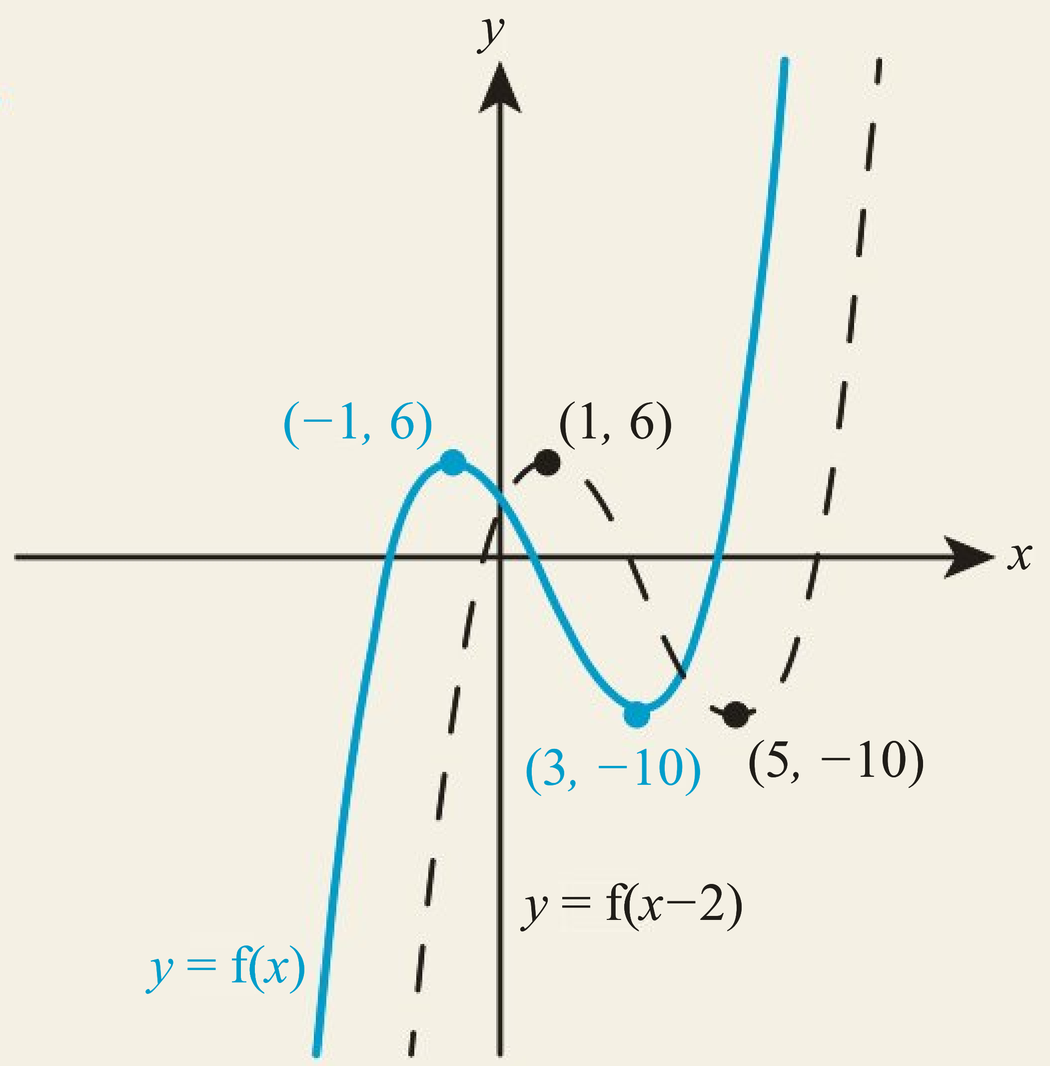
c $y = \mathrm{f}\left(\frac{1}{2}x + 1\right)$



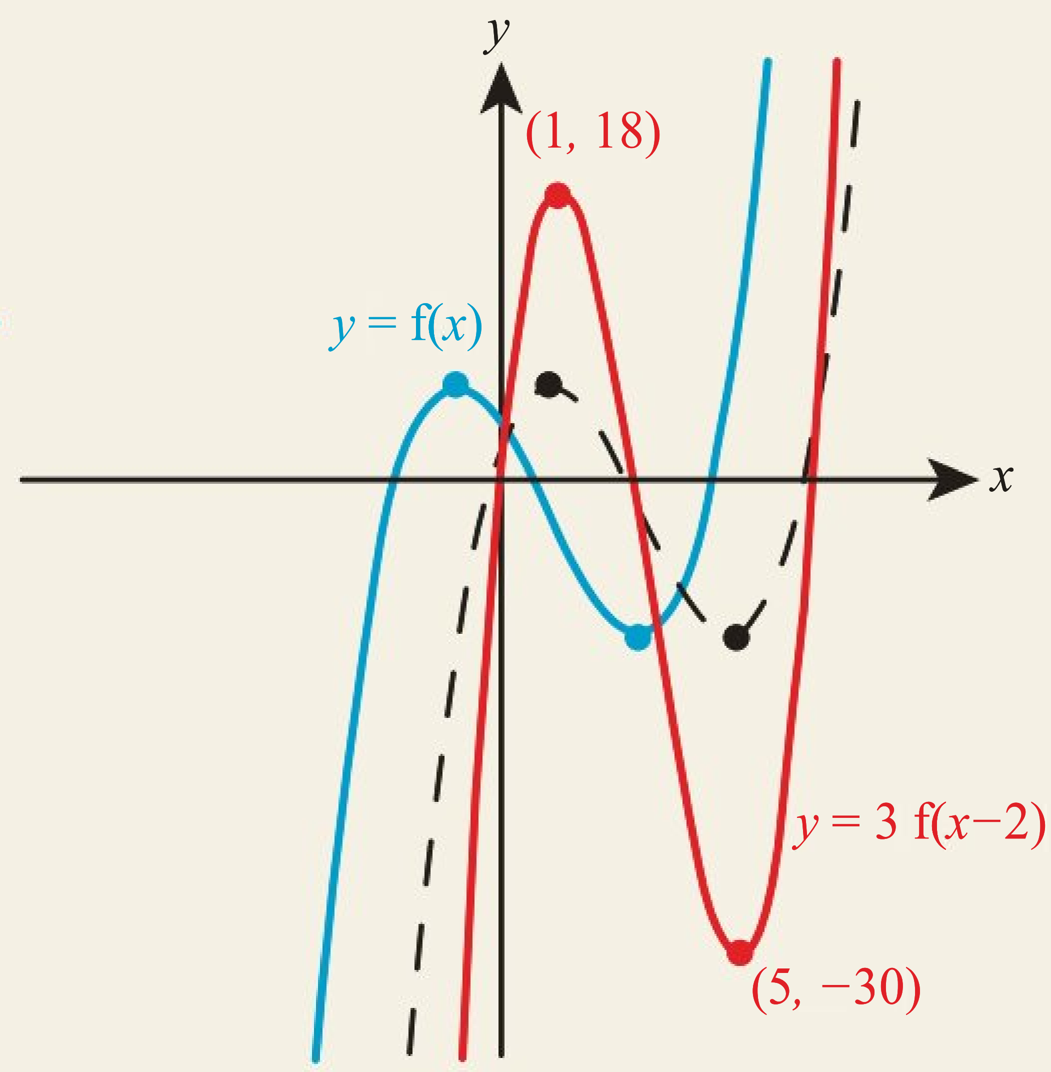
There is one vertical and one horizontal transformation, so it doesn't matter in which order you apply them

Translate horizontally by +2...

a



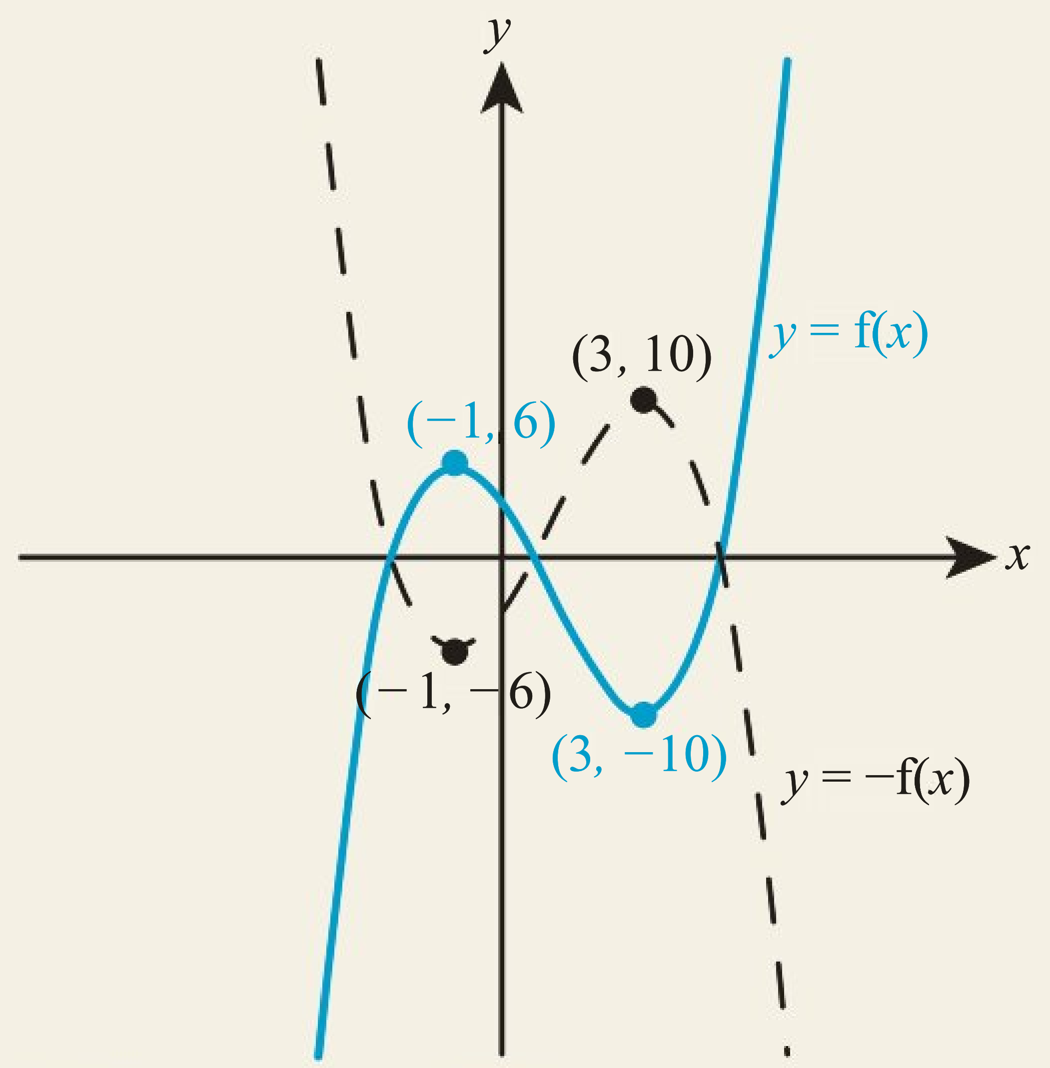
... and then stretch vertically with scale factor 3



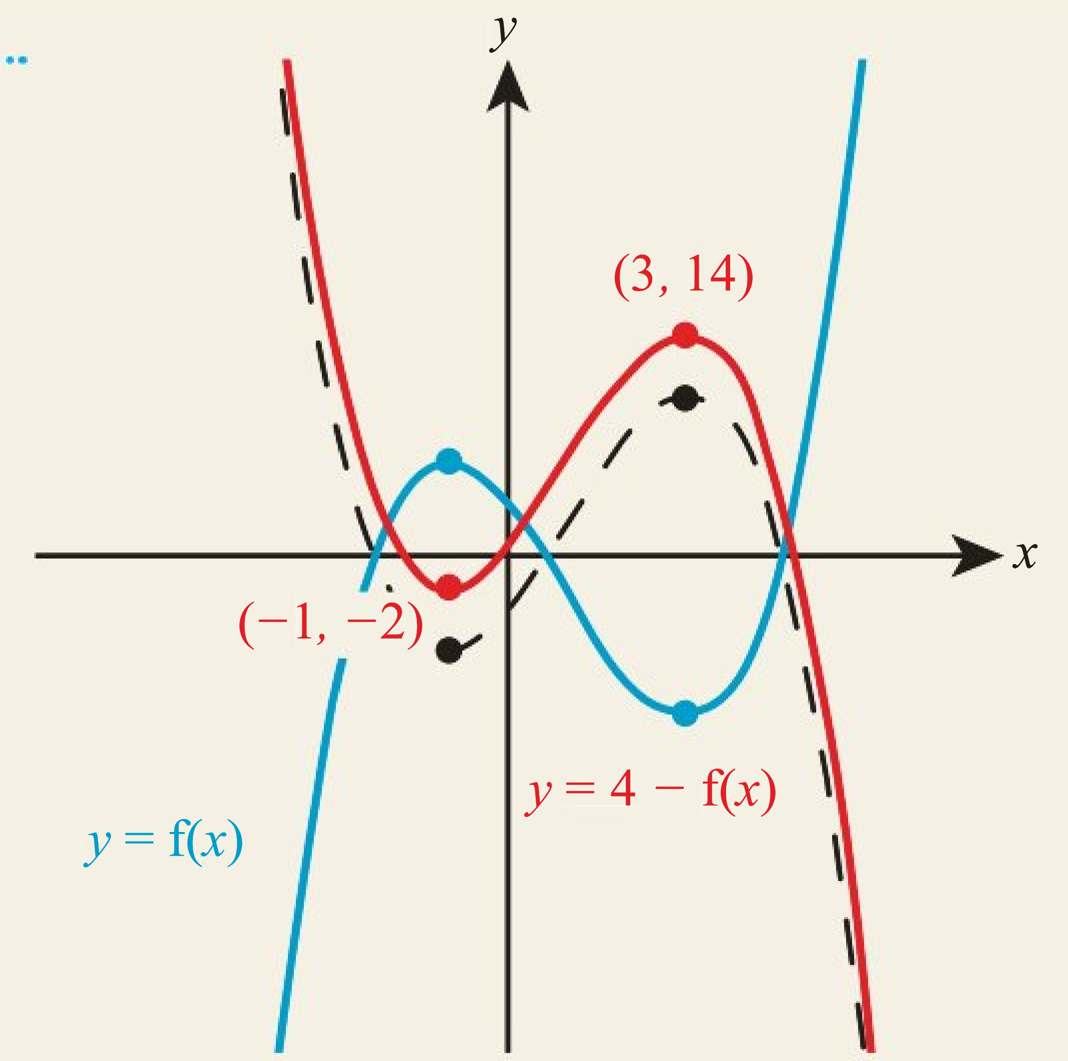
There are two vertical transformations so the order is important

First reflect in the x-axis...

b



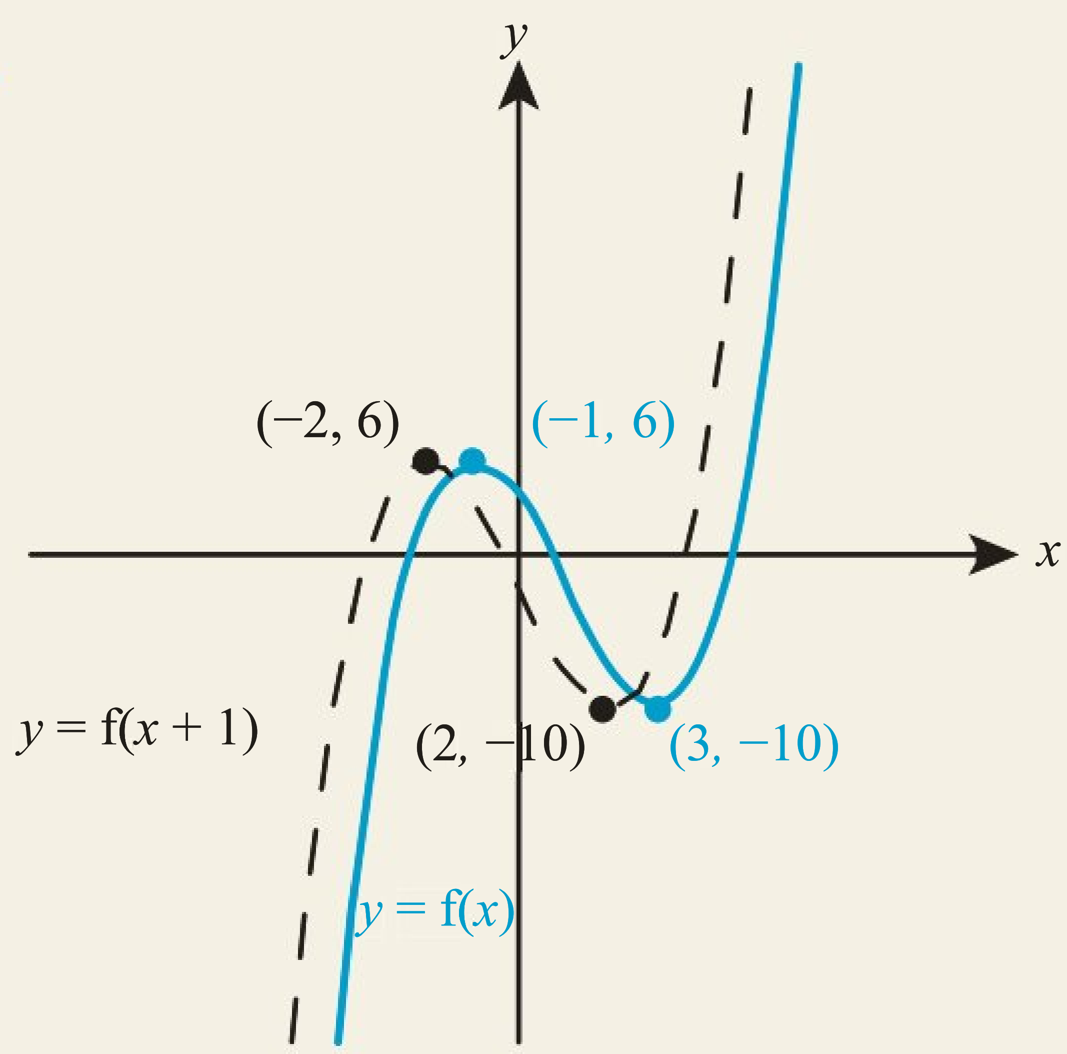
... and then translate vertically by 4



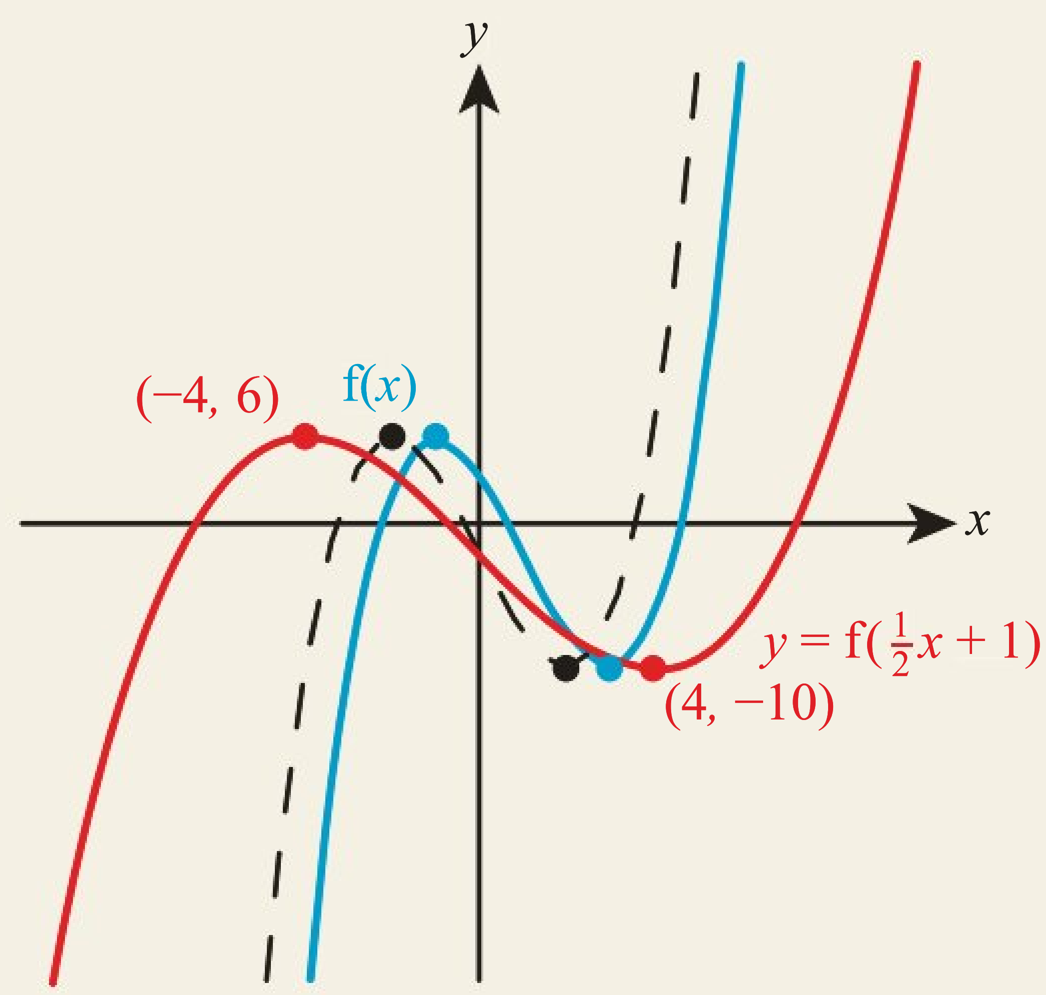
There are two horizontal transformations so the order is important

c

First translate horizontally by −1…



… and then stretch horizontally by a factor of 2 (as x is multiplied by $\frac{1}{2}$)



WORKED EXAMPLE 5.11

Describe a sequence of transformations that maps the graph of $y = x^3 - x$ to the graph of $y = 2x^3 - 2x - 1$.

Express the second equation in function notation, related to the first

Let $f(x) = x^3 - x$

Then,

$$\begin{aligned} y &= 2x^3 - 2x - 1 \\ &= 2(x^3 - x) - 1 \\ &= 2f(x) - 1 \end{aligned}$$

State the transformation, making sure the stretch comes before the translation

Vertical stretch with scale factor 2, followed by vertical translation by −1.

You are the Researcher

Some graphs do not change under transformations. For example, the graph $y = x$ looks exactly the same after a horizontal stretch factor 2 and a vertical stretch with scale factor 2. Graphs with these properties are studied in an area of maths that you might be interested in researching called fractals. Fractals have found a wide variety of applications from data storage to art. This could be a suitable area of research for a sophisticated and rigorous mathematical exploration.

Be the Examiner 5.3

The graph of $y = f(x)$ is stretched horizontally with scale factor 2 and translated horizontally by 3.

Find the equation of the transformed graph.

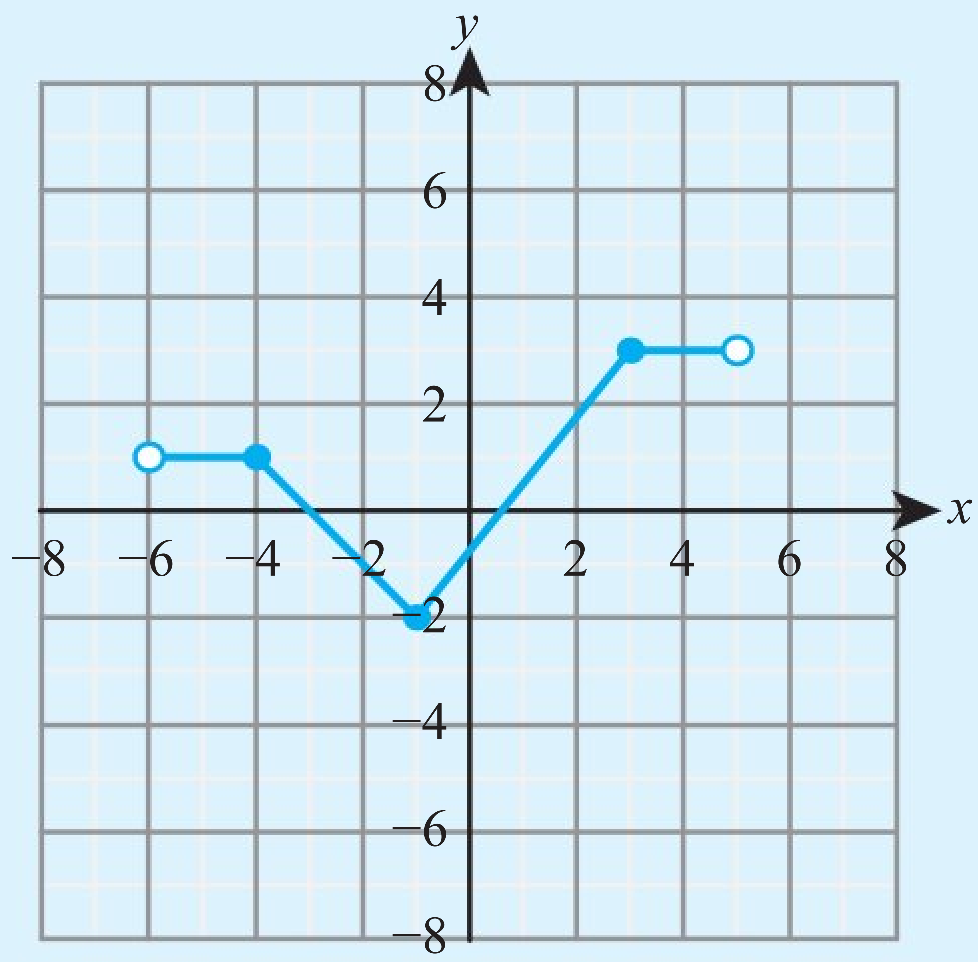
Which is the correct solution? Identify the errors in the incorrect solutions.

Solution 1	Solution 2	Solution 3
$y = f\left(\frac{1}{2}x - 3\right)$	$y = f\left(\frac{x-3}{2}\right)$	$y = f(2x - 6)$

Exercise 5C

For questions 1 to 4, use the method demonstrated in Worked Example 5.5 to sketch the required graph.

The graph of $y = \mathrm{f}(x)$ is given alongside.



1 **a** $y = \mathrm{f}(x) + 3$
b $y = \mathrm{f}(x) + 4$

2 **a** $y = \mathrm{f}(x) - 2$
b $y = \mathrm{f}(x) - 5$

3 **a** $y = \mathrm{f}(x + 4)$
b $y = \mathrm{f}(x + 1)$

4 **a** $y = \mathrm{f}(x - 3)$
b $y = \mathrm{f}(x - 2)$

For questions 5 to 8, use the method demonstrated in Worked Example 5.6 to find the equation of the graph after the given transformation is applied.

5 **a** $y = 3x^2$ after a translation of 3 units vertically up
b $y = 2x^3$ after a translation of 5 units vertically up

6 **a** $y = 8x^2 - 7x + 1$ after a translation of 5 units vertically down
b $y = 8x^2 - 7x + 1$ after a translation of 2 units vertically down

7 **a** $y = 4x^2$ after a translation of 3 units to the right
b $y = 3x^3$ after a translation of 6 units to the right

8 **a** $y = x^2 + 6x + 2$ after a translation of 3 units to the left
b $y = x^2 + 5x + 4$ after a translation of 2 units to the left

For questions 9 to 12, use the method demonstrated in Worked Example 5.7 to sketch the required graph. The graph of $y = \mathrm{f}(x)$ is given above.

9 **a** $y = 2\mathrm{f}(x)$
b $y = 3\mathrm{f}(x)$

11 **a** $y = \mathrm{f}(3x)$
b $y = \mathrm{f}(2x)$

10 **a** $y = \frac{1}{2}\mathrm{f}(x)$
b $y = \frac{1}{3}\mathrm{f}(x)$

12 **a** $y = \mathrm{f}\left(\frac{x}{2}\right)$
b $y = \mathrm{f}\left(\frac{x}{3}\right)$

For questions 13 to 16, use the method demonstrated in Worked Example 5.8 to describe the transformation that maps the graph of $y = 12x^2 - 6x$ to the graph with the given equation.

13 **a** $y = 36x^2 - 18x$
b $y = 24x^2 - 12x$

15 **a** $y = 48x^2 - 12x$
b $y = 108x^2 - 18x$

14 **a** $y = 6x^2 - 3x$
b $y = 4x^2 - 2x$

16 **a** $y = 3x^2 - 3x$
b $y = \frac{4}{3}x^2 - 2x$

For questions 17 to 20, use the method demonstrated in Worked Example 5.9.

17 **a** The graph of $y = \mathrm{f}(x)$ has a vertex at (2, 3). Find the coordinates of the vertex of the graph of $y = -\mathrm{f}(x)$.
b The graph of $y = \mathrm{f}(x)$ has a vertex at (5, 1). Find the coordinates of the vertex of the graph of $y = -\mathrm{f}(x)$.

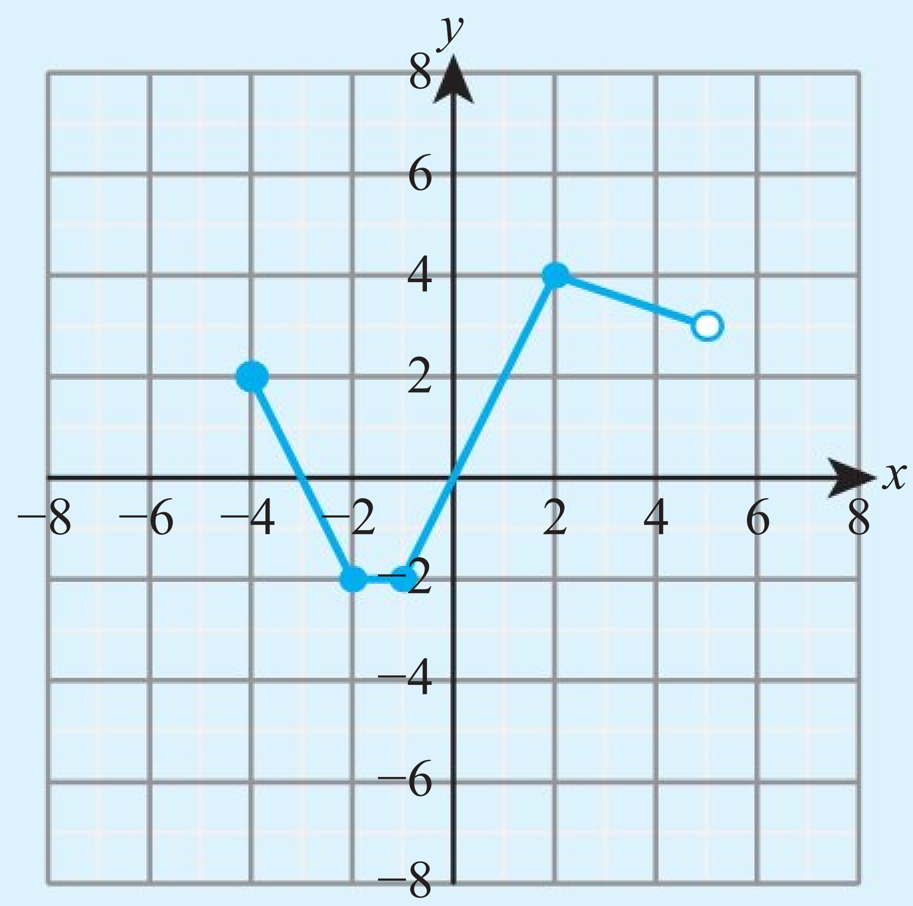
18 **a** The graph of $y = \mathrm{f}(x)$ has a vertex at (–2, –4). Find the coordinates of the vertex of the graph of $y = -\mathrm{f}(x)$.
b The graph of $y = \mathrm{f}(x)$ has a vertex at (–3, –1). Find the coordinates of the vertex of the graph of $y = -\mathrm{f}(x)$.

19 **a** The graph of $y = \mathrm{f}(x)$ has a vertex at (2, 3). Find the coordinates of the vertex of the graph of $y = \mathrm{f}(-x)$.
b The graph of $y = \mathrm{f}(x)$ has a vertex at (1, 5). Find the coordinates of the vertex of the graph of $y = \mathrm{f}(-x)$.

20 **a** The graph of $y = \mathrm{f}(x)$ has a vertex at (–2, –3). Find the coordinates of the vertex of the graph of $y = \mathrm{f}(-x)$.
b The graph of $y = \mathrm{f}(x)$ has a vertex at (–5, –1). Find the coordinates of the vertex of the graph of $y = \mathrm{f}(-x)$.

For questions 21 to 26, use the method demonstrated in Worked Example 5.10 to sketch the required graph.

The graph of $y = f(x)$ is given alongside.



21 **a** $y = f(x - 2) + 3$
b $y = f(x + 2) - 3$

22 **a** $y = 3f(2x)$
b $y = 2f(3x)$

23 **a** $y = 4f(-x)$
b $y = 2f(-x)$

24 **a** $y = 5 - 3f(x)$
b $y = 3 - 2f(x)$

25 **a** $y = f\left(\frac{1}{3}x + 1\right)$
b $y = f\left(\frac{1}{2}x + 2\right)$

26 **a** $y = f(2x - 1)$
b $y = f(2x - 2)$

For questions 27 to 30, use the method demonstrated in Worked Example 5.11 to describe a sequence of transformations that maps the graph of $y = 4x^2 + x$ to the graph with the given equation.

27 **a** $y = 12x^2 + 3x - 2$
b $y = 20x^2 + 5x + 1$

28 **a** $y = x^2 + \frac{1}{2}x - 1$
b $y = x^2 + \frac{1}{2}x + 3$

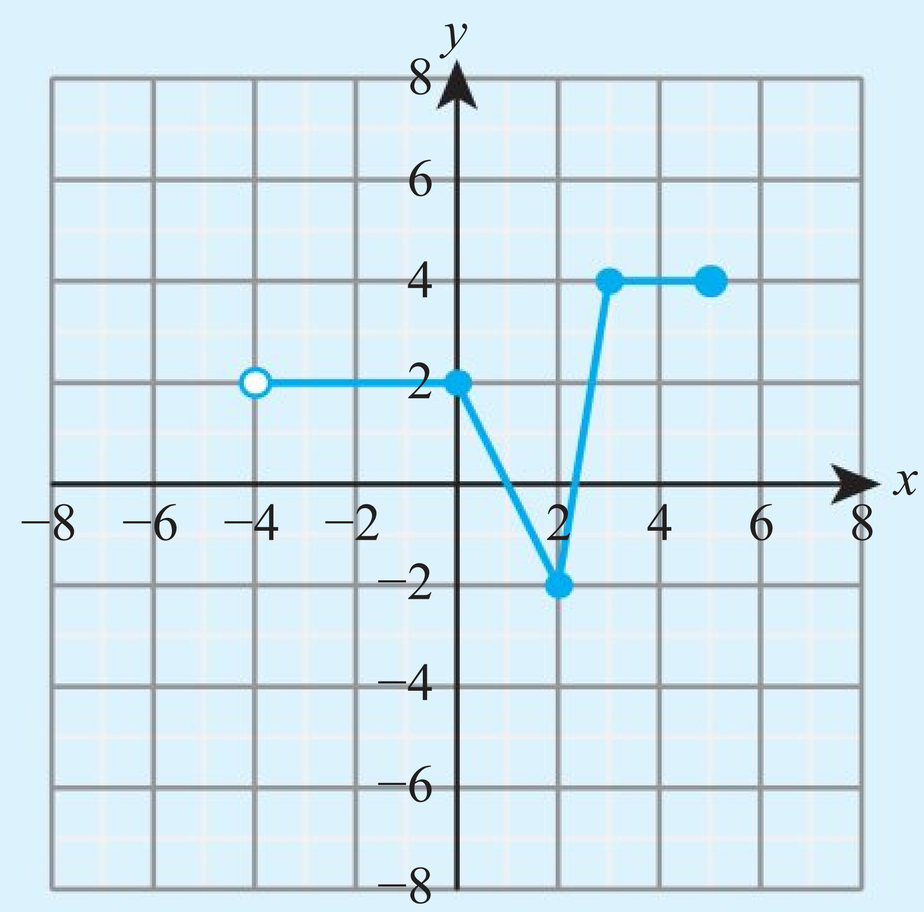
29 **a** $y = 4(x - 2)^2 + (x - 2) + 5$
b $y = 4(x + 3)^2 + (x + 3) - 4$

30 **a** $y = 36x^2 - 3x$
b $y = 16x^2 - 2x$

31 The graph of $y = f(x)$ is given alongside.

On separate diagrams, sketch the graph of

a $y = f(x - 2)$ **b** $y = 2f(x)$ **c** $y = f\left(\frac{1}{2}x\right) - 1$



32 The graph of $y = 3x^2 - 4x$ is translated 3 units in the negative x-direction and then stretched vertically with scale factor 4. Find the equation of the resulting graph, giving your answer in the form $y = ax^2 + bx + c$.

33 The graph of $y = e^x$ is translated horizontally 2 units in the positive direction and then stretched vertically with scale factor 3. Find the equation of the resulting graph.

34 **a** Write $x^2 - 10x + 11$ in the form $(x - h)^2 + k$.
b Hence describe a sequence of two transformations that map the graph of $y = x^2$ to the graph of $y = x^2 - 10x + 11$.

35 **a** Write $5x^2 + 30x + 45$ in the form $a(x + h)^2$.
b Hence describe a sequence of two transformations that map the graph of $y = x^2$ to the graph of $y = 5x^2 + 30x + 45$.

36 The graph of $y = x^2$ is stretched vertically with scale factor 9.
a Write down the equation of the resulting graph.
b Find the scale factor of a horizontal stretch that maps the graph from part **a** back to the graph of $y = x^2$.

37 **a** Find the equation of the graph obtained from the graph of $y = 2x^3$ by stretching it vertically with scale factor 8.
b Find the scale factor of a horizontal stretch that has the same effect on the graph of $y = 2x^3$.

38 The graph of $y = 2x^3 - 5x^2$ is reflected in the x-axis and then reflected in the y-axis. Find the equation of the resulting graph.

39 **a** Describe the transformations that maps the graph of $y = \sqrt{x}$ to $y = \sqrt{2x - 1}$.
b Describe the transformations that maps the graph of $y = \sqrt{x}$ to $y = \sqrt{2(x - 1)}$.

40 The graph of $y = ax^2 + bx + c$ is transformed by the following sequence:
- Translation by 2 units in the positive x direction
- Horizontal stretch with scale factor 3.

The resulting graph is $y = x^2 + cx + 14$.

Find the values of a, b and c.

41 Find two possible sequences of two transformations which will transform the graph of $y = \mathrm{f}(x)$ to the graph of $y = \mathrm{f}(4x + 8)$.

42 The graph of $y = \mathrm{f}(x)$ is stretched vertically with scale factor 5 and then translated 3 units up forming the graph of $y = \mathrm{g}(x)$.

a Write down the equation of the resulting graph.

b The graph of $y = \mathrm{g}(x)$ is instead formed from the graph of $y = \mathrm{f}(x)$ by translating upwards first and then stretching vertically. Describe fully the translation and the stretch.

43 The graph of $y = \mathrm{f}(x)$ is stretched vertically by a factor of 2 away from the line $y = 1$ (that is, $y = 1$ is kept invariant rather than the x-axis). Find the equation of the resulting graph.

5D Further modelling

Exponential models to calculate half-life

Exponential models are used to describe many situations – cooling, population growth and radioactive decay, amongst many others.

You have already met exponential models in Chapter 13 of Mathematics: applications and interpretation SL.

A form of an exponential model which shows how a quantity, y, changes with time, t, is:

$$y = y_0 e^{kt}$$

In this model, y_0 is the initial amount above the background level and k is a measure of how quickly the quantity is changing. If k is positive, there is exponential growth. If k is negative, there is exponential decay.

One important feature of these models is that there is a constant half-life, $t_{0.5}$. This is the time for the value of y to reach half of its current value.

KEY POINT 5.11

$$t_{0.5} = -\frac{\ln 2}{k}$$

Proof 5.2

Prove that if $y = y_0 e^{kt}$, then the half-life of y given by $t_{0.5} = -\frac{\ln 2}{k}$

Replace y with $\frac{y_0}{2}$. At this point $t = t_{0.5}$

$$\frac{y_0}{2} = y_0 e^{kt_{0.5}}$$

Divide both sides by y_0

$$\frac{1}{2} = e^{kt_{0.5}}$$

Multiply both sides by 2 and divide by $e^{kt_{0.5}}$. Use the laws of indices to write $\frac{1}{e^{-kt_{0.5}}}$ as $e^{-kt_{0.5}}$

$$e^{-kt_{0.5}} = 2$$

Take natural logs of both sides

$$-kt_{0.5} = \ln 2$$

Isolate $t_{0.5}$

$$t_{0.5} = -\frac{\ln 2}{k}$$

Then we have to argue that this half-life is not just the time it takes to first halve, but will be the case no matter when we start observing

Since the answer is independent of y_0, this result holds for any starting value so will be true at any time during the decay of y.

A more general exponential model takes into account a background level d – for example the temperature of the room which the cooling object is in. The form changes to

$$y = Ae^{kt} + d$$

In this situation, the half-life measures how long the object takes to reach half of the way to the background level.

WORKED EXAMPLE 5.12

The background radiation in a room is 50 mSv per year. A radioactive sample is introduced and the radiation level (R) increases to 80 mSv per year. If the half-life of the radioactive sample is 20 minutes, form an exponential model to estimate the radiation level after t minutes.

The background level is 50 so $d = 50$. The initial value is $A + d$ so $A = 30$

$$R = 50 + 30e^{kt}$$

Rearranging the equation for the half-life allows k to be found

$$k = -\frac{\ln 2}{t_{0.5}} = -\frac{\ln 2}{20} \approx -0.0347$$

So,

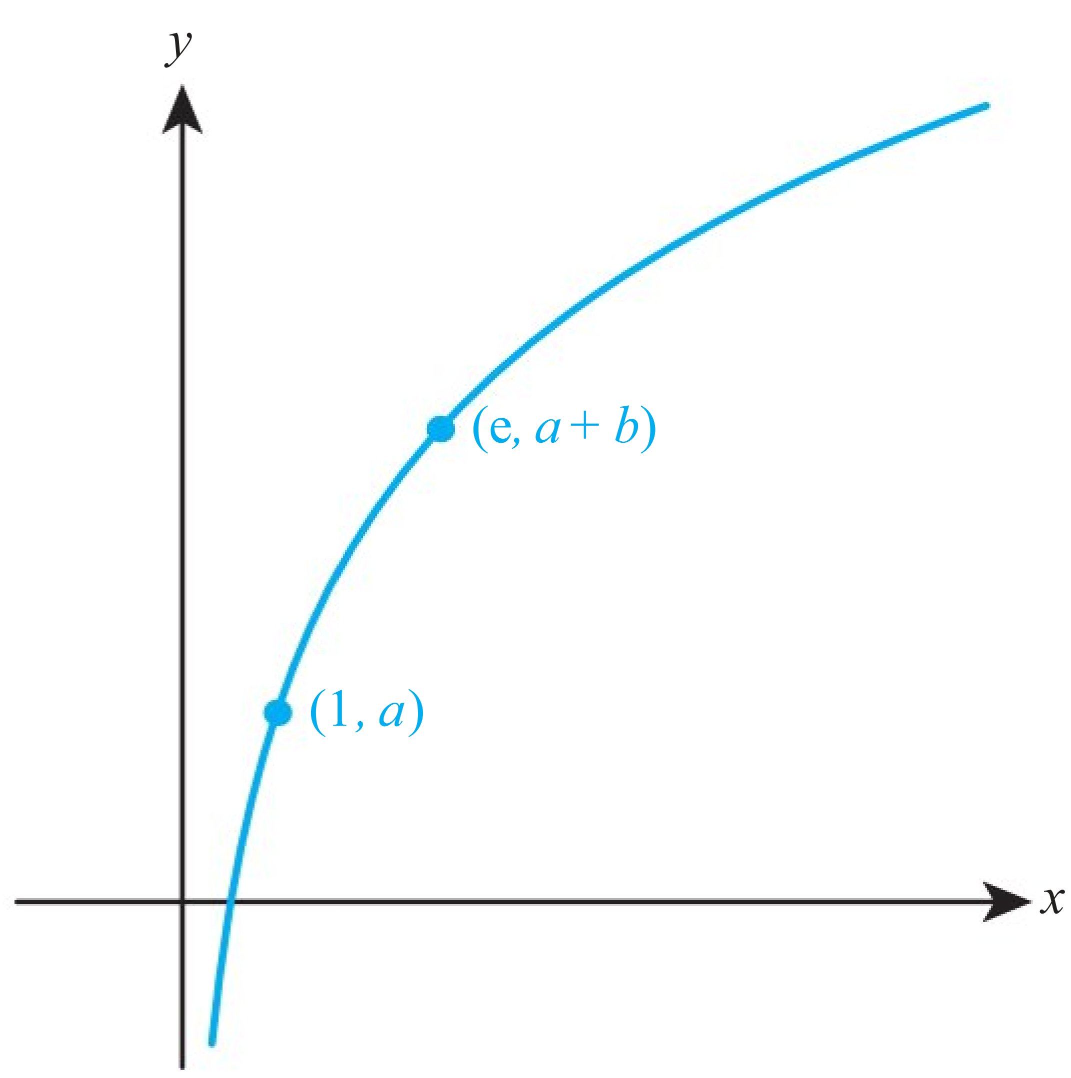
$$R \approx 50 + 30e^{-0.0347t}$$

Natural logarithmic models

A natural logarithmic model is of the form

$$y = a + b\ln x$$

These have the general shape shown below.



WORKED EXAMPLE 5.13

The pH of a chemical follows a logarithmic model based on the concentration of hydrogen ions, $[H^+]$, in units of moles per litre (M). When the concentration of hydrogen ions is 10^{-7} M, the pH is 7. When the concentration is 10^{-2} M, the pH is 2.

Use a logarithmic model of the form $\text{pH} = a + b\ln\left[H^+\right]$ to estimate the value of the pH when the concentration is 0.05 M.

There are two unknown parameters in the model so we need to use two equations
The first is formed using the information for the pH of 7

$$7 = a + b\ln\left(10^{-7}\right)$$
$$\approx a - 16.12b$$

The second is formed using the information for the pH of 2

$$2 = a + b\ln\left(10^{-2}\right)$$
$$\approx a - 4.605b$$

We can use technology to solve these equations simultaneously to find the parameters of the model

From the GDC:
$a = 0,\ b \approx -0.434$

We need to substitute $[H^+] = 0.05$ into the model to answer the question

Therefore, when $[H^+] = 0.05$
$\text{pH} \approx -0.434\ln(0.05) \approx 1.30$

Sinusoidal models with phase shifts

Basic sinusoidal models were covered in Chapter 13 of Mathematics: applications and interpretation SL.

You already know that a sinusoidal model of the form

$$y = a\sin bx + d$$

is an oscillating function with amplitude a oscillating around the central line $y = d$. When working in degrees, the period was $\frac{360}{b}$ but the equivalent when working in radians is that the period is $\frac{2\pi}{b}$. In Higher level papers, if you are not told otherwise, you should assume that the angles are measured in radians.

Using the graph transformations from Section 5C, you can now also move this function left and right. This is called a **phase shift**, denoted by c, and it results in a similar function.

KEY POINT 5.12

A sinusoidal model is of the form

$$y = a\sin(b(x - c)) + d$$

where a is the amplitude, the period is $\frac{2\pi}{b}$, c is the phase shift and $y = d$ is the central line.

TOOLKIT: Modelling

The phrase 'period' suggests that x is a measure of time, but actually the oscillation might be occurring over space, in which case the 'period' is actually the distance between peaks over space – more commonly called the wavelength. In reality, waves vary across both space and time. This is modelled by something called the wave equation. You might like to research this very important differential equation. It has solutions which look like $\sin(x - ct)$, where x is a position from an origin and t is a time. Try using technology to explore this function and see if you can find out the meaning of c in this equation.

Tip

It is a good idea to check your answer by sketching it on your GDC.

WORKED EXAMPLE 5.14

The air pressure (P kPa) close to a tuning fork, at a time t seconds after the fork is struck, is modelled by a sinusoidal function. The average pressure is 100 kPa and the amplitude is 2 kPa. The period is 0.005 seconds. The first time that the pressure passes through 100 kpa is after 0.001 seconds and at this point the pressure is increasing. Find a model relating P and t.

First, use the period to find the value of b. This is $\frac{2\pi}{T}$

The period is 0.005 seconds so

$$b = \frac{2\pi}{0.005} \approx 1260$$

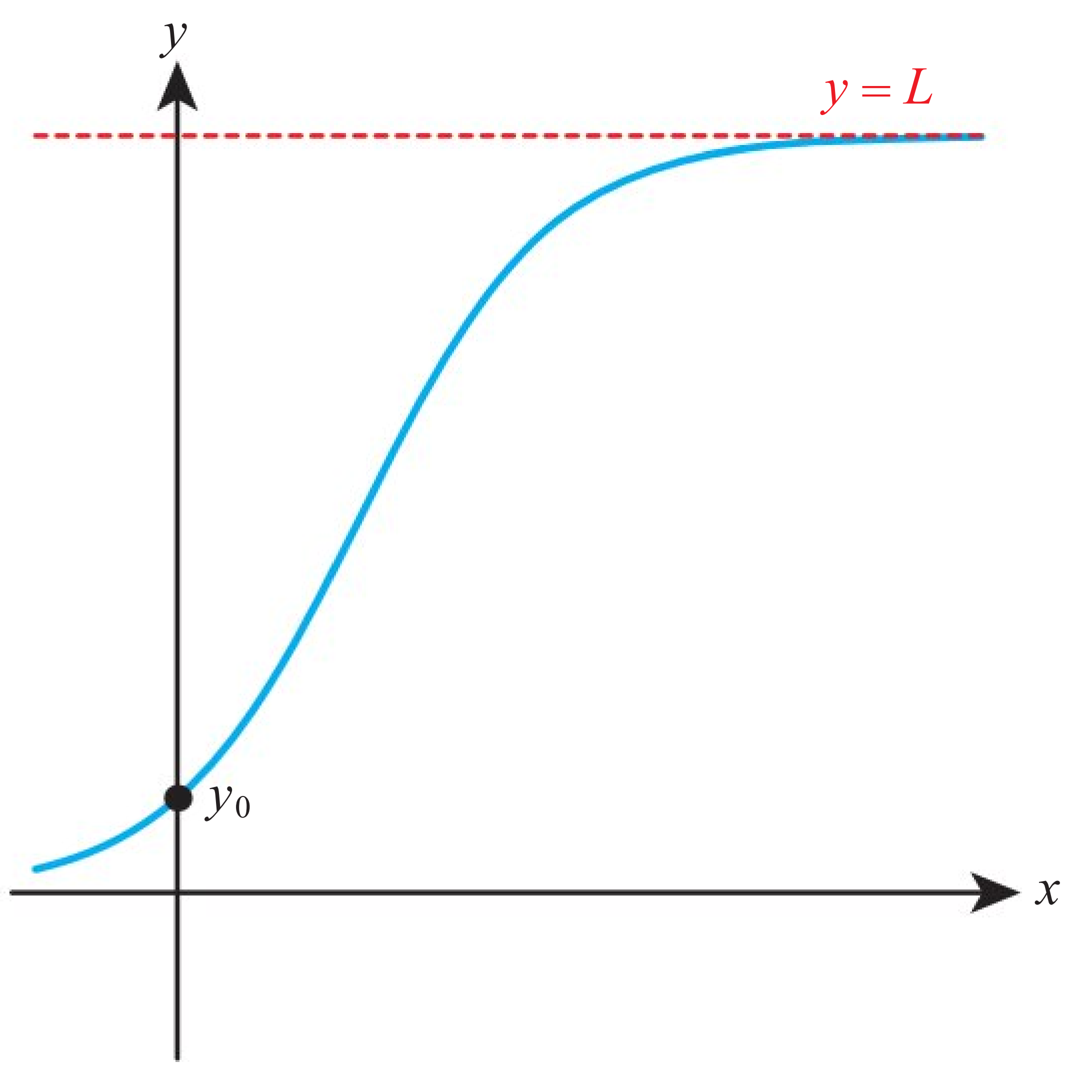
The amplitude is 2, the central line is 100 and the phase shift is 0.001

$$P \approx 2\sin(1260(t - 0.001)) + 100$$

Logistic models

Many populations do not continue growing for ever. For example, the number of bacteria in a petri dish will initially grow slowly when the population is small, then it will accelerate but will eventually be constrained by the size of the petri dish or the nutrients available.

The type of curve we would expect is shown below.



This shape is often modelled using a logistic function.

KEY POINT 5.13

A logistic model is of the form

$$y = \frac{L}{1 + Ce^{-kx}}$$

where L is the eventual population size (also known as the carrying capacity), $\frac{L}{1+C}$ is the initial population and k measures how quickly the population is growing.

WORKED EXAMPLE 5.15

10 rabbits are introduced onto an island which was previously empty. The capacity of the island is 1000 rabbits. After one year, there are 40 rabbits. Use a logistic model to estimate the population after 2 years.

First find L using the fact that it is the carrying capacity

$$L = 1000$$

Find C using the initial population – i.e. when $x = 0$

$$\frac{1000}{1+C} = 10$$

Rearrang the equation

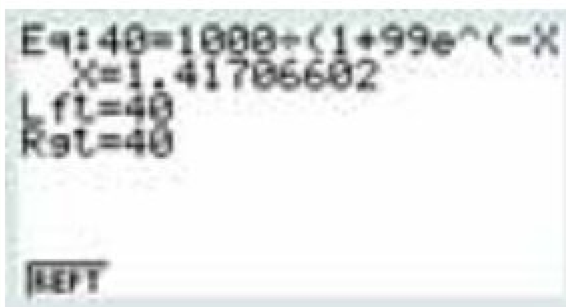
$$100 = 1 + C$$
$$C = 99$$

Substitute $x = 1$

$$40 = \frac{1000}{1+99e^{-k}}$$

Solve this equation using technology

From the GDC, $k \approx 1.417$



To answer the question, we need to estimate the population when $x = 2$

So, when $x = 2$,

$$y \approx \frac{1000}{1+99e^{-1.417\times 2}} \approx 146$$

Piecewise models

You have already met piecewise linear models. However, it is possible to have more general models with different rules in different regions. Often, in realistic contexts, these models will be continuous, meaning that there are no jumps in the graphs.

WORKED EXAMPLE 5.16

Find a to make f(x) continuous where

$$f(x) = \begin{cases} 1 + x & 0 \leqslant x < 2 \\ ax^2 + x & x \geqslant 2 \end{cases}$$

Look at the value of f(x) on the boundary of the first branch

$$f(2) = 3$$

Look at the value of f(x) on the boundary of the second branch

$$f(2) = 4a + 2$$

If the function is continuous, f(2) must be the same on both branches so we can equate the two values found

$$3 = 4a + 2$$

Solve this equation

$$1 = 4a$$
$$a = \frac{1}{4}$$

Tip

Try sketching the solution to this function on your calculator. (You might need to look up how to do a piece-wise function in your manual.) Does it make sense that this is called a continuous function? Compare it to a graph with a different value of a.

Exercise 5D

In questions 1 to 3, use the method demonstrated in Worked Example 5.12 to create an exponential model for y as a function of t, given the following information.

1 **a** Initial population 10, background level 0, half-life 2

b Initial population 7, background level 0, half-life 10

2 **a** Initial population 12, background level 10, half-life 5

b Initial population 20, background level 15, half-life 7

3 **a** Initial population 1, background level 1.5, half-life 0.1

b Initial population 100, background level 120, half-life 0.05

In questions 4 to 6, use the method demonstrated in Worked Example 5.13 to create a natural logarithmic model for y as a function of x, given that it passes through the two points stated.

4 **a** (1, 3) and (e, 5)

b (1, 10) and (e, 15)

5 **a** (2, 3) and (3, 5)

b (4, 4) and (5, 5)

6 **a** (3, 6) and (10, 0)

b (2, 4) and (4, 2)

In questions 7 to 9, use the method demonstrated in Worked Example 5.14 to find a sinusoidal model with the information given.

7 **a** Amplitude 5, central line $y = 3$, period 4, phase shift 1

b Amplitude 10, central line $y = 0$, period 3, phase shift 2

8 **a** Amplitude 6, average value 2, wavelength 10, phase shift −2

b Amplitude 10, average value −4, wavelength 6, phase shift −1

9 **a** Maximum value 3, minimum value 0, period 8, phase shift 4

b Maximum value 2, minimum value −4, period 2, phase shift 0.5

In questions 10 to 12, use the method demonstrated in Worked Example 5.15 to form a suitable logistic model from the given information.

10 **a** Carrying capacity 500, initial population 100, population of 200 after 1 year

b Carrying capacity 800, initial population 40, population of 500 after 1 year

11 **a** Carrying capacity 2000, initial population 500, population of 1000 after 4 years

b Carrying capacity 70, initial population 25, population of 40 after 3 years

12 **a** Carrying capacity 200, initial population 40, population of 150 after 2 years

b Carrying capacity 200, initial population 100, population of 150 after 2 years

In questions 13 to 18, use the method demonstrated in Worked Example 5.16 to find the values of a which make the following functions continuous at all points.

13 **a** $f(x) = \begin{cases} -x & x \leqslant 1 \\ x + a & x > 1 \end{cases}$

b $f(x) = \begin{cases} 3x & x \leqslant 2 \\ x + a & x > 2 \end{cases}$

14 **a** $f(x) = \begin{cases} x + 1 & x \leqslant 2 \\ ax & x > 2 \end{cases}$

b $f(x) = \begin{cases} x + 1 & x \leqslant 2 \\ ax + 2 & x > 2 \end{cases}$

15 **a** $f(x) = \begin{cases} x^2 & x \leqslant 3 \\ x + a & x > 3 \end{cases}$

b $f(x) = \begin{cases} x^2 - 4 & x \leqslant 2 \\ x + a & x > 2 \end{cases}$

16 **a** $f(x) = \begin{cases} x + 6 & x \leqslant 2 \\ x^a & x > 2 \end{cases}$

b $f(x) = \begin{cases} 16x + 16 & x \leqslant 1 \\ (x + 1)^a & x > 1 \end{cases}$

17 **a** $f(x) = \begin{cases} e^{ax} & x \leqslant 2 \\ \dfrac{1}{x} & x > 2 \end{cases}$

b $f(x) = \begin{cases} e^{ax} & x \leqslant 1 \\ x + 3 & x > 1 \end{cases}$

18 **a** $f(x) = \begin{cases} x^2 & x \leqslant 1 \\ \ln(ax) & x > 1 \end{cases}$

b $f(x) = \begin{cases} x^2 & x \leqslant 2 \\ \ln(ax) & x > 2 \end{cases}$

19 A radioactive substance decays so that its mass, m mg, at time t days is given by the equation $m = 1.2e^{-0.26t}$. Find the number of complete days required for the mass to decrease to below

a half of its initial value

b 0.1 mg.

20 Ice cream mixture, initially at the room temperature of 23°C, is placed in a large bowl of ice. The temperature of the mixture (T°C) after t minutes is modelled by the equation $T = Ae^{-kt}$.

- **a** Given that, 15 minutes later, the temperature of the mixture is half of its initial value, find the value of k.
- **b** How long does it take the mixture to cool down to 8°C?
- **c** State one modelling assumption that was used in this model.

21 The strength of an earthquake on the Richter scale, R, depends on the amplitude of the measurement on the seismograph, A mm, and is thought to follow a logarithmic model of the form $R = p \ln A + q$. When the amplitude is 2 mm, the strength of the earthquake is 3. When the amplitude is 25 mm, the strength of the earthquake is 4.

- **a** Find the values of p and q.
- **b** Estimate the strength of an earthquake for which the measurement of the seismograph is 100 mm.
- **c** The seismograph needs to be able to measure earthquakes of strength between 2 and 5 on the Richter scale. Find the range of amplitudes the seismograph needs to be able to measure.

22 The volume of water (V cm^3) in a bucket, at time t seconds, is modelled by

$$V = \begin{cases} pt & 0 \leqslant t < 5 \\ qt^2 & t \geqslant 5 \end{cases}$$

After 4 seconds the volume is 100 cm^3.

- **a** Find the value of p.
- **b** Find the value of q.
- **c** The capacity of the bucket is 4500 cm^3. Use the model to predict how long it takes for the bucket to fill.

23 The time taken for a bacteria population to double is 3 hours. The initial population size is P_0.

Find an expression for the size of the population, P, in terms of the time, t hours, in the form

$$P = Ae^{kt}$$

24 The size of semiconductor chips, S microns, is thought to follow the model $S = 400 \times 4^{-t}$, where t is the number of decades since the semiconductor chips were invented.

- **a** Find the time it takes for the size of semiconductor chips to halve.
- **b** According to this model, how long will it take for the size of semiconductor chips to reach 1 micron?

You are the Researcher

Find out about Moore's law for the development of transistor circuits.

25 A cup of coffee has a temperature of 90°C when it is first poured. The temperature of the surrounding air is 20°C. The temperature of the coffee is modelled by the function $T = B + Ae^{-kt}$, where T°C is the temperature of the coffee t minutes after it was poured.

- **a** Find the values of A and B.
- **b** It takes 4 minutes for the temperature difference between the coffee and the surrounding air to decrease to 40°C. Find the value of k.
- **c** The coffee is safe to drink when its temperature decreases below 50°C. How long after first being poured is the coffee safe to drink?

26 A cake, initially at the room temperature of 22°C, is placed in the oven. The temperature of the cake, T°C, is modelled by the equation $T = 200 - Ae^{-kt}$, where t is the time, in minutes, since the cake was placed in the oven. You may assume that the temperature of the oven remains constant.

- **a** Write down the temperature of the oven, and the initial difference between the temperature of the cake and the oven.
- **b** Find the value of A.
- **c** It takes 4 minutes for the temperature of the cake to increase to 111°C. Find the value of k.
- **d** How long does it take for the temperature of the cake to reach 180°C?

27 The mass of a radioactive substance (M g) follows an exponential model of the form $M = M_0 e^{-kt}$, where t is the time measured in years. It takes ten years for the mass of the substance to decrease to one quarter of its initial value.

a Write down the half-life of the substance.

b Find the value of k.

c The mass of the substance after 20 years is 0.6 g. Find the initial amount of the substance.

28 A mobile phone factory has an assembly unit and a packing unit. The power consumption of both units is modelled using a sinusoidal function with period 1 day. The assembly unit has maximum power consumption 50 MW and minimum power consumption 10 MW. When $t = 0$, the power consumption is 30 MW and increasing.

a Find a model of the form $P_A = a\sin bt + d$ for the power consumption of the assembly unit at time t hours.

The maximum and minimum power consumptions of the packing unit are half of those for the assembly unit. The first maximum occurs four hours later than the maximum for the assembly unit.

b Find a model of the form $P_P = A\sin B(t - C) + D$ for the power consumption of the packing unit at time t hours.

c Find the first time at which the two units have the same power consumption.

d Use technology to sketch the total power consumption of the two units over a 24-hour period, and find its maximum value.

29 **a** On the same diagram, sketch the graphs of $y_1 = 2\sin x$ and $y_2 = \sin\left(x - \frac{\pi}{3}\right)$ for $0 \leqslant x \leqslant 4\pi$.

b Use technology to sketch the graph of $y_3 = y_1 + y_2$ for $0 \leqslant x \leqslant 4\pi$, labelling all maximum and minimum points.

c Hence suggest an expression for y_1 in the form $a\sin b(x - c)$.

30 A new species of bird is introduced to an island, which can sustain a maximum of 500 birds. The number of birds, (N), t years after the initial introduction, is modelled by a logistic function of the form $N = \frac{L}{1 + Ce^{-kt}}$.

a Given that after two years there are 300 birds on the island, show that $C = \frac{2}{3}e^{2k}$.

b Given that after *another* three years there are 400 birds on the island, find another similar equation for C in terms of k.

c Using graphs, or otherwise, find the values of k and C.

d How many birds were initially introduced to the island?

e Find the number of *whole* years required for the number of birds to exceed 480.

31 Rabbits are introduced to an island at a rate of 100 rabbits per year, for three years.

a State any assumptions required and find the population of rabbits at the end of three years.

After three years, rabbits are no longer introduced and the population is thought to follow a logistic model with a carrying capacity of 1000 rabbits. Four years after the rabbits were first introduced, the population is 450.

b Using this model, estimate the population five years after the rabbits were first introduced.

32 A petri dish has the carrying capacity of 2000 bacteria. At time $t = 0$, 2500 bacteria are added to the empty dish. The number (N) of the bacteria in the dish, at time t hours, is modelled by the logistic function $N = \frac{L}{1 + Ce^{-kt}}$.

a Write down the value of L, and find the value of C.

b Given that after one hour there are around 2200 bacteria in the dish, estimate the value of k correct to one decimal place.

c Sketch the graph showing the number of bacteria as a function of time.

33 The initial population of worms in a garden is N_0. In the first half of the year, the population grows exponentially, so that the time taken for the population to double is $\frac{\ln 2}{\alpha}$. In the second half of the year, the population decays exponentially, and the time taken for the population to halve is $\frac{\ln 2}{\beta}$.

a Find an expression for $N(t)$, where N is the number of worms, t is the time in years, and $0 \leqslant t \leqslant 1$.

b This behaviour repeats every year. Find the condition on α and β for the annual population to increase on average.

34 A population of tigers in a wildlife reserve is modelled as decaying exponentially with a half-life of two years. The initial population is 200 tigers.

- **a** Find an expression for the population, N tigers, after t years.
- **b** State a condition on N that can be interpreted as tigers becoming extinct.
- **c** Use the condition from part **b** to estimate the time for the population to become extinct.
- **d** Explain why this model is likely to break down before the point of extinction.

35 The temperature, T°C of water in a kettle t minutes after boiling is modelled by an exponential decay. The room temperature stays at a constant value of 20°C. The temperature at boiling point is 100°C and it takes q minutes to reach 30°C.

- **a** Find an equation linking T and t in the form

$$T = a + be^{-kt}$$

where k is written in terms of q.

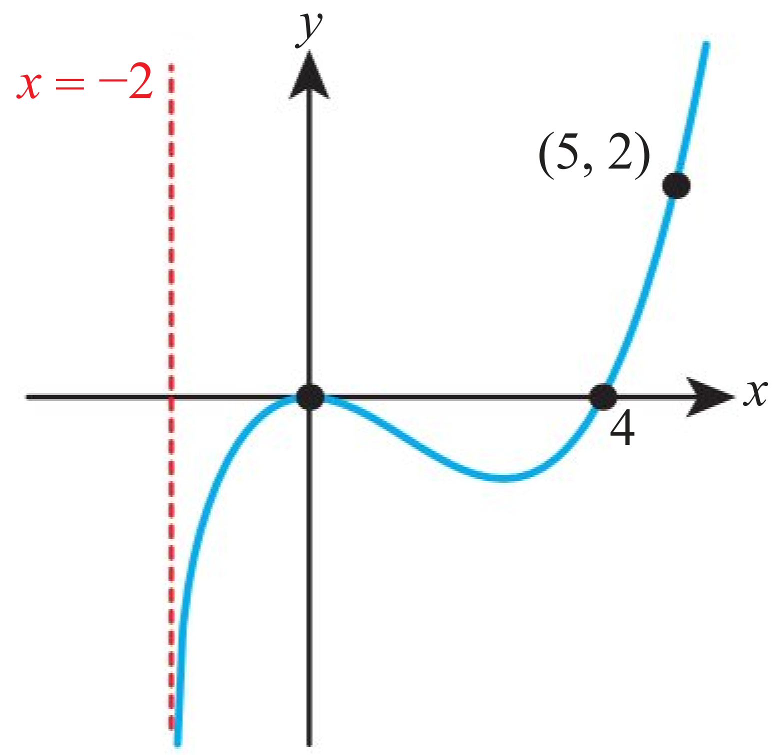
- **b** Use your answer to part **a** to estimate the time it takes for the water in the kettle to reach 50°C, giving your answer in terms of q.

Checklist

- You should be able to form composite functions.
- You should know when composite functions exist.
- You should be able to find inverse functions.
  - $(\mathrm{f} \circ \mathrm{f}^{-1})(x) = (\mathrm{f}^{-1} \circ \mathrm{f})(x) = x$
  - To find an expression for $\mathrm{f}^{-1}(x)$:
    1. Let $y = \mathrm{f}(x)$
    2. Rearrange to make x the subject
    3. State $\mathrm{f}^{-1}(x)$ by replacing any ys with xs.
- You should know when inverse functions exist.
  - A function has to be one-to-one to have an inverse.
- You should be able to apply single transformations to graphs.
  - $y = \mathrm{f}(x) + b$ is a vertical translation by b
  - $y = \mathrm{f}(x + c)$ is a horizontal translation by $-c$
  - $y = p\mathrm{f}(x)$ is a vertical stretch with scale factor p
  - $y = \mathrm{f}(qx)$ is a horizontal stretch with scale factor $\frac{1}{q}$
  - $y = -\mathrm{f}(x)$ is a reflection in the x-axis
  - $y = \mathrm{f}(-x)$ is a reflection in the y-axis
- You should be able to apply composite transformations to graphs.
  - When two vertical transformations are applied, the order matters: $y = p\mathrm{f}(x) + b$ is a horizontal stretch with scale factor p followed by a vertical translation of b.
  - When two horizontal transformations are applied, the order matters: $y = \mathrm{f}(qx + c)$ is a horizontal translation of $-c$ followed by a horizontal stretch with scale factor $\frac{1}{q}$.
  - When one vertical and one horizontal transformation are applied, the order does not matter.
- You should be able to use exponential, logarithmic, sinusoidal, logistic and piecewise models.
  - For an exponential model of the form $y = Ae^{-kt}$, the half-life is $t_{0.5} = -\frac{\ln 2}{k}$.
  - The sinusoidal model of the form $y = a\sin(b(x - c)) + d$ has amplitude a, period $\frac{2\pi}{b}$, phase shift c and central line $y = d$.
  - In a logistic model, $y = \frac{L}{1 + Ce^{-kx}}$, L is the carrying capacity.

Mixed Practice

1 The graph of $y = \mathrm{f}(x)$ is shown.



Sketch the following graphs, indicating the positions of asymptotes and x-intercepts.

a $y = 2\mathrm{f}(x-3)$

b $y = -\mathrm{f}(2x)$

c $y = \mathrm{f}\left(\frac{x}{3}\right) - 2$

2 **a** For the function $\mathrm{f}(x) = 3x - 1$, find the inverse function, $\mathrm{f}^{-1}(x)$.

b Verify that $(\mathrm{f} \circ \mathrm{f}^{-1})(x) = x$ for all x.

3 A function is defined by $\mathrm{h}(x) = \sqrt{5-x}$ for $x \leqslant a$.

a State the largest possible value of a.

b Find $\mathrm{h}^{-1}(3)$.

4 Given that $\mathrm{f}(x) = \mathrm{e}^{3x}$, evaluate $\mathrm{f}^{-1}(4)$.

5 The table shows some values of the function $\mathrm{f}(x)$.

x	0	1	2	3	4
$\mathrm{f}(x)$	3	4	0	1	2

a Find $(\mathrm{f} \circ \mathrm{f})(2)$.

b Find $\mathrm{f}^{-1}(4)$.

6 Let $\mathrm{f}(x) = x + 3$ and $\mathrm{g}(x) = \mathrm{e}^{2x}$. Solve the equation $(\mathrm{g} \circ \mathrm{f})(x) = 1$.

7 Given that $\mathrm{g}(x) = 3\ln(x-2)$,

a find the largest possible domain for g

b using the domain for part **a**, find an expression for $\mathrm{g}^{-1}(x)$ and state its range.

8 Given the functions $\mathrm{f}(x) = 3x + 1$ and $\mathrm{g}(x) = x^3$, find $(\mathrm{f} \circ \mathrm{g})^{-1}(x)$.

9 Let $\mathrm{f}(x) = 2x + 3$ and $\mathrm{g}(x) = x^3$.

a Find $(\mathrm{f} \circ \mathrm{g})(x)$.

b Solve the equation $(\mathrm{f} \circ \mathrm{g})(x) = 0$.

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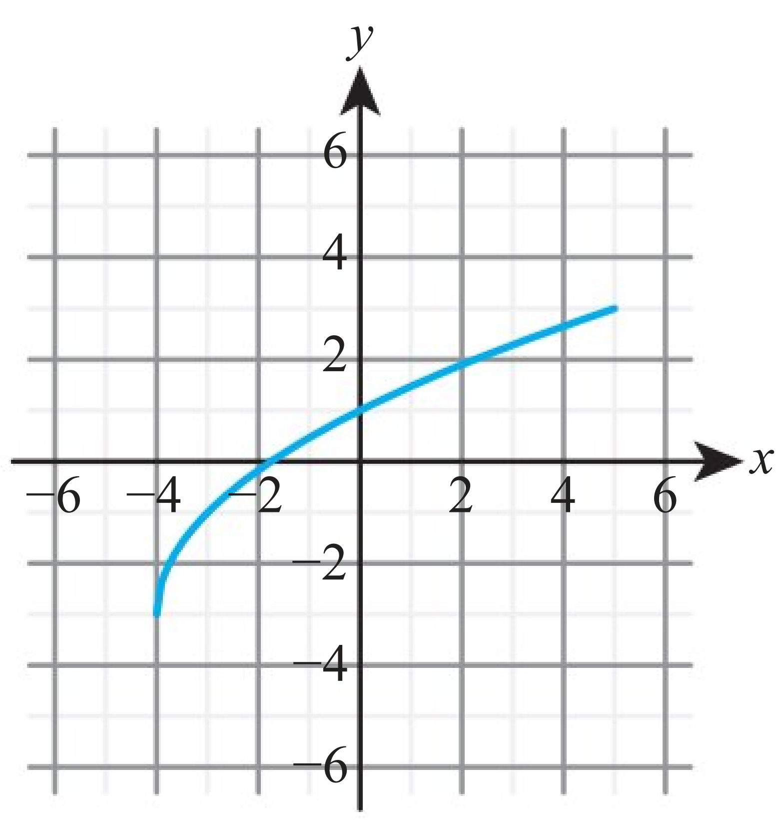
10 The following diagram shows the graph of $y = \mathrm{f}(x)$, for $-4 \leqslant x \leqslant 5$.



a Write down the value of

i $\mathrm{f}(-3)$ **ii** $\mathrm{f}^{-1}(1)$.

b Find the domain of f^{-1}.

c Sketch the graph of f^{-1}.

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11 For the function $\mathrm{f}(x) = x - 2$ and $\mathrm{g}(x) = \frac{1}{x-1}$ $(x \neq 1)$,

a find $(\mathrm{f} \circ \mathrm{g})(x)$ and state its domain

b verify that $(\mathrm{f} \circ \mathrm{g})^{-1}(x) = (\mathrm{g}^{-1} \circ \mathrm{f}^{-1})(x)$ for all x.

12 Given that $\mathrm{f}(x) = \frac{3x-1}{x+4}$ for $x \neq -4$, find an expression for $\mathrm{f}^{-1}(x)$.

13 Let the function $\mathrm{g}(x) = x + \mathrm{e}^x$ be defined for all real numbers x.

a By sketching a graph, or otherwise, show that g has an inverse function.

b Solve the equation $\mathrm{g}^{-1}(x) = 2$.

14 Let $f(x) = \sqrt{x}$ $(x \geqslant 0)$ and $g(x) = 9^x$ $(x \in \mathbb{R})$.

a Evaluate $(g \circ f)\left(\frac{1}{4}\right)$.

b Solve the equation $(f^{-1} \circ g)(x) = \frac{1}{3}$.

15 Let $h(x) = \ln(x - 2)$ for $x > 2$, and $g(x) = e^x$ for $x \in \mathbb{R}$.

a Find $h^{-1}(x)$ and state its range.

b Find $(g \circ h)(x)$, giving your answer in the form $ax + b$, where $a, b \in \mathbb{Z}$.

16 Function f is defined by $f(x) = (2x + 1)^2$ for $x \leqslant a$.

a By using a graph, or otherwise, find the largest value of a for which f has an inverse function.

b Find an expression for $f^{-1}(x)$.

17 Let $f(x) = x^2 + 3$, $x \geqslant 1$ and $g(x) = 12 - x$.

a Evaluate f(5).

b Find and simplify an expression for $gf(x)$.

c **i** State the geometric relationship between the graphs of $y = f(x)$ and $y = f^{-1}(x)$.

ii Find an expression for $f^{-1}(x)$.

iii Find the range of $f^{-1}(x)$.

18 Let $f(x) = 3x + 1$, $x \in \mathbb{R}$ and $g(x) = \frac{x+4}{x-1}$, $x \neq 1$.

a Find and simplify

i f(7)

ii $fg(x)$

iii $ff(x)$.

b State the range of f. Hence explain why $gf(x)$ does not exist.

c **i** Show that $g^{-1}(x) = g(x)$ for all $x \neq 1$.

ii State the range of $g^{-1}(x)$.

19 Let $f(x) = \sqrt{x - 5}$, for $x \geqslant 5$.

a Find $f^{-1}(2)$.

b Let g be a function such that g^{-1} exists for all real numbers. Given that $g(30) = 3$, find $(f \circ g^{-1})(3)$.

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20 Let $f(x) = 3x - 2$ and $g(x) = \frac{5}{3x}$, for $x \neq 0$.

a Find $f^{-1}(x)$.

b Show that $(g \circ f^{-1})(x) = \frac{5}{x+2}$.

Let $h(x) = \frac{5}{x+2}$, for $x \geqslant 0$. The graph of h has a horizontal asymptote at $y = 0$.

c **i** Find the y-intercept of the graph of h.

ii Hence sketch the graph of h.

d For the graph of h^{-1},

i write down the x-intercept

ii write down the equation of the vertical asymptote.

e Given that $h^{-1}(a) = 3$, find the value of a.

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21 For the functions $f(x) = e^{2x}$ and $g(x) = \ln(x - 2)$, verify that $(f \circ g)^{-1}(x) = (g^{-1} \circ f^{-1})(x)$.

22 Let $f(x) = \frac{1}{1 + \sqrt{x}}$ and $g(x) = x + 7$. Solve $f^{-1}(g^{-1}(x)) = 9$.

23 Let $f(x) = 2 + x - x^3$ for $x \geqslant a$.

a Find the smallest value of a so that f has an inverse function.

b State the geometric relationship between the graphs of $y = f(x)$ and $y = f^{-1}(x)$.

c Find the exact solution of the equation $f^{-1}(x) = f(x)$.

24 Consider the functions given below.

$f(x) = 2x + 3$

$g(x) = \dfrac{1}{x}, x \neq 0$

a **i** Find $(g \circ f)(x)$ and write down the domain of the function.

ii Find $(f \circ g)(x)$ and write down the domain of the function.

b Find the coordinates of the point where the graph of $y = f(x)$ and the graph of $y = (g^{-1} \circ f \circ g)(x)$ intersect.

Mathematics HL May 2011 Paper 1 TZ1 Q8

25 The graph of $y = x^3 - 2x$ is translated 3 units to the right and then stretched vertically with scale factor 2. Find the equation of the resulting graph in the form $y = ax^3 + bx^2 + cx + d$.

26 **a** Write $x^2 + 4x + 9$ in the form $(x + h)^2 + k$.

b Hence describe a sequence of two transformations which transform the graph of $y = x^2$ to the graph of $y = x^2 + 4x + 9$.

27 The graph of $y = \dfrac{1}{x}$ is translated 2 units in the negative x direction and then stretched vertically with scale factor 3.

a Write down the equation of the resulting graph.

b Sketch the graph, indicating any asymptotes and intercepts.

28 **a** Show that $2 + \dfrac{1}{x-5} = \dfrac{2x-9}{x-5}$.

b Hence describe two transformations which map the graph of $y = \dfrac{1}{x}$ to the graph of $y = \dfrac{2x-9}{x-5}$.

c State the equations of the asymptotes of the graph of $y = \dfrac{2x-9}{x-5}$.

29 Sketch the following graphs. In each case, indicate clearly the positions of the vertical asymptote and the x-intercept.

a $y = \ln x$

b $y = 3\ln(x - 2)$

c $y = 5 - \ln(3x)$

30 The graph of $y = ax + b$ is transformed using the following sequence of transformations:

- translation 3 units to the right
- vertical stretch with scale factor 7
- reflection in the x-axis.

The resulting graph has equation $y = 35 - 21x$. Find the values of a and b.

31 Find two transformations which transform the graph of $y = 9(x - 3)^2$ to the graph of $y = 3(x + 2)^2$.

32 The graph of $y = \ln x$ is translated 2 units to the right, then translated 3 units up and finally stretched vertically with scale factor 2. Find the equation of the resulting graph, giving your answer in the form $y = \ln(g(x))$.

33 For the graph of $y = \dfrac{4x-3}{2x+7}$,

a write down the equations of the asymptotes
b find the axis intercepts.
c Hence sketch the graph.

34 Let $f(x) = \dfrac{3x-1}{x+5}$.

a Write down the equations of the asymptotes of the graph of $y = f(x)$.
b Hence state the domain and the range of $f(x)$.

35 **a** Describe fully the transformation which maps the graph of $y = \ln x$ to the graph of $y = \ln(x+3)$.
b On the same diagram, sketch the graphs of $y = \ln(x+3)$ and $y = \ln(x^2+6x+9)$ for $x > -3$.

36 The quadratic function $f(x) = p + qx - x^2$ has a maximum value of 5 when $x = 3$.

a Find the value of p and the value of q.
b The graph of $f(x)$ is translated 3 units in the positive direction parallel to the x-axis. Determine the equation of the new graph.

Mathematics HL May 2011 Paper 1 TZ1 Q1

37 Let $f(x) = p + \dfrac{9}{x-q}$, for $x \neq q$. The line $x = 3$ is a vertical asymptote to the graph of f.

a Write down the value of q.

The graph of f has a y-intercept at (0, 4).

b Find the value of p.
c Write down the equation of the horizontal asymptote of the graph of f.

Mathematics SL November 2014 P1 Q5

38 The number of fish, N, in a pond is decreasing according to the model

$$N(t) = ab^{-t} + 40\ ,\ t \geqslant 0$$

where a and b are positive constants, and t is the time in months since the number of fish in the pond was first counted.
At the beginning, 840 fish were counted.

a Find the value of a.

After 4 months, 90 fish were counted.

b Find the value of b.

The number of fish in the pond will **not** decrease below p.

c Write down the value of p.

Mathematical Studies May 2015 Paper 1 TZ2 Q14

39 Let $f(x) = \ln x$. The graph of f is transformed into the graph of the function g by a translation of 3 to the right and 2 down, followed by a reflection in the x-axis. Find an expression for $g(x)$, giving your answer as a single logarithm.

Mathematics HL May 2012 Paper 2 TZ1 Q6

40 Let $f(x)=\dfrac{2x-9}{x-5}$.

a State the equations of the vertical and horizontal asymptotes on the graph of $y = f(x)$.

b Find the values of α and β if $f(x) \equiv \alpha + \dfrac{\beta}{x-5}$.

c State two consecutive transformations which can be applied to the graph of $y=\dfrac{1}{x}$ to create the graph $y = f(x)$.

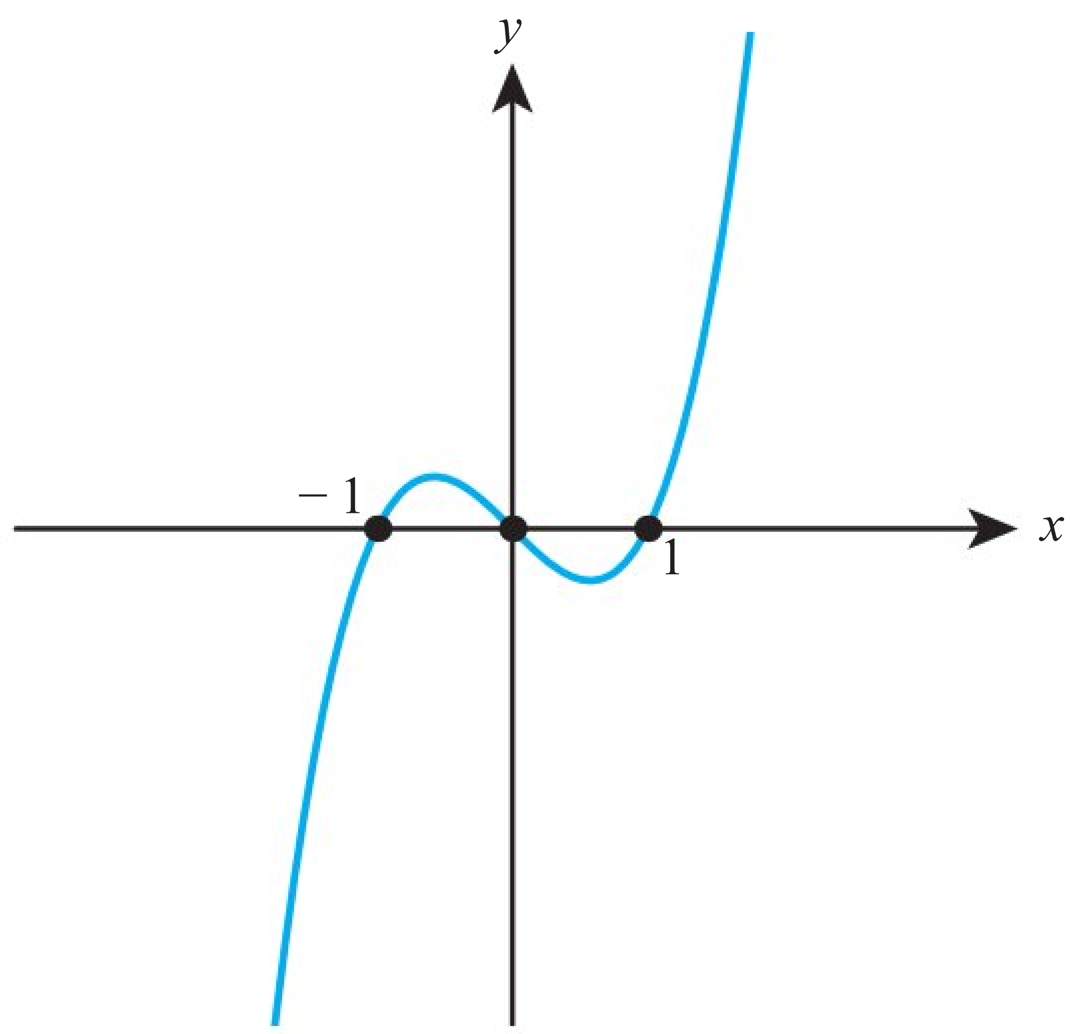
d Find an expression for $f^{-1}(x)$ and state its domain.

e Describe the translation which maps $y = f(x)$ to $y = f^{-1}(x)$.

41 Prove that the graph of $y = 2^x$ can be created by stretching the graph of $y = 4^x$ and describe fully the stretch.

42 Describe fully the stretch which maps the graph $y = \ln x$ to the graph $y \log_{10} x$.

43 The graph below shows $y = f(x)$.



Sketch the graph of $y = xf(x)$.

44 The graph of $y = f(x)$ is reflected in the line $y = 1$. Find the equation of the new graph.

45 Prove that if $f(x^2) = x^2f(x)$ for all x, then $y = f(x)$ has the y-axis as a line of reflection symmetry.

6 Complex numbers

ESSENTIAL UNDERSTANDINGS

- Algebra is an abstraction of numerical concepts and employs variables to solve mathematical problems.

In this chapter you will learn...

- how to work with the imaginary number i
- how to find sums, products and quotients of complex numbers in Cartesian form
- how to represent complex numbers geometrically on the complex plane (Argand diagram)
- how to find the modulus and argument of a complex number
- how to write a complex number in modulus–argument form
- how to find sums, products, quotients and powers of complex numbers in modulus–argument form
- how to write a complex number in exponential form
- how to find sums, products, quotients and powers of complex numbers in exponential form
- how to apply complex numbers to combine two sinusoidal functions
- how to interpret operations with complex numbers geometrically.

CONCEPTS

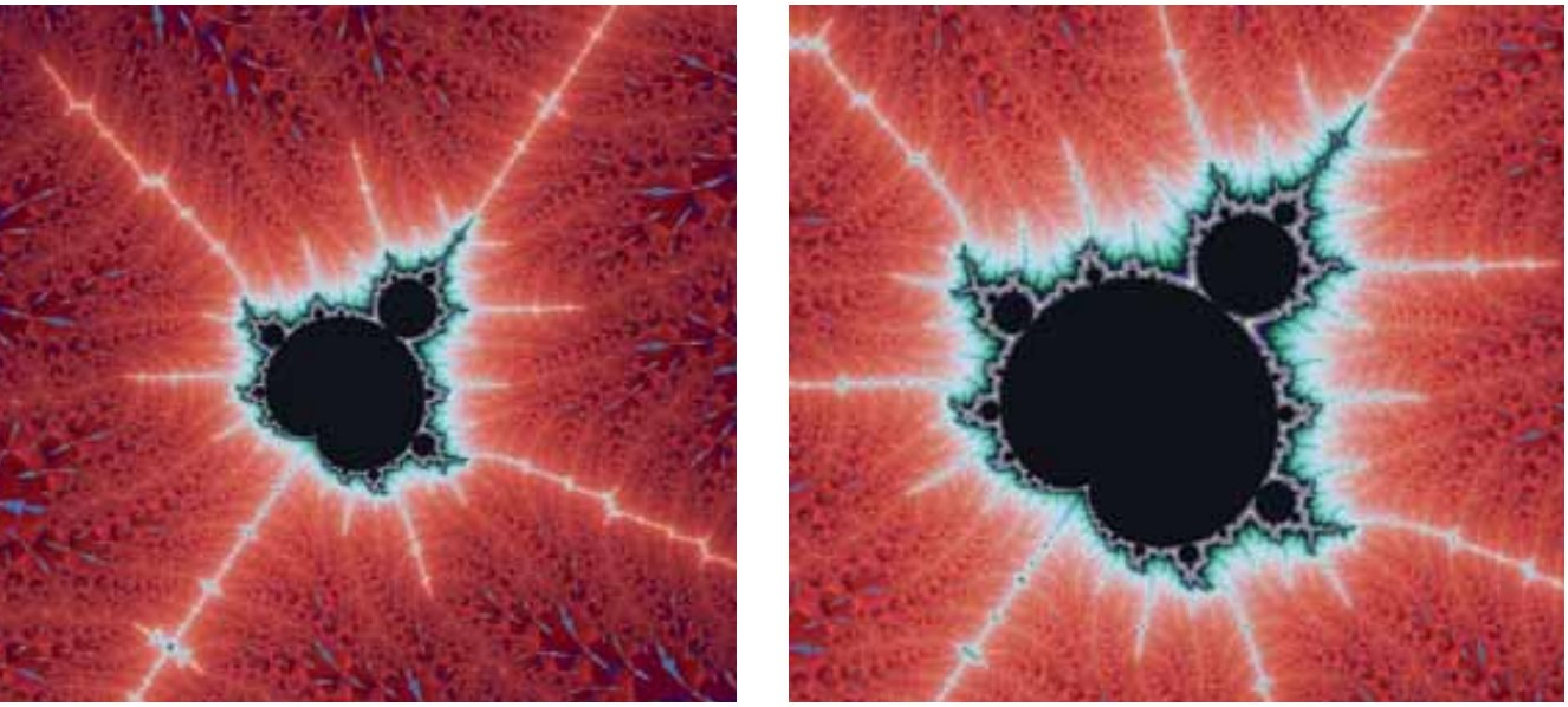
The following concepts will be addressed in this chapter:

- Utilising complex numbers provides a **system** to efficiently simplify and solve problems.
- **Representing** abstract quantities using complex numbers in different forms enables solutions of real-life problems.

LEARNER PROFILE – Balanced

What is the optimum amount of time to spend on a problem before taking a break? What are your strategies for what to do when you are stuck? We often find just trying to describe what your issue is to someone else activates new ideas. Do not expect to always do a problem straight away – sometimes if you come back to it the next day the problem suddenly cracks!

Figure 6.1 Can you imagine a shape which has a finite area but infinite perimeter?



PRIOR KNOWLEDGE

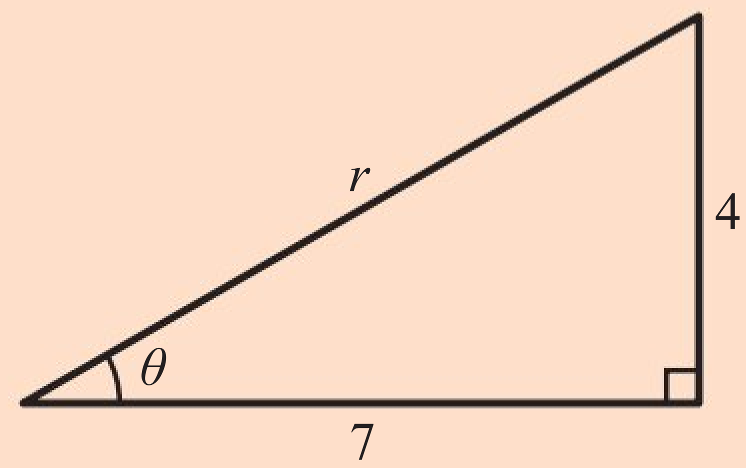
Before starting this chapter, you should already be able to complete the following:

1 Solve the equation $3x^2 - 6x + 2 = 0$, giving your answers to three significant figures.

2 Write these angles in radians.

a $30°$ b $225°$

3 Find the length marked r and the angle marked θ (in radians) in this triangle.



4 Simplify

a $2e^{7x} \times 5e^{-3x}$

b $8e^{12x} \div e^{-2x}$

c $\left(2e^{2x}\right)^5$

Starter Activity

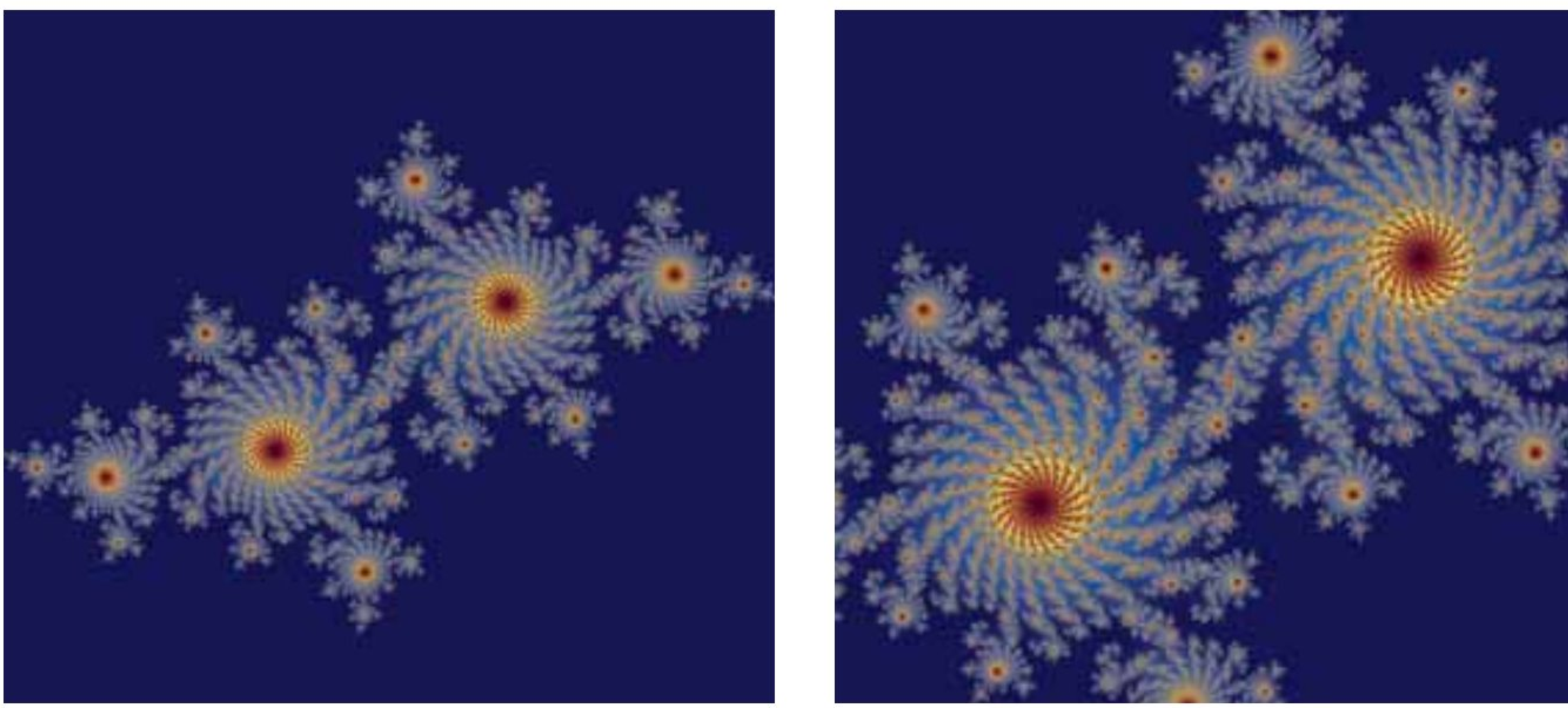
Can you have half of a drop of water? Can you show someone minus two pens? Can you imagine a shape which has a finite area but infinite perimeter? Can you draw a line with an irrational length? Do you know a number which squares to give minus one?

All of these are problems were once considered impossible, but their study has opened up new areas of mathematics with sometimes surprising applications.

In this chapter we shall extend the number line to another dimension! The new number, i, allows us to solve equations that we had previously said ‘have no real solution’. However, if that were the only purpose of complex numbers, they probably would have been discarded as a mathematical curiosity. We shall also explore how they can be used to analyse combinations of waves, and how their geometric representations give us a new way to describe transformations.

You are the Researcher

The pictures in Figure 6.1 are called fractals. They are shapes of a fractional dimension. You are used to 2D and 3D shapes, but what does it mean for the shapes to have fractional dimension?



6A Cartesian form

The number i

There are no real numbers that solve the equation $x^2 = -1$. But there is an imaginary number that solves this equation: i.

KEY POINT 6.1

$\mathrm{i} = \sqrt{-1}$

This number behaves just like a constant.

WORKED EXAMPLE 6.1

Simplify

a i^3 **b** i^4 **c** $2\mathrm{i} \times 3\mathrm{i}$

Consider 3 as 2 + 1, then use a rule of exponents to isolate i^2

$\mathrm{i}^2 = \left(\sqrt{-1}\right)^2 = -1$

a $\mathrm{i}^3 = \mathrm{i}^2 \times \mathrm{i}$
$= (-1)\mathrm{i}$
$= -\mathrm{i}$

Consider 4 as 2×2, then use a rule of exponents to isolate i^2

Again, $\mathrm{i}^2 = -1$

b $\mathrm{i}^4 = \left(\mathrm{i}^2\right)^2$
$= (-1)^2$
$= 1$

Group real and imaginary terms

Again, $\mathrm{i}^2 = -1$

c $2\mathrm{i} \times 3\mathrm{i} = (2 \times 3) \times (\mathrm{i} \times \mathrm{i})$
$= 6 \times (-1)$
$= -6$

Imaginary numbers can be added to real numbers to form **complex numbers**.

The variable z (rather than x) is often used for complex numbers and the set of all complex numbers is given the symbol $\mathbb{C}$.

KEY POINT 6.2

A complex number z can be written in **Cartesian form** as

$$z = x + \mathrm{i}y$$

where $x, y \in \mathbb{R}$

The **real part** of the complex number $z = x + \mathrm{i}y$ is x: this is denoted by $\mathrm{Re}(z)$. The **imaginary part** of z is y: this is denoted by $\mathrm{Im}(z)$.

So, for example, if $z = 3 - 2\mathrm{i}$ then $\mathrm{Re}(z) = 3$ and $\mathrm{Im}(z) = -2$.

Sums, products and quotients in Cartesian form

Adding, subtracting and multiplying complex numbers works in ways you might expect, with real and imaginary parts being grouped together.

Many calculators can do arithmetic with complex numbers, so make sure you know how to use this feature. However, you also need to be able to do calculations by hand in order to manipulate algebraic expressions.

WORKED EXAMPLE 6.2

$z = 2 + \mathrm{i}$ and $w = 5 - 3\mathrm{i}$.

Find

a $z + w$ **b** $z - w$ **c** zw

Group real and imaginary parts

a $z + w = 2 + \mathrm{i} + 5 - 3\mathrm{i}$
$= 7 - 2\mathrm{i}$

Group real and imaginary parts

b $z - w = 2 + \mathrm{i} - (5 - 3\mathrm{i})$
$= 2 + \mathrm{i} - 5 + 3\mathrm{i}$
$= -3 + 4\mathrm{i}$

Expand the brackets as usual

c $zw = (2 + \mathrm{i})(5 - 3\mathrm{i})$
$= 10 - 6\mathrm{i} + 5\mathrm{i} - 3\mathrm{i}^2$

$\mathrm{i}^2 = -1$

$= 10 - 6\mathrm{i} + 5\mathrm{i} + 3$

Group real and imaginary parts

$= 13 - \mathrm{i}$

Division is a little more involved and requires the idea of a **complex conjugate** of a complex number.

KEY POINT 6.3

If $z = x + \mathrm{i}y$, then its complex conjugate, z^*, is

$$z^* = x - \mathrm{i}y$$

It satisfies

$$zz^* = x^2 + y^2$$

This means that the product of a complex number and its complex conjugate is always real.

Proof 6.1

Prove that zz^* is always real.

Let $z = x + \mathrm{i}y$, where $x, y \in \mathbb{R}$
Then $z^* = x - \mathrm{i}y$.
So, $zz^* = (x + \mathrm{i}y)(x - \mathrm{i}y)$

$\mathrm{i}^2 = -1$

$= x^2 - \mathrm{i}xy + \mathrm{i}yx - \mathrm{i}^2y^2$
$= x^2 - (-y^2)$
$= x^2 + y^2$

Remember that x and y are real numbers

which is real.

KEY POINT 6.4

To divide two complex numbers, multiply the top and bottom by the complex conjugate of the denominator.

WORKED EXAMPLE 6.3

$z = 7 + 11\mathrm{i}$ and $w = 4 - \mathrm{i}$

Find $\frac{z}{w}$.

$$\frac{z}{w} = \frac{7+11\mathrm{i}}{4-\mathrm{i}}$$

The denominator can be made real by multiplying by its complex conjugate. The numerator must be multiplied by this too

$$= \frac{7+11\mathrm{i}}{4-\mathrm{i}} \times \frac{4+\mathrm{i}}{4+\mathrm{i}}$$

Expand the numerator: $(7+11\mathrm{i})(4+\mathrm{i})$
The denominator is given by $zz^* = x^2 + y^2$

$$= \frac{28 + 7\mathrm{i} + 44\mathrm{i} + 11\mathrm{i}^2}{4^2 + 1^2}$$

$\mathrm{i}^2 = -1$

$$= \frac{28 + 7\mathrm{i} + 44\mathrm{i} - 11}{17}$$

$$= \frac{17 + 51\mathrm{i}}{17}$$

$$= 1 + 3\mathrm{i}$$

Tip

This procedure is like rationalising the denominator with surds.

The idea of separating real and imaginary parts is very useful when solving equations.

WORKED EXAMPLE 6.4

Find the complex number z such that $5z + 3z^* = 8 - 4\mathrm{i}$.

Let $z = x + \mathrm{i}y$.

Then

Expand and group together real and imaginary parts on the LHS

$$5z + 3z^* = 8 - 4\mathrm{i}$$
$$5(x + \mathrm{i}y) + 3(x - \mathrm{i}y) = 8 - 4\mathrm{i}$$
$$5x + 5y\mathrm{i} + 3x - 3y\mathrm{i} = 8 - 4\mathrm{i}$$
$$8x + 2y\mathrm{i} = 8 - 4\mathrm{i}$$

$$\text{Re}: 8x = 8$$
$$x = 1$$

Equate real and imaginary parts on either side

$$\text{Im}: 2y = -4$$
$$y = -2$$

So, $z = 1 - 2\mathrm{i}$

The complex plane

While real numbers can be represented on a one-dimensional number line, complex numbers need two-dimensional coordinates. The x-axis represents the real part of the number and the y-axis the imaginary part.

This is referred to as the **complex plane** or an **Argand diagram**.

WORKED EXAMPLE 6.5

$z = 3 + 4\text{i}$

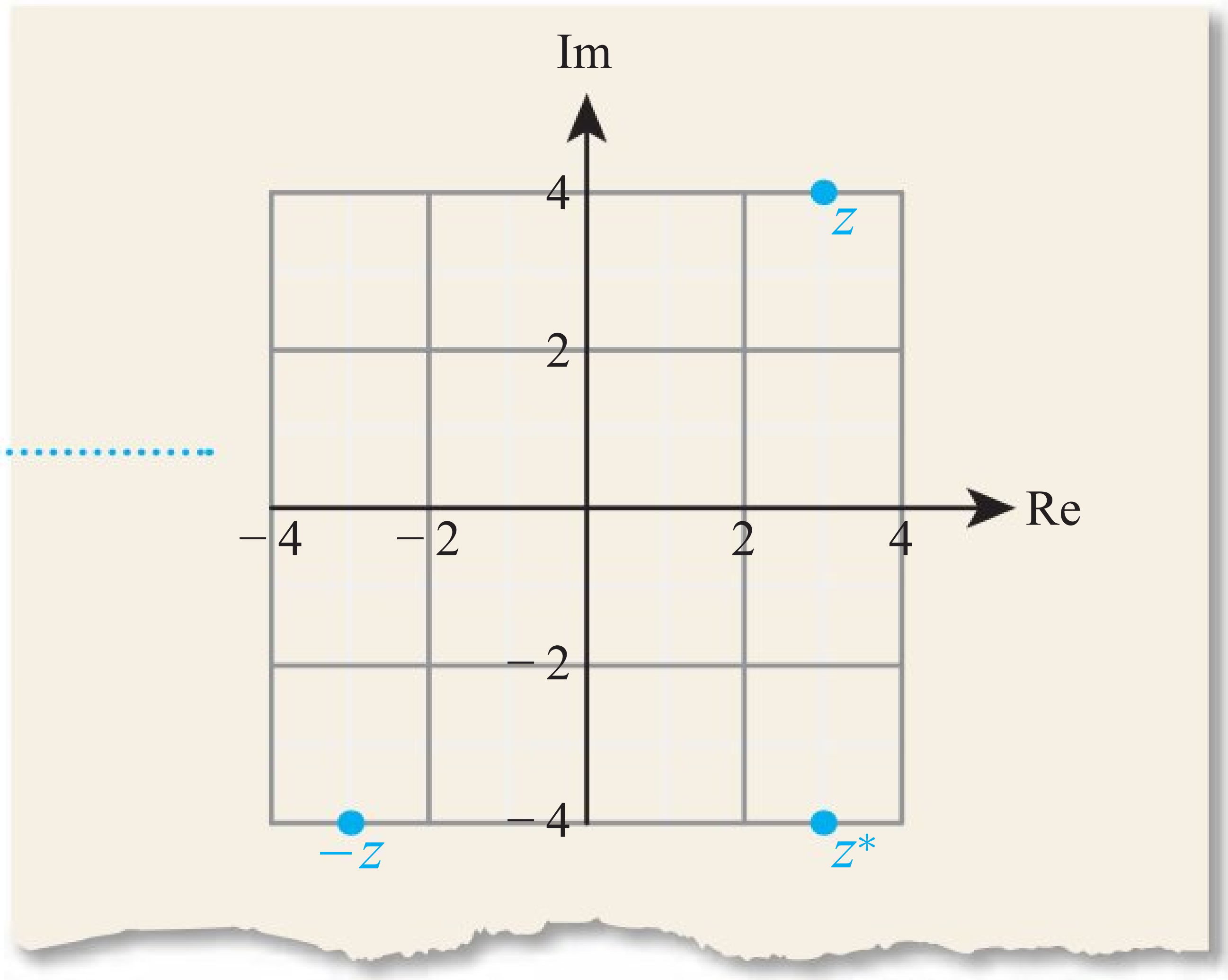
Represent these on the complex plane.

a z **b** $-z$ **c** z^*

z has coordinates (3, 4)

$-z = -(3+4\text{i}) = -3-4\text{i}$
So this has coordinates $(-3, -4)$

$z^* = 3 - 4\text{i}$
So this has coordinates $(3, -4)$



Who was Argand? What contribution did he make to our understanding of complex numbers?

CONCEPTS – PATTERNS AND REPRESENTATIONS

When you first saw a number line, it probably started at the origin and extended to the right. You might have been surprised when somebody suggested that it could also go to the left to represent negative numbers, but after a while you probably got used to that. We are now extending this **pattern** to going up and down. As before, you might start off unsure about that **representation**, but you will eventually get used to it. You might ask whether this pattern continues and if there will be another dimension added in your future work. Mathematicians have shown that we do not need another 'new' type of number to solve all the current types of equations studied, but that does not mean that a use would not be found for them in future research.

Quadratic equations

Historically, one of the motivations for introducing complex numbers was to solve equations, in particular polynomial equations, which do not have real solutions. For example, the equation $x^2 = -1$ has no real solutions. However, you now know that both $x = \text{i}$ and $x = -\text{i}$ are complex solutions of this equation.

In general, a quadratic equation has the form $ax^2 + bx + c = 0$. The solutions of this equation are given by the x-intercepts of the parabola $y = ax^2 + bx + c$.

In Section 13B of Mathematics: applications and interpretation SL you learnt that the vertex of the parabola has the x-coordinate $-\frac{b}{2a}$ and is located halfway between the x-intercepts.

It is possible for the y-coordinate of the vertex to be such that the graph does not cross the x-axis at all. It turns out that this happens when $b^2 - 4ac < 0$. In this situation, the quadratic equation has no real solutions. However, using complex numbers still allows us to find two solutions.

WORKED EXAMPLE 6.6

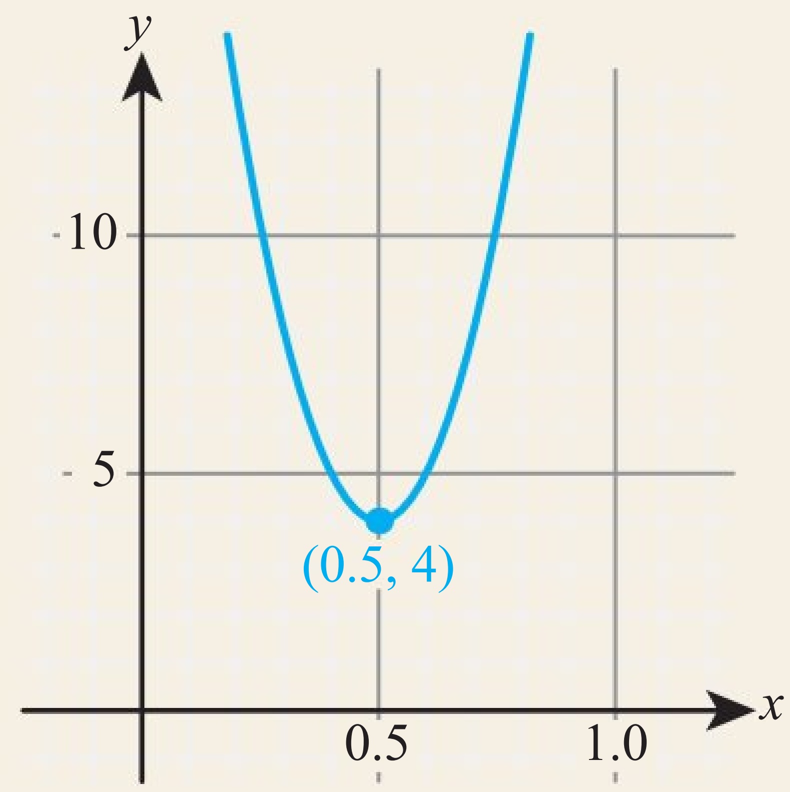
Use technology to answer these questions.

a Draw the graph of $y = 100x^2 - 100x + 29$ and label the coordinates of the vertex.

b Solve the equation $100x^2 - 100x + 29 = 0$.

c Show the solutions on an Argand diagram.

a



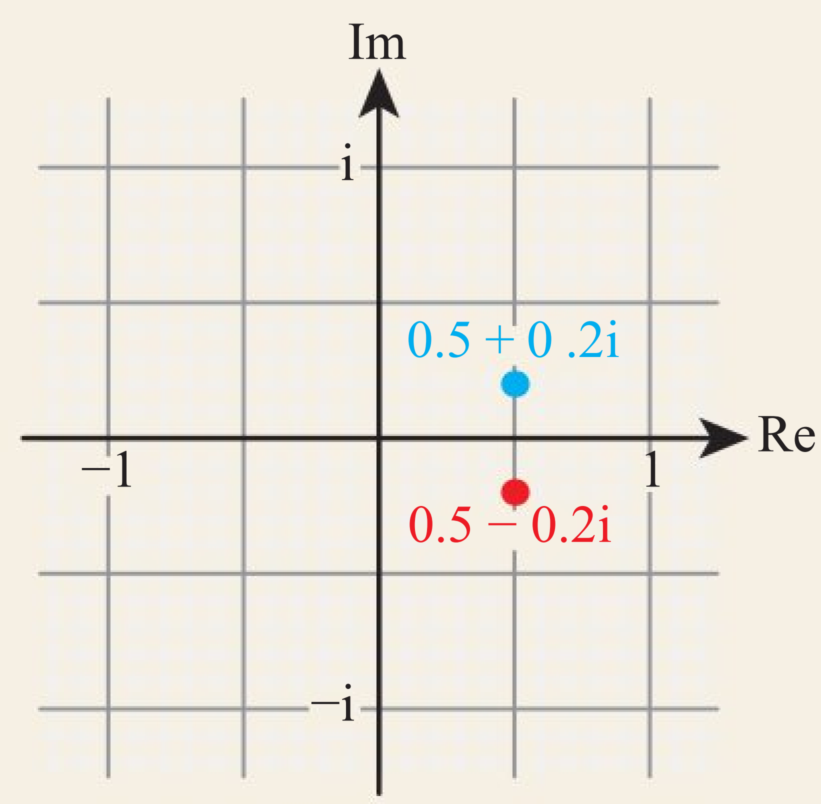
Use the quadratic equation solver. Make sure your calculator is in complex mode

b $100x^2 - 100x + 29 = 0$

From the GDC, $x = 0.5 \pm 0.2\text{i}$

The two points have coordinates $(0.5, 0.2)$ and $(0.5, -0.2)$

c



Notice that the two solutions have equal real parts and differ only in the sign of the imaginary part – they form a complex conjugate pair. The real part is equal to the x-coordinate of the vertex of the parabola.

You are the Researcher

Find out about the quadratic formula for solving quadratic equations. You can use it to explain the observations above, as well as why $b^2 - 4ac < 0$ leads to complex solutions. It turns out that, for all real polynomial equations, complex roots come in conjugate pairs. This is a part of an important result called The Fundamental Theorem of Algebra.

The development of negative numbers and complex numbers shows an interesting interplay between mathematics and historical events. The notion of 'debts' as negative numbers was formalized and understood by Arabian mathematicians such as Al-Samawal (1130–1180) in Baghdad. The Indian mathematician Brahmagupta (598–670) and Chinese mathematician Liu Hui (225–295) might have claims to similar ideas predating Al-Samawal. Until that point, equations such as $x+1=0$ were said to have no solution, just like you might have said that $x^2+1=0$ has no solution until you met complex numbers.

Figure 4.2 Girolamo Cardano

As the influence of the Islamic world spread – initially through conquest, then trade – these ideas also spread. It is no surprise that the jump from negative numbers to imaginary numbers happened in a country with major trade links to the Islamic world – Italy. Girolamo Cardano (1501–1576) found that he was having to find square roots of negative numbers when he was solving cubic equations. He called these numbers 'fictitious' and although they appeared in the middle of his working, they disappeared by the end and he got the correct answer so he accepted them. Probably the relatively recent introduction of equally puzzling negative numbers helped him to overcome his scepticism of this other new type of number.

In countries less influenced by the Islamic world, such as Northern Europe, it took several centuries more for negative numbers to achieve widespread acceptance. Even by the eighteenth century in England, there were respected mathematicians, such as Maseres (1731–1824) who decided that negative numbers could only be used as long as they did not appear in the final answer.

Exercise 6A

For questions 1 to 11, work without a calculator first and then check your answers using technology.

For questions 1 to 4, use the technique demonstrated in Worked Example 6.1 to simplify the expression.

1 **a** i^5 **b** i^6

2 **a** $-4i^3$ **b** $5i^4$

3 **a** $(3i)^2$ **b** $(2i)^3$

4 **a** $5i \times 4i$ **b** $-2i \times 3i$

For questions 5 to 8, use the technique demonstrated in Worked Example 6.2 to simplify the expression.

5 **a** $(2-i)+(9+5i)$ **b** $(-3-7i)+(-1+9i)$

6 **a** $(2+i)-(1+3i)$ **b** $(-4+7i)-(2-3i)$

7 **a** $(2+3i)(1-2i)$ **b** $(3+i)(5-i)$

8 **a** $(3+i)^2$ **b** $(4-3i)^2$

For questions 9 to 11, use the technique demonstrated in Worked Example 6.3 to write each expression in the form $x+iy$.

9 **a** $\frac{10}{2+i}$ **b** $\frac{6i}{1-i}$

10 **a** $\frac{10-5i}{1-2i}$ **b** $\frac{7+i}{3+4i}$

11 **a** $\frac{3+2i}{5-i}$ **b** $\frac{5-4i}{2+3i}$

For questions 12 to 15, use the technique of equating real and imaginary parts demonstrated in Worked Example 6.4 to find the following.

12 **a** $a, b \in \mathbb{R}$ such that $(a+3i)(1-2i) = 11-bi$

b $a, b \in \mathbb{R}$ such that $(4-ai)(3+i) = b+13i$

13 **a** $a, b \in \mathbb{R}$ such that $(7-ai)(2-i) = b-4i$

b $a, b \in \mathbb{R}$ such that $(2a-i)(3-5i) = -2+bi$

14 **a** $z \in \mathbb{C}$ such that
$z + 3\mathrm{i} = 2z^* + 4$

b $z \in \mathbb{C}$ such that
$3z + 2z^* = 5 + 2\mathrm{i}$

15 **a** $z \in \mathbb{C}$ such that
$z + 2z^* = 2 - 7\mathrm{i}$

b $z \in \mathbb{C}$ such that
$2z + \mathrm{i} = -3 - \mathrm{i}z^*$

For questions 16 to 19, use the technique demonstrated in Worked Example 6.5 to represent each complex number on an Argand diagram.

16 **a** **i** $z = 5 + 2\mathrm{i}$ **ii** $-z$ **iii** z^*
b **i** $z = -2 + 3\mathrm{i}$ **ii** $-z$ **iii** z^*

17 **a** **i** $z = 3 - 4\mathrm{i}$ **ii** $2z$ **iii** $\mathrm{i}z$
b **i** $z = -4 - 5\mathrm{i}$ **ii** $2z$ **iii** $\mathrm{i}z$

18 **a** **i** $z = -3 + 2\mathrm{i}$ **ii** $w = 5 + 3\mathrm{i}$ **iii** $z + w$
b **i** $z = 2 + 2\mathrm{i}$ **ii** $w = 1 + 3\mathrm{i}$ **iii** $z + w$

19 **a** **i** $z = -4 - \mathrm{i}$ **ii** $w = -2 - 3\mathrm{i}$ **iii** $z - w$
b **i** $z = -2 + 5\mathrm{i}$ **ii** $w = 6 - 2\mathrm{i}$ **iii** $z - w$

For questions 20 to 23, use the technique demonstrated in Worked Example 6.6 to solve the equation.

20 **a** $x^2 + 9 = 0$
b $x^2 + 36 = 0$

21 **a** $x^2 + 8 = 0$
b $x^2 + 75 = 0$

22 **a** $x^2 - 2x + 5 = 0$
b $x^2 - 4x + 13 = 0$

23 **a** $2x^2 + 4x + 3 = 0$
b $3x^2 - 2x + 2 = 0$

24 **a** Solve the equation $5x^2 + 6x + 5 = 0$. **b** Show the solutions on an Argand diagram.

25 Given that $z = 7 + 3\mathrm{i} - \dfrac{10\mathrm{i}}{2+\mathrm{i}}$, find z^*.

26 Find the complex number z such that $4z - 23 = 5\mathrm{i}z + 2\mathrm{i}$.

27 Let $z = 2 + \mathrm{i}$.
a Use technology to find z^2 and z^3. **b** Show z, z^2 and z^3 on a single Argand diagram.

28 Let $z = 1 + \frac{1}{2}\mathrm{i}$. Show z, z^2 and z^4 on a single Argand diagram.

29 Given that $z = a + 2\mathrm{i}$ and $w = -a\mathrm{i}$, write the following in terms of a.
a $z + 2w$ **b** zw

30 Let $z = a - 3\mathrm{i}$ and $w = 1 + b\mathrm{i}$. Given that $zw = 12 - 3\mathrm{i}$, find the possible values of a and b.

31 Find the complex number z such that $3\mathrm{i}z - 2z^* = \mathrm{i} - 4$.

32 Find the complex number z such that $2z + \mathrm{i}z^* = 1 - 7\mathrm{i}$.

33 Find two possible complex numbers z which satisfy $zz^* - z^2 = 18 + 12\mathrm{i}$.

34 Write $\dfrac{k+\mathrm{i}}{k-\mathrm{i}}$, where $k \in \mathbb{R}$, in the form $x + \mathrm{i}y$.

35 Let $z = \dfrac{a+3\mathrm{i}}{a-3\mathrm{i}}$, $a \in \mathbb{R}$.
Find the values of a for which $\mathrm{Re}(z) = 0$.

36 By writing $z = x + \mathrm{i}y$, prove that $(z^*)^2 = (z^2)^*$.

37 Solve the simultaneous equations $\begin{cases} 3z + \mathrm{i}w = 5 - 11\mathrm{i} \\ 2\mathrm{i}z - 3w = -2 + \mathrm{i} \end{cases}$ where $z, w \in \mathbb{R}$.

38 Solve the simultaneous equations
$$\begin{cases} 2z - 3\mathrm{i}w = 9 + \mathrm{i} \\ (1+\mathrm{i})z + 4w = 1 + 10\mathrm{i} \end{cases}$$
where $z, w \in \mathbb{C}$.

39 Find the possible values of $a, b \in \mathbb{C}$ such that $(1 + a\mathrm{i})(1 + b\mathrm{i}) = b - a + 9\mathrm{i}$.

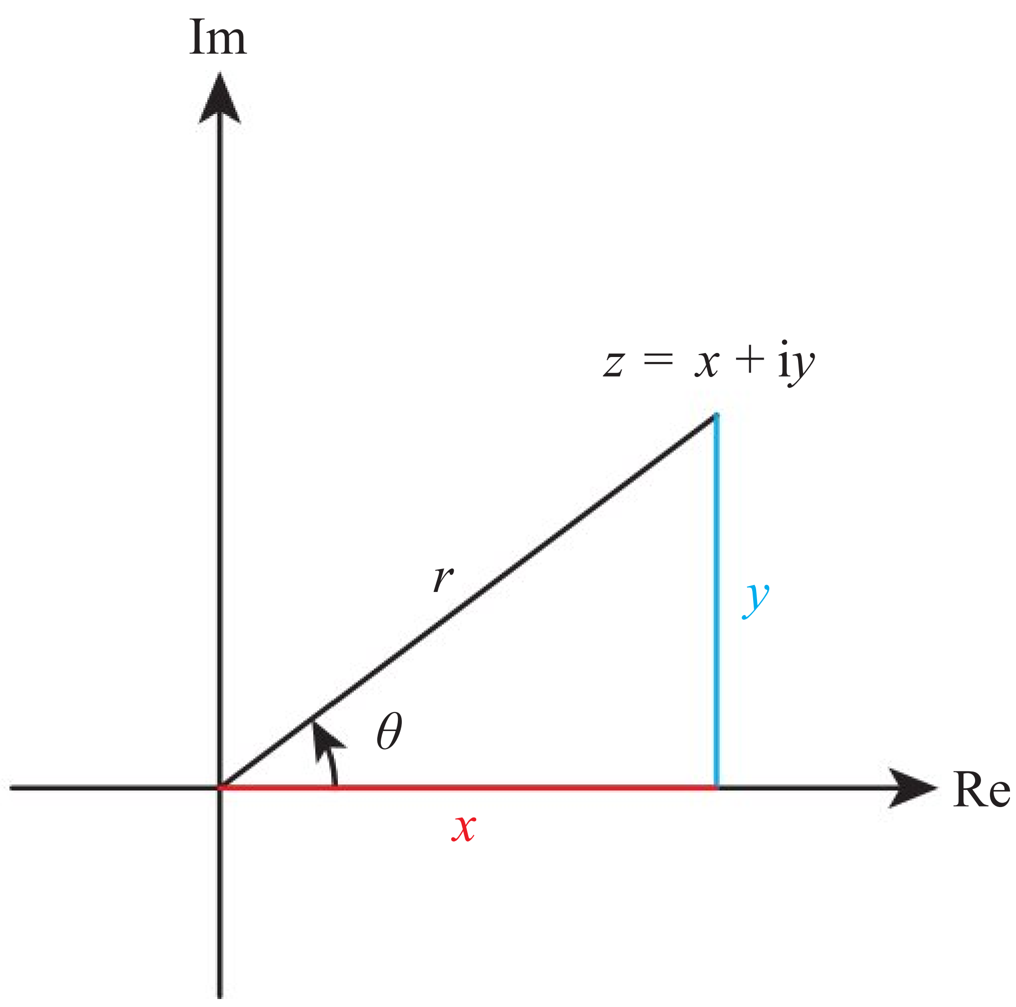
40 Let $z = \dfrac{7+\mathrm{i}}{2-\mathrm{i}} - \dfrac{3+\mathrm{i}}{a+2\mathrm{i}}$.
Find the values of $a \in \mathbb{C}$ such that $\mathrm{Re}(z) = \mathrm{Im}(z)$.

6B Modulus–argument form and exponential form

Modulus–argument form

When a complex number, z, is represented on an Argand diagram, its distance from the origin is called the **modulus**, denoted by $|z|$ or r.

The angle made with the positive x-axis (measured anticlockwise and in radians) is called the **argument**, denoted by $\arg z$ or θ.



You can find the modulus and argument of a complex number from the diagram above.

KEY POINT 6.5

If $z = x + \mathrm{i}y$, then the modulus, r, and argument, θ, are given by

- $r = \sqrt{x^2 + y^2}$
- $\tan\theta = \dfrac{y}{x}$

Tip

Always draw the complex number in an Argand diagram before finding the argument. It is important to know which angle you need to find.

The modulus must be positive. The argument can either be measured between 0 and 2π or between $-\pi$ and π. It will be made clear in the question which is required.

Using trigonometry, we can write complex numbers in terms of their modulus and argument:

$$\begin{aligned} z &= x + \mathrm{i}y \\ &= r\cos\theta + \mathrm{i}r\sin\theta \\ &= r(\cos\theta + \mathrm{i}\sin\theta) \end{aligned}$$

This form is so common that a shorthand is often used: $\operatorname{cis}\theta = \cos\theta + \mathrm{i}\sin\theta$.

KEY POINT 6.6

A complex number z can be written in **modulus–argument (polar) form** as

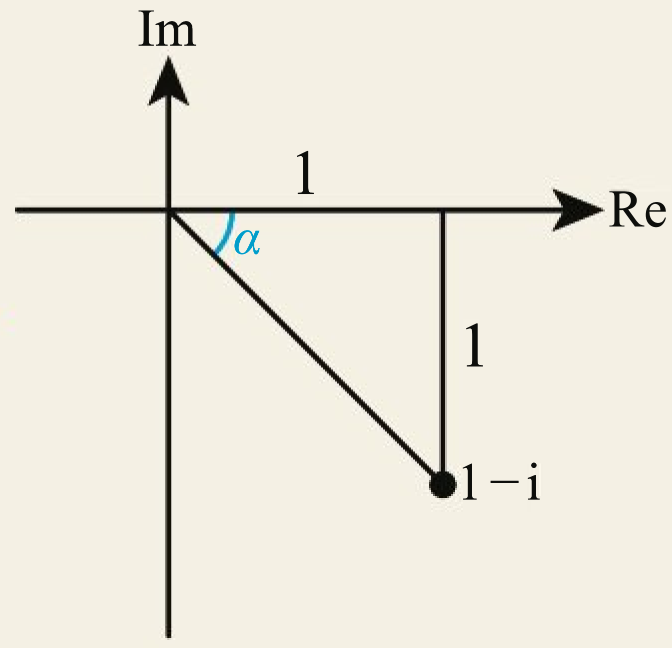
$$z = r(\cos\theta + \mathrm{i}\sin\theta) = r\operatorname{cis}\theta$$

WORKED EXAMPLE 6.7

Write $1 - \mathrm{i}$ in modulus–argument form, with the argument between 0 and 2π.

Use $r = \sqrt{x^2 + y^2}$ ······ $r = \sqrt{1^2 + (-1)^2} = \sqrt{2}$

To find the argument, draw a diagram to see where the complex number actually is ······



Find the angle α ······ $\tan\alpha = \frac{1}{1}$

$\alpha = \frac{\pi}{4}$

From the diagram, $\theta = 2\pi - \alpha$ ······ $\theta = 2\pi - \frac{\pi}{4} = \frac{7\pi}{4}$

Modulus–argument form is $z = r(\cos\theta + \mathrm{i}\sin\theta)$ ······ So, $z = \sqrt{2}\left(\cos\frac{7\pi}{4} + \mathrm{i}\sin\frac{7\pi}{4}\right)$

WORKED EXAMPLE 6.8

A complex number z has modulus 4 and argument $\frac{\pi}{6}$.

Write the number in Cartesian form.

Use $x = r\cos\theta$ and $y = r\sin\theta$ ······ $x = 4\cos\frac{\pi}{6} = \frac{4\sqrt{3}}{2} = 2\sqrt{3}$

$y = 4\sin\frac{\pi}{6} = \frac{4}{2} = 2$

Cartesian form is $z = x + \mathrm{i}y$ ······ So, $z = 2\sqrt{3} + 2\mathrm{i}$

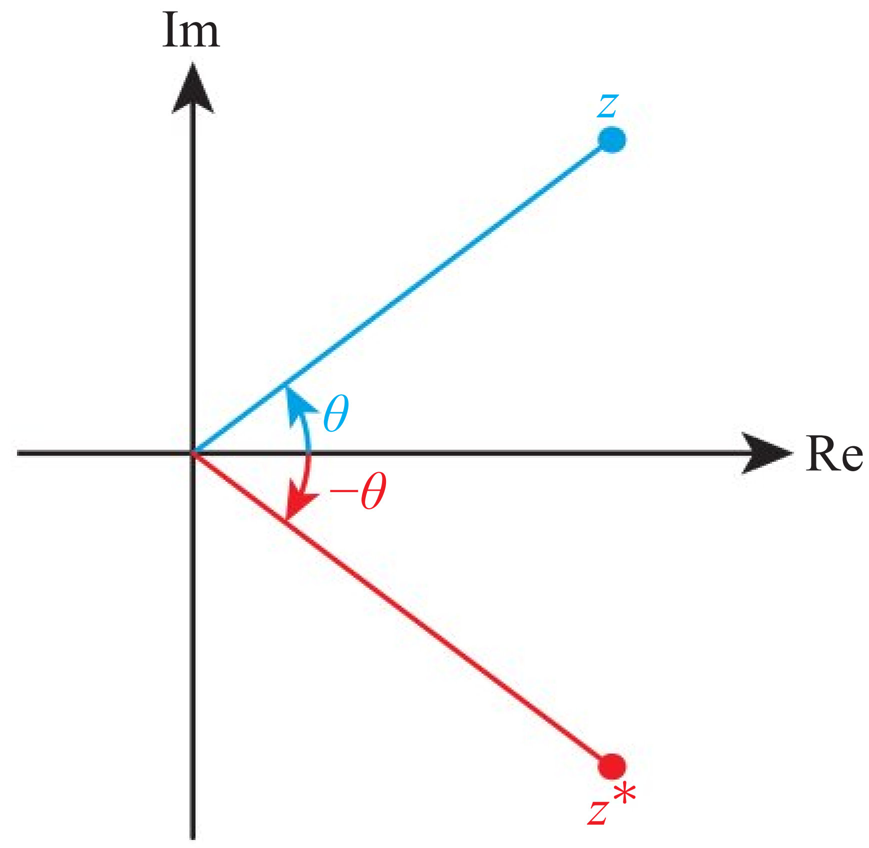
Be the Examiner 6.1

Find the argument of $z = -5 - 2\mathrm{i}$, where $-\pi < \arg z \leqslant \pi$.

Which is the correct solution? Identify the errors in the incorrect solutions.

Solution 1	Solution 2	Solution 3
$\arctan\left(\frac{2}{5}\right) = 0.381$ $\pi - 0.381 = 2.76$ So, $\arg z = -2.76$	$\arg z = -\arctan\left(\frac{2}{5}\right)$ $= -2.76$	$\arctan\left(\frac{-2}{-5}\right) = 0.381$ $\arg z = 2\pi - 0.381$ $= 5.90$

You have already seen that the complex conjugate is represented in an Argand diagram by reflection in the x-axis. Modulus–argument form, with $-\pi < \theta \leqslant \pi$, therefore gives a nice form for the conjugate.



KEY POINT 6.7

If $z = r\,\text{cis}\,\theta$, then $z^* = r\,\text{cis}(-\theta)$.

Products, quotients and powers in modulus–argument form

Addition and subtraction are straightforward in Cartesian form, but multiplication and division are more difficult. However, these operations are much easier in modulus–argument form.

Tip

You should be able to use your calculator for operations in modulus–argument form. However, you also need to know these rules in order to manipulate algebraic expressions.

KEY POINT 6.8

- $|zw| = |z||w|$
- $\arg(zw) = \arg z + \arg w$
- $|z^n| = |z|^n$
- $\arg(z^n) = n\arg z$
- $\left|\frac{z}{w}\right| = \frac{|z|}{|w|}$
- $\arg\left(\frac{z}{w}\right) = \arg z - \arg w$

You are the Researcher

The rules in Key Point 6.8 can be proved using compound angle formulae, which allow you to expand expressions like $\cos(x + y)$.

WORKED EXAMPLE 6.9

$z = 2\,\text{cis}\left(\frac{\pi}{3}\right)$ and $w = 5\,\text{cis}\left(\frac{\pi}{4}\right)$

Write zw in the form $r\,\text{cis}\,\theta$.

Multiply the moduli and add the arguments:

$|z| = 2$ and $\arg z = \frac{\pi}{3}$

$|w| = 5$ and $\arg w = \frac{\pi}{4}$

$$|zw| = 2 \times 5 = 10$$

$$\arg(zw) = \frac{\pi}{3} + \frac{\pi}{4} = \frac{7\pi}{12}$$

So, $zw = 10\,\text{cis}\left(\frac{7\pi}{12}\right)$

WORKED EXAMPLE 6.10

$z = 8\left(\cos\frac{2\pi}{3} + \mathrm{i}\sin\frac{2\pi}{3}\right)$ and $w = 4\left(\cos\frac{\pi}{6} + \mathrm{i}\sin\frac{\pi}{6}\right)$

Write $\frac{z^2}{w^3}$ in Cartesian form.

First find the modulus and argument of z^2 and w^3 using

$|z| = 8$ and $\arg z = \frac{2\pi}{3}$

$|w| = 4$ and $\arg w = \frac{\pi}{6}$

$$|z^2| = 8^2 = 64,\ \arg(z^2) = 2\left(\frac{2\pi}{3}\right) = \frac{4\pi}{3}$$

$$|w^3| = 4^3 = 64,\ \arg(w^3) = 3\left(\frac{\pi}{6}\right) = \frac{\pi}{2}$$

Divide the moduli and subtract the arguments

$$\left|\frac{z^2}{w^3}\right| = \frac{64}{64} = 1$$

$$\arg\left(\frac{z^2}{w^3}\right) = \frac{4\pi}{3} - \frac{\pi}{2} = \frac{5\pi}{6}$$

So,

$$\frac{z^2}{w^3} = 1\left(\cos\frac{5\pi}{6} + \mathrm{i}\sin\frac{5\pi}{6}\right)$$

Evaluate the trigonometric functions (your calculator may give you exact values or decimals)

$$= -\frac{\sqrt{3}}{2} + \frac{1}{2}\mathrm{i}$$

Exponential form

The rules for finding the argument when multiplying or dividing complex numbers are just the same as the rules of indices, that is, add the arguments when multiplying and subtract the arguments when dividing.

You might wonder why we choose e to be the base in exponential form. When you study differentiation of exponential and trigonometric functions in Chapter 10, you will find some justification for this decision.

This suggests that complex numbers can be written in an exponential form with the argument as the exponent.

KEY POINT 6.9

A complex number z can be written in **exponential (Euler) form** as

$$z = r\mathrm{e}^{\mathrm{i}\theta}$$

where r is the modulus of z and θ is the argument of z, so that $\mathrm{e}^{\mathrm{i}\theta} = \cos\theta + \mathrm{i}\sin\theta$

Conventionally the arguments in exponential form satisfy $0 \leqslant \theta < 2\pi$, however numbers are not uniquely represented in exponential form and you should be able to deal with arguments outside of that range. For example, 1 can be written as $\mathrm{e}^{0\mathrm{i}}$ or $\mathrm{e}^{2\pi}$ or $\mathrm{e}^{-4\pi}$.

TOK Links

How is new knowledge created in mathematics? Is analogy a valid way of creating knowledge? If I define $\mathrm{e}^{\mathrm{i}x}$ to be $\cos x + \mathrm{i}\sin x$ because it seems to have similar properties, does that make it true?

This allows us to use the rules of indices on complex numbers.

WORKED EXAMPLE 6.11

$z = 12\mathrm{e}^{\mathrm{i}\frac{2\pi}{5}}$ and $w = 4\mathrm{e}^{\mathrm{i}\frac{\pi}{3}}$

Find, in exponential form,

a zw **b** $\dfrac{z}{w}$

a $zw = 12\mathrm{e}^{\mathrm{i}\frac{2\pi}{5}} \times 4\mathrm{e}^{\mathrm{i}\frac{\pi}{3}}$

Follow the rules of exponents by adding the powers

$= 48\mathrm{e}^{\mathrm{i}\left(\frac{2\pi}{5}+\frac{\pi}{3}\right)}$

$= 48\mathrm{e}^{\mathrm{i}\left(\frac{6\pi}{15}+\frac{5\pi}{15}\right)}$

$= 48\mathrm{e}^{\mathrm{i}\frac{11\pi}{15}}$

b $\dfrac{z}{w} = \dfrac{12\mathrm{e}^{\mathrm{i}\frac{2\pi}{5}}}{4\mathrm{e}^{\mathrm{i}\frac{\pi}{3}}}$

Follow the rules of exponents by subtracting the powers

$= 3\mathrm{e}^{\mathrm{i}\left(\frac{2\pi}{5}-\frac{\pi}{3}\right)}$

$= 3\mathrm{e}^{\mathrm{i}\frac{\pi}{15}}$

Exponential form is particularly useful for finding powers of complex numbers.

WORKED EXAMPLE 6.12

$z = 2\mathrm{e}^{\mathrm{i}\frac{\pi}{12}}$

Find z^3 in Cartesian form.

Follow the laws of exponents by applying the power 3 to both terms of the product…

$z^3 = \left(2\mathrm{e}^{\mathrm{i}\frac{\pi}{12}}\right)^3$

$= 2^3\left(\mathrm{e}^{\mathrm{i}\frac{\pi}{12}}\right)^3$

…and then multiplying $\mathrm{i}\frac{\pi}{12}$ by 3

$= 8\mathrm{e}^{\mathrm{i}\frac{3\pi}{12}}$

$= 8\mathrm{e}^{\mathrm{i}\frac{\pi}{4}}$

Convert to Cartesian form by first changing to modulus–argument form using $\mathrm{e}^{\mathrm{i}\theta} = \cos\theta + \mathrm{i}\sin\theta$ and then evaluating

$= 8\left(\cos\frac{\pi}{4} + \mathrm{i}\sin\frac{\pi}{4}\right)$

$= 8\left(\frac{\sqrt{2}}{2} + \mathrm{i}\frac{\sqrt{2}}{2}\right)$

$= 4\sqrt{2} + 4\sqrt{2}\mathrm{i}$

You are the Researcher

The use of laws of exponents can be extended to negative and fractional powers as well, using a result called De Moivre's theorem. It is then possible to define general complex powers such as 2^{1+3i}, as well as trigonometric functions of complex numbers.

TOK Links

From exponential (Euler) form we find the famous mathematical equation $e^{i\pi} + 1 = 0$. This is often described as mathematical poetry – it has the fundamental constant of arithmetic (1), calculus (e), geometry (π) and imaginary numbers (i). It uses all the fundamental operations: addition, multiplication and raising to a power. When these are combined together in just the right way, the answer is nothing!
Is there a role for aesthetics in mathematics? Could someone with no understanding of calculus and complex numbers appreciate this result in the same way that you now can? Does our previous experience change what we find to be beautiful?

If a complex number is given in Cartesian form, the easiest way to raise it to a power is to convert it to modulus–argument (or exponential) form first.

WORKED EXAMPLE 6.13

a Find the modulus and argument of $1 + i$.

b Hence find $(1 + i)^6$ in Cartesian form.

a $|1+i| = \sqrt{1^2 + 1^2} = \sqrt{2}$

$\arg(1+i) = \arctan\frac{1}{1} = \frac{\pi}{4}$

Raising to a power is easiest when written in exponential form

b $\left(\sqrt{2}e^{i\frac{\pi}{4}}\right)^6 = (\sqrt{2})^6 e^{i\frac{6\pi}{4}}$

$= 8e^{i\frac{3\pi}{2}}$

Write in terms of sin and cos, then evaluate

$= 8\left(\cos\frac{3\pi}{2} + i\sin\frac{3\pi}{2}\right)$

$= 8(0 + i(-1))$

$= -8i$

KEY CONCEPT – REPRESENTATION

When adding two complex numbers together, which **representation** is easiest to work with, Cartesian form or modulus–argument form? What about when raising a complex number to a large power?

Application to adding sinusoidal functions

TOOLKIT: Problem Solving

There are many situations where you may want to add two sine or cosine functions; for example, interference of waves or combining voltages in an AC circuit.
The two sine waves will often have the same frequency but different amplitudes and phase shifts. In other words, we are interested in simplifying a sum of the form $A_1\sin(\omega x) + A_2\sin(\omega x + c)$.

Use graphing software to investigate sums of this form. For example, sketch the graph of $y = 2\sin x + 3\sin(x+5)$ and find its amplitude, frequency and phase shift; hence write the sum in the form $A\sin(x+\alpha)$. Investigate other similar sums. Can they always be written in the form $A\sin(x+\alpha)$? What about adding two cosine functions?

Tip

You can use the same technique to find the sum of two cosine functions, just take the real part at the end.

It is possible to add two sinusoidal functions by using the modulus–argument form of complex numbers. This is because $A\sin\theta$ is the imaginary part of $z = Ae^{i\theta}$. So instead of working with $A\sin\theta$ you can work with $Ae^{i\theta}$ and take the imaginary part at the end. This is best illustrated with an example.

WORKED EXAMPLE 6.14

Write $2\sin\theta + 3\sin(\theta+5)$ in the form $A\sin(\theta+\alpha)$.

Use the fact that $r\sin\theta = \mathrm{Im}(re^{i\theta})$ to express the problem in terms of complex numbers

Let $z_1 = 2e^{i\theta}$, $z_2 = 3e^{i(\theta+5)}$

Then $2\sin\theta + 3\sin(\theta+5) = \mathrm{Im}(z_1+z_2)$

We want the imaginary part of z_1+z_2, so aim to write this sum in exponential form

$z_1 + z_2 = 2e^{i\theta} + 3e^{i(\theta+5)}$

Take out a common factor $e^{i\theta}$

$= e^{i\theta}(2+3e^{5i})$

Convert $2+3e^{5i}$ to exponential form (you can use technology)

$= e^{i\theta}(4.05e^{5.49i})$

Use rules of exponents to add the powers

$= 4.05e^{i(\theta+5.49)}$

Now take the imaginary part

$\therefore\ 2\sin\theta + 3\sin(\theta+5) = 4.05\sin(\theta+5.49)$

Tip

Remember that this method only works when the two functions you are adding have the same frequency, for example, you can use it to simplify $2\sin(8x) + 3\sin(8x+3)$ but not $2\sin(8x) + 3\sin(5x+3)$.

TOK Links

Adding sinusoidal functions is an example of the application of complex numbers to derive results about real functions. Other examples include a formula for solving cubic equations and a method for deriving new trigonometric identities. Do these 'real' applications change your opinion about whether complex numbers 'exist'?

Exercise 6B

For questions 1 to 15, work without a calculator first and then use a calculator to check your answer.

In questions 1 to 3, use the technique demonstrated in Worked Example 6.7 to write the following in modulus–argument form, with $0 \leqslant \theta < 2\pi$.

1 a -5i
b -7i

2 a $3-3\sqrt{3}\text{i}$
b $4-4\text{i}$

3 a $-2-2\text{i}$
b $-1-\sqrt{3}\text{i}$

For questions 4 to 7, use the technique demonstrated in Worked Example 6.8 to write z in Cartesian form.

4 a $|z| = 10, \arg z = \left(-\frac{\pi}{2}\right)$
b $|z| = 8, \arg z = \frac{\pi}{2}$

5 a $|z| = 4, \arg z = \frac{\pi}{3}$
b $|z| = \sqrt{2}, \arg z = \frac{\pi}{4}$

6 a $|z| = 4\sqrt{3}, \arg z = \frac{3\pi}{4}$
b $|z| = 2, \arg z = \frac{2\pi}{3}$

7 a $|z| = 8, \arg z = \frac{11\pi}{6}$
b $|z| = 2, \arg z = \left(-\frac{\pi}{4}\right)$

For questions 8 and 9, use the techniques demonstrated in Worked Example 6.9 to write the following in the form $r\operatorname{cis}\theta$, where $0 < \theta \leqslant 2\pi$.

8 a $\left(2\operatorname{cis}\frac{\pi}{3}\right)\left(6\operatorname{cis}\frac{\pi}{5}\right)$
b $\left(\frac{1}{2}\operatorname{cis}\frac{3\pi}{4}\right)\left(10\operatorname{cis}\frac{11\pi}{8}\right)$

9 a $\dfrac{15\operatorname{cis}\frac{\pi}{9}}{5\operatorname{cis}\frac{\pi}{6}}$
b $\dfrac{\operatorname{cis}\frac{5\pi}{7}}{3\operatorname{cis}\frac{\pi}{7}}$

For questions 10 and 11, use the techniques demonstrated in Worked Example 6.10 to write the following in Cartesian form.

10 a $\left(9\operatorname{cis}\frac{2\pi}{5}\right)^2\left(\frac{2}{3}\operatorname{cis}\frac{\pi}{10}\right)^3$
b $\left(4\operatorname{cis}\frac{17\pi}{9}\right)^2\left(\operatorname{cis}\frac{4\pi}{9}\right)^3$

11 a $\dfrac{\left(\operatorname{cis}\frac{5\pi}{18}\right)^5}{\left(\operatorname{cis}\left(-\frac{7\pi}{18}\right)\right)^6}$
b $\dfrac{\left(8\operatorname{cis}\frac{5\pi}{12}\right)^2}{\left(4\operatorname{cis}\frac{\pi}{6}\right)^3}$

For questions 12 and 13, use the technique demonstrated in Worked Example 6.11 to evaluate these in exponential form.

12 a $3\text{e}^{0.1\text{i}} \times 5\text{e}^{-0.2\text{i}}$
b $\sqrt{2}\text{e}^{0.5\text{i}} \times \sqrt{2}\text{e}^{1.5\text{i}}$

13 a $4\text{e}^{\pi\text{i}} \div 2\text{e}^{\frac{\pi\text{i}}{4}}$
b $12\text{e}^{\frac{\pi\text{i}}{6}} \div 3\text{e}^{\frac{\pi\text{i}}{4}}$

For questions 14 and 15, use the techniques demonstrated in Worked Example 6.12 to evaluate these in Cartesian form.

14 a $\left(\text{e}^{\frac{\text{i}\pi}{6}}\right)^2$
b $\left(\text{e}^{\frac{\text{i}\pi}{8}}\right)^2$

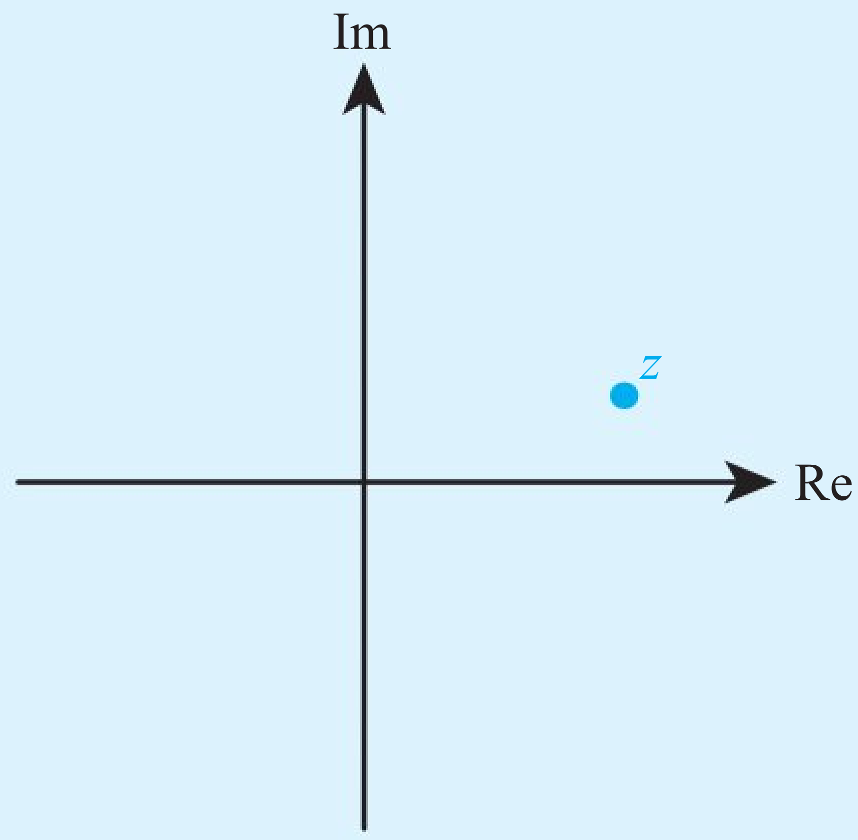
15 a $\left(\text{e}^{\frac{\text{i}\pi}{6}}\right)^3$
b $\left(\text{e}^{\frac{\text{i}\pi}{3}}\right)^4$

16 The complex number z is plotted on the Argand diagram alongside.

On a copy of this diagram sketch and label

a z^2

b $\mathrm{i}z$.



17 You are given that $z = -2 + 2\mathrm{i}$.

a Find $|z|$.

b Find $\arg z$.

c Hence write down the modulus and argument of z^2.

d Hence write z^2 in Cartesian form.

18 Simplify $\operatorname{cis} 0.6 \times \operatorname{cis} 0.4$.

19 In this question, $z = 1 + \mathrm{i}$ and $w = 1 + \sqrt{3}\mathrm{i}$.

a Find $\arg(w)$.

b Find $\arg(zw)$.

20 Write these in Cartesian form.

a $\operatorname{cis}\frac{\pi}{3} \times \operatorname{cis}\frac{\pi}{6}$

b $\operatorname{cis}\frac{\pi}{3} + \operatorname{cis}\frac{\pi}{6}$

21 a Express $z = 2 - 2\mathrm{i}$ in modulus–argument form.

b Hence find, in Cartesian form, z^5.

22 a Express $z = \sqrt{3} + \mathrm{i}$ in modulus–argument form.

b Hence find, in Cartesian form, z^3.

23 a Express $w = -\sqrt{2} - \sqrt{2}\mathrm{i}$ in modulus–argument form.

b Given that $z = \cos\left(-\frac{\pi}{7}\right) + \mathrm{i}\sin\left(-\frac{\pi}{7}\right)$, find, in Cartesian form, $w^6 z^7$.

24 Given that $w = 3 - 3\mathrm{i}$ and $z = \cos\frac{3\pi}{8} + \mathrm{i}\sin\frac{3\pi}{8}$, find, in Cartesian form, $w^4 z^6$.

25 Let $z = 2\operatorname{cis}\frac{7\pi}{24}$.

Find the smallest positive integer value of n for which z^n is a real number.

26 Let $z = \operatorname{cis}\frac{5\pi}{18}$.

Find the smallest positive integer value of n for which $z^n = \mathrm{i}$.

27 The complex numbers z and w have arguments between 0 and π.

Given that $zw = -4\sqrt{2} + 4\sqrt{2}\mathrm{i}$ and $\frac{z}{w} = 1 + \sqrt{3}\mathrm{i}$, find the modulus and argument of z and the modulus and argument of w.

28 The complex numbers z and w are defined by $z = \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}}\mathrm{i}$ and $w = 1 + \sqrt{3}\mathrm{i}$.

a Find

i $|z|$

ii $\arg z$

iii $|w|$

iv $\arg w$.

b Find $\frac{w}{z}$ in modulus–argument form.

c Find $\frac{w}{z}$ in Cartesian form.

d Hence find an exact surd expression for $\cos\frac{\pi}{12}$.

29 a Find the modulus and the argument of $5 + 2\mathrm{e}^{3\mathrm{i}}$.

b Hence write $5\sin\theta + 2\sin(\theta + 3)$ in the form $A\sin(\theta + \alpha)$.

30 a Write $3 - 2\mathrm{e}^{10\mathrm{i}}$ in exponential form.

b Hence write $3\cos t - 2\cos(t + 10)$ in the form $R\cos(t + c)$.

31 Two AC voltage sources are connected in a circuit. The two voltages vary according to the equations

$V_1 = 8\cos(30t)$ and $V_2 = 10\cos(30t + 5)$

a On the same diagram, sketch the graphs of V_1 and V_2 for $0 \leqslant t \leqslant 0.5$.

b Find an expression for the total voltage, $V = V_1 + V_2$, in the form $V = A\cos(30t + c)$.

32 A population of birds varies during the year and is modelled by the equation $B = 200 + 40\sin\left(\frac{2\pi}{365}t\right)$, where t is the time, in days, since 1st December. A population of owls can be modelled by the equation $W = 60 + 20\sin\left(\frac{2\pi}{365}t + 2\right)$. Find an expression for the total number of animals of the two species, in the form $C + A\sin(bt + c)$.

33 a Write i in exponential form, and hence show that i^{i} is a real number.

b Use the approximation $\mathrm{e} \approx \pi \approx 3$ to estimate the value of i^{i} to one decimal place.

34 a Write -2 in exponential form.

b Hence suggest a value for $\ln(-2)$.

35 a Write i in exponential form.

b Hence suggest a value for $\ln(\mathrm{i})$.

c Explain why there is more than one plausible value for $\ln(\mathrm{i})$.

You learnt how to add and subtract vectors in Chapter 2.

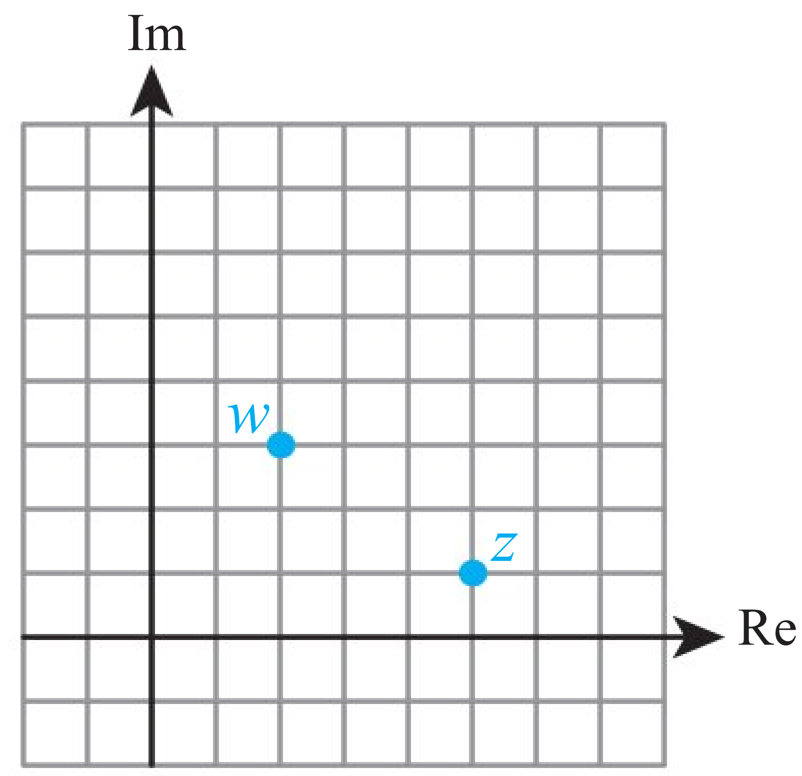
6C Geometric representation

Addition and subtraction can be represented neatly on the complex plane. When represented on an Argand diagram, you can think of the real and imaginary parts of a complex number as the components of a position vector.

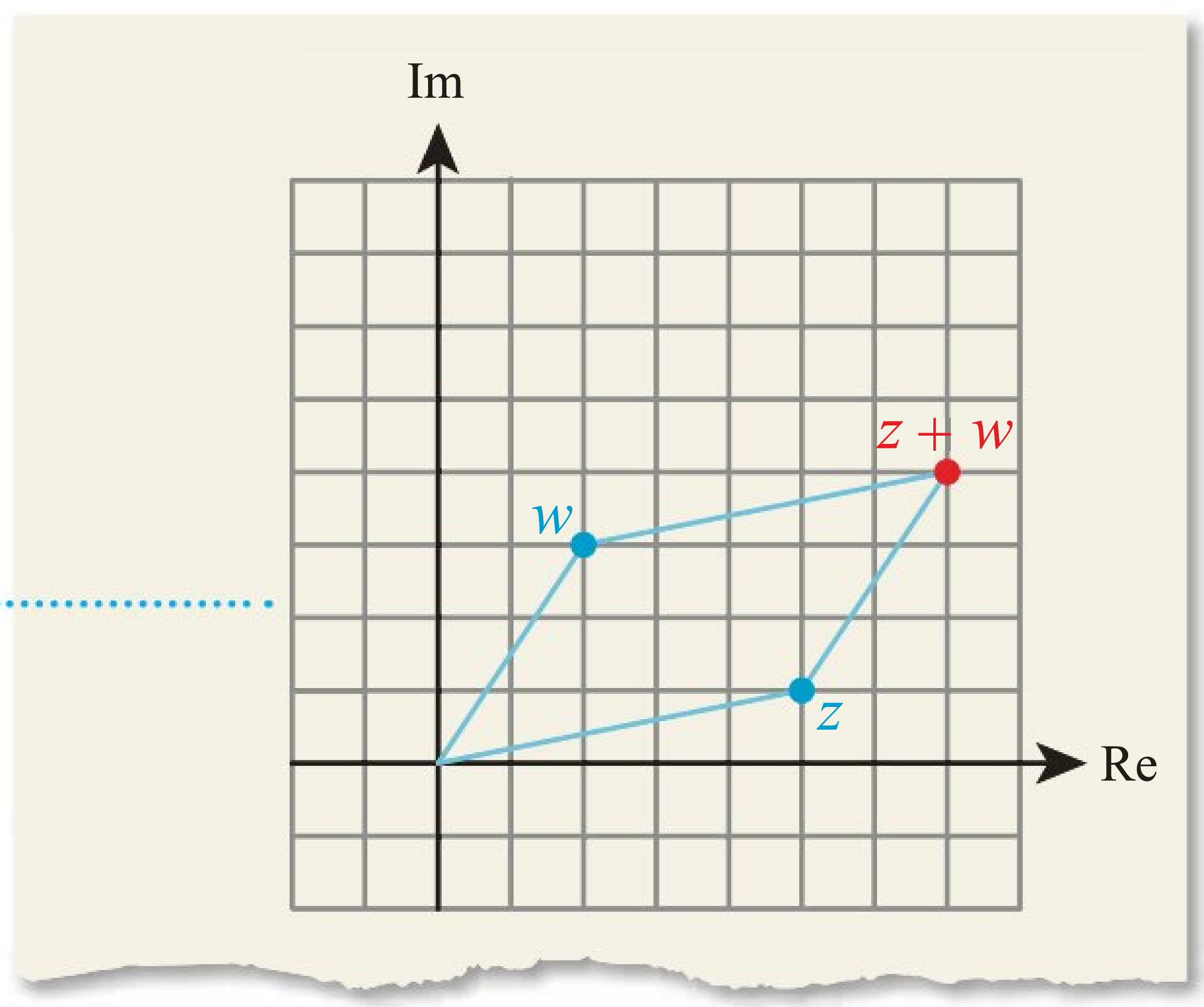
WORKED EXAMPLE 6.15

The complex numbers z and w are shown on the Argand diagram below.

Mark on the complex number $z + w$.



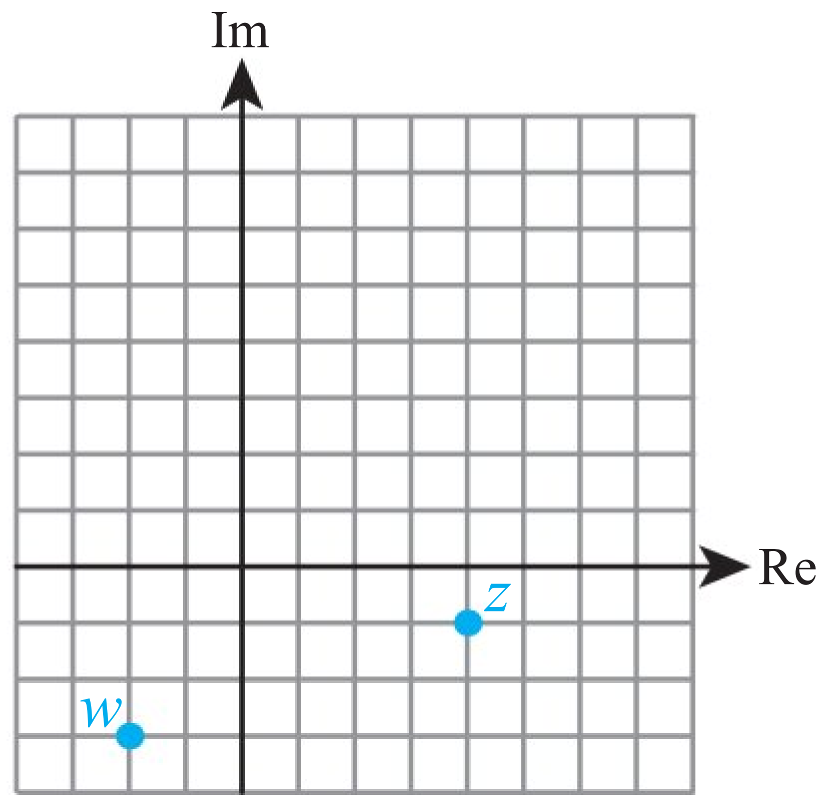
The point representing $z + w$ can be found by creating a parallelogram with the relevant points as vertices



WORKED EXAMPLE 6.16

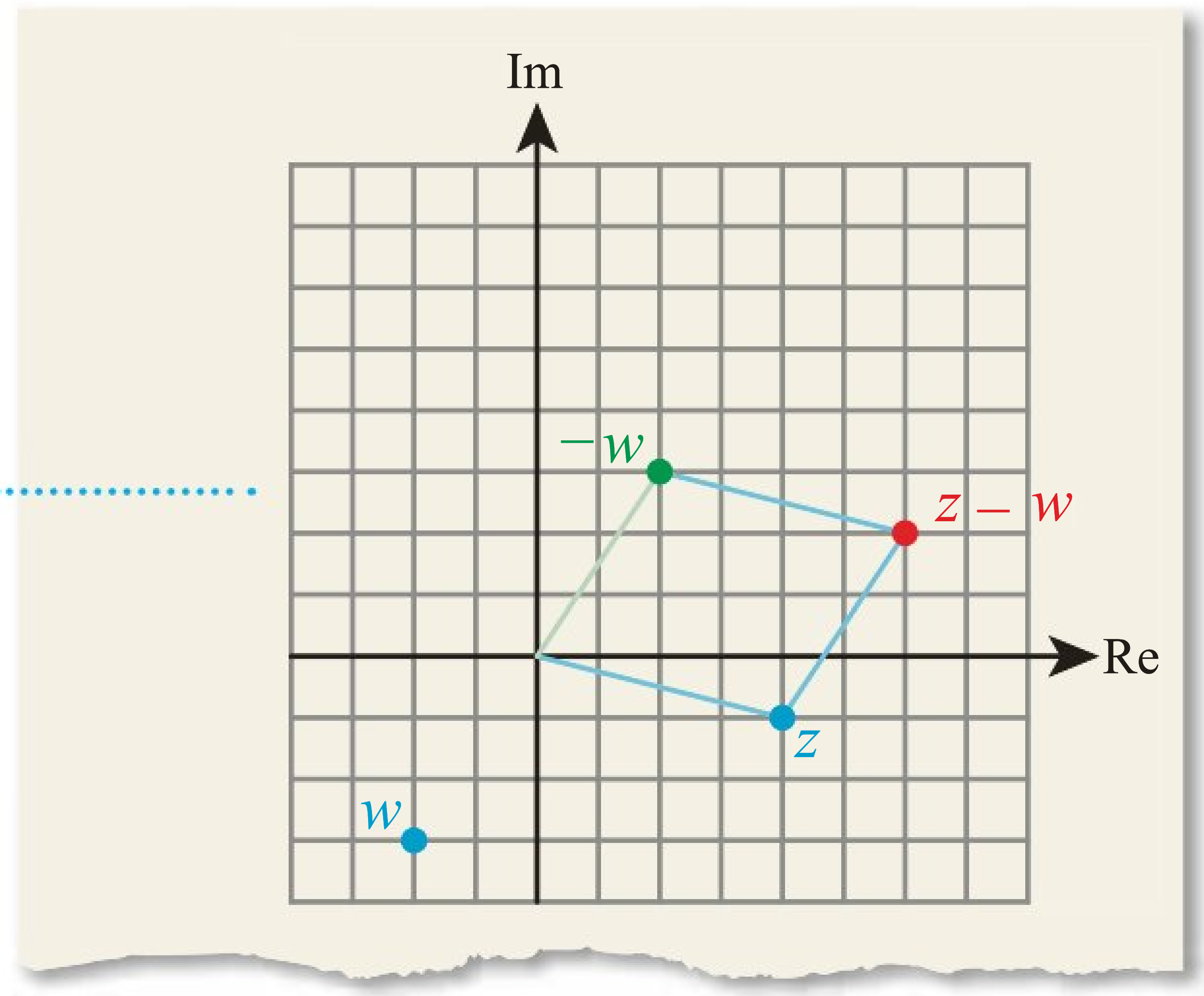
The complex numbers z and w are shown on the Argand diagram below.

Mark on the complex number $z - w$.

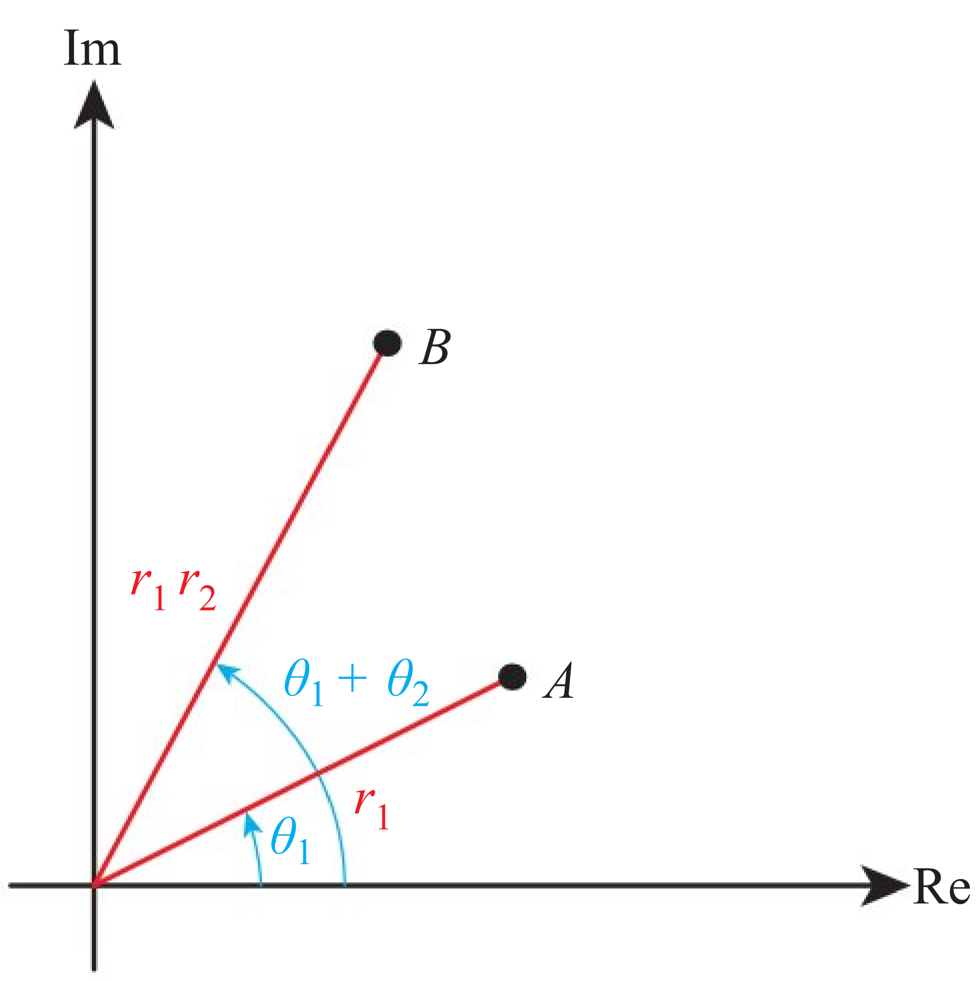


The point representing $z - w$ can be found as for addition by thinking of the process as $z + (-w)$

The complex number $-w$ is w rotated by 180° around the origin



Multiplication of complex numbers also has an interesting geometrical interpretation. Let A be the point corresponding to the complex number $z_1 = r_1 \text{cis}\,\theta_1$, and let B be the point corresponding to the complex number $z_1 \times (r_2 \text{cis}\,\theta_2) = r_1 r_2 \text{cis}\,(\theta_1 + \theta_2)$.

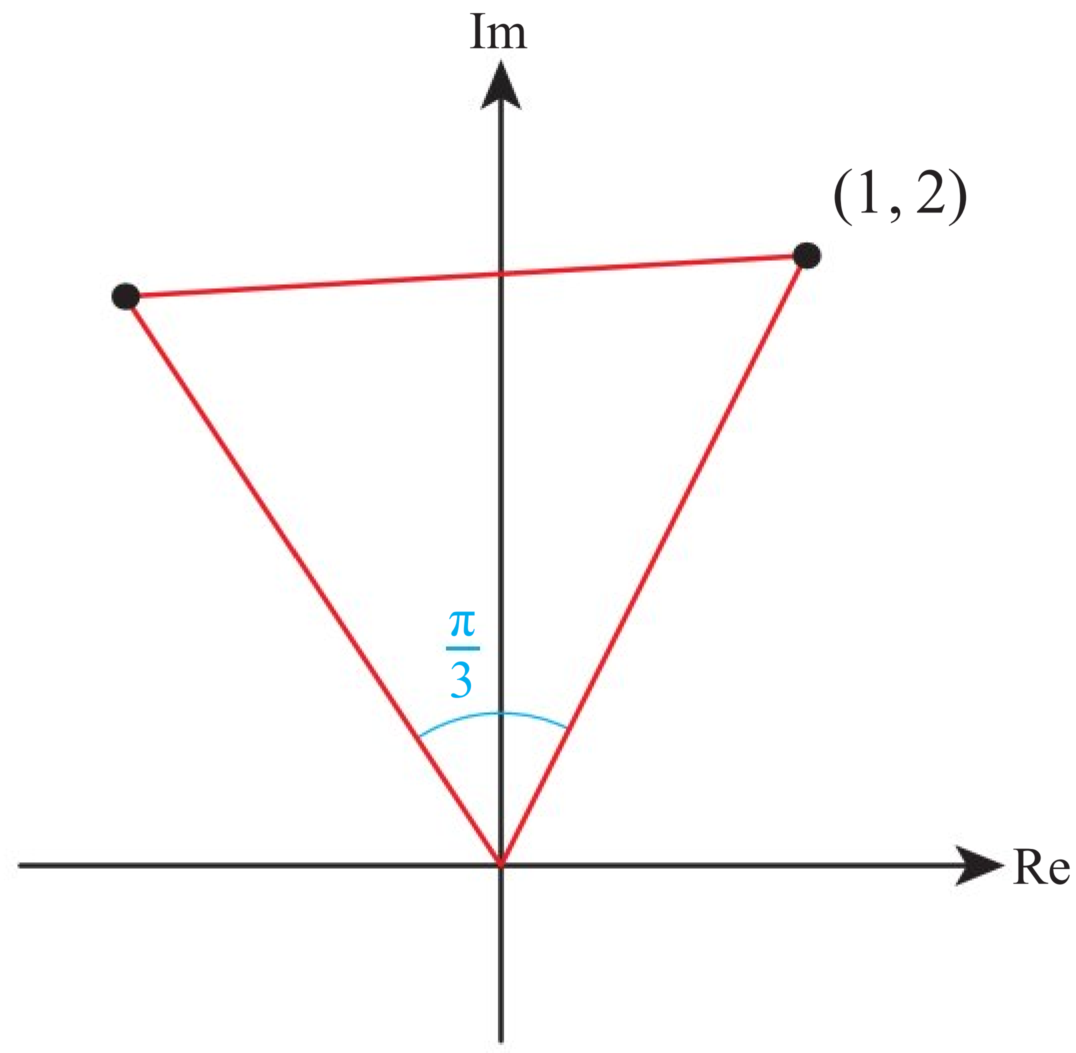


Then $OA = r_1$, $OB = r_1 r_2$ and $\angle AOB = (\theta_1 + \theta_2) - \theta_1 = \theta_2$. Hence, multiplication by $r_2 \text{cis}\, \theta_2$ corresponds to rotation around the origin though angle θ_2 and a stretch with scale factor r_2.

This result is remarkably useful in some situations that have nothing to do with complex numbers.

WORKED EXAMPLE 6.17

An equilateral triangle shown in the diagram has one vertex at the origin and another at (1, 2). Find the coordinates of the third vertex.



The third vertex can be obtained by rotation through 60° anticlockwise around the origin and no enlargement. This corresponds to multiplication by the complex number with modulus 1 and argument $\frac{\pi}{3}$

Point (1, 2) corresponds to complex number $1 + 2\text{i}$.

Rotation through 60° around the origin corresponds to multiplication by $\text{cis}\,\frac{\pi}{3}$.

The complex number corresponding to the third vertex is

$$(1+2\text{i})\left(\text{cis}\,\frac{\pi}{3}\right) = (1+2\text{i})\left(\cos\frac{\pi}{3} + \text{i}\sin\frac{\pi}{3}\right)$$

$$= (1+2\text{i})\left(\frac{1}{2} + \frac{\sqrt{3}}{2}\text{i}\right)$$

$$= \left(\frac{1}{2} - \sqrt{3}\right) + \left(\frac{\sqrt{3}}{2} + 1\right)\text{i}$$

This complex number gives the coordinates of the third vertex

So, the coordinates are

$$\left(\frac{1}{2} - \sqrt{3}, \frac{\sqrt{3}}{2} + 1\right)$$

TOOLKIT: Problem Solving

You could try solving the problem in Worked Example 6.17 using coordinate geometry or trigonometry – the calculations are much more complicated than this, as you may be able to guess by looking at the answer!

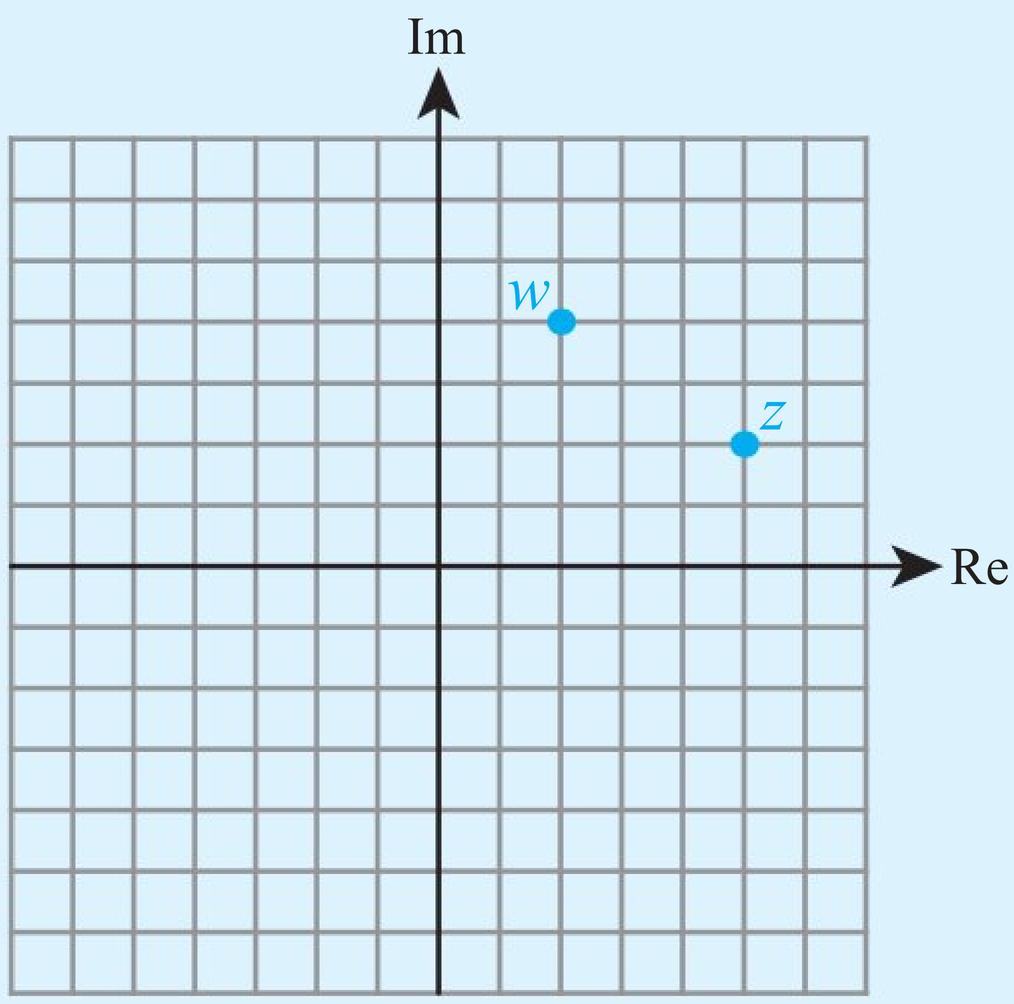
KEY CONCEPT – REPRESENTATION

In Chapter 3 you learnt that rotations can also be represented using matrices. For example, the rotation through angle θ anticlockwise around the origin can be represented by the matrix $\begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$, and also as multiplication by the complex number $\text{cis}\,\theta = \cos\theta + \text{i}\sin\theta$. Can you see the link between the two **representations**? Which one do you find easier to work with?

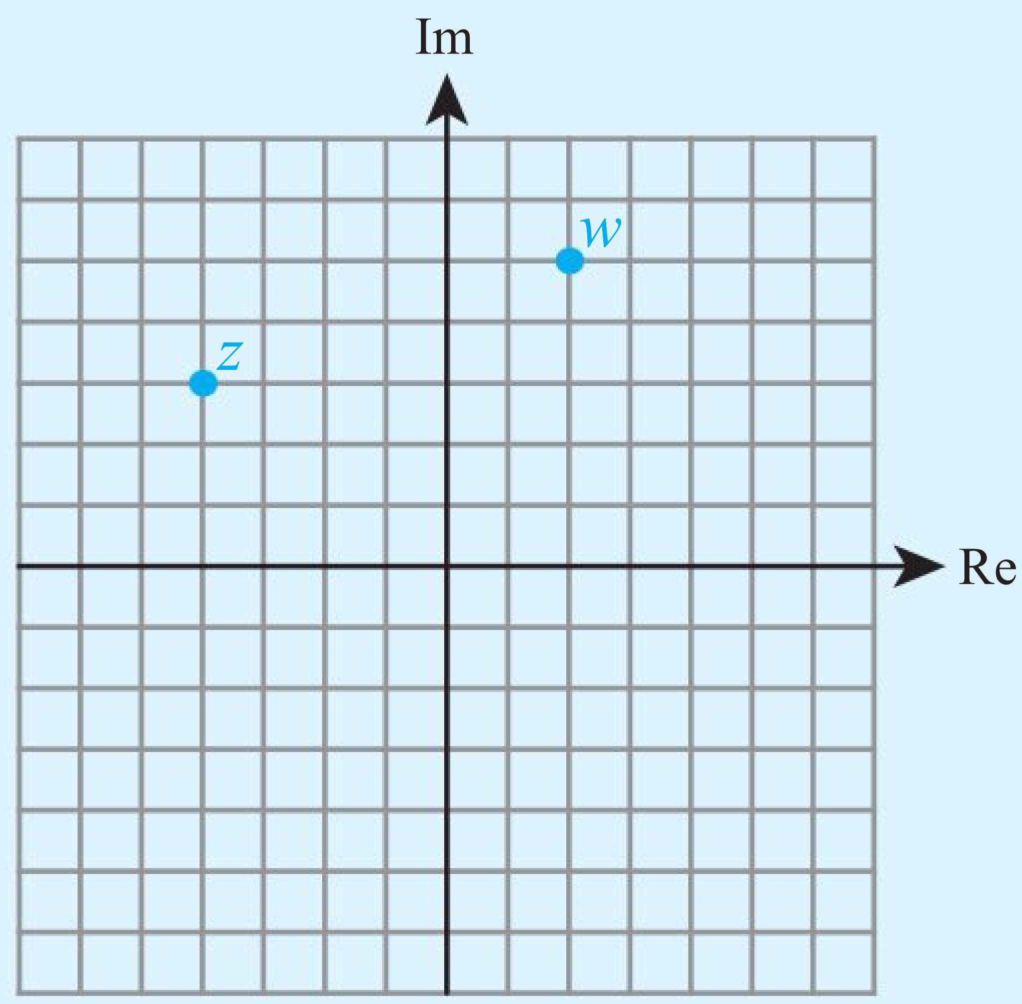
Exercise 6C

For questions 1 and 2, use the techniques demonstrated in Worked Examples 6.15 and 6.16 to add the points corresponding to the stated complex numbers to a copy of each complex plane.

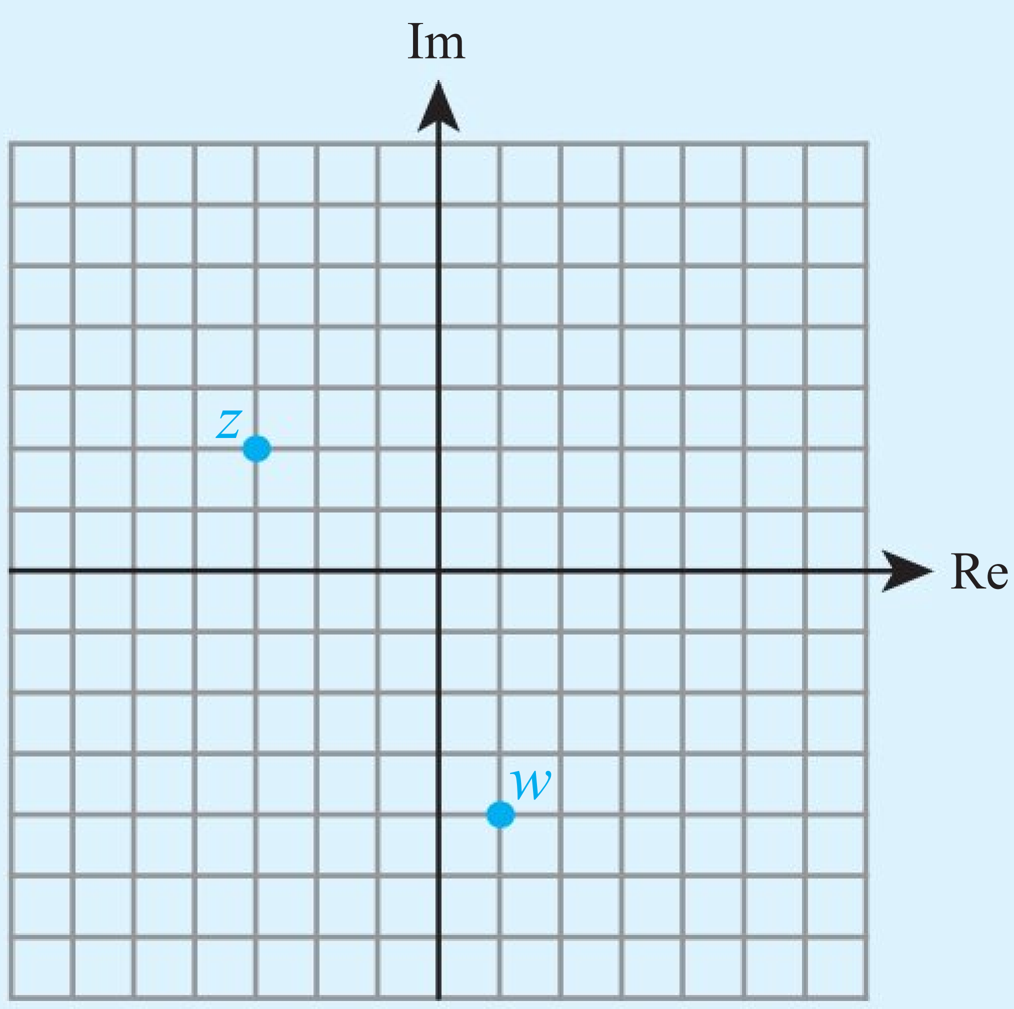
1 a i z^* ii $z + w$



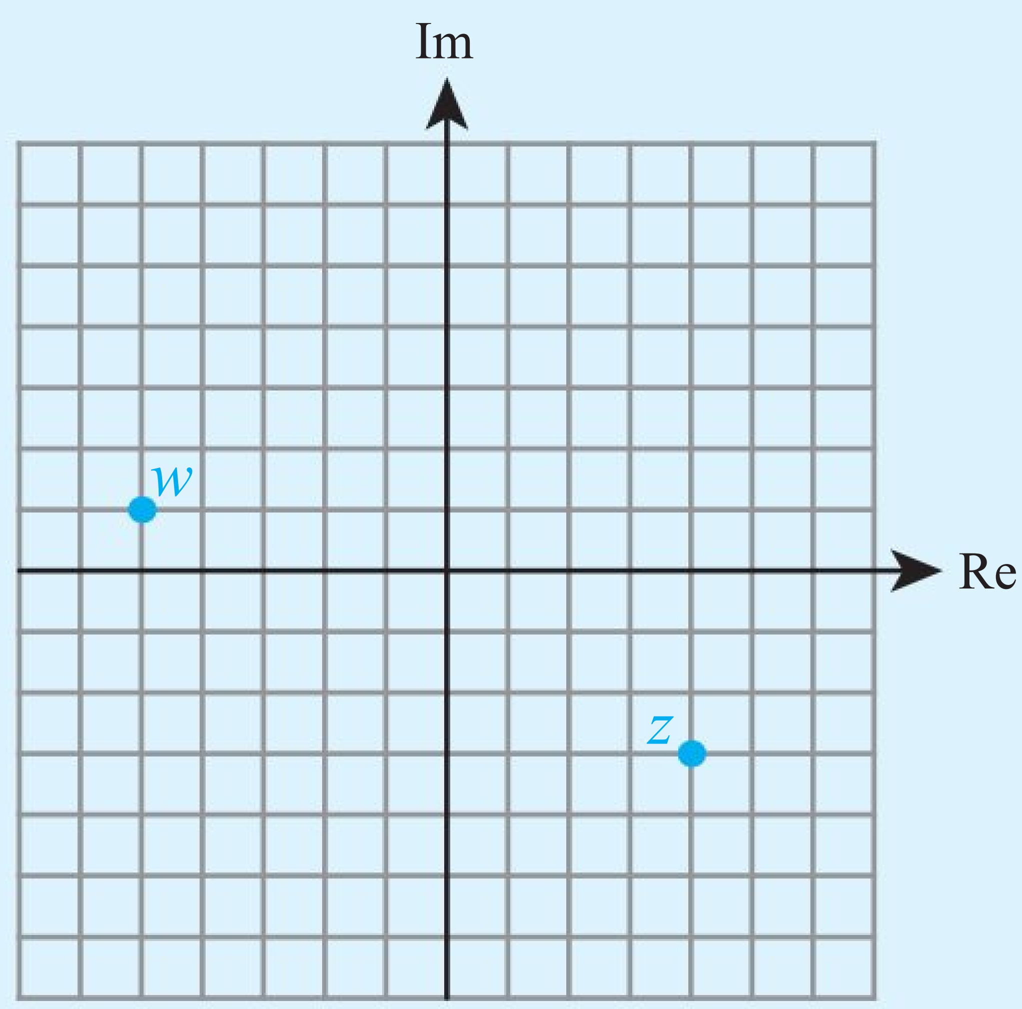
b i z^* ii $z + w$



2 a i $-w$ ii $z - w$



b i $-w$ ii $z - w$



3 Points A and B represent complex numbers $a = 4 + \mathrm{i}$ and $b = 5 + 3\mathrm{i}$ on an Argand diagram.

a Find the modulus and argument of a and b.

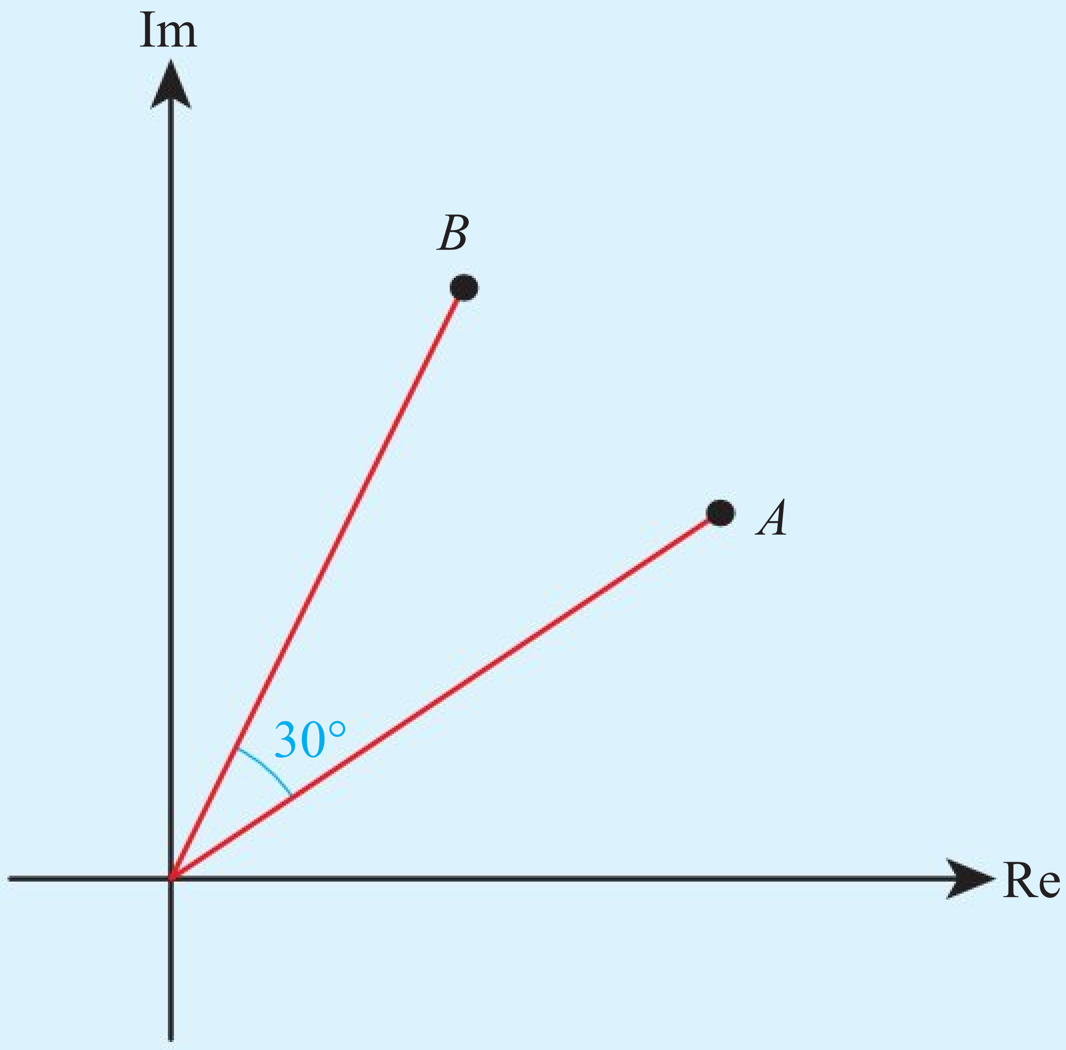
b Point A is mapped to point B by a combination of an enlargement and a rotation. Find the scale factor of the enlargement and the angle of rotation.

4 Points P and Q represent complex numbers $p = 3 + 5\mathrm{i}$ and $q = -\sqrt{30} + 2\mathrm{i}$, respectively.

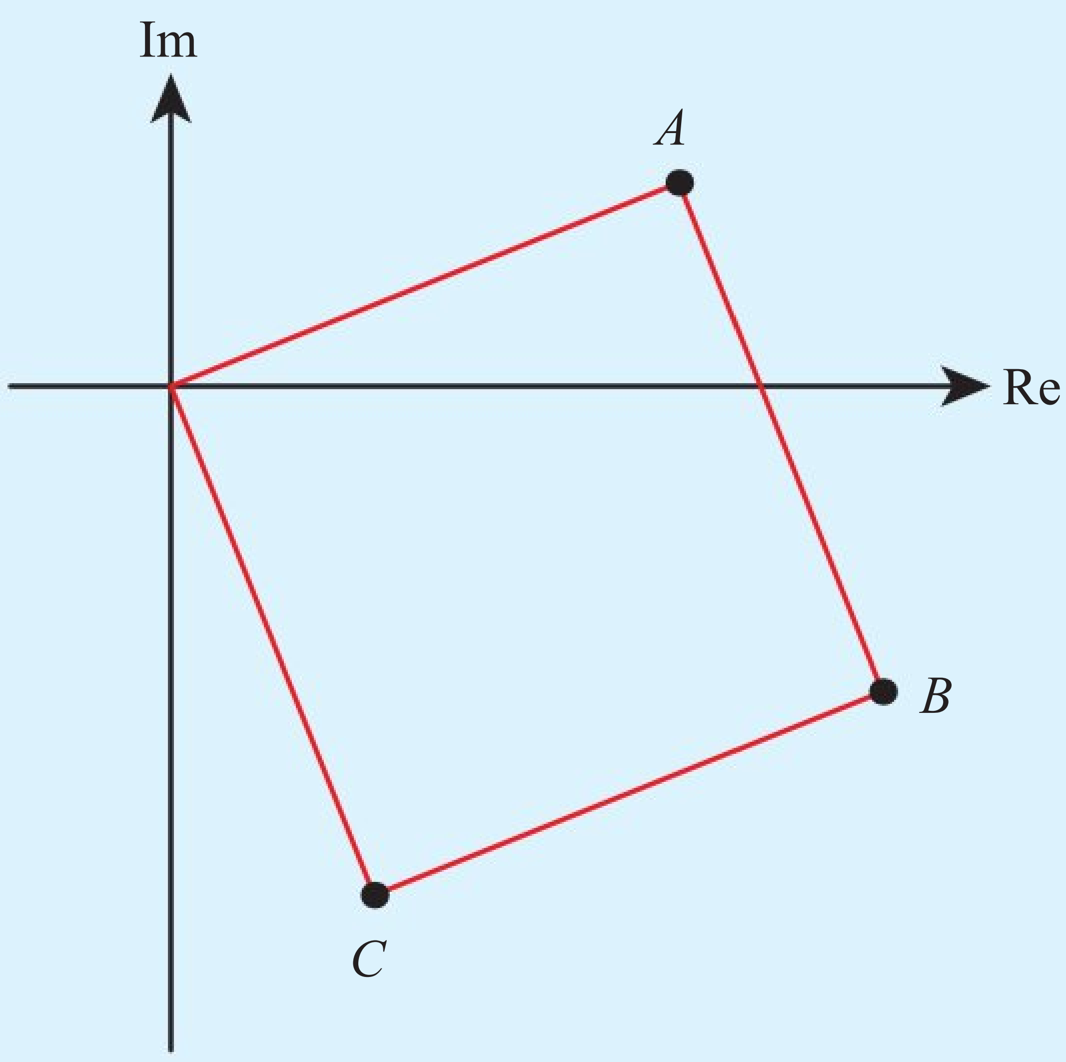
a Show that $|p| = |q|$.

b Describe a single transformation that maps P to Q.

5 The complex number corresponding to the point A in the diagram is $z_1 = 3 + 2\mathrm{i}$. The distance $OB = OA$. Find, in surd form, the complex number corresponding to the point B.

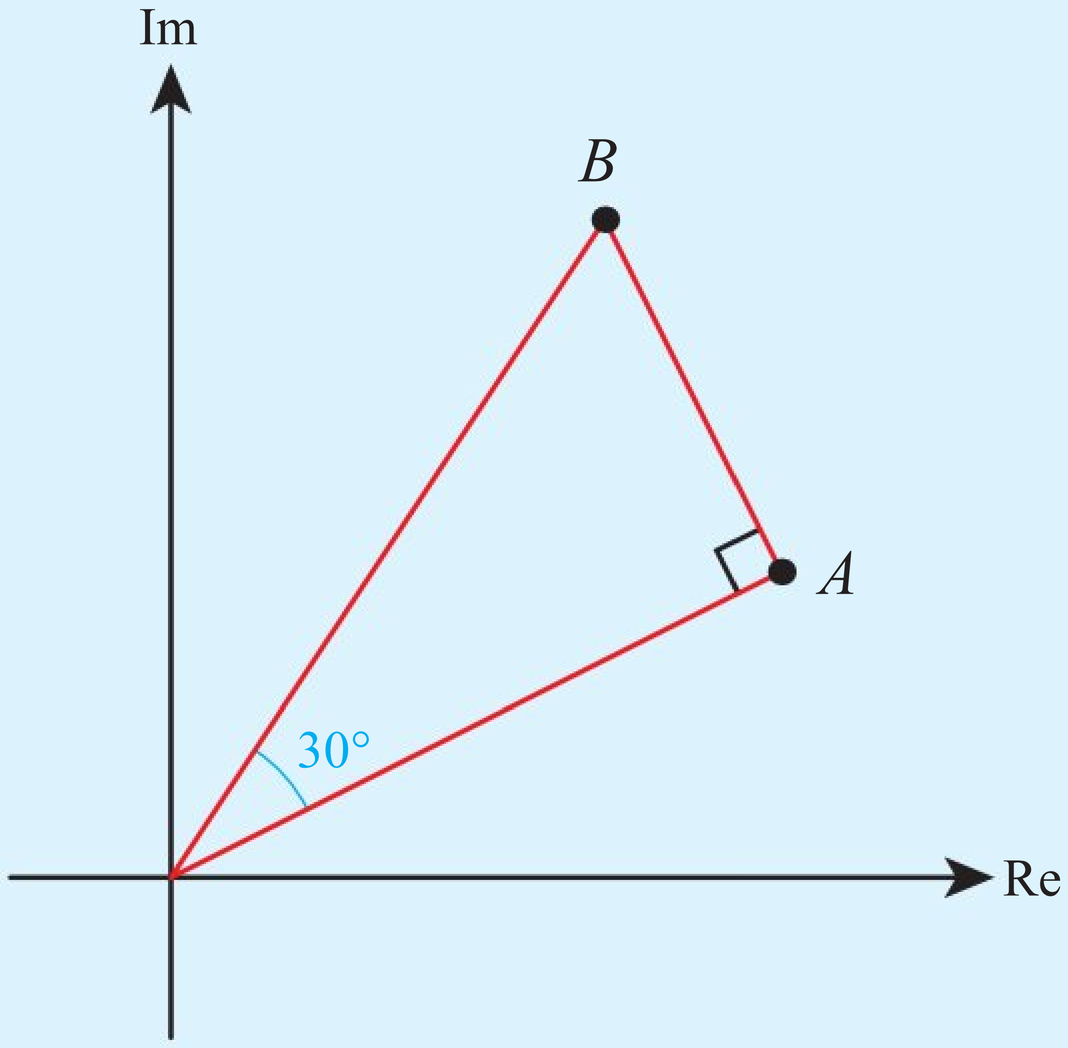


6 The diagram shows a square $OABC$, where A has coordinates (5, 2).



Find the exact coordinates of B and C.

7 The diagram shows a right-angled triangle OAB with angle $AOB = 30°$. The coordinates of A are (6, 3).



a Find the exact length OB.

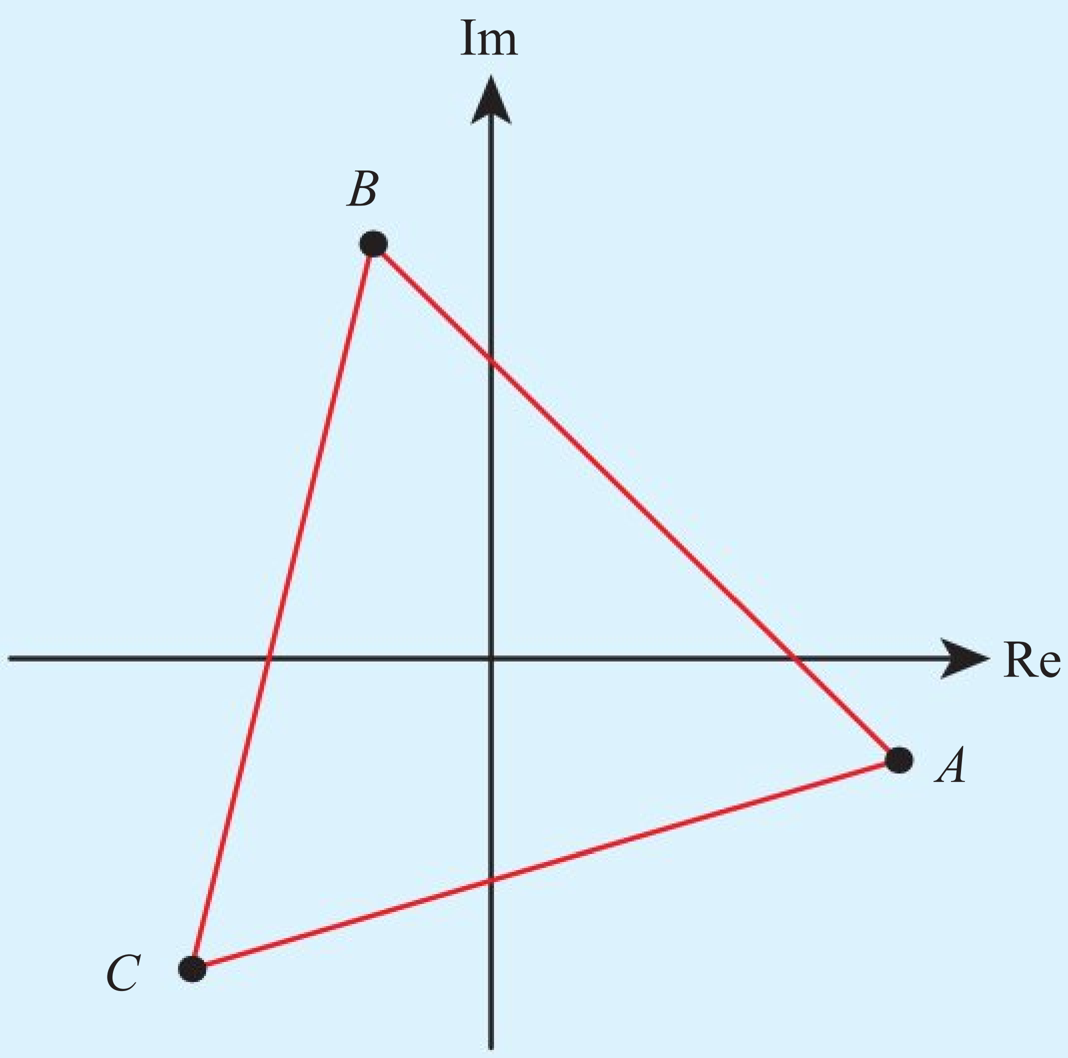
b Using complex numbers, or otherwise, find the coordinates of B.

8 Let $z = 0.6 + 0.8\text{i}$.

a Represent z, z^2 and z^3 on an Argand diagram.

b Describe fully the transformation mapping z to z^3.

9 The diagram shows an equilateral triangle with its centre at the origin and one vertex $A(4, -1)$.



a Write down the complex number corresponding to the vertex A.

b Hence find the coordinates of the other two vertices.

10 a The point representing the complex number q on an Argand diagram is rotated through angle θ about the point representing the complex number a. The resulting point represents complex number q. Explain why $q - a = (p - a)\text{e}^{\text{i}\theta}$.

b Find the exact coordinates of the image when the point $P\ (1, 3)$ is rotated 60° anticlockwise about the point $A\ (2, -1)$.

Checklist

- You should be able to work with the imaginary number i.
  - $\mathrm{i} = \sqrt{-1}$
- You should be able to find sums, products and quotients of complex numbers in Cartesian form.
  - A complex number z can be written in Cartesian form as
    $z = x + \mathrm{i}y$
    where $x,\ y \in \mathbb{R}$
  - Its complex conjugate, z^*, is
    $z^* = x - \mathrm{i}y$
  - The product of a complex number with its conjugate is real:
    $zz^* = x^2 + y^2$
- You should be able to represent complex numbers geometrically on the complex plane (Argand diagram).
- You should be able to find the modulus, r, and argument, θ, of a complex number.
  - If $z = x + \mathrm{i}y$, then
    – $r = \sqrt{x^2 + y^2}$
    – $\tan\theta = \dfrac{y}{x}$
- You should be able to write complex numbers in modulus–argument (polar) form.
  - $z = r(\cos\theta + \mathrm{i}\sin\theta) = r\,\mathrm{cis}\,\theta$
  - $z^* = r\,\mathrm{cis}\,(-\theta)$
- You should be able to find sums, products and quotients of complex numbers in modulus–argument form.
  - $|zw| = |z||w|$
  - $\arg(zw) = \arg z + \arg w$
  - $|z^n| = |z|^n$
  - $\arg(z^n) = n\arg z$
  - $\left|\dfrac{z}{w}\right| = \dfrac{|z|}{|w|}$
  - $\arg\left(\dfrac{z}{w}\right) = \arg z - \arg w$
- You should be able to write complex numbers in exponential (Euler) form.
  - $z = r\mathrm{e}^{\mathrm{i}\theta}$ where $\mathrm{e}^{\mathrm{i}\theta} = \cos\theta + \mathrm{i}\sin\theta$
  - $z^* = r\mathrm{e}^{-\mathrm{i}\theta}$
- You should be able to find sums, products and quotients of complex numbers in exponential form using the usual rules of algebra and exponents.
- You should be able to find powers of complex numbers.
  - $\left(r(\cos\theta + \mathrm{i}\sin\theta)\right)^n = r^n(\cos n\theta + \mathrm{i}\sin n\theta)$ for $n \in \mathbb{N}$
- You should know that, on an Argand diagram:
  - the sum of two complex numbers can be represented by the sum of the corresponding vectors
  - multiplication by a complex number $r\,\mathrm{cis}\,\theta$ corresponds to a rotation through angle θ around the origin and a stretch with scale factor r

Mixed Practice

1 Plot and label the following points on a single Argand diagram.

a $z_1 = \frac{1}{i}$ b $z_2 = (1+i)(2-i)$ c $z_3 = z_2^*$

2 a Solve the equation $x^2 - 2x + 2 = 0$.
b Show the solutions on an Argand diagram.

3 a Solve $x^2 - 6x + 12 = 0$.
b Show the solutions on an argand diagram.

4 If $z = \frac{1+i}{1+2i}$, find z^* in the form $a + ib$.

5 Solve the equation $\frac{z}{z+i} = 1 + 2i$.

6 Solve the equation $z + i = 2z^*$.

7 Solve the equation $z + 4 = iz$.

8 a If $z = 1 + i$, find the modulus and argument of z.

b If $w = \cos\frac{3\pi}{5} + i\sin\frac{3\pi}{5}$, find and simplify z^6w^5.

9 If $z = 2 - 2i$, find the modulus and argument of $(z^*)^3$.

10 If $ip + 4q = 2 + 3i$, find the values of p and q if they are
a real
b conjugate complex numbers.

11 Write these in Cartesian form.

a $\operatorname{cis}\frac{\pi}{2} \div \operatorname{cis}\frac{\pi}{6}$ b $\operatorname{cis}\frac{\pi}{2} - \operatorname{cis}\frac{\pi}{6}$

12 Write $\frac{1}{a+i}$ in Cartesian form where $a \in \mathbb{R}$.

13 Solve $\frac{1}{z+i} + \frac{2}{z-i} = 0$.

14 Solve the simultaneous equations

$$z + z^* = 8$$
$$z - z^* = 6i$$

15 The complex numbers z and w both have arguments between 0 and π.

Given that $zw = -\sqrt{3} + i$ and $\frac{z}{w} = -\frac{i}{2}$, find the modulus and argument of z.

16 If both b and $\frac{2}{2+i} - \frac{1}{b+i}$ are real numbers, find the possible values of b.

17 Find the possible values of z if $\text{Re}(z) = 2$ and $\text{Re}(z^2) = 3$.

18 Solve $z + |z| = 18 + 12i$.

19 One root of the equation $x^2 + bx + c = 0$ is $1 + 2\text{i}$. Find the values of a and b, given that they are real.

20 Find the complex number z such that

$$|z| + z = 8 + 4\text{i}$$

21 **a** Find the modulus and the argument of $0.3 + 0.4\text{e}^{2\text{i}}$.

b The heights of waves, at time t, are modelled by equations $h_1 = 0.3\cos 3t$ and $h_2 = 0.4\cos(3t + 2)$. Find an expression for the combined wave in the form $h_1 + h_2 = A\cos(3t + c)$.

22 **a** Find the modulus and argument of $-4 + \text{i}$.

Point P has coordinates $(-4, 1)$. Point Q is obtained by rotating P around the origin through $\frac{\pi}{6}$ radians anticlockwise.

b Write, in exponential form, the complex number with modulus 1 and argument $\frac{\pi}{6}$.

c Hence or otherwise, find the coordinates of Q.

23 Find the eigenvalues of the matrix $\begin{pmatrix} 2 & 1 \\ -5 & 2 \end{pmatrix}$.

24 If $z = x + \text{i}y$ and $2|z| = |z + 3|$, find the relationship between x and y.

25 Let $z = 1 + \text{i}$ and $\text{w} = 1 + \sqrt{3}\text{i}$.

a Write zw in Cartesian form.

b Write z and w in modulus–argument form. Hence find the modulus and argument of zw.

c Use your answers from parts **a** and **b** to find the exact value of $\sin\left(\frac{7\pi}{12}\right)$.

26 **a** Find the modulus and argument of $\sqrt{3} + \text{i}$.

b Simplify $(\sqrt{3} + \text{i})^7 + (\sqrt{3} - \text{i})^7$.

27 If $z = 6 + 8\text{i}$ and $|w| = 5$, find w if $|z + w| = |z| + |w|$.

28 The point $A(1, 3)$ is transformed to the point $B(-6, 4)$ using a rotation around the origin followed by an enlargement. Using complex numbers, find the angle of rotation and scale factor of the enlargement.

29 The complex numbers $z_1 = 2 - 2\text{i}$ and $z_2 = 1 - \sqrt{3}\text{i}$ are represented by the points A and B, respectively on an Argand diagram. Given that O is the origin,

a find AB, giving your answer in the form $a\sqrt{b - \sqrt{3}}$, where $a, b \in \mathbb{Z}^+$

b calculate $A\hat{O}B$ in terms of π.

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7 Graphs and algorithms

ESSENTIAL UNDERSTANDINGS

- Geometry and trigonometry allow us to quantify the physical world, enhancing our spatial awareness in two and three dimensions.
- This branch provides us with the tools for analysis, measurement and transformation of quantities, movements and relationships.

In this chapter you will learn…

- how to represent a network as a graph, or using an adjacency matrix
- about different types of graphs and networks
- about different ways of moving around a graph
- how to use an adjacency matrix or a transition matrix for a graph
- about the PageRank algorithm
- how to find a minimum spanning tree for a graph
- how to solve the Chinese postman problem
- how to solve the travelling salesman problem.

CONCEPTS

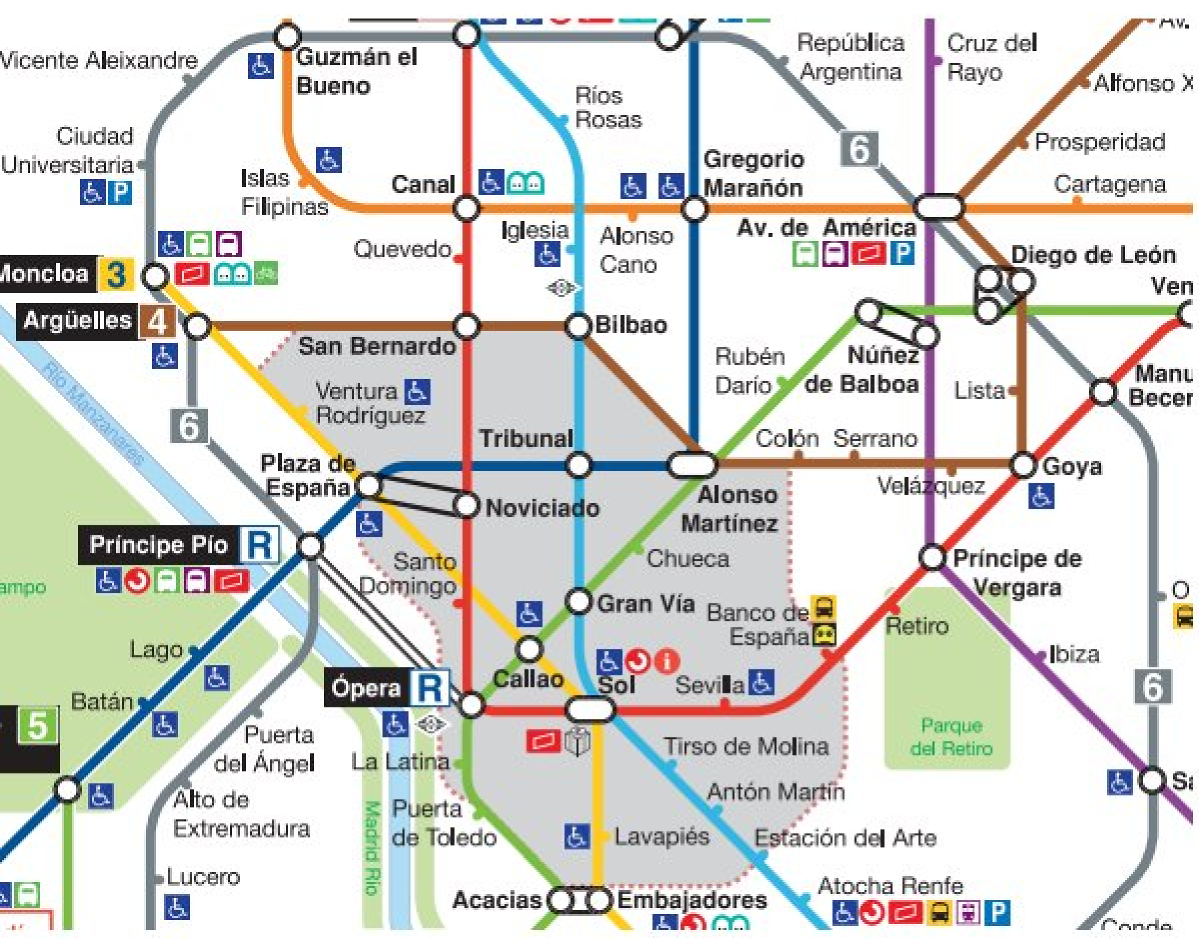
The following concepts will be addressed in this chapter:

- Graph theory algorithms allow us to **represent** networks and to **model** complex real-world problems.

LEARNER PROFILE – Open-minded

How useful is academic education? Is the mathematics chosen to be assessed in this course appropriate and beneficial?

■ **Figure 7.1** What information is being represented in each of these images? What information has been left out?



PRIOR KNOWLEDGE

Use your calculator to find $\mathbf{M}^5$, where

$$\mathbf{M} = \begin{pmatrix} 1 & 0 & 0.5 & 1.2 \\ 2 & 1.5 & 1.3 & 0 \\ 0 & 2 & 1 & 0.6 \\ 1.2 & 0.8 & 0 & 1 \end{pmatrix}$$

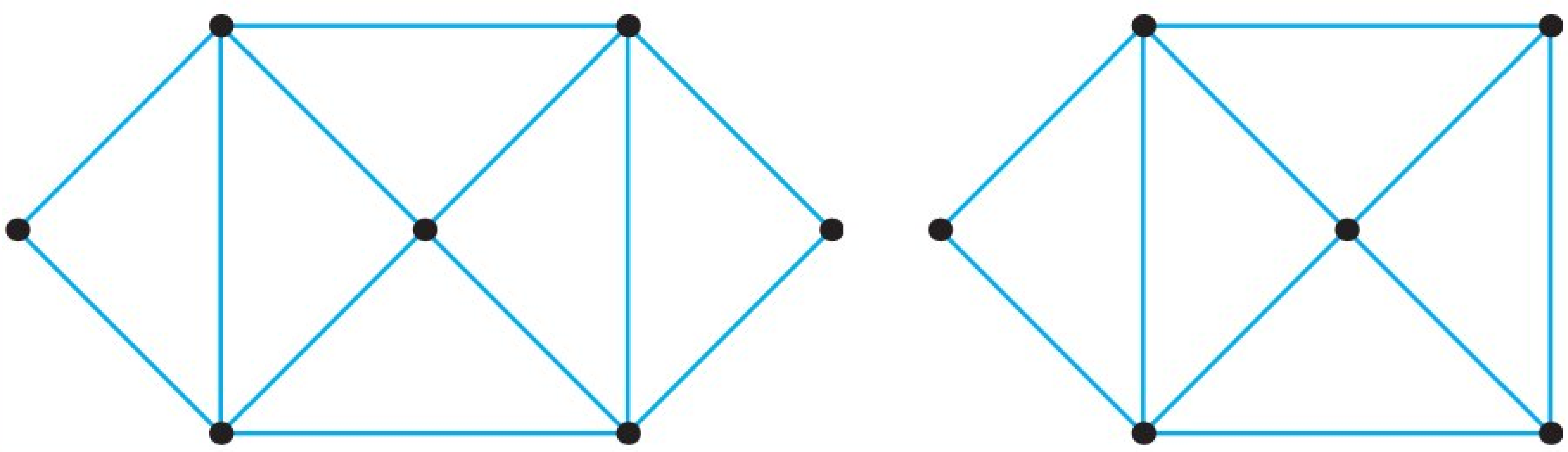
Many real-life systems can be represented as a network of connections between individual components. The mathematical study of such networks is called graph theory, and involves studying the number of connections, ways of moving around a network and the costs or distances involved. It has many applications, from finding optimal travel routes to designing internet search engines.

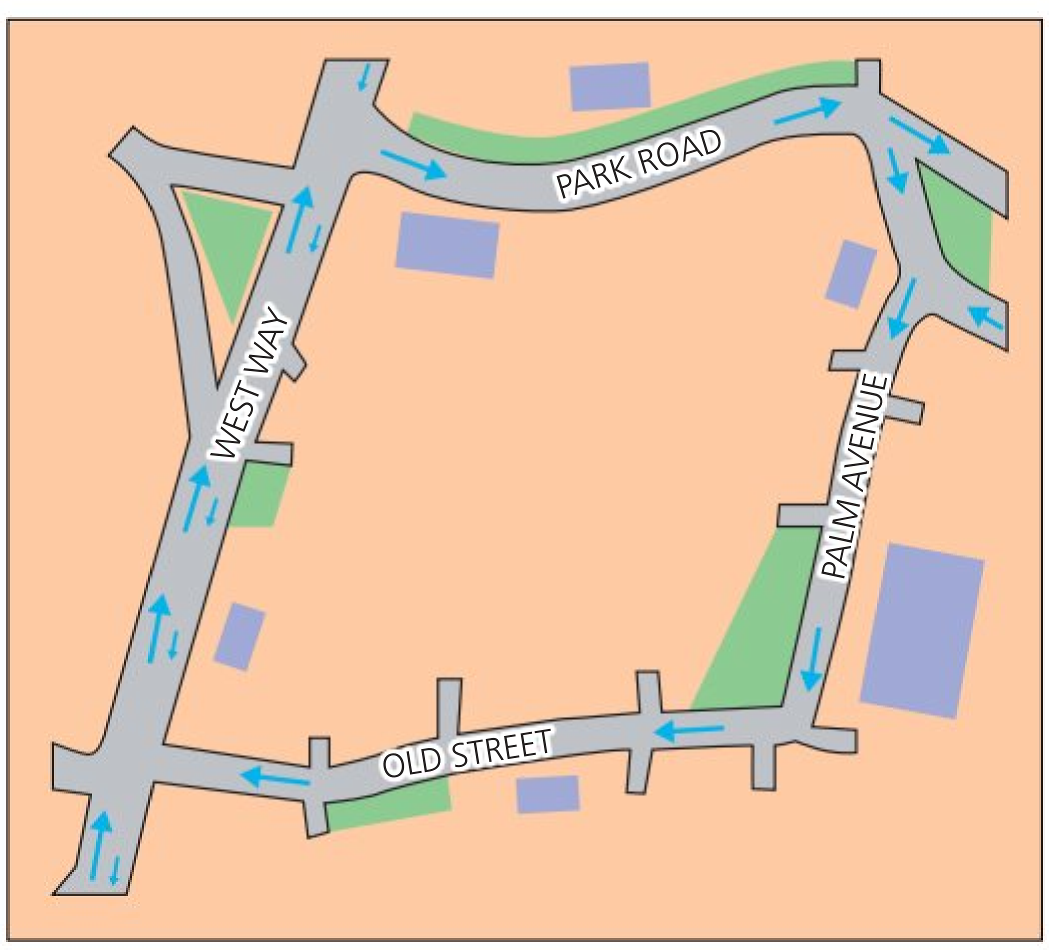
Starter Activity

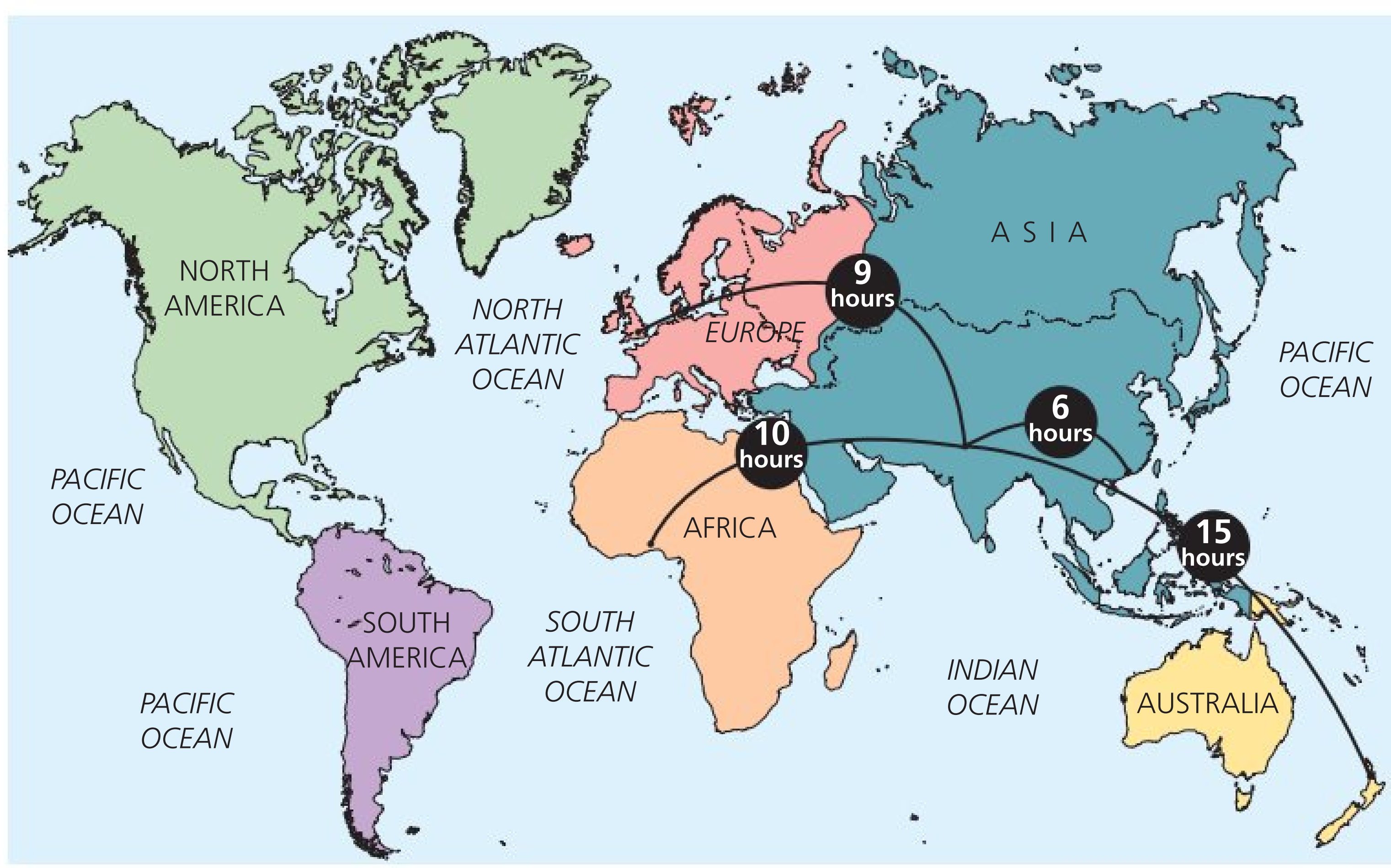
Look at the images in Figure 7.1. Discuss what information is shown in each image. Are there any other ways in which you could represent this information? What information that could be relevant to the situation is not shown in the image?

Now look at this problem:

Can you draw the following shapes without lifting the tip of your pen off the paper and without going over any lines twice? What if you have to start and finish at the same point?



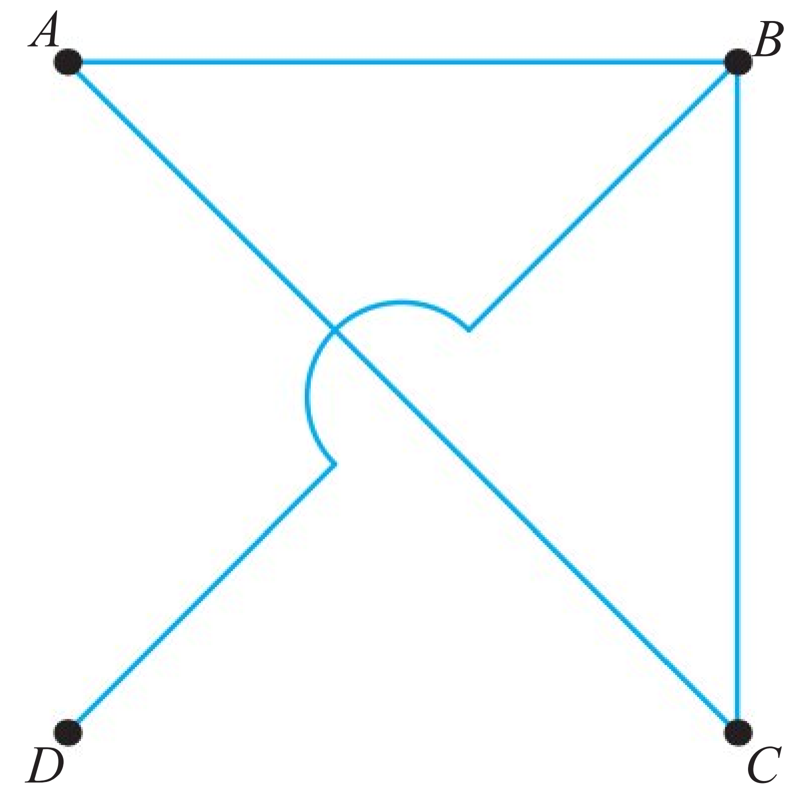




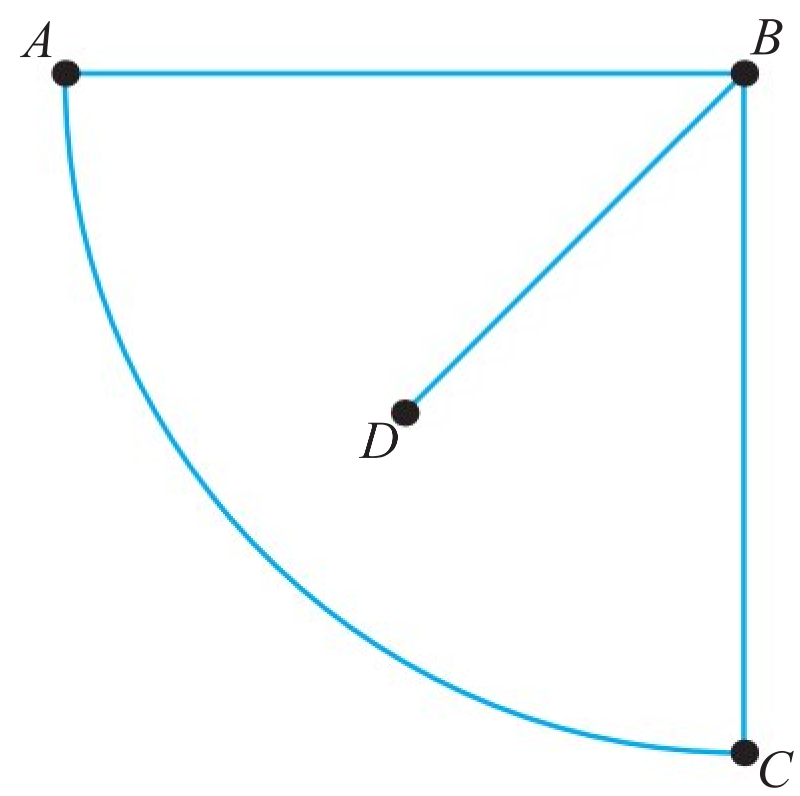
7A Introduction to graph theory

Definitions

A **graph** consists of **vertices** connected by **edges**. Two vertices are called **adjacent** if they are joined by an edge. Two edges are called adjacent if they share a common vertex.



Graphs are often represented by showing vertices as point and edges as lines between them. When we draw a graph, the position of the vertices and the shape of the edges is not relevant; the only thing that matters is which vertices are adjacent. So the two diagrams you see here represent the same graph.



The vertices of the graph are usually labelled by capital letters. The graph has vertices A, B, C and D and edges AB, AC, BC and BD.

It is often important to know the number of edges coming out of each vertex. This is called the **degree** of a vertex. For example, in the graph shown,

$$\deg(A) = 2,\ \deg(B) = 3,\ \deg(C) = 2,\ \deg(D) = 1$$

Tip

The edges may intersect at points other than vertices, so it is important to label the vertices clearly.

Graph theory is a relatively new branch of mathematics, with its beginnings in the eighteenth century and many important developments in the early twentieth century. Its development was mainly motivated by applications and it has had contributions from many different communities – mathematicians, computer scientists, chemists, even linguists. For those reasons some of the terminology is not firmly established and standardized yet, so you may find slightly different definitions in other books. For example, vertices and edges are also referred to as nodes and arcs. The terminology we use here is what will be used in your IB examination.

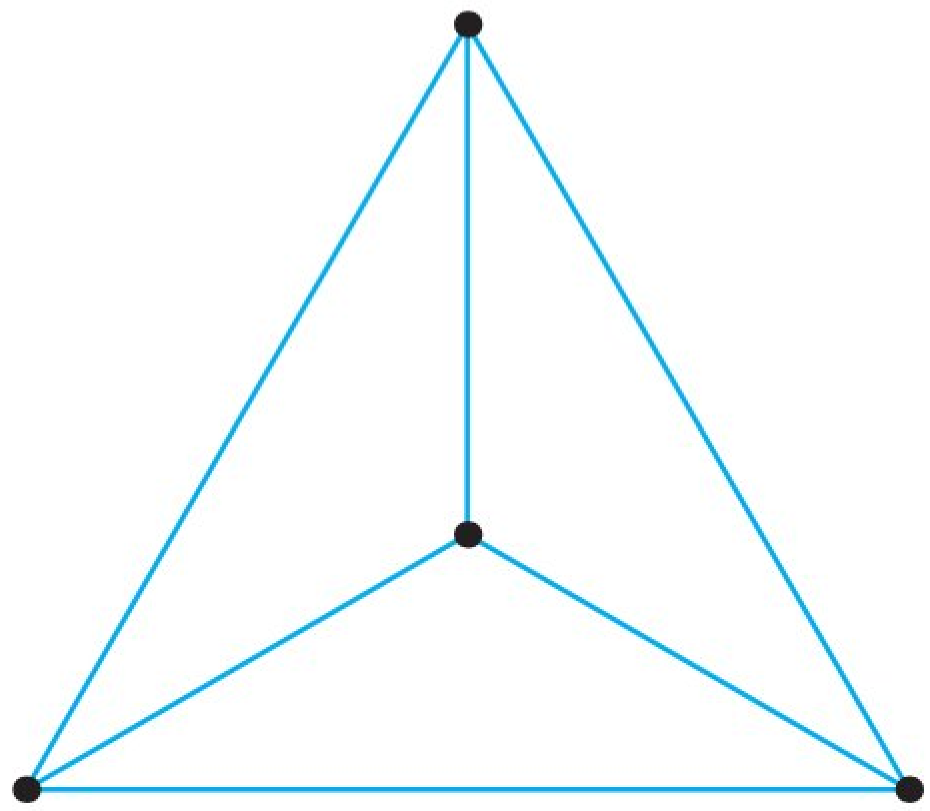
You are the Researcher

There are many interesting relationships connecting the number of edges, number of vertices and their degrees in a graph. One of the most important is the handshaking lemma, which states that the sum of the degrees of all the vertices in a graph is equal to twice the number of edges.

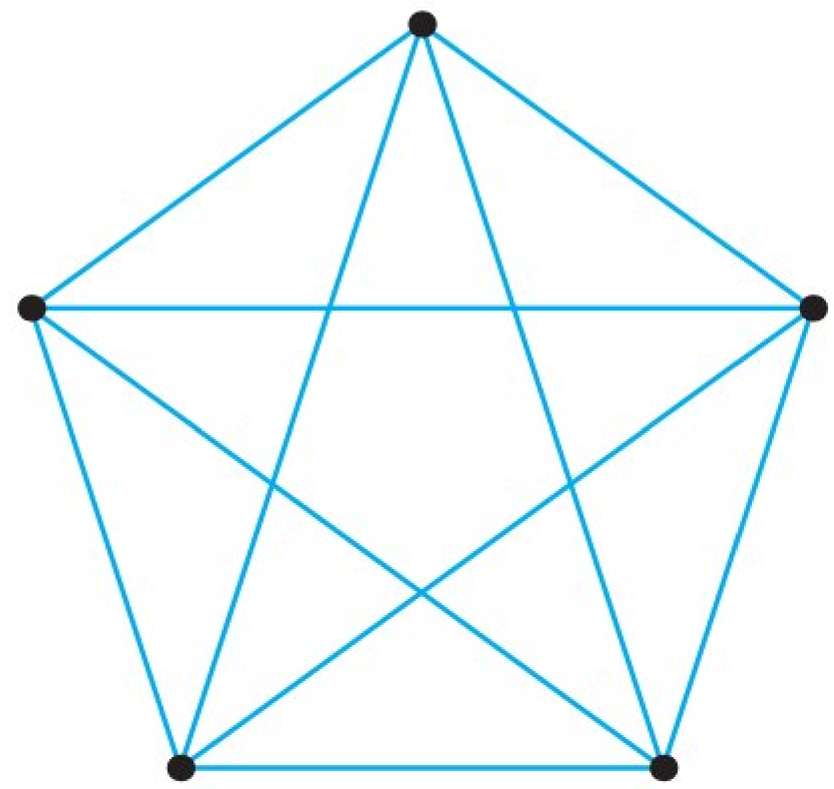
Types of graphs

Different types of graphs are appropriate for different practical situations.

Sometimes every vertex is connected to every other vertex by exactly one edge. This is called a **complete graph**. A symbol for a complete graph with n vertices is K_n.



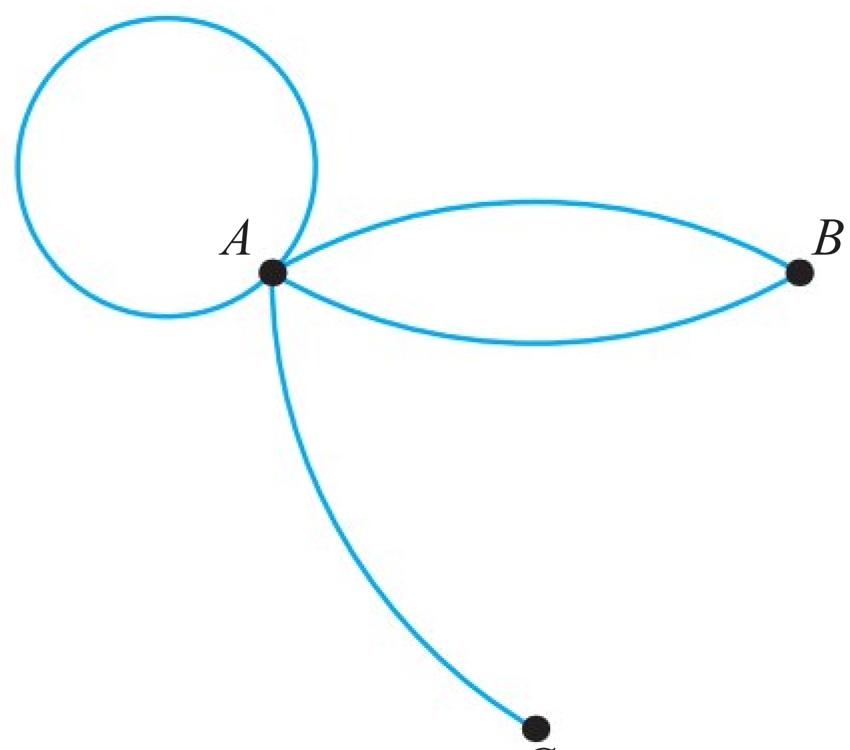
K_3



K_5

> You can use your knowledge of counting selections to show that the number of edges in K_n is $\binom{n}{2}$.

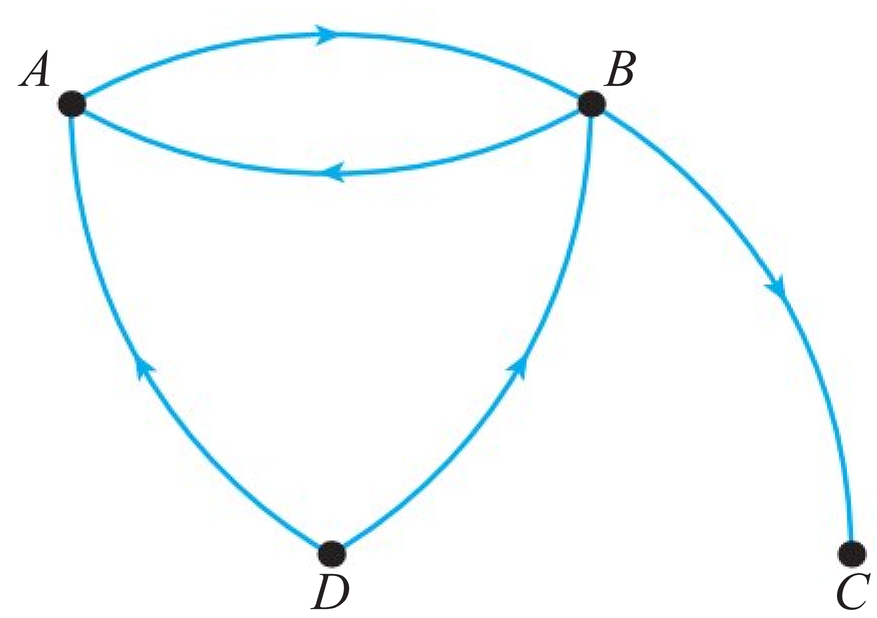
In some situations, it is possible to have more than one edge between two vertices, or for a vertex to be joined to itself, whereas in other situations this is not allowed. A **simple graph** has no multiple edges and no vertex is joined to itself. Graph 1, shown alongside, is not simple, but the graphs K_3 and K_5 above are simple graphs.



Graph 1

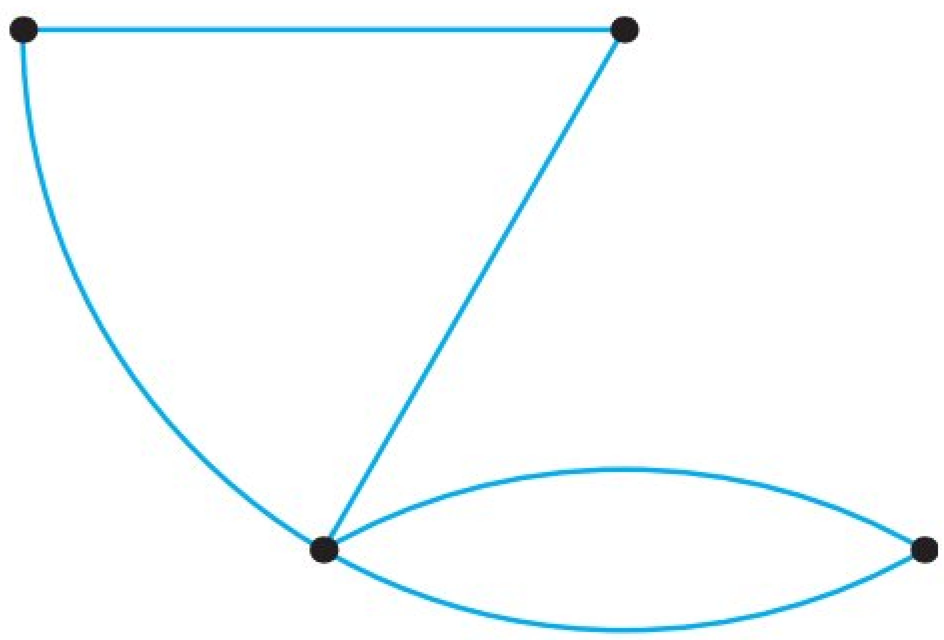
Sometimes we can only move in one direction between vertices, for example when modelling a road network with one-way roads. In that case, we can put arrows on the edges to indicate the allowed direction. The resulting graph is called a **directed graph**, or **digraph**. In Graph 2, shown on the right, it is possible to get from D to A but not from A to D, while both directions between A and B are allowed.

For a directed graph we can distinguish between **in-degree** and **out-degree** of each vertex. For example, vertex A in Graph 2 has in-degree of 2 and out-degree of 1.

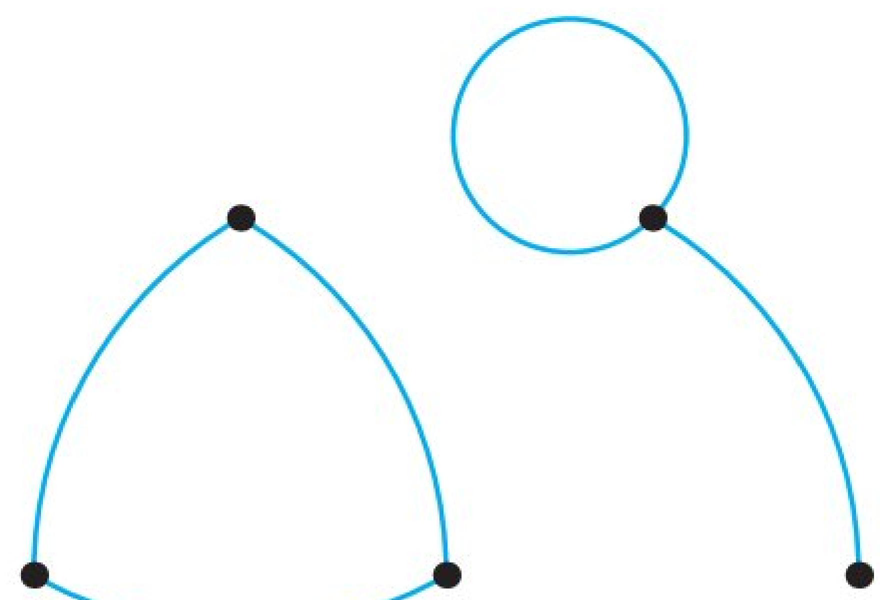


Graph 2

Graphs are often used to model situations where it is important to know how to get from one vertex to another. A graph is called **connected** if all pairs of vertices are connected (directly or indirectly). This means that the graph cannot be split into two parts.

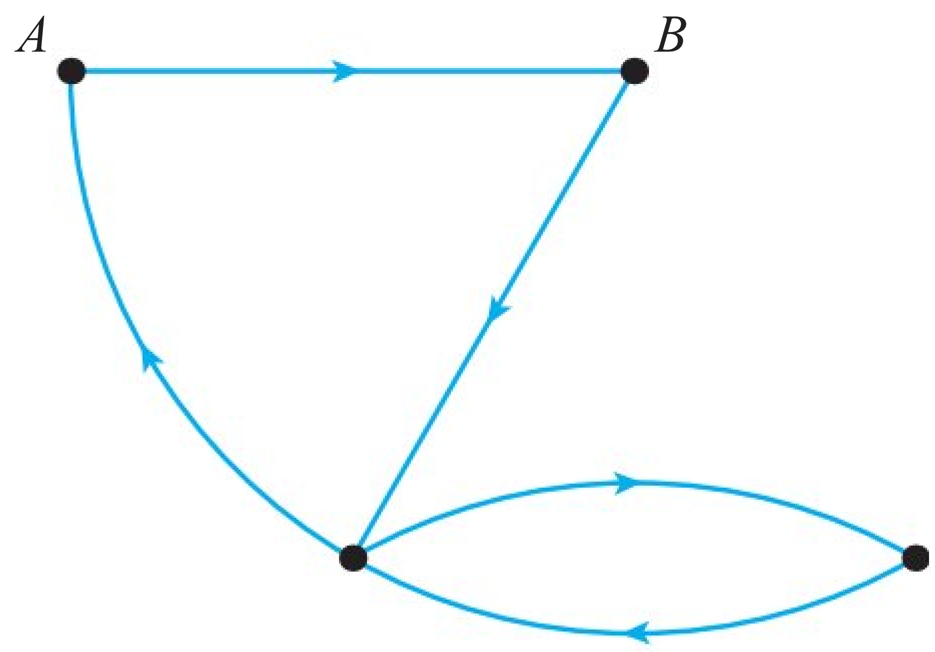


■ Connected

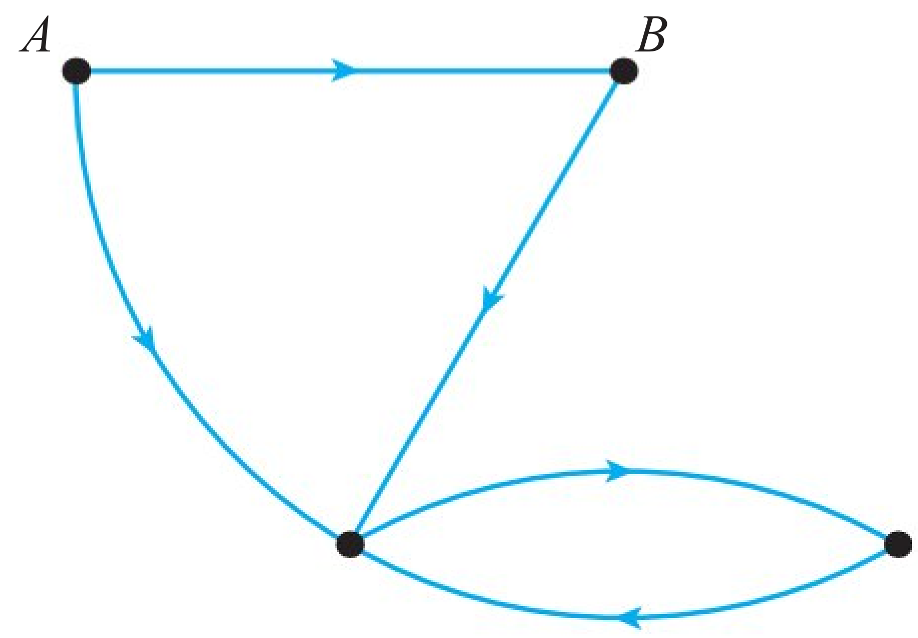


■ Not connected

In a directed graph it may be possible to get from A to B, but not from B to A. If all pairings of any two vertices are connected (in both directions) the graph is called **strongly connected**.



■ Strongly connected



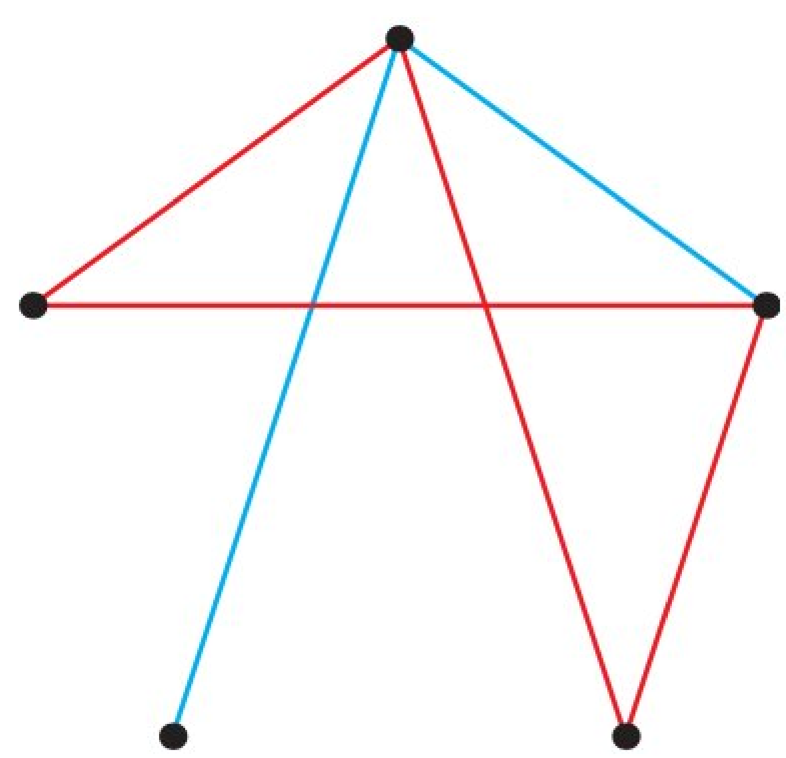
■ Not strongly connected (no path from B to A)

You are the Researcher

Another important type of graph is planar graphs, which can be drawn without edges crossing. There are many interesting results concerning the possible number of vertices and edges in a planar graph, including the Euler characteristic that also applies to polyhedra.

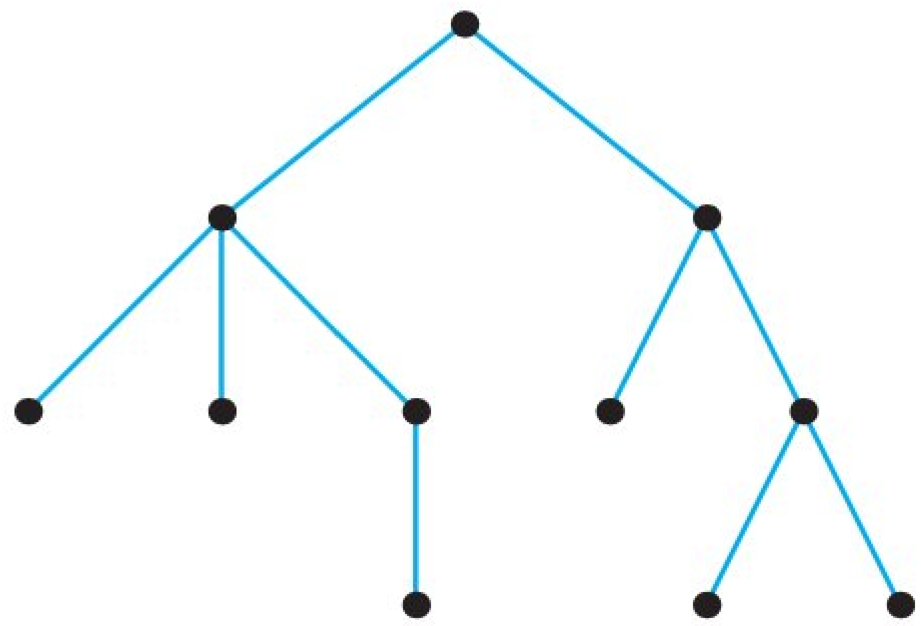
Subgraphs and trees

A **subgraph** of a given graph is a new graph formed by using only some of the edges of the original graph. In the diagram shown a subgraph is indicated by the red edges. Note that every simple graph with n vertices is a subgraph of the complete graph K_n.

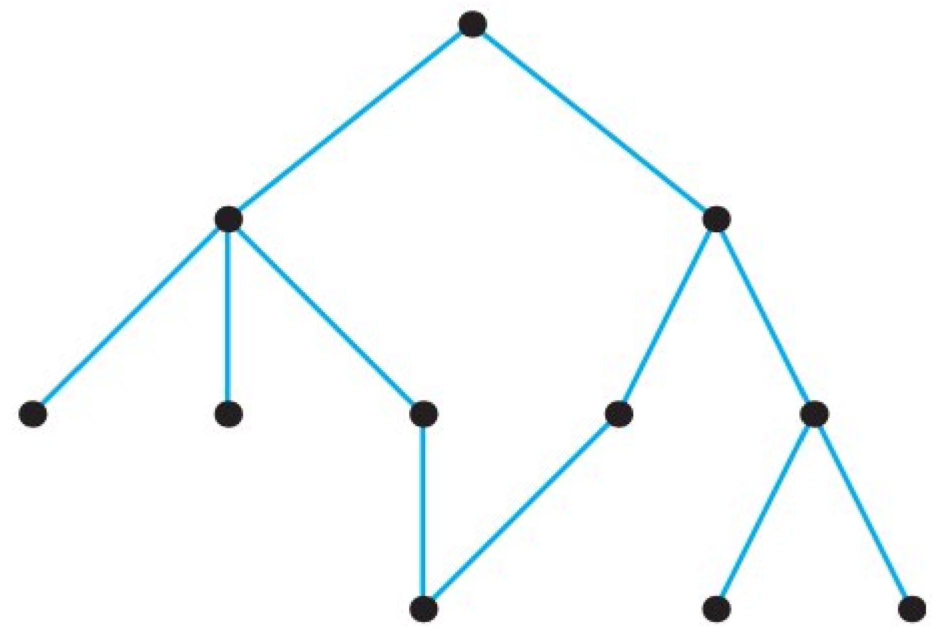


■ A subgraph is shown in red

A **tree** is a connected graph for which it is not possible to find a sequence of distinct edges that returns to the starting vertex (i.e. there are no closed paths).



A tree



Not a tree

Finding a subgraph which is a tree has many applications, for example in the minimum connector problem which you will meet in Section C.

KEY POINT 7.1

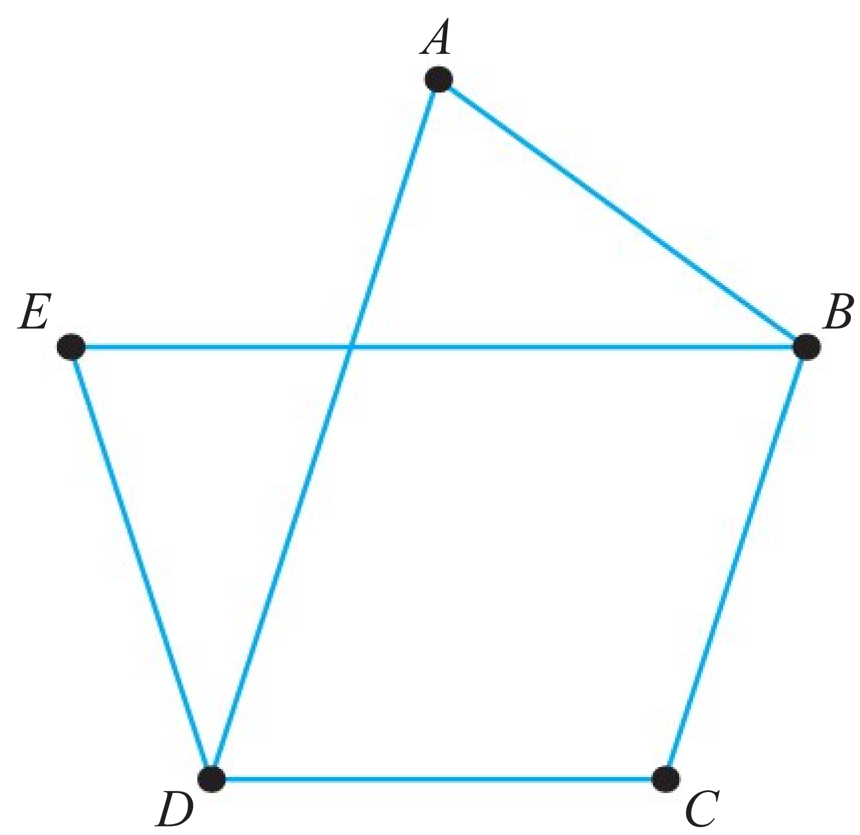
A tree with n vertices has $n-1$ edges.

Note that this is the smallest possible number of edges needed to make a graph with n vertices connected, so if any one edge is removed the graph is no longer connected.

WORKED EXAMPLE 7.1

Which of the following terms describe the graph shown in the diagram? For those that do not, give a reason for your answer.

- **a** Simple
- **b** Connected
- **c** Complete
- **d** Tree



The graph contains no multiple edges between vertices, and no edges connecting a vertex to itself

a The graph is simple.

A connected graph cannot be split into two separate components

b The graph is connected.

A complete graph has an edge between each vertex and every other vertex

c The graph is not complete. For example, there is no edge between A and C.

A tree contains no sequences of vertices that return to the starting vertex

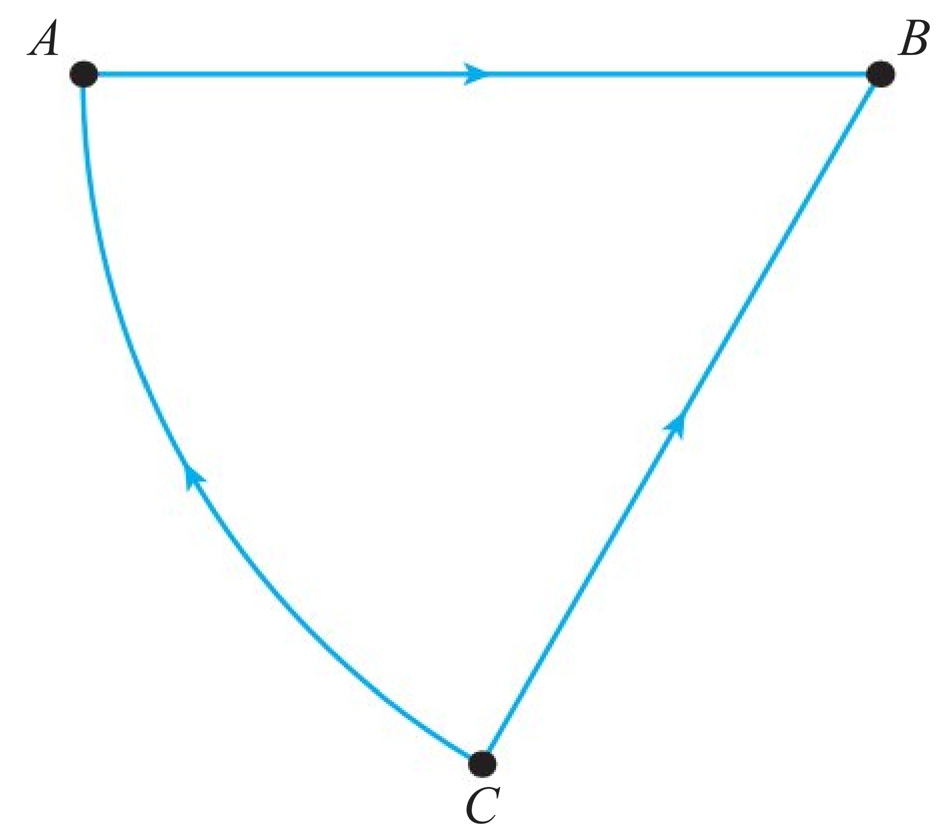
d The graph is not a tree, because $ABCDA$ returns to the starting vertex.

WORKED EXAMPLE 7.2

Which of the following terms describe the graph shown in the diagram? For those that do not, give a reason for your answer.

a Directed

b Strongly connected



There are arrows on the edges, indicating allowed directions

a The graph is directed.

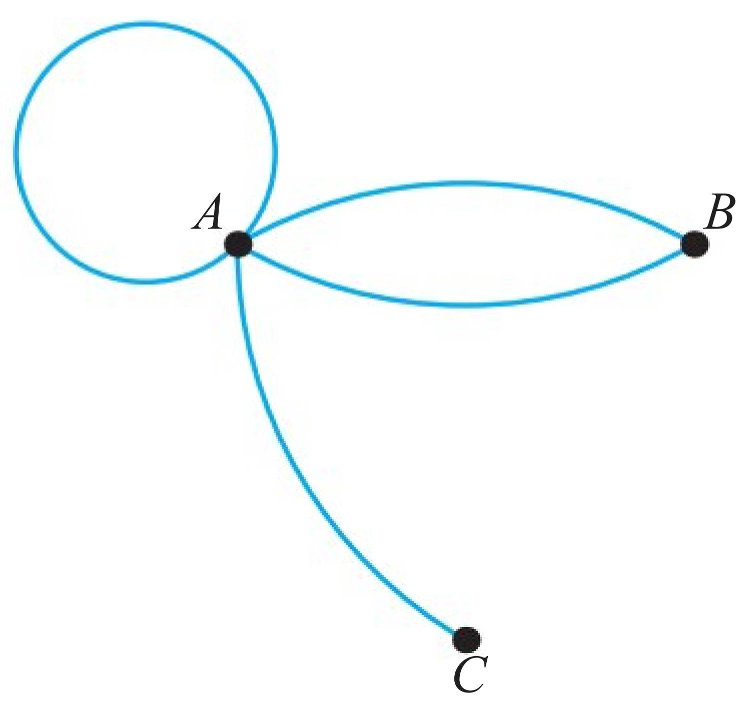
For a graph to be strongly connected, there must be a path between each and every two vertices, in both directions

b The graph is not strongly connected. For example, there is no path from B to A.

Adjacency matrices

If a graph is very large it is impractical to draw it. Instead, you can use a matrix to show the number of edges between each pair of vertices. This is called the **adjacency matrix** for the graph.

For example, the adjacency matrix for the graph below is shown alongside.



$$\begin{array}{c} A \\ B \\ C \end{array} \begin{pmatrix} 2 & 2 & 1 \\ 2 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}$$

Tip

Notice that the loop connecting A to itself contributes two connections in the adjacency matrix: the entry (A, A) is 2.

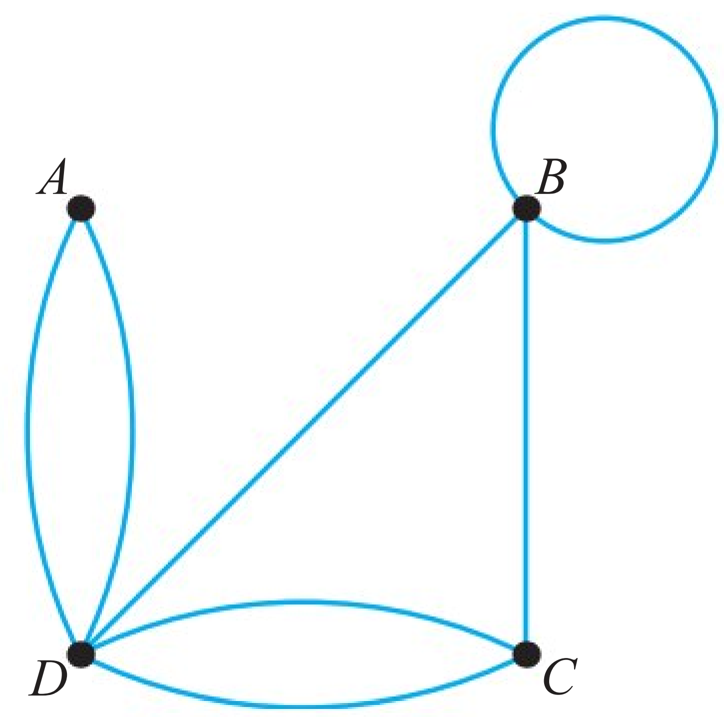
Each row/column of the matrix represents a vertex. The edges go from the vertex on the side to the vertex on top. This graph is undirected, so the matrix is symmetrical. For example, there are two edges from B to A, and one edge from C to A. The numbers on the diagonal show the number of edges from each vertex to itself.

WORKED EXAMPLE 7.3

a Write down the adjacency matrix for the following graph.

b Draw the graph represented by the following adjacency matrix:

$$\begin{array}{c} A \\ B \\ C \\ D \end{array} \begin{pmatrix} 0 & 1 & 3 & 0 \\ 1 & 0 & 0 & 1 \\ 3 & 0 & 0 & 2 \\ 0 & 1 & 2 & 2 \end{pmatrix}$$



Look at each vertex in turn

From the first row, there are no edges from A to itself, B or C; there are two edges from A to D. So the first row is 0, 0, 0, 2

From the second row, there are edges from B to B, C and D. Remember that the loop at B counts as two edges from B to B

From C, there is one edge to B and two edges to D

Finally, from D, there are 2 edges to A, one edge to B, two edges to C and no edges to D

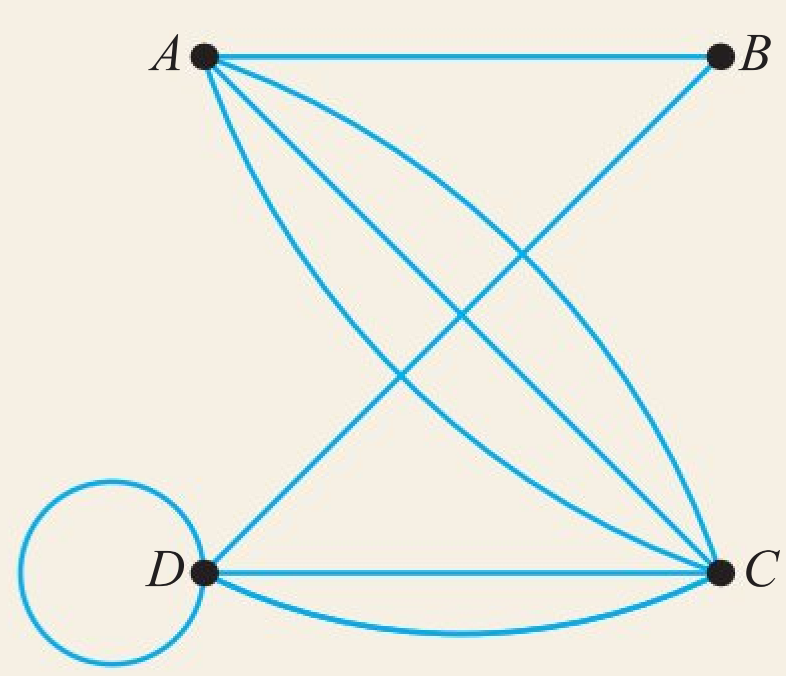
You should check that your matrix is symmetrical (as the graph is undirected)

a

$$\begin{array}{c} A \\ B \\ C \\ D \end{array} \begin{pmatrix} 0 & 0 & 0 & 2 \\ 0 & 2 & 1 & 1 \\ 0 & 1 & 0 & 2 \\ 2 & 1 & 2 & 0 \end{pmatrix}$$

Taking each vertex in turn, connect it to the other vertices by the number of edges indicated

b



You can work out the degree of each vertex by looking at the corresponding row (or column) of the adjacency matrix, and summing the elements.

KEY POINT 7.2

For an undirected graph:

- the adjacency matrix is symmetrical about the leading diagonal
- the degree of a vertex equals the sum of the entries in the corresponding row (or column).

Additionally, if the graph is simple:

- the adjacency matrix contains only zeros and ones.

WORKED EXAMPLE 7.4

A graph has the following adjacency matrix:

$$\begin{array}{c} A \\ B \\ C \\ D \end{array} \begin{pmatrix} 2 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 2 \\ 1 & 0 & 2 & 0 \end{pmatrix}$$

a List all the vertices adjacent to B.

b Find the degree of vertex A.

Row B has 1s in the columns corresponding to A and C

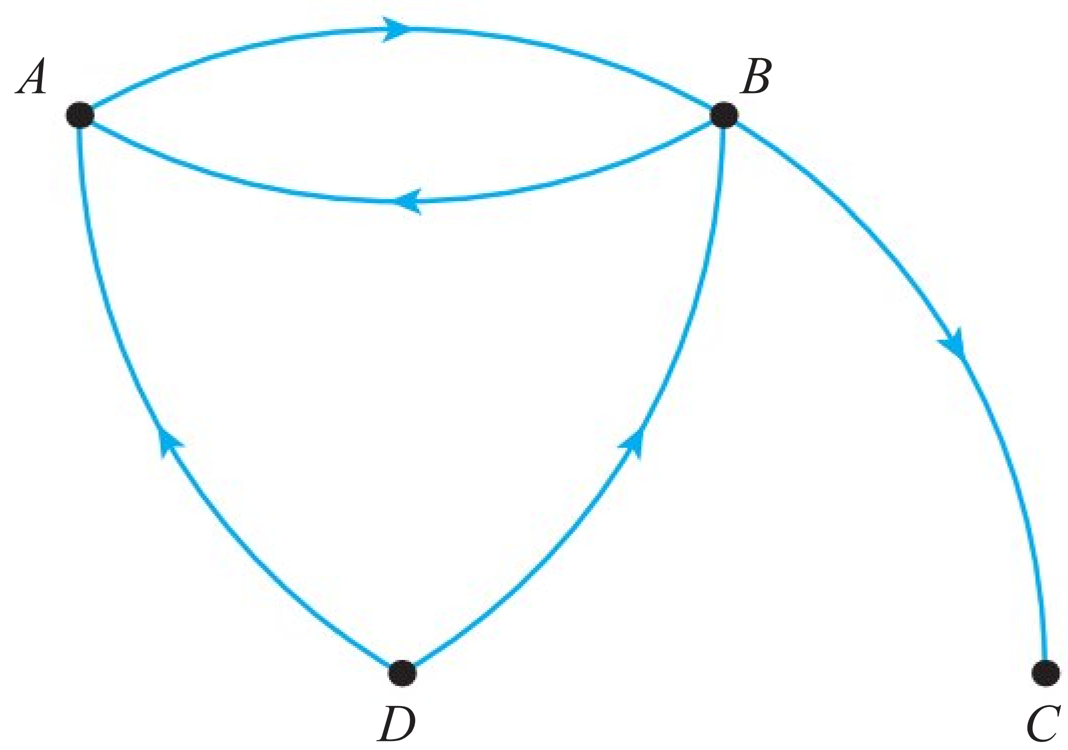
a A, C

This is the sum of all the numbers in row A

b $2 + 1 + 0 + 1 = 4$

Adjacency matrices can also be used to represent directed graphs. In this case, the matrix will not be symmetrical on the leading diagonal. The rows give the starting vertex and the columns the end vertex.

Graph 2 and its adjacency matrix are shown below.



$$\text{from} \quad \begin{array}{c} \\ A \\ B \\ C \\ D \end{array} \begin{array}{c} \text{to} \\ \begin{array}{cccc} A & B & C & D \end{array} \\ \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \end{pmatrix} \end{array}$$

The out-degree of a vertex is the sum of the entries in its row, and the in-degree is the sum of the entries in its column.

TOK Links

The decision to use rows for the starting vertex and columns for the end vertex of an edge is purely a convention, with no underlying mathematical reason. Do such conventions help or hinder the learning of a new concept?

WORKED EXAMPLE 7.5

A directed graph is represented by the following adjacency matrix:

$$\begin{array}{c} A \\ B \\ C \\ D \end{array} \begin{pmatrix} 0 & 1 & 0 & 2 \\ 0 & 0 & 0 & 2 \\ 0 & 3 & 0 & 1 \\ 1 & 1 & 0 & 0 \end{pmatrix}$$

a Determine

i the out-degree of vertex A

ii the in-degree of vertex D.

b Draw the graph.

The edges starting from A are shown in the first row of the matrix: there is one edge from A to B and there are two edges from A to D

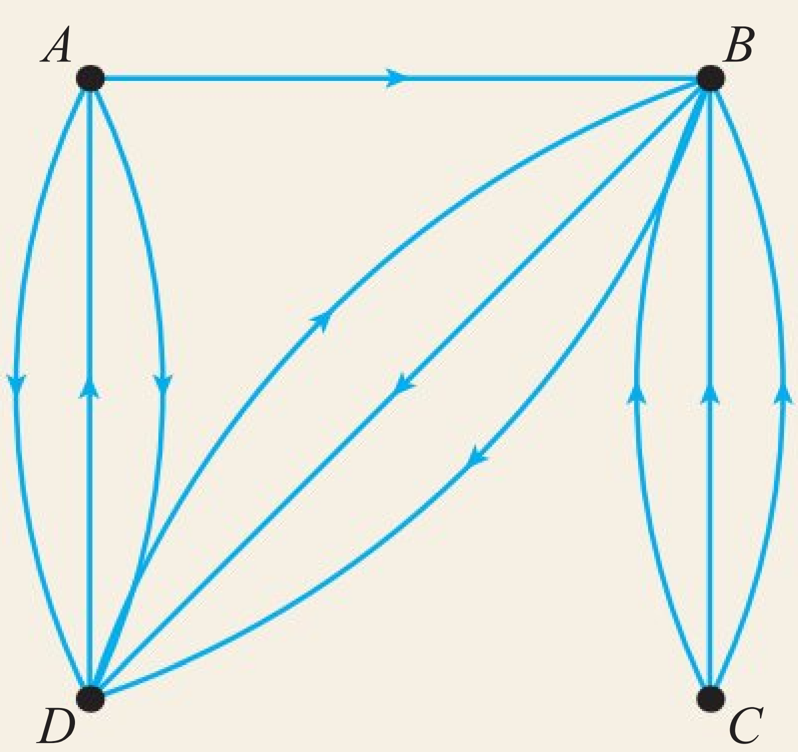
a i The out-degree of A is $0 + 1 + 0 + 2 = 3$.

The edges going into D are shown in the fourth column of the graph. There are two edges coming from A, two from B and one from C

ii The in-degree of D is $2 + 2 + 1 + 0 = 5$.

Start by marking the four vertices. Then use each row of the table to draw the edges from the corresponding vertex

b

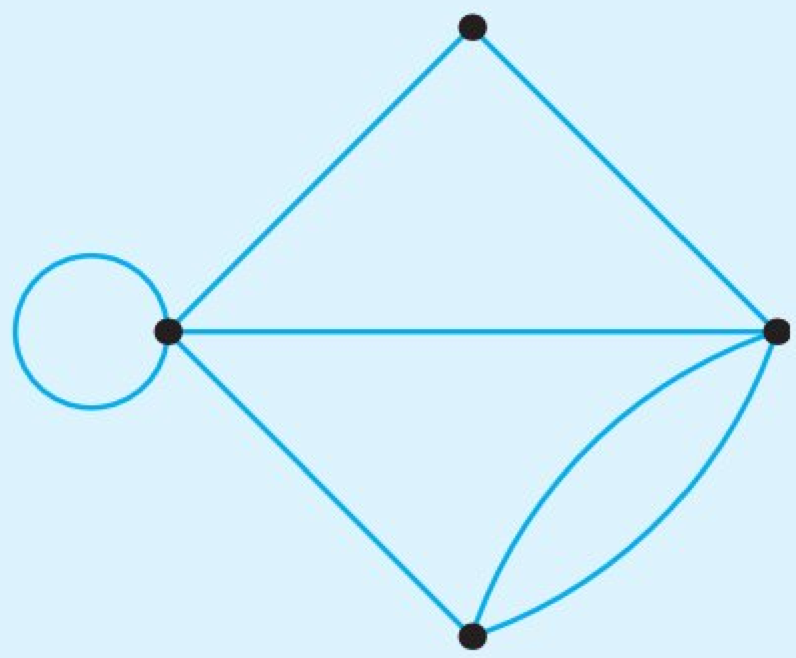


Exercise 7A

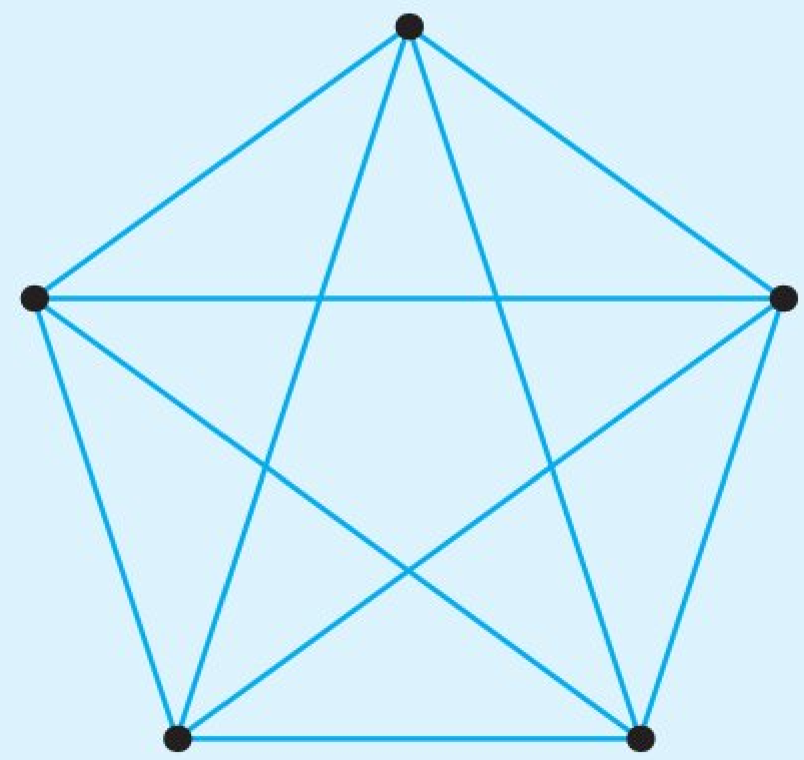
For questions 1 and 2, use the method demonstrated in Worked Example 7.1 to say which of the following terms describe each graph.

i complete **ii** connected **iii** simple **iv** tree

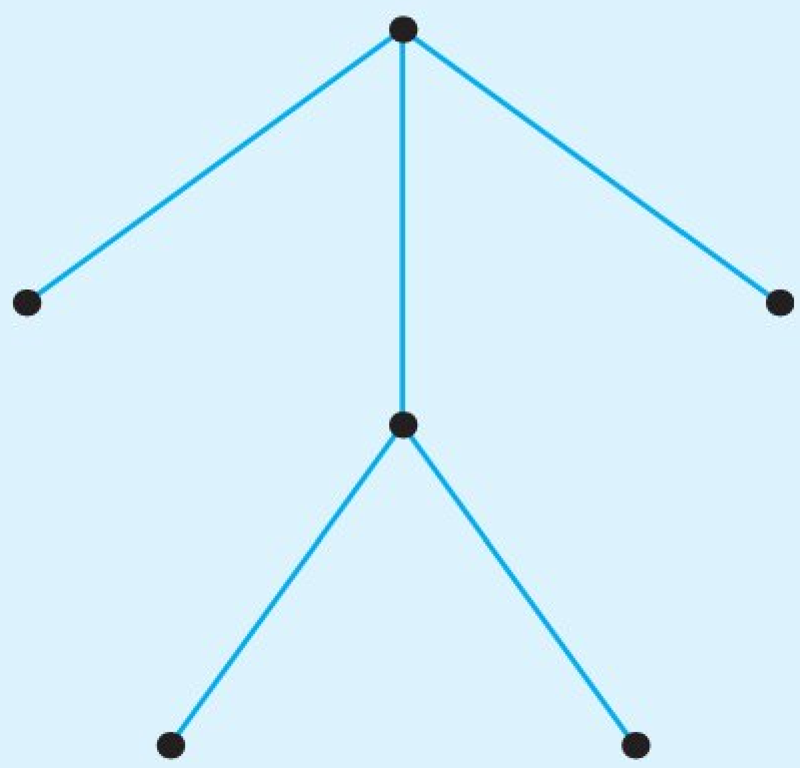
1 a



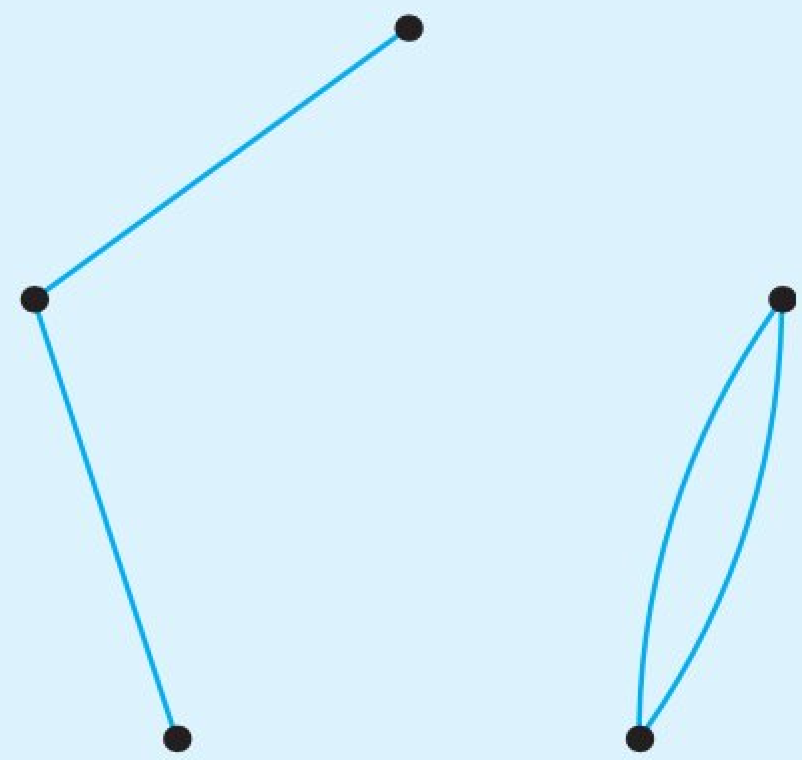
b



2 a

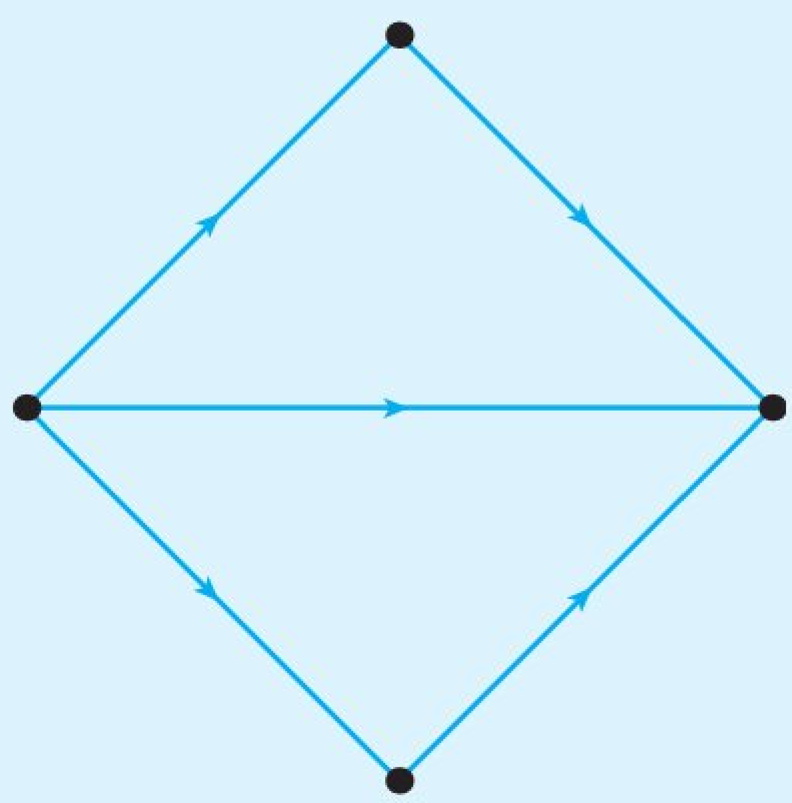


b

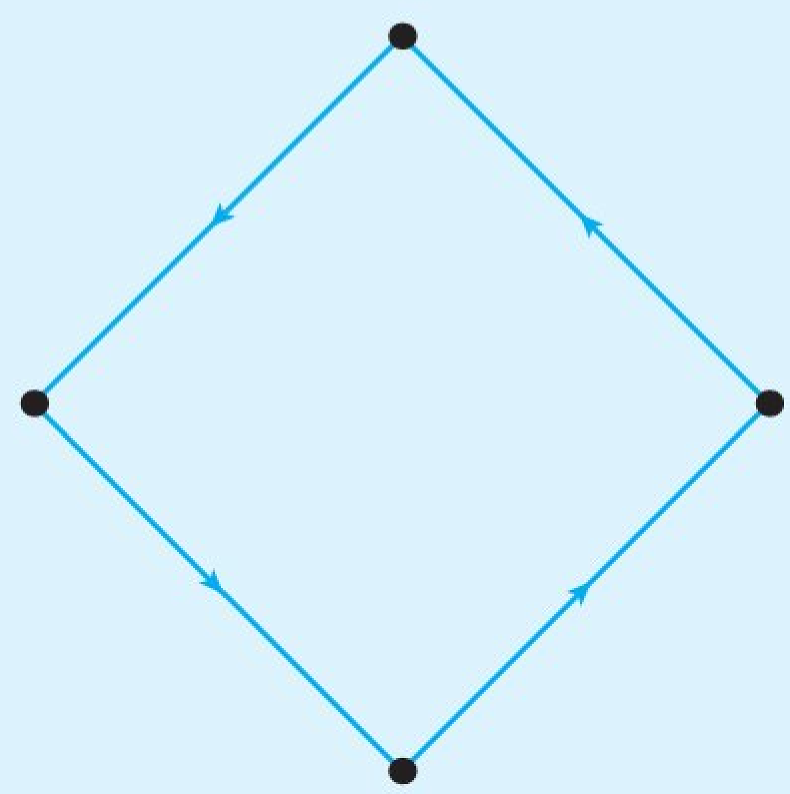


For questions 3 and 4, use the method demonstrated in Worked Example 7.2 to decide whether these directed graphs are strongly connected.

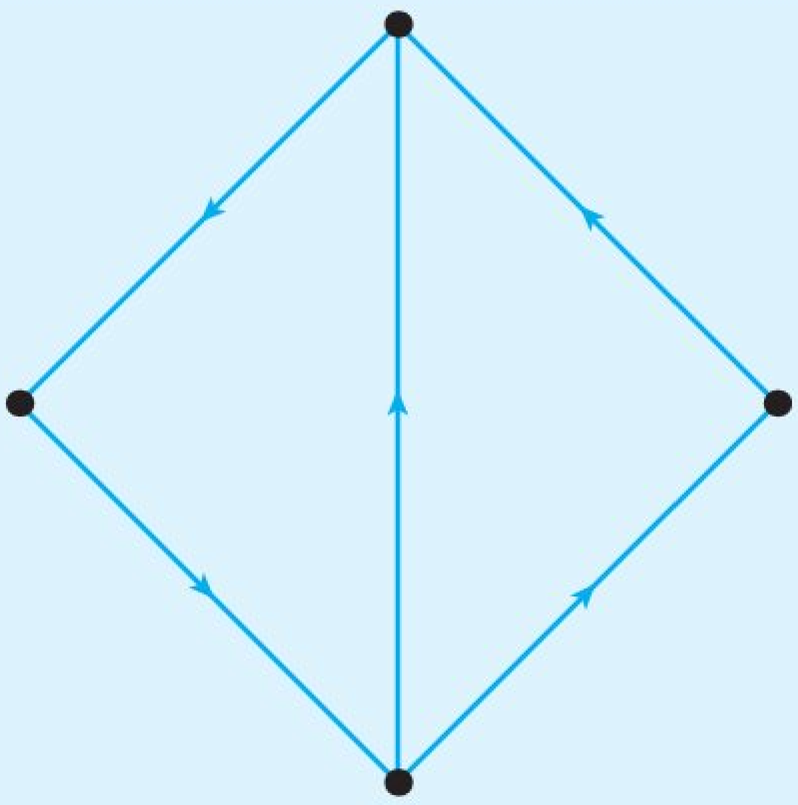
3 a



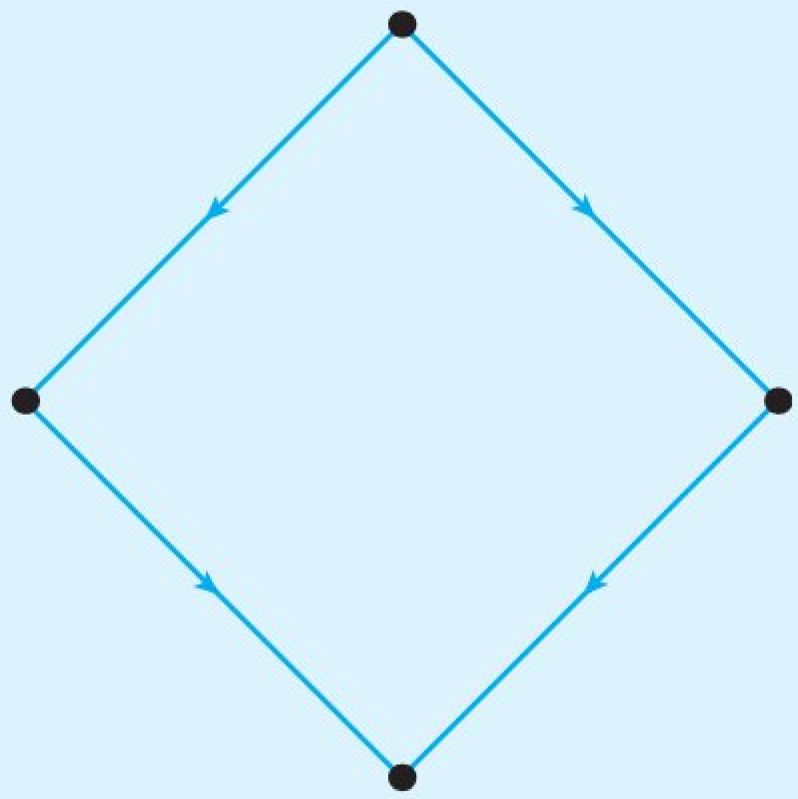
b



4 a

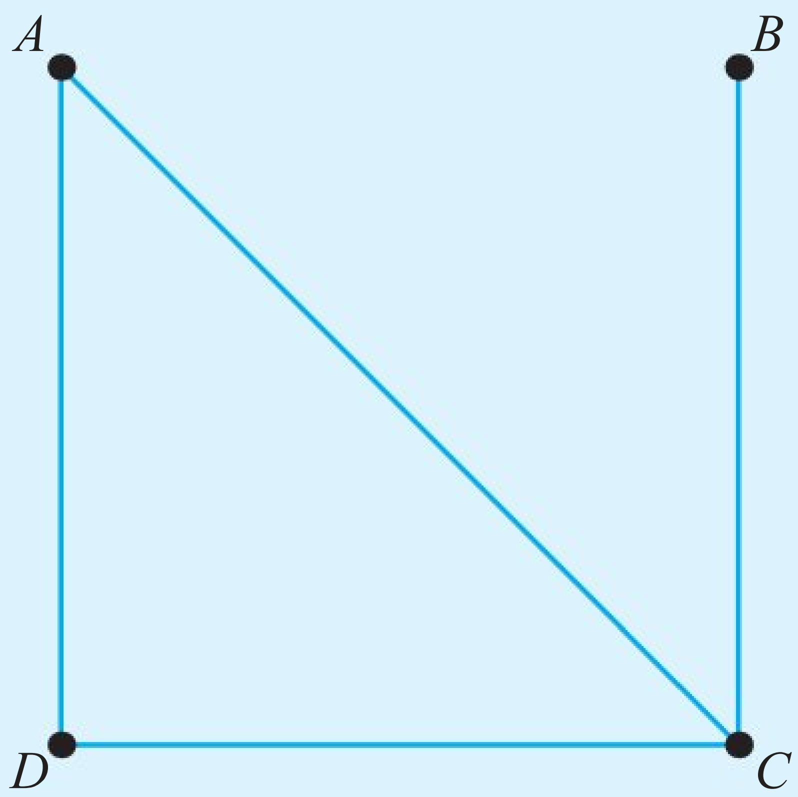


b

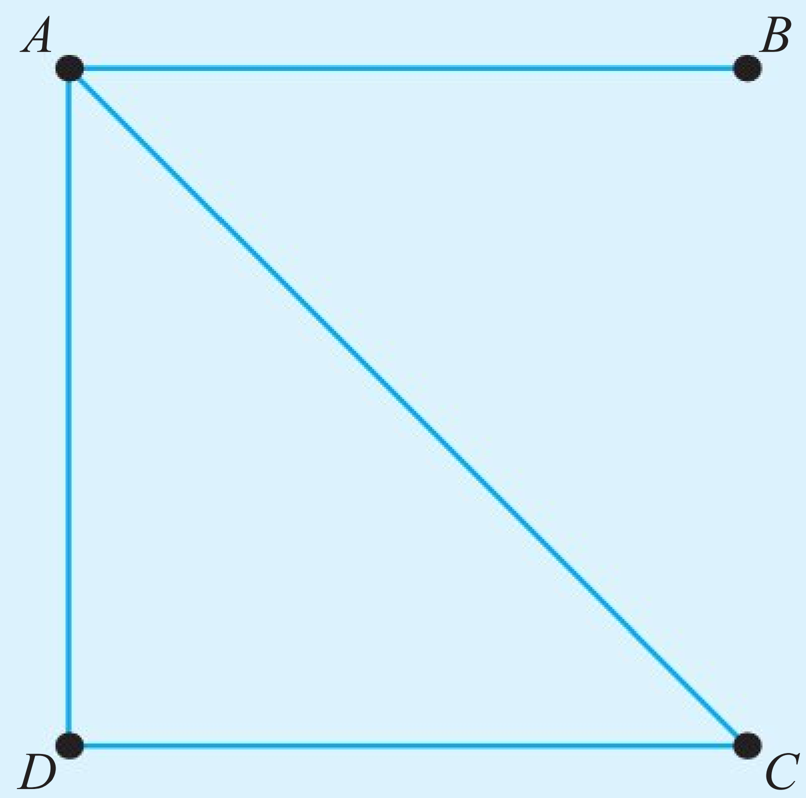


For questions 5 and 6, use the method demonstrated in Worked Example 7.3a to construct an adjacency matrix for each graph.

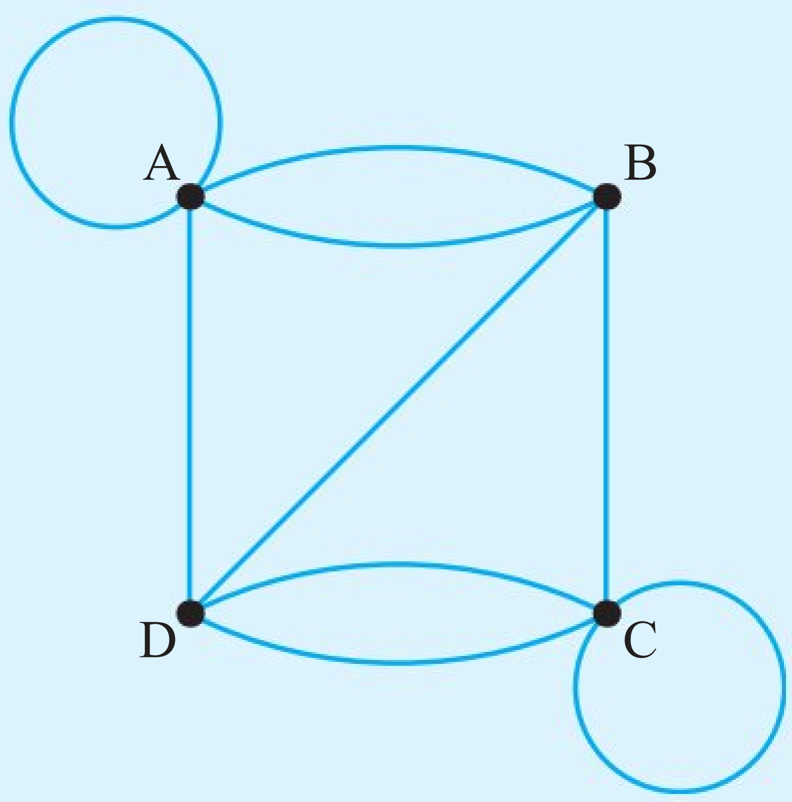
5 a



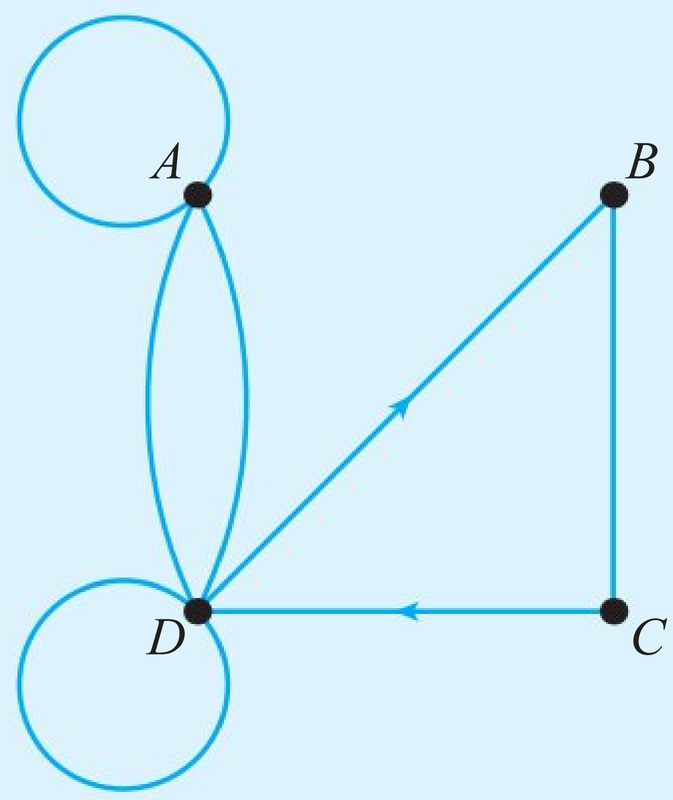
b



6 a



b



For questions 7 and 8, use the method demonstrated in Worked Example 7.3b to draw a graph with the given adjacency matrix.

7 a

$$\begin{array}{c} A \\ B \\ C \\ D \end{array} \begin{pmatrix} 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \end{pmatrix}$$

b

$$\begin{array}{c} A \\ B \\ C \\ D \end{array} \begin{pmatrix} 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 1 \\ 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 \end{pmatrix}$$

8 **a** $\begin{array}{c} A \\ B \\ C \\ D \\ E \end{array} \begin{pmatrix} 2 & 1 & 0 & 1 & 2 \\ 1 & 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & 2 & 1 \\ 1 & 0 & 2 & 0 & 0 \\ 2 & 1 & 1 & 0 & 2 \end{pmatrix}$

b $\begin{array}{c} A \\ B \\ C \\ D \\ E \end{array} \begin{pmatrix} 2 & 2 & 0 & 1 & 0 \\ 2 & 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & 2 & 1 \\ 1 & 0 & 2 & 0 & 0 \\ 0 & 1 & 1 & 0 & 2 \end{pmatrix}$

For questions 9 and 10, use the method demonstrated in Worked Example 7.4 to answer the questions related to the graph represented by the given adjacency matrix.

9 **a** $\begin{array}{c} A \\ B \\ C \\ D \\ E \end{array} \begin{pmatrix} 0 & 1 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 1 \\ 0 & 1 & 1 & 1 & 0 \end{pmatrix}$

i List all the vertices adjacent to A.

ii List all the vertices adjacent to C.

iii State the degree of vertex B.

iv State the degree of vertex C.

b $\begin{array}{c} A \\ B \\ C \\ D \\ E \end{array} \begin{pmatrix} 0 & 1 & 0 & 1 & 1 \\ 1 & 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & 1 & 1 \\ 1 & 0 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 \end{pmatrix}$

i List all the vertices adjacent to D.

ii List all the vertices adjacent to C.

iii State the degree of vertex D.

iv State the degree of vertex A.

10 **a** $\begin{array}{c} A \\ B \\ C \\ D \\ E \end{array} \begin{pmatrix} 0 & 3 & 0 & 1 & 0 \\ 3 & 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & 2 & 1 \\ 1 & 0 & 2 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 \end{pmatrix}$

i List all the vertices adjacent to A.

ii List all the vertices adjacent to C.

iii State the degree of vertex D.

iv State the degree of vertex B.

b $\begin{array}{c} A \\ B \\ C \\ D \\ E \end{array} \begin{pmatrix} 0 & 0 & 2 & 1 & 0 \\ 0 & 0 & 1 & 1 & 1 \\ 2 & 1 & 0 & 2 & 3 \\ 1 & 1 & 2 & 0 & 0 \\ 0 & 1 & 3 & 0 & 0 \end{pmatrix}$

i List all the vertices adjacent to B.

ii List all the vertices adjacent to D.

iii State the degree of vertex C.

iv State the degree of vertex A.

For questions 11 and 12, you are given an adjacency matrix for a directed graph. Use the method demonstrated in Worked Example 7.5 to state

i the out-degree of vertex C

ii the in-degree of vertex A.

11 **a** $\begin{array}{c} A \\ B \\ C \\ D \end{array} \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 \\ 1 & 1 & 1 & 0 \end{pmatrix}$

b $\begin{array}{c} A \\ B \\ C \\ D \end{array} \begin{pmatrix} 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \end{pmatrix}$

12 **a** $\begin{array}{c} A \\ B \\ C \\ D \end{array} \begin{pmatrix} 0 & 1 & 0 & 2 \\ 3 & 0 & 1 & 0 \\ 2 & 1 & 0 & 1 \\ 1 & 0 & 2 & 0 \end{pmatrix}$

b $\begin{array}{c} A \\ B \\ C \\ D \end{array} \begin{pmatrix} 0 & 2 & 1 & 1 \\ 1 & 0 & 1 & 3 \\ 0 & 1 & 0 & 1 \\ 2 & 1 & 0 & 0 \end{pmatrix}$

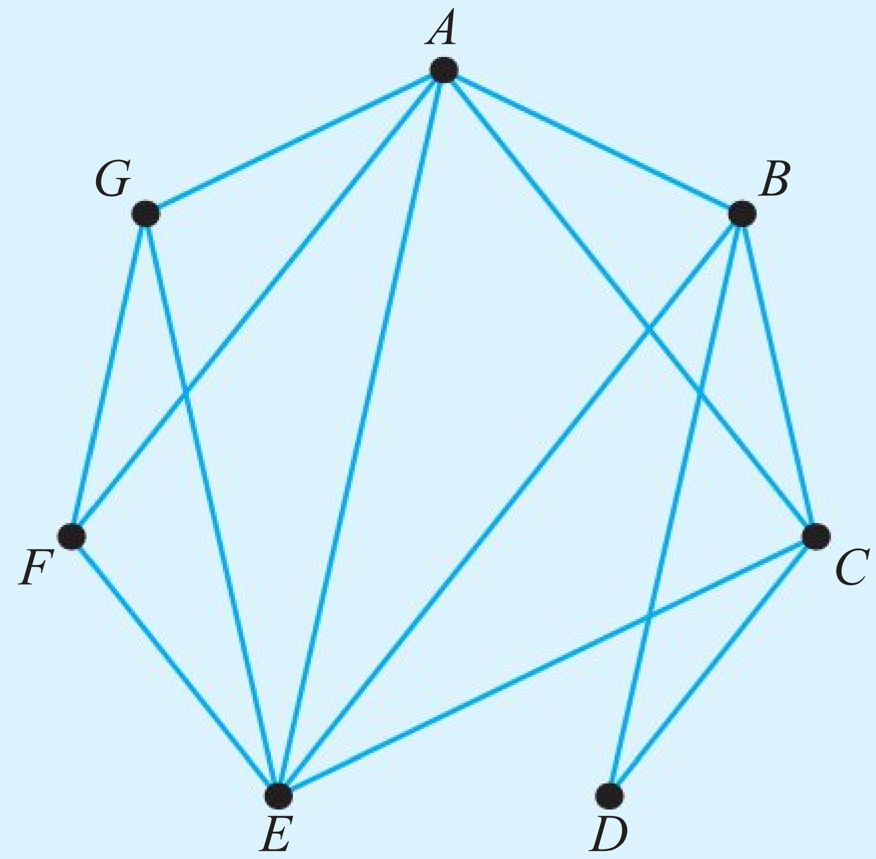
13 In the graph below, the vertices represent seven members of a social network and there is an edge between two vertices if they are friends.

a Write down an adjacency matrix for the graph.

b State the degree of vertex C, and interpret it in context.

c Interpret the meaning of the term ‘complete graph’ in this context.

d How many edges need to be added to make the above graph complete?

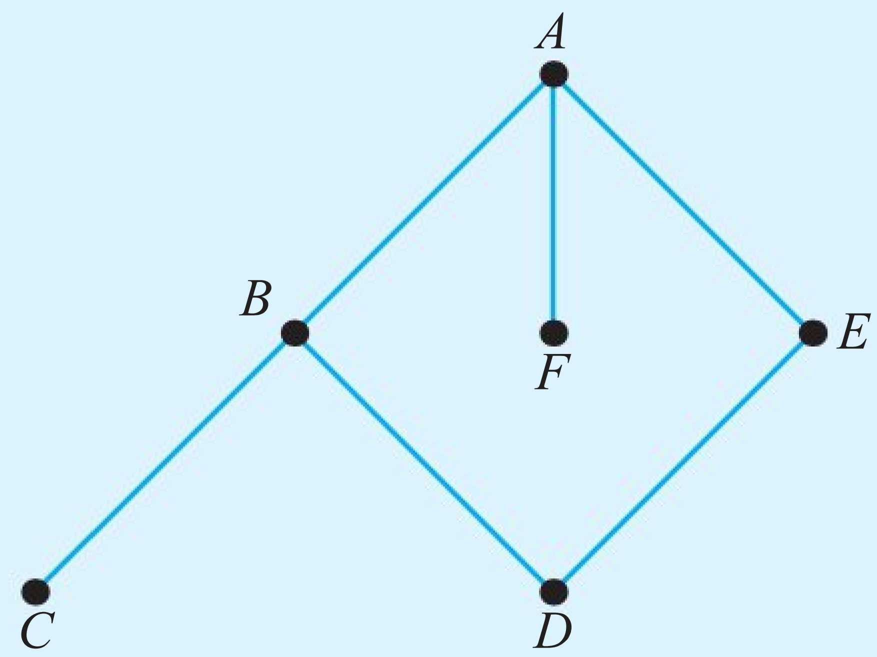


14 The graph shows power cables which connect a power source at A to devices B to F.

a Explain why the graph is not a tree.

b Explain why it is important for this graph to be connected.

c State one possible edge that can be removed for the graph to stay connected.



15 A network of one-way streets, connecting junctions A to E in a town, is represented by the following adjacency matrix:

$$\begin{array}{c} A \\ B \\ C \\ D \\ E \end{array} \begin{pmatrix} 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

a Draw the graph to represent this situation.

b Show that the graph is not strongly connected.

c Explain what that means in this context.

d The town council proposes to make one of the streets two-way in order to make the graph strongly connected. Which street should it be?

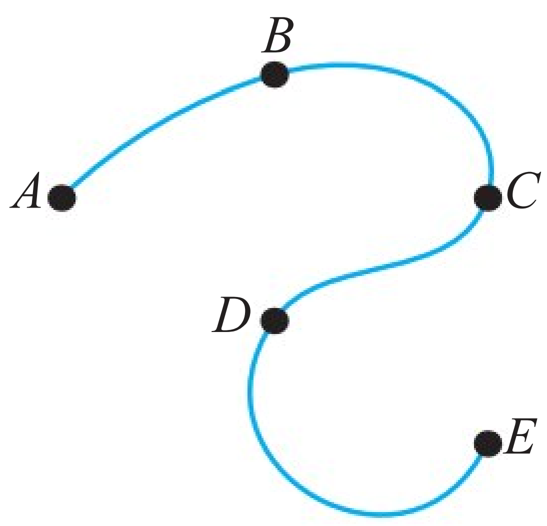
7B Moving around a graph

In many applications of graph theory we are interested in different ways of getting from one vertex to another. We may be required to return to the starting vertex, visit every vertex, or there may be restrictions on whether we are allowed to use each edge more than once. All these different ways of moving around a graph are given names.

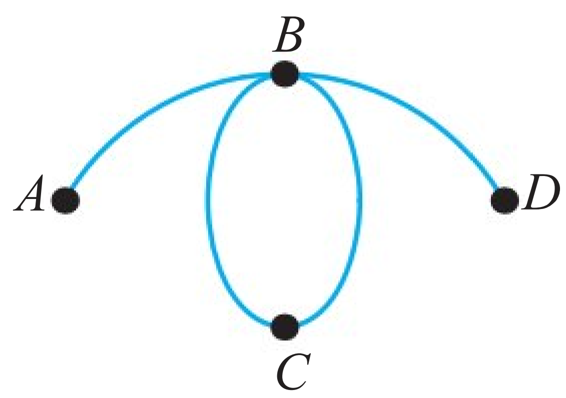
A **walk** is any sequence of adjacent edges.

A **trail** is a walk with no repeated edges.

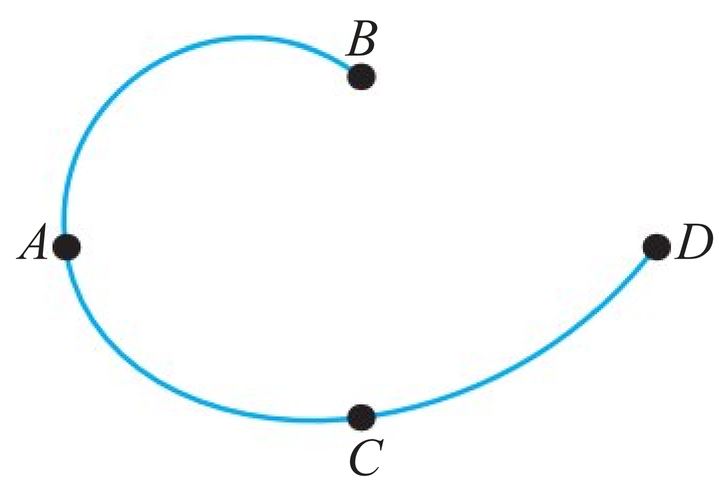
A **path** is a walk with no repeated vertices



■ *ABCDE* is a path (and therefore also a walk and a trail)



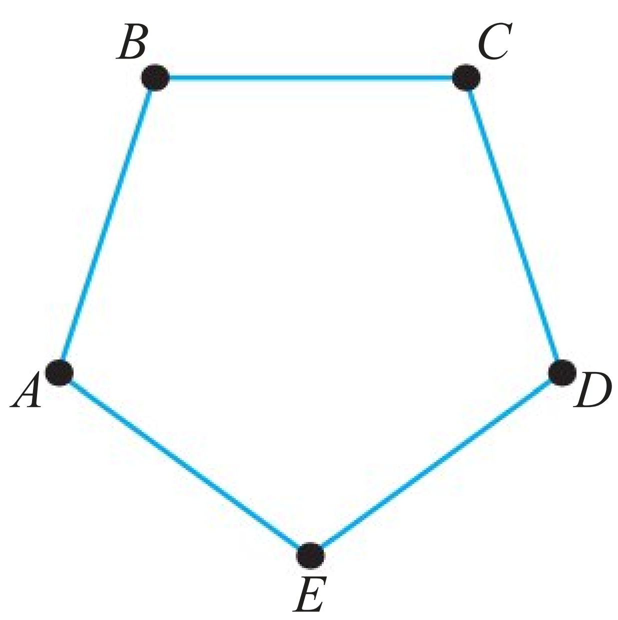
■ *ABCBD* is a trail (and therefore also a walk), but not a path



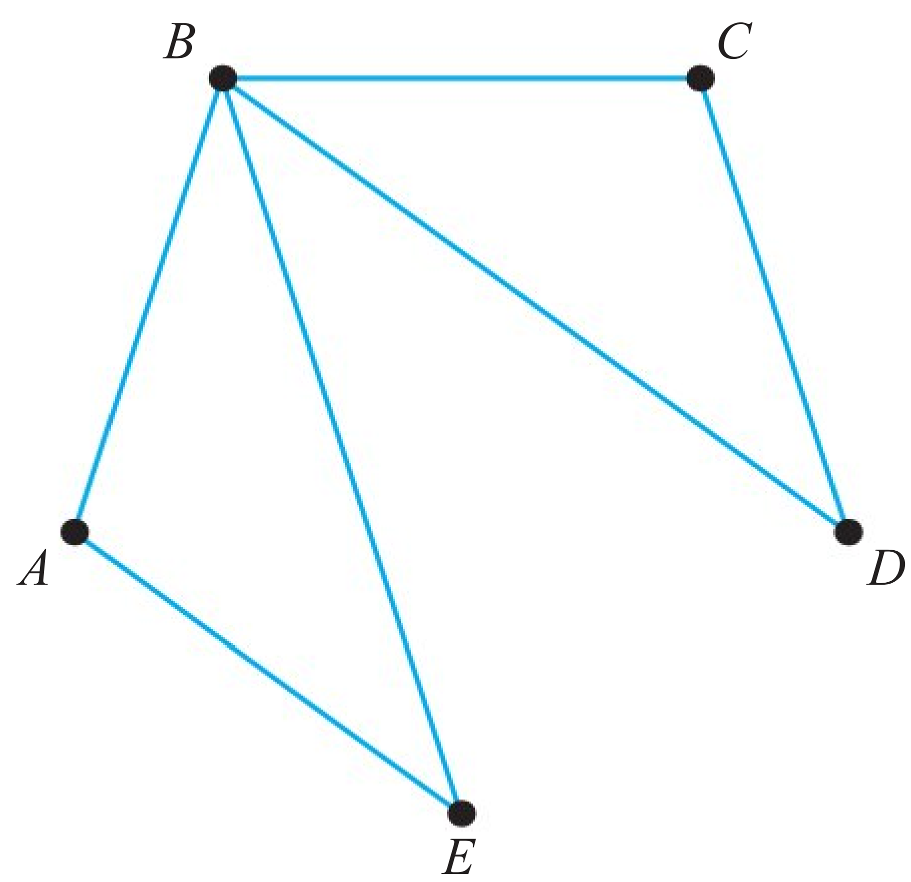
■ *ABACD* is a walk, but not a trail (and therefore also not a path)

A **cycle** is a walk that starts and ends at the same vertex and has no other repeated vertices (so it is a closed path). A tree has no cycles.

A **circuit** is a walk that starts and ends at the same vertex and has no repeated edges (so it is a closed trail).



■ *ABCDEA* is a cycle (and therefore also a circuit)



■ *ABCDBEA* is a circuit but not a cycle

The length of a walk is the number of its edges.

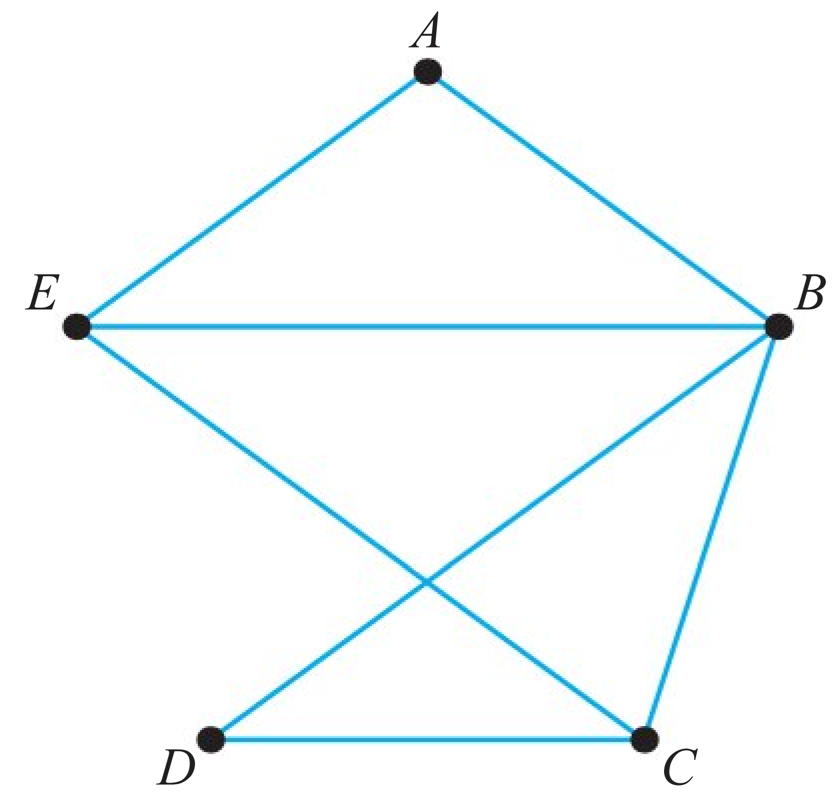
WORKED EXAMPLE 7.6

Consider this graph.

a List all paths of length 3 from E to C.

b List all the cycles of length 4 starting and ending at C.

c Find an example of a trail that uses each edge of the graph exactly once.



This can only be done by inspection. Think where we can go from E

A path of length 3 has four different vertices

a $EABC$, $EBDC$

A cycle of length 4 has four different vertices

Note that, in an undirected graph, $CBEAC$ is the same cycle as $CEABC$

b $CEABC$

Some vertices will appear more than once

c $EABECBDC$

Using an adjacency matrix to count walks

The entries in an adjacency matrix tell you the number of edges between two adjacent vertices. Do you think it is also possible to count the number of ways of moving between two vertices which are not adjacent? It turns out that this can be done using powers of the adjacency matrix.

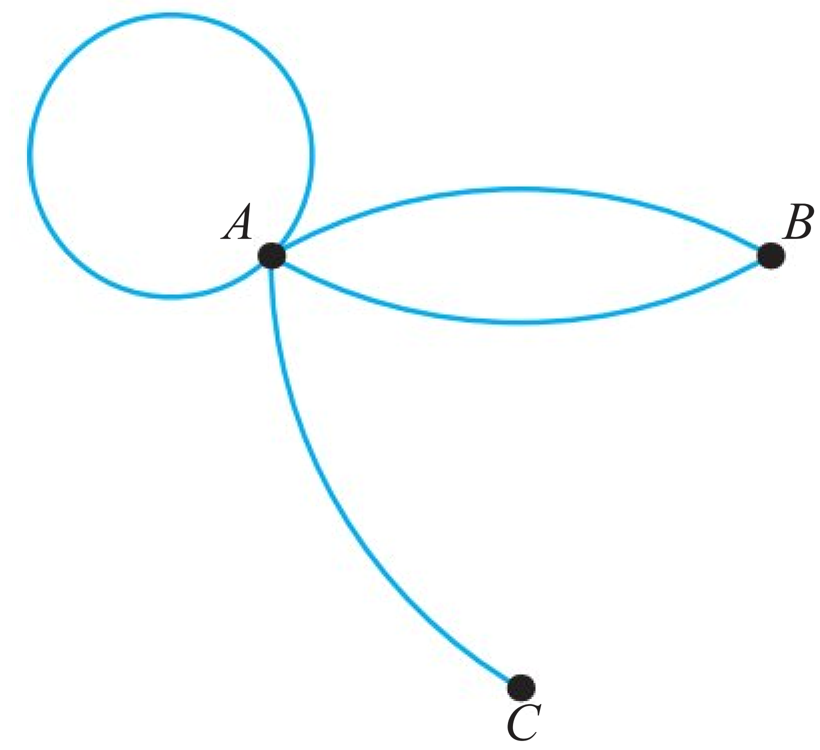
KEY POINT 7.3

For a graph with adjacency matrix A, the number of walks of length k between two vertices is given by the corresponding entry in the matrix $\mathbf{A}^k$.

There is one adjustment you may need to make to the adjacency matrix. If a vertex is connected to itself, there will be a '2' on the diagonal of the matrix. In most contexts this needs to be changed to '1' before raising the matrix to the required power. The only time when you would leave it as a '2' would be if you wanted to treat clockwise and anticlockwise movements around the path as different walks.

WORKED EXAMPLE 7.7

Find the number of walks of length 5 between vertices A and B in this graph.



You need the fifth power of the adjacency matrix — The adjacency matrix is

Remember to write the entry in the top left corner as '1' rather than '2'

$$\mathbf{A} = \begin{pmatrix} 1 & 2 & 1 \\ 2 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}$$

Use your GDC to evaluate the power of the matrix — Then

$$\mathbf{A}^5 = \begin{pmatrix} 96 & 82 & 41 \\ 82 & 44 & 22 \\ 41 & 22 & 11 \end{pmatrix}$$

The number of walks between A and B is the entry in the first row and second column — The required number of walks is 82.

Tip

You may want to list all the walks of length 2 from A to A and compare their number to the relevant entry in $\mathbf{A}^2$.

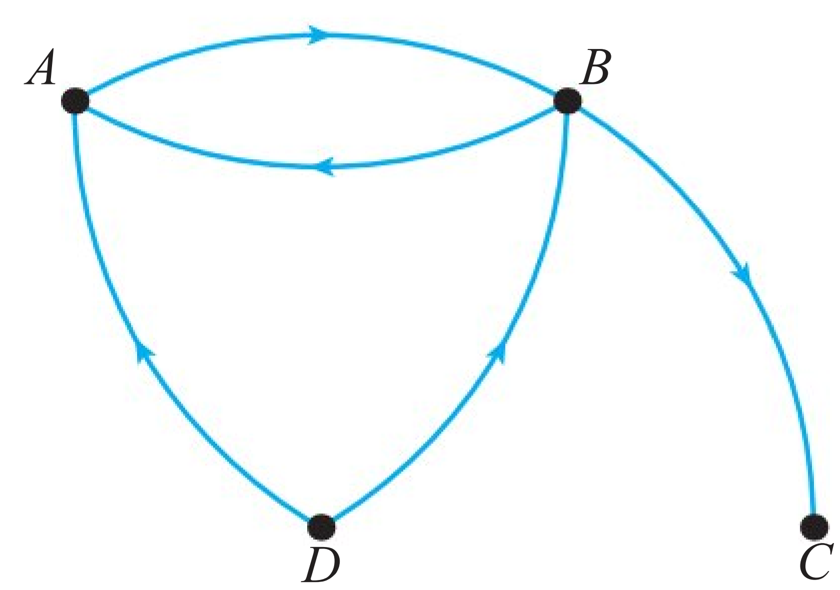
The result also applies to directed graphs.

WORKED EXAMPLE 7.8

For the graph shown, find the number of walks of length less than 4

a from A to D

b from D to A.



First write down the adjacency matrix. The rows are the 'from' vertices and the columns 'to' vertices

$$\mathbf{A} = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \end{pmatrix}$$

You need the number of walks of length 1, 2 and 3. Therefore, you need to look at $\mathbf{A}$, $\mathbf{A}^2$ and $\mathbf{A}^3$

$$\mathbf{A}^2 = \begin{pmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 \end{pmatrix}, \mathbf{A}^3 = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 \end{pmatrix}$$

Walks from A to D are given by the first row and fourth column

a From A to D: $0 + 0 + 0 = 0$

Walks from B to A are given by the second row and first column

b From B to A: $1 + 1 + 1 = 3$

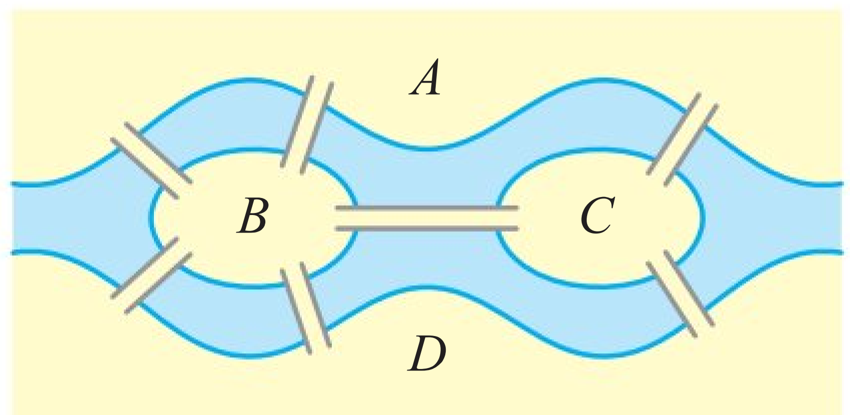
Tip

You can see the number of all walks of length less than 4 by considering the matrix $\mathbf{A} + \mathbf{A}^2 + \mathbf{A}^3$.

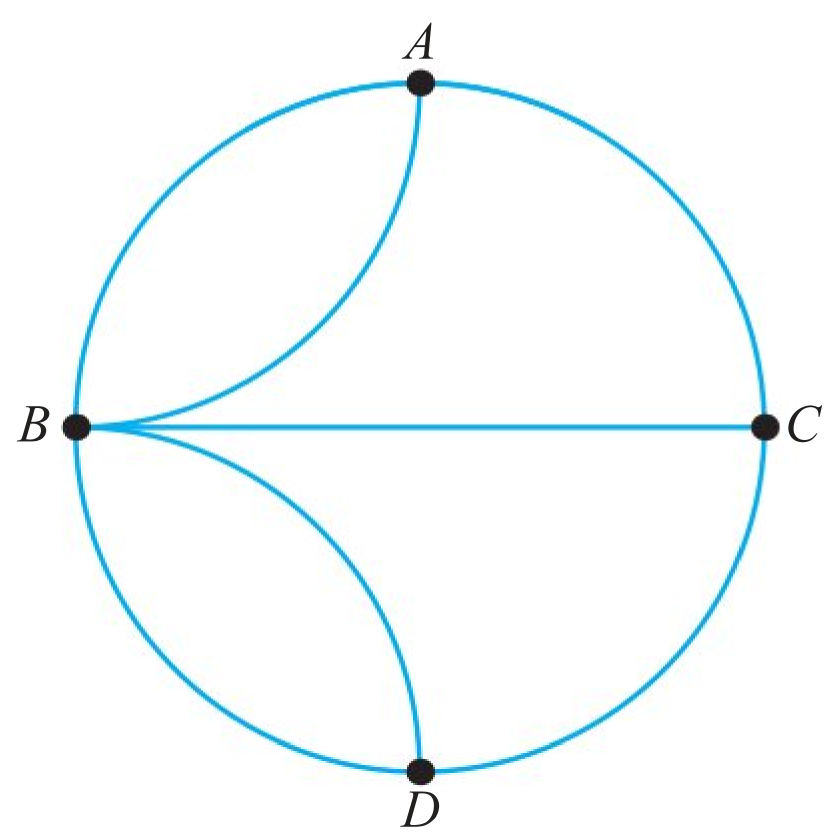
Eulerian trails and circuits

This problem was first studied by Leonhard Euler in 1836.

The following is the Königsberg bridges problem, considered to be the first published problem in graph theory.



Two islands are connected to each other, and to the two river banks, by bridges shown in the diagram. Is it possible to have a walk which crosses each bridge exactly once and returns to the starting point?



You already looked at some similar problems in the Starter Activity.

The problem is equivalent to trying to find a circuit which includes every edge of the graph shown. It turns out that, in this case, this is not possible.

A circuit which includes every edge of the graph exactly once is called an **Eulerian circuit**. A graph which has an Eulerian circuit is called an **Eulerian graph**. So the graph in the example above is not Eulerian. There is a simple way to check whether a graph is Eulerian.

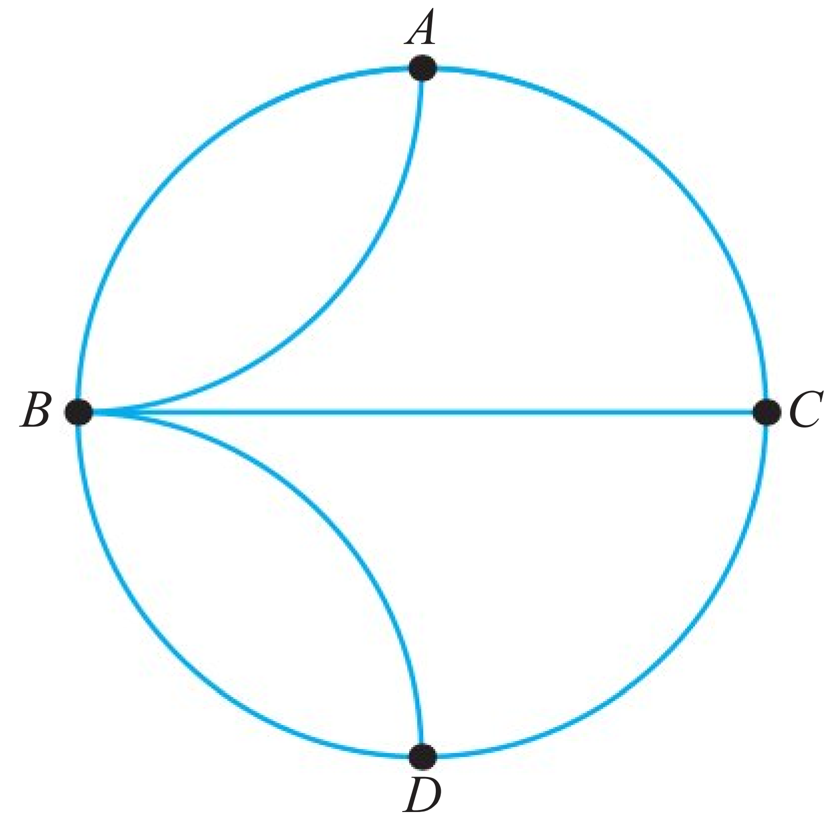
KEY POINT 7.4

In an Eulerian graph every vertex has even degree.

This result can be explained as follows: in an Eulerian circuit, every time we visit a vertex we must go in and out of it, which uses up two edges. This also applies to the starting vertex, as we return to it at the end. So the number of edges at each vertex must be even.

WORKED EXAMPLE 7.9

Show that the graph in the Königsberg bridges problem is not Eulerian.



Determine the degree of each vertex — Degrees of vertices: 3, 5, 3, 3

In an Eulerian graph all vertices have even degree — All vertices have odd degree, so the graph is not Eulerian.

When an Eulerian circuit does not exist, it may still be possible to find a walk which uses each edge exactly once (without returning to the starting point). This is called an **Eulerian trail** and the graph is then called **semi-Eulerian**. The argument about going 'in and out' of each vertex still applies to all vertices except for the start and end ones.

KEY POINT 7.5

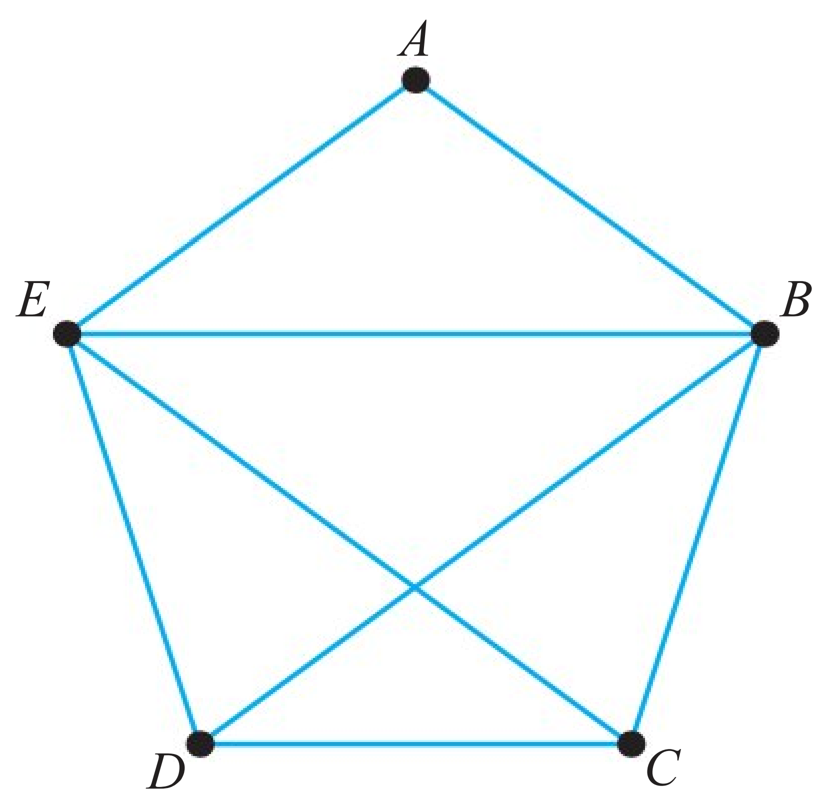
A semi-Eulerian graph has exactly two vertices of odd degree. Every Eulerian trail starts at one of the odd vertices and ends at the other.

Tip

Finding an Eulerian trail or circuit is equivalent to drawing the graph without picking up your pen and without going over any edge more than once. You can use this to check that your trail or circuit is correct.

WORKED EXAMPLE 7.10

Show that the following graph is semi-Eulerian and find an Eulerian trail.



You need the degrees of all the vertices

Degrees of the vertices:

$$A = 2,\ B = 4,\ C = 3,\ D = 3,\ E = 4$$

There are exactly two vertices of odd degree, so the graph is semi-Eulerian.

The Eulerian trail starts and ends at the odd vertices, so you can start at C and end at D

Note that each time you pass through a vertex, you 'use up' two edges. So the trail should visit vertex A once and all the other vertices twice

Eulerian trail:

CDECBAEBD

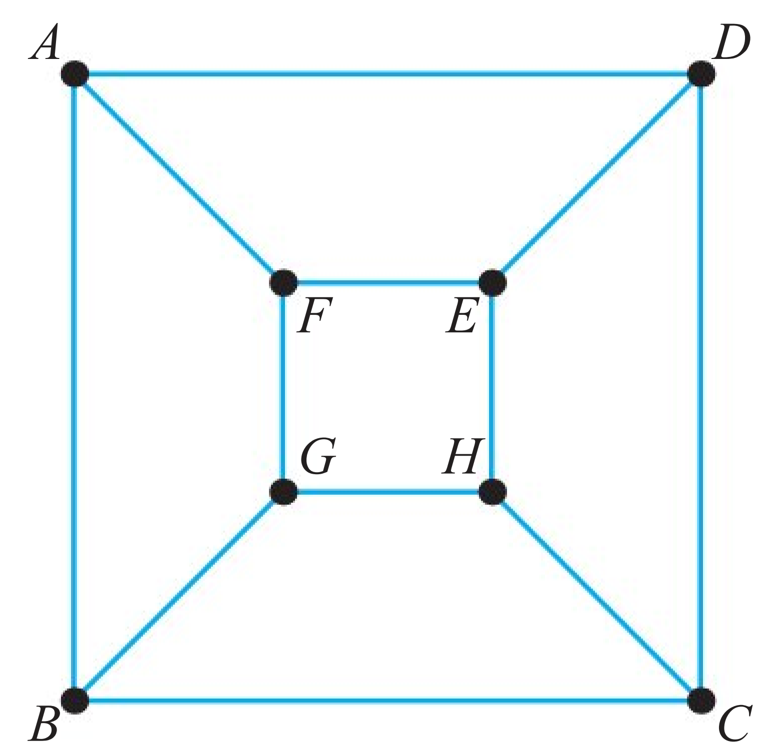
Eulerian circuits and trails will be used when solving the Chinese postman problem in Section 7E.

Hamiltonian paths and cycles

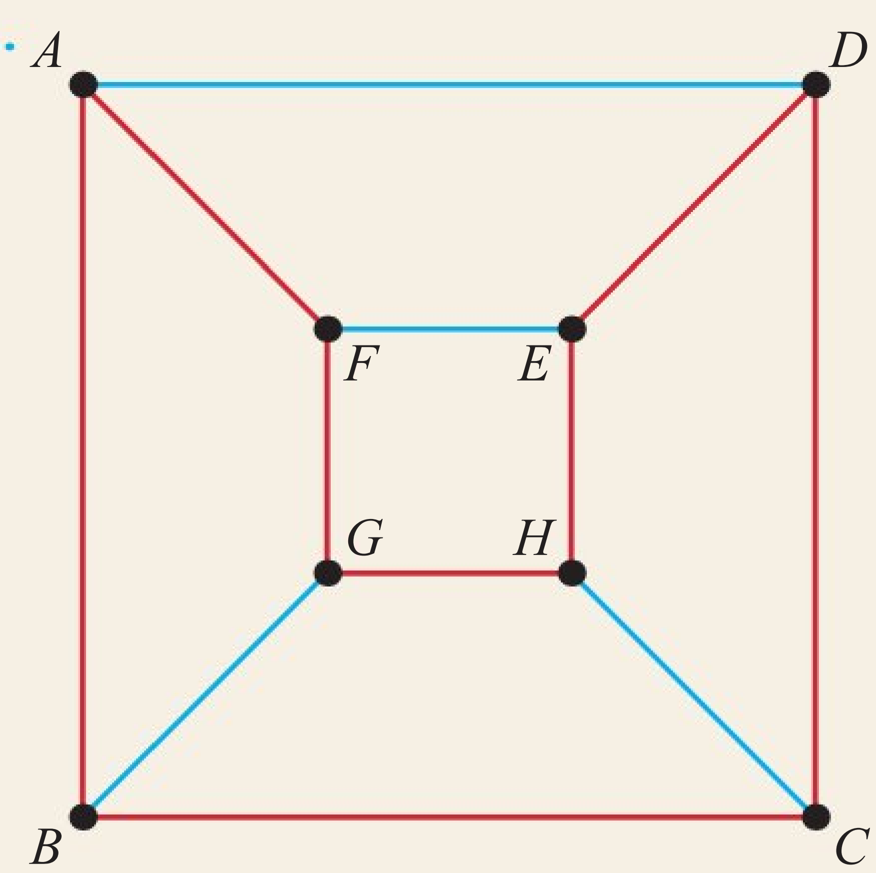
A **Hamiltonian path** in a graph visits each vertex exactly once. A **Hamiltonian cycle** visits each vertex exactly once and returns to the starting vertex. Not every graph has a Hamiltonian cycle; if it does it is called a **Hamiltonian graph**.

WORKED EXAMPLE 7.11

Show that this graph is Hamiltonian.



Try to find a Hamiltonian cycle: can you visit each vertex without repeating any edges?



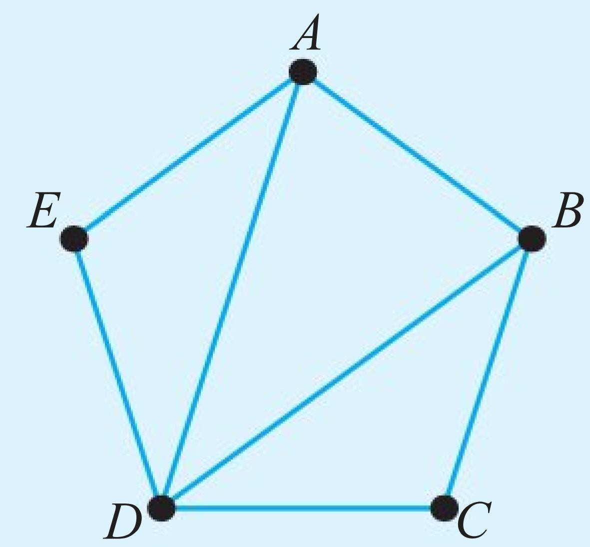
Hamiltonian cycle: *ABCDEHGFA*
Hence, the graph is Hamiltonian.

There is no quick way to identify whether a graph is Hamiltonian. This is what makes some problems with graphs very difficult to solve.

Hamiltonian cycles will be used in solving the travelling salesman problem in Section 7F.

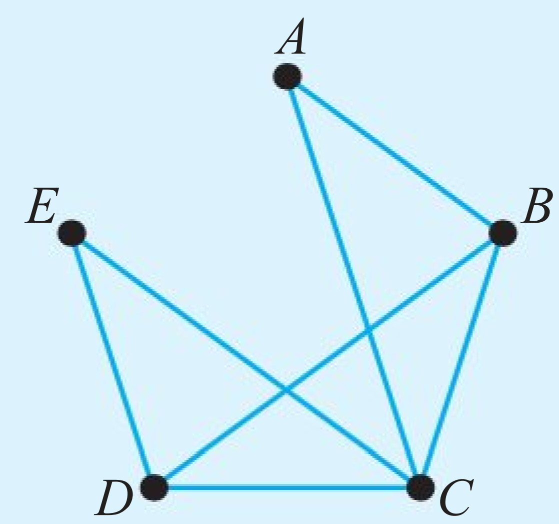
Exercise 7B

Questions 1 and 2 refer to the graph shown alongside. Use the method demonstrated in Worked Example 7.6.



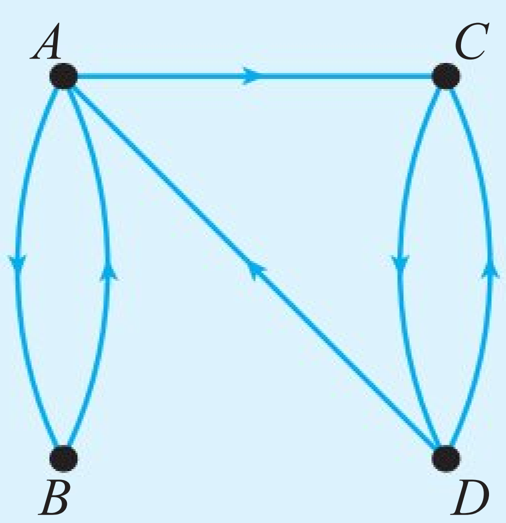
1 Find all the paths of length 3
   a from A to B
   b from C to E.
2 Find all the cycles of length 4 starting and ending at
   a A
   b C.

Questions 3 and 4 refer the graph shown alongside. Use the method demonstrated in Worked Example 7.7.



3 Find the number of walks of length 3 between
   a A and C
   b B and C.
4 Find the number of walks of length 4 between
   a A and D
   b C and D.

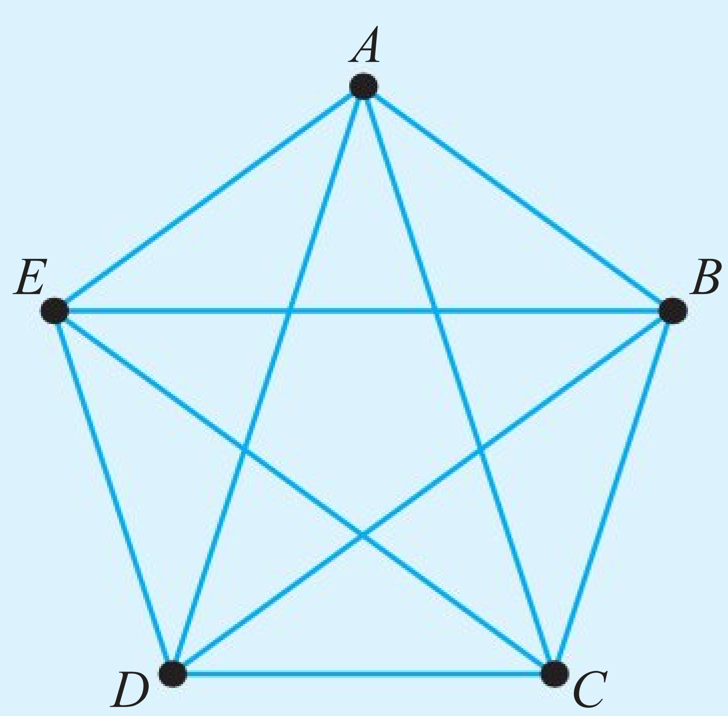
Questions 5 and 6 refer the graph shown alongside. Use the method demonstrated in Worked Example 7.8.



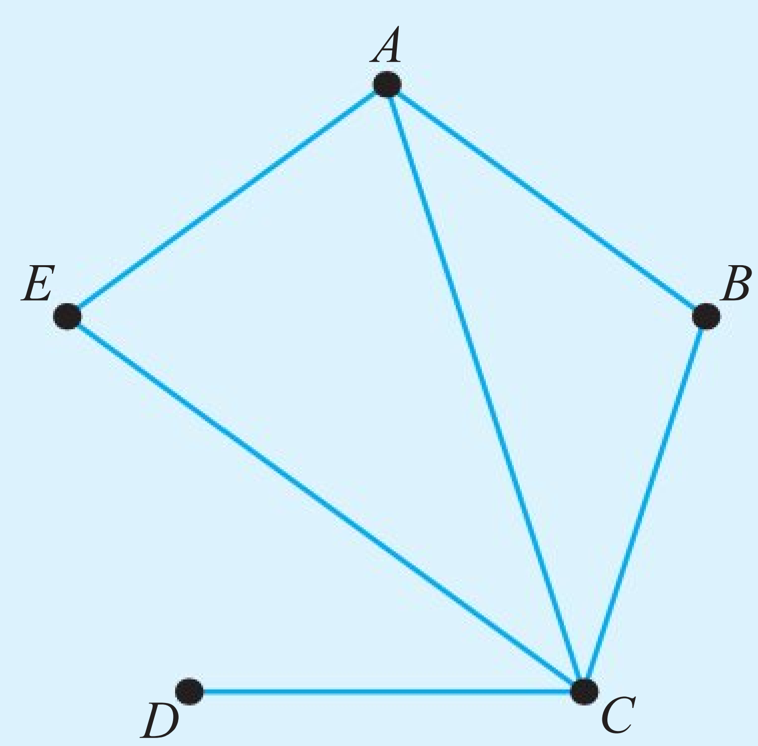
5 Find the number of walks of length 3
   a from A to D
   b from B to D.
6 Find the number of walks of length 4 between
   a from A to C
   b from C to D.

For questions 7 and 8, use the methods demonstrated in Worked Examples 7.9 and 7.10 to determine whether each graph is Eulerian or semi-Eulerian. If it is, find an Eulerian cycle or an Eulerian trail.

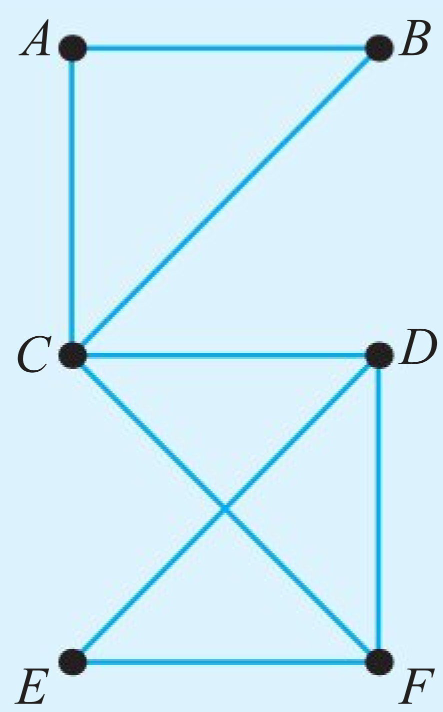
7 a



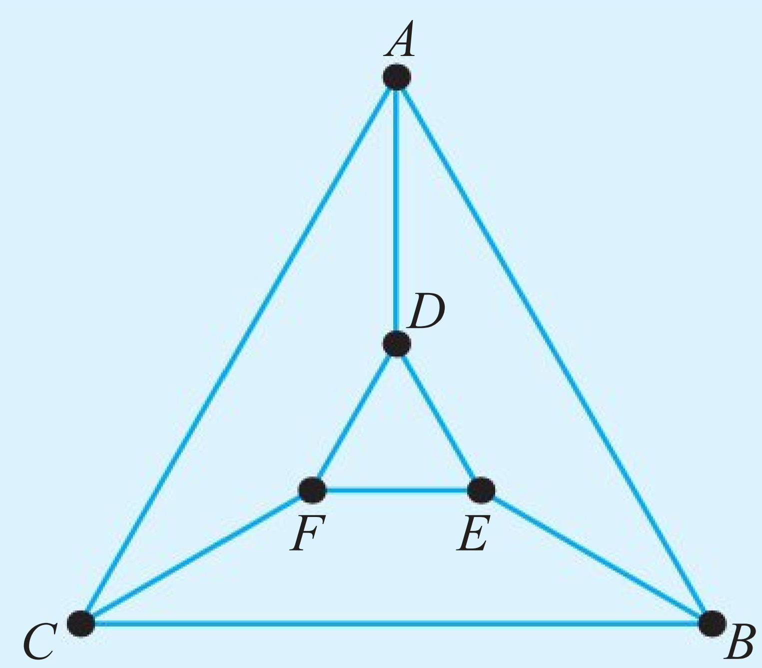
b



8 **a**

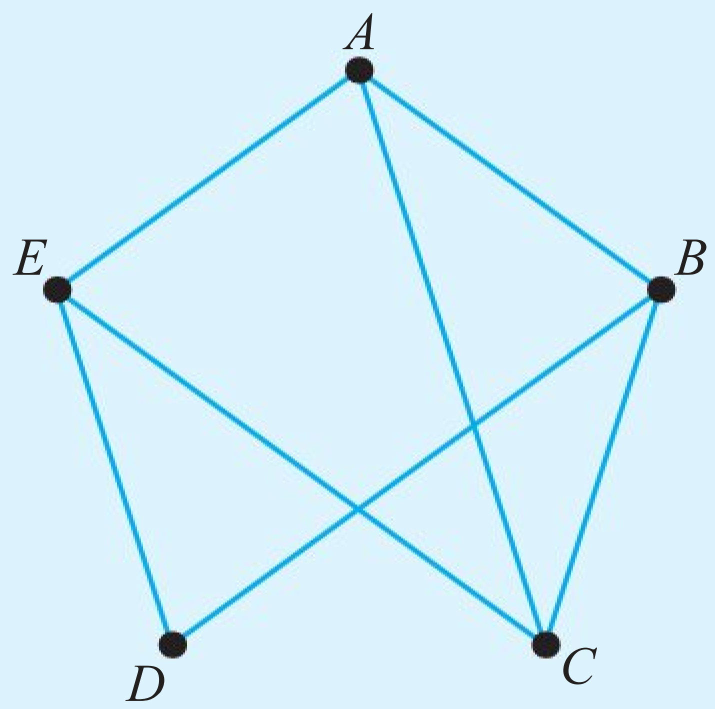


b

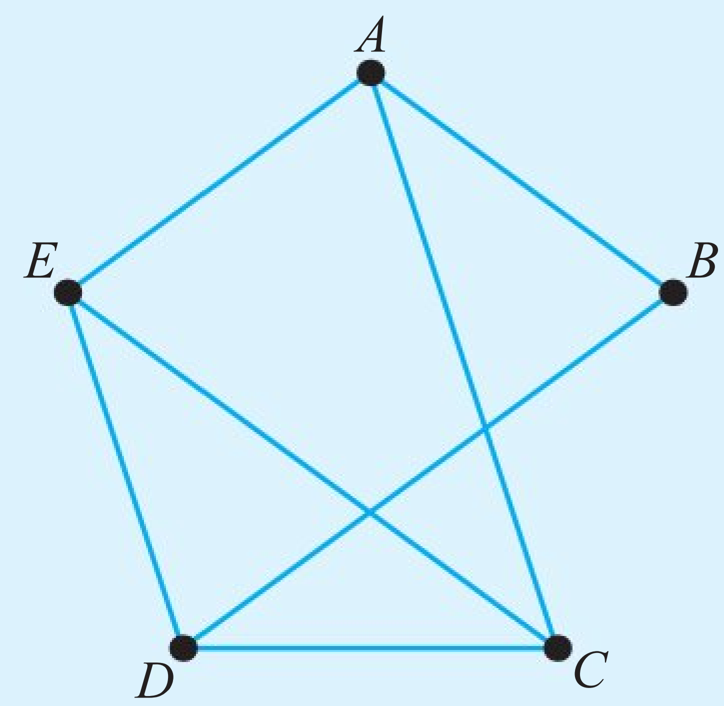


For questions 9 and 10, use the method demonstrated in Worked Example 7.11 to find a Hamiltonian cycle in each of the following graphs.

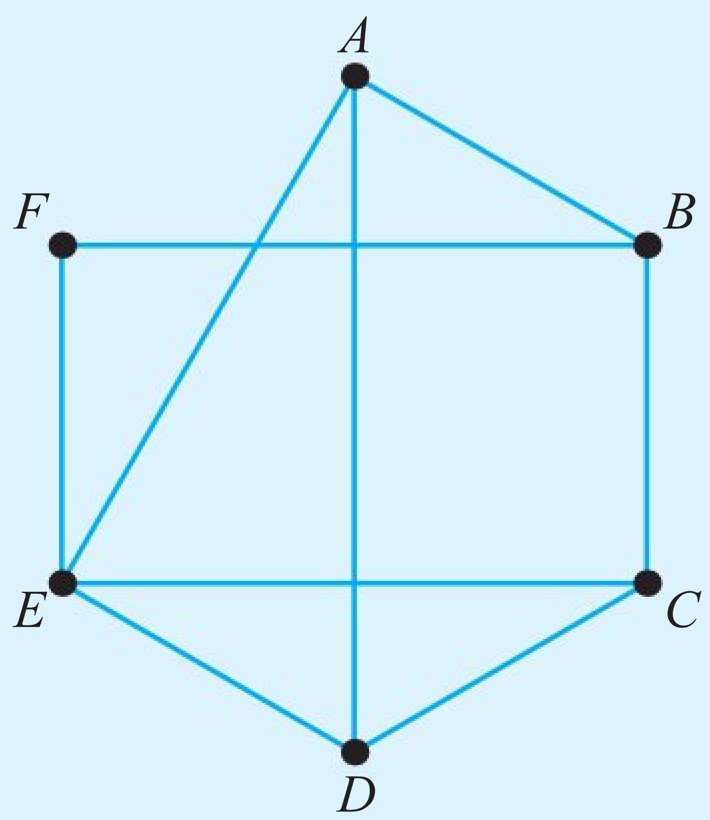
9 **a**



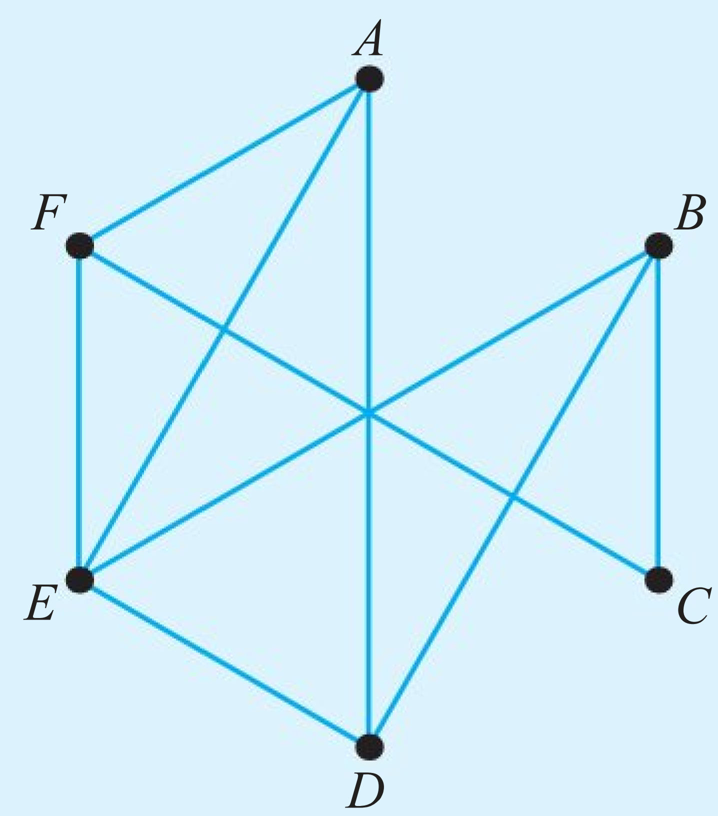
b



10 **a**

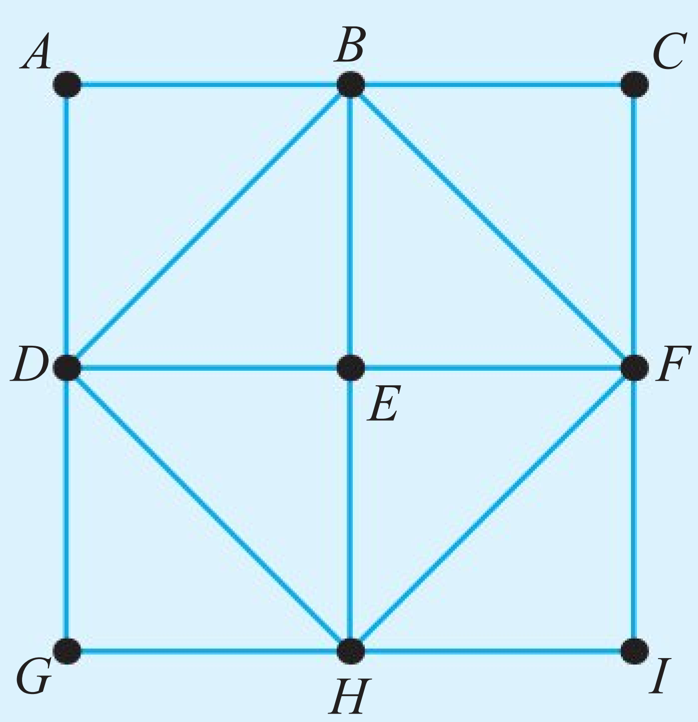


b

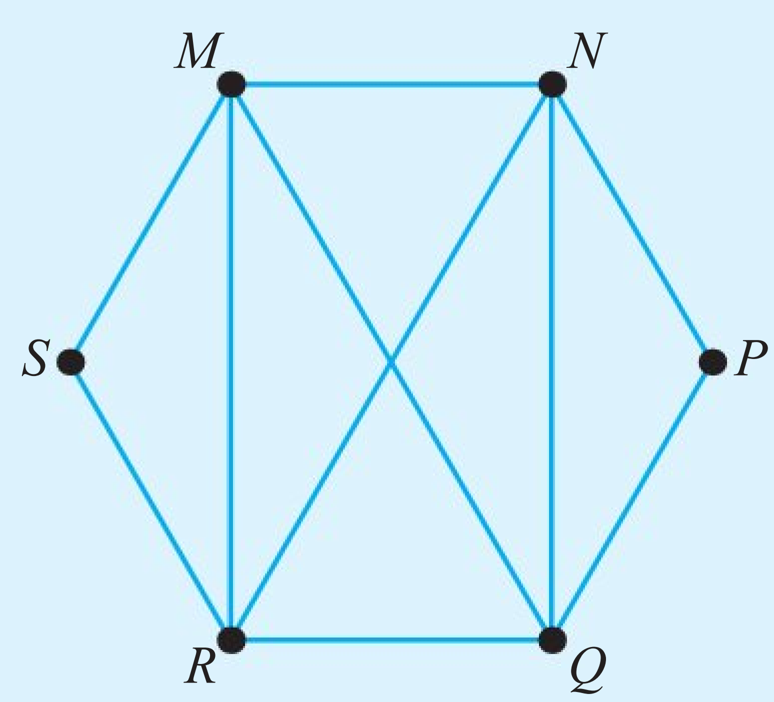


11 Two graphs, G and H, are shown below.

Graph G



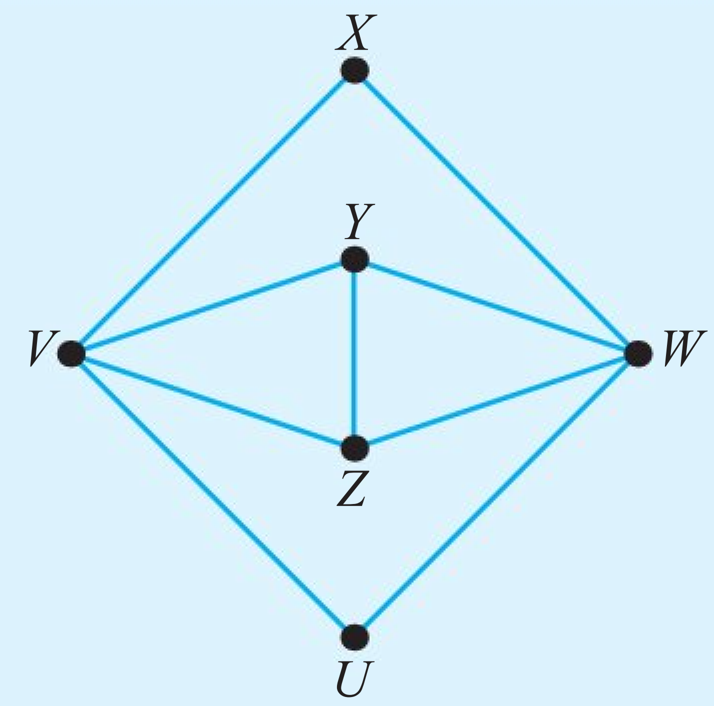
Graph H



a Explain why G is not Eulerian.

b Find an Eulerian circuit in H.

12 Explain why the graph shown alongside is semi-Eulerian, and find an Eulerian trail.



13 **a** Which of these three adjacency matrices represent Eulerian graphs?

$$\boldsymbol{A}=\begin{pmatrix}0&1&0&1&1\\1&0&0&0&1\\0&0&0&0&1\\1&0&0&0&1\\1&1&1&1&0\end{pmatrix},\ \boldsymbol{B}=\begin{pmatrix}0&0&1&1\\0&0&1&1\\1&1&0&0\\1&1&0&0\end{pmatrix},\ \boldsymbol{C}=\begin{pmatrix}0&1&1&0&0&0\\1&0&1&1&1&0\\1&1&0&1&0&1\\0&1&1&0&0&0\\0&1&0&0&0&1\\0&0&1&0&1&0\end{pmatrix}$$

b Draw the graph represented by the matrix B. Label the vertices X, Y, Z and W, so that the first column of the matrix corresponds to the vertex X and so on.

c For the graph represented by the matrix B, find the number of walks of length 5 between X and Z.

14 A graph is represented by the following adjacency matrix. The vertices are labelled A to E.

$$\begin{pmatrix}0&1&1&0&1\\1&0&0&1&1\\1&0&0&1&1\\0&1&1&0&1\\1&1&1&1&0\end{pmatrix}$$

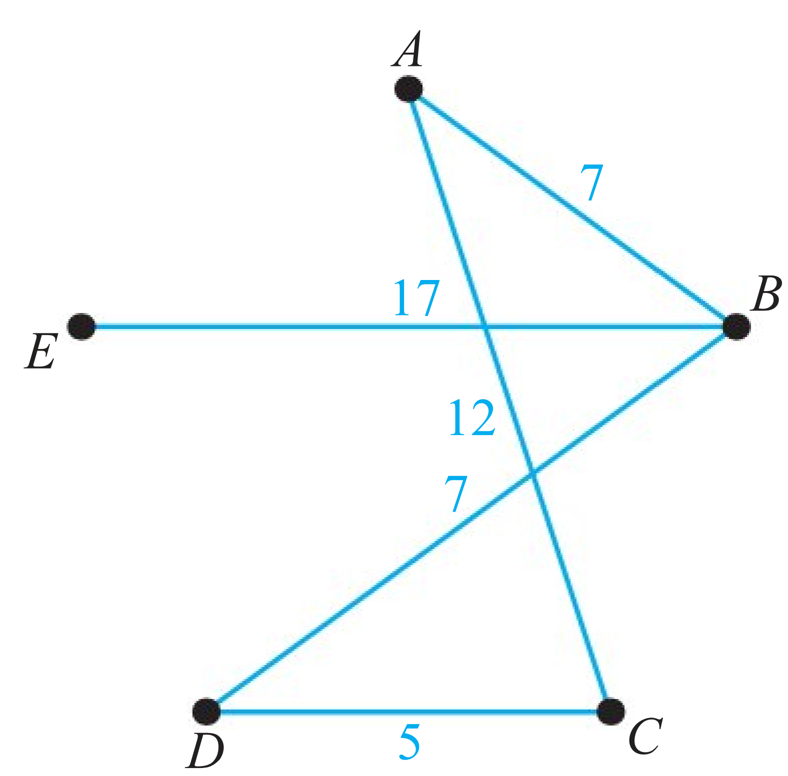
a Show that the graph is Hamiltonian.

b Find the number of walks of length 5 from A to E.

7C Weighted graphs

In a **weighted graph** each edge has a number associated with it. This number is called the **weight**, and can represent the distance, time or cost of travel between the two vertices. Note that this is not the actual length of the edge, and that there is no need to try and draw the graph to scale. Also remember that not all intersections of edges are vertices of the graph.

For example, the graph alongside could represent a road network and the numbers could be times (in minutes) taken to travel between different junctions. So it takes 17 minutes to travel between junctions B and E. To get from B to C to you can go either via A (which takes 19 minutes) or via D (which takes 12 minutes).



When a graph is large, it is not convenient to draw it out. Instead, we can represent it using a **weighted adjacency table**. The numbers in the table represent the weights of the edges. The table shown here represents the above graph.

Weighted graphs can also be directed, and the weight of an edge from A to B can be different from the weight of an edge from B to A.

	A	**B**	**C**	**D**	**E**
A	–	7	12	–	–
B	7	–	–	7	17
C	12	–	–	5	–
D	–	7	5	–	–
E	–	17	–	–	–

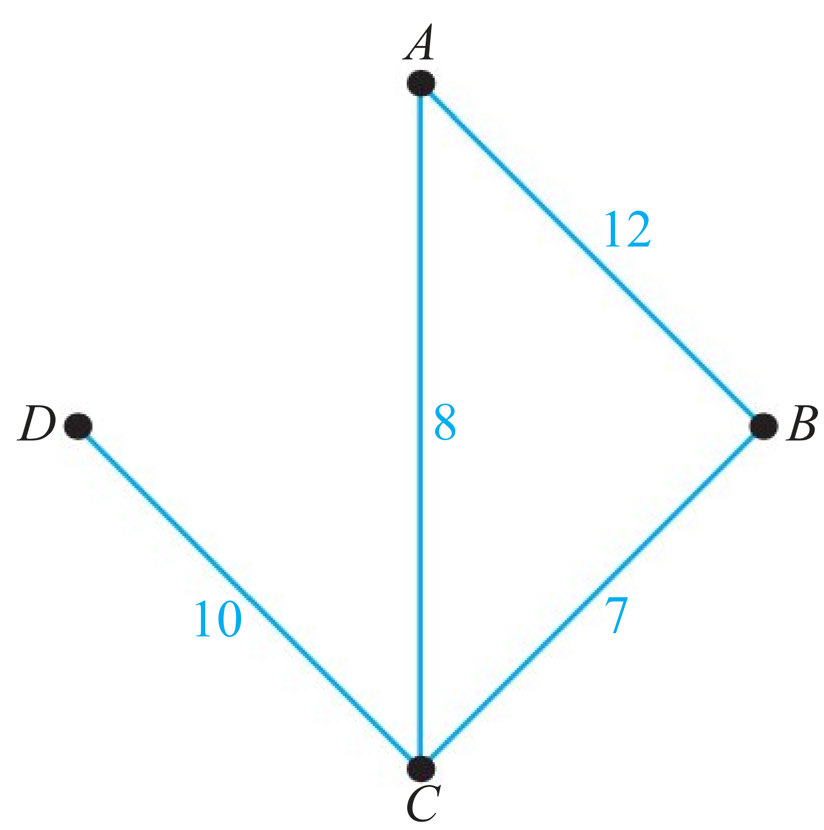
TOOLKIT: Modelling

Think of situations that could be modelled using weighted graphs. For each situation:

- Decide whether a directed or an undirected graph would be more appropriate.
- Discuss the modelling assumptions and any factors that are ignored when representing it as a graph.
- Come up with practical questions that could be answered using your graph. Can you also think of some questions for which your graph would not be the most useful model?

WORKED EXAMPLE 7.12

a Construct a weighted adjacency table for this graph.



b Draw the graph with the following weighted adjacency table.

	A	**B**	**C**	**D**
A	–	4	–	6
B	6	–	5	7
C	–	5	–	3
D	–	–	5	–

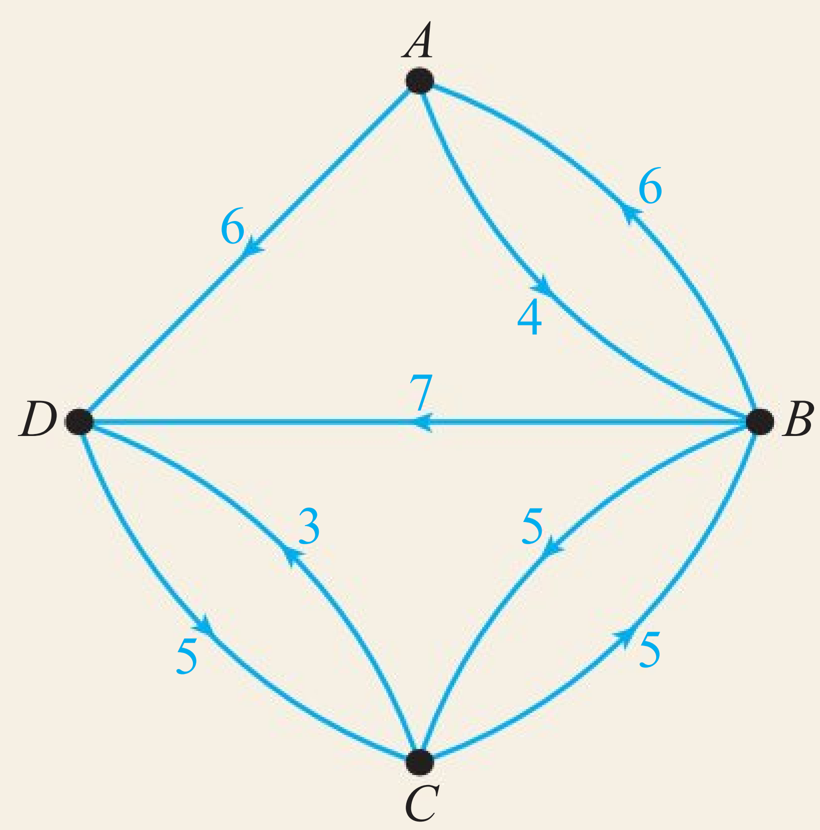
The graph is undirected, so the matrix should be symmetrical

a

	A	*B*	*C*	*D*
A	–	12	8	–
B	12	–	7	–
C	8	7	–	10
D	–	–	10	–

Be careful to get the directions correct. For example, the edge from A to B has weight 4, but the edge from B to A has weight 6

b



Transition matrices

Suppose you are moving around a graph so that, whenever you arrive at a vertex, you choose at random one of the possible edges to take next. This is called a **random** walk on the graph. You can represent this situation by assigning to each edge a weight equal to the probability of selecting it; this will be the reciprocal of the degree of the starting vertex. The resulting weighted graph will be directed. The **transition matrix** shows the probabilities of moving from one vertex to another.

Tip

Unlike an adjacency matrix, a transition matrix has the 'from' vertices listed along the top.

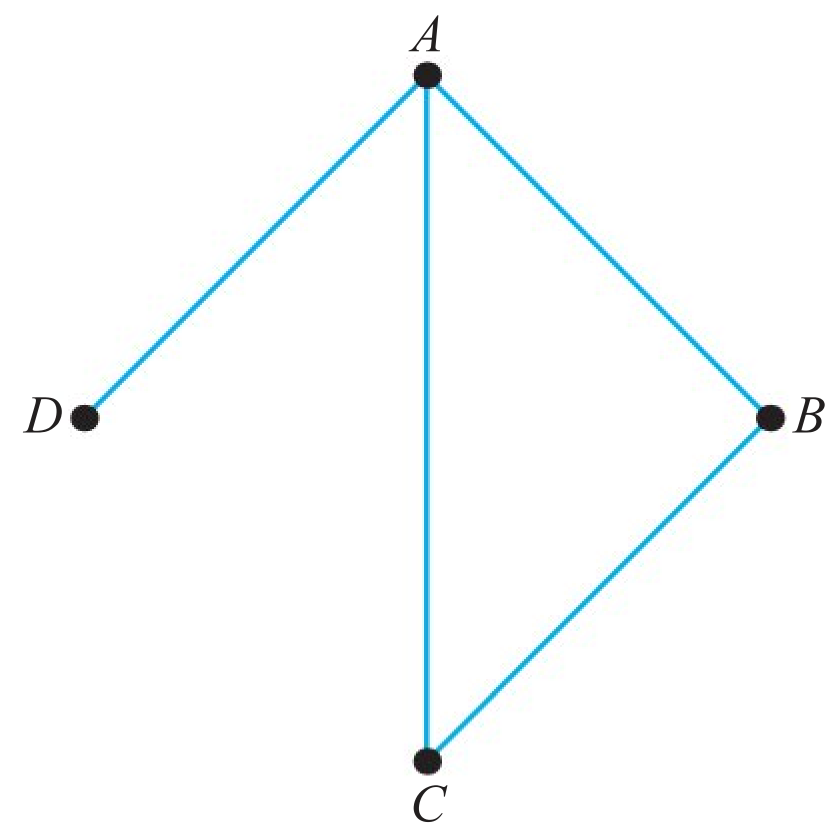
KEY POINT 7.6

In a transition matrix:

- the probability of moving from vertex A to vertex B is given by the entry in column A and row B
- the entries in each column must add up to 1.

WORKED EXAMPLE 7.13

Write down the transition matrix for the following graph.



All diagonal entries are zero (no vertex is connected to itself)

$$\begin{array}{c} \begin{array}{cccc} A & B & C & D \end{array} \\ \begin{pmatrix} 0 & 1/2 & 1/2 & 1 \\ 1/3 & 0 & 1/2 & 0 \\ 1/3 & 1/2 & 0 & 0 \\ 1/3 & 0 & 0 & 0 \end{pmatrix} \end{array}$$

From A, there is one edge to each of the other vertices, so the remaining three entries in column A are all $\frac{1}{3}$

The non-zero entries in columns B and C are $\frac{1}{2}$, as those vertices have degree 2

The only edge from D is DA, so the first entry in the D column is 1

You will learn in Section 8D how transition matrices can be used in a variety of other situations, and how the long-term behaviour is determined by the eigenvalues and eigenvectors of the matrix.

Raising a transition matrix to a power gives the probabilities of being at various vertices after a certain number of moves. For example, for the graph from Worked Example 7.13,

$$\mathbf{M}^3 = \begin{pmatrix} 1/6 & 11/24 & 11/24 & 2/3 \\ 11/36 & 1/6 & 7/24 & 1/6 \\ 11/36 & 7/24 & 1/6 & 1/6 \\ 2/9 & 1/12 & 1/12 & 0 \end{pmatrix}$$

If you start at vertex A, the probability of being at B after three moves is $\frac{11}{36}$ (the number highlighted in the matrix). If the power is very large, the probabilities are the proportion of time spent at each vertex if you continue to move around the graph for a long time. You treat the time taken to move between vertices as negligible.

TOOLKIT: Problem Solving

You could use a tree diagram to find the probability of getting from A to B after three moves. Reflect whether the tree diagram or the matrix method would be more efficient for a larger number of moves.

WORKED EXAMPLE 7.14

This is the transition matrix for the graph from the previous worked example.

$$\begin{array}{c} \begin{array}{cccc} A & B & C & D \end{array} \\ \begin{pmatrix} 0 & 1/2 & 1/2 & 1 \\ 1/3 & 0 & 1/2 & 0 \\ 1/3 & 1/2 & 0 & 0 \\ 1/3 & 0 & 0 & 0 \end{pmatrix} \end{array}$$

If a random walk is performed on this graph, find the percentage of time spent at vertex B. Ignore the transition time between vertices.

Look at a high power of the matrix – it should be high enough that the numbers don't change to three decimal places when you go up another power

$$\mathbf{M}^{40} = \begin{pmatrix} 0.375 & 0.375 & 0.375 & 0.375 \\ 0.250 & 0.250 & 0.250 & 0.250 \\ 0.250 & 0.250 & 0.250 & 0.250 \\ 0.125 & 0.125 & 0.125 & 0.125 \end{pmatrix}$$

Look at the row corresponding to vertex B – all the entries are 0.250

The random walk will spend 25% of the time at vertex B.

TOOLKIT: Problem Solving

In the above example, raising the transition matrix to a high power resulted in all four columns being identical. This means that the long-term proportion of time spent on each vertex does not depend on the starting point. Will this always be the case?

Investigate large powers of transition matrices of connected and unconnected graphs. For directed graphs, investigate the difference between graphs which are strongly connected and those which are not.

The PageRank algorithm

Google's PageRank algorithm ranks the pages in an internet search based on the number of links to them from other pages, but also taking into account how well linked those pages are. So a page receives a high ranking if it has links from many other pages, but also if it has links from a few high-ranking pages.

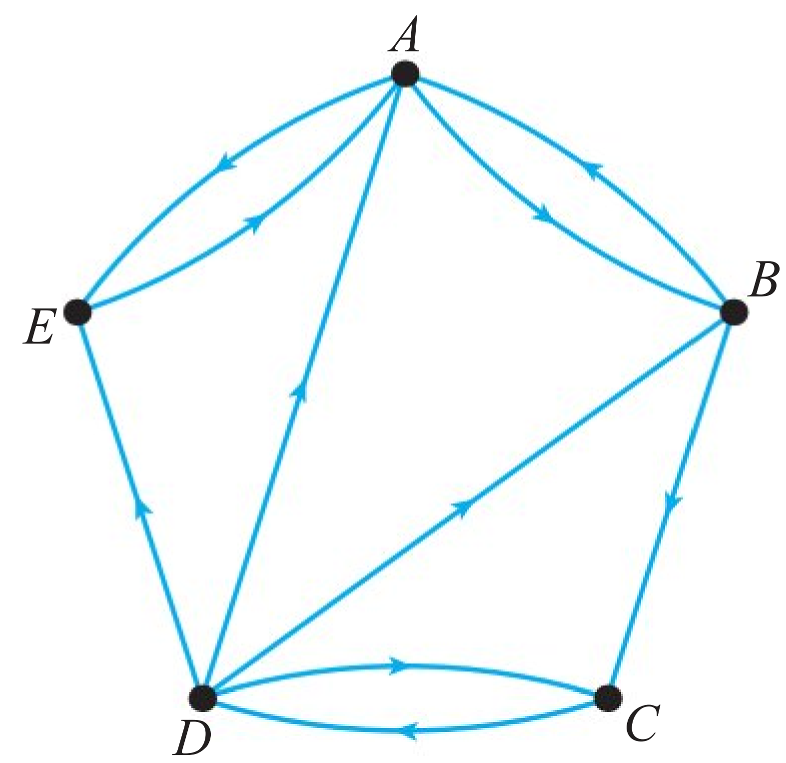
A simplified version of the algorithm assigns each page a ranking according to the probability that a person randomly clicking on links would end up on that page. If links between pages are represented by a directed graph, then the transition matrix can be used to determine the proportion that a random walk spends at each vertex.

The PageRank algorithm was developed by Larry Page and Sergey Brin in 1996 and has since formed the basis of all Google's search algorithms. More sophisticated versions of the algorithm take into account whether any of the sites are from a particularly authoritative source, as well as the probability of the user abandoning the search.

WORKED EXAMPLE 7.15

The graph shows the links between five web pages.

Use a transition matrix to determine which page should come top in a search.



Write down the transition matrix for the graph

The transition matrix is:

$$\mathbf{M} = \begin{pmatrix} 0 & 0.5 & 0 & 0.25 & 1 \\ 0.5 & 0 & 0 & 0.25 & 0 \\ 0 & 0.5 & 0 & 0.25 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0.5 & 0 & 0 & 0.25 & 0 \end{pmatrix}$$

Raise the matrix to a large power

In $\mathbf{M}^n$, each column is

$$\begin{pmatrix} 0.333 \\ 0.2 \\ 0.133 \\ 0.133 \\ 0.2 \end{pmatrix}$$

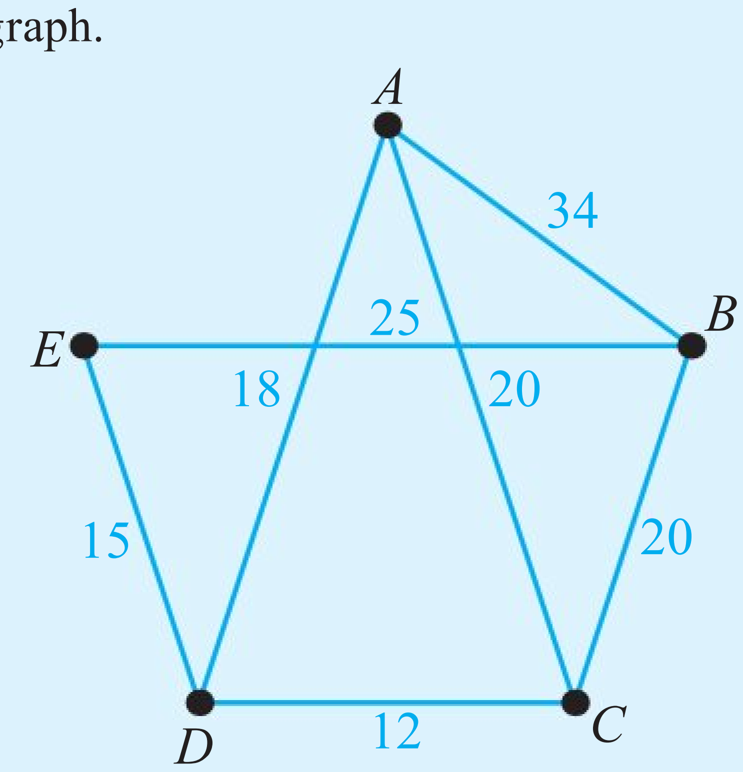
The vertex with the largest entry should come at the top of the search

Page A should come at the top of the search.

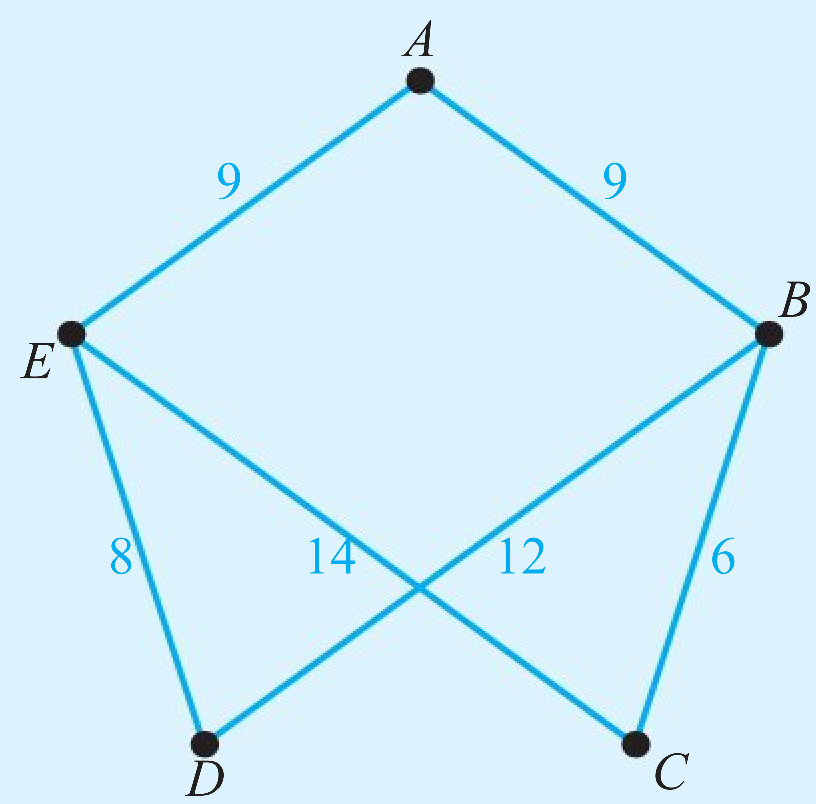
Exercise 7C

For questions 1 and 2, use the method demonstrated in Worked Example 7.12a to construct the adjacency table for each graph.

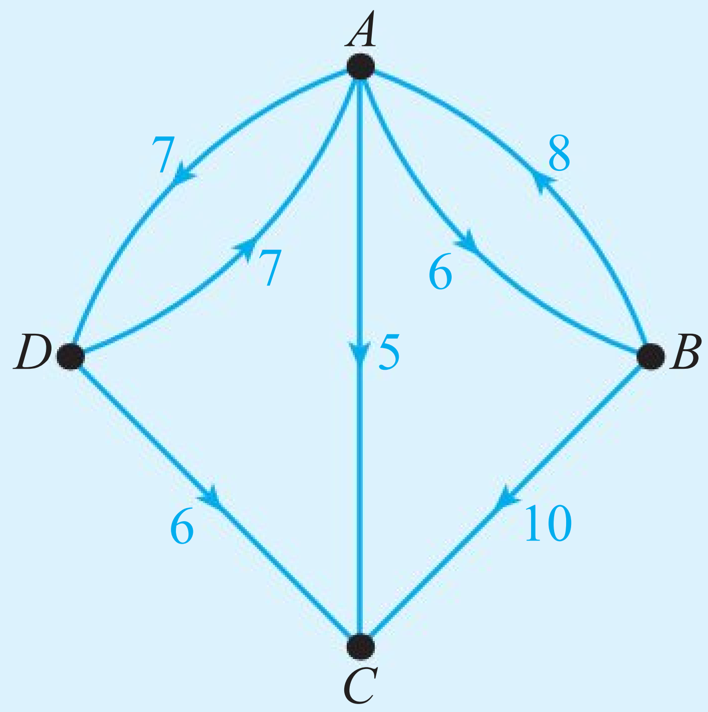
1 **a**



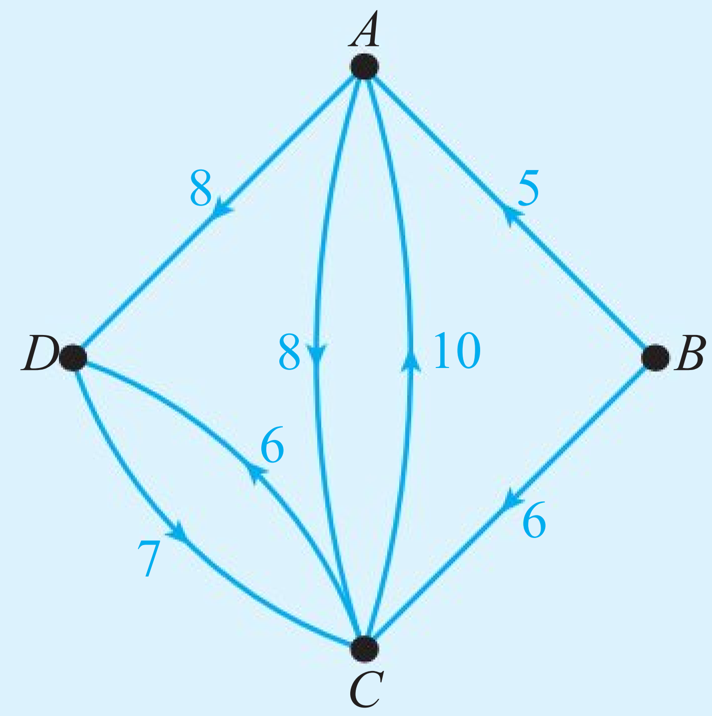
b



2 a



b



For questions 3 and 4, use the method demonstrated in Worked Example 7.12b to draw a weighted graph with the given adjacency table.

3 a

	A	**B**	**C**	**D**
A	–	–	–	7
B	6	–	7	–
C	–	8	–	5
D	–	8	–	–

b

	A	**B**	**C**	**D**
A	–	12	–	–
B	17	–	15	–
C	–	15	–	–
D	25	–	–	–

4 a

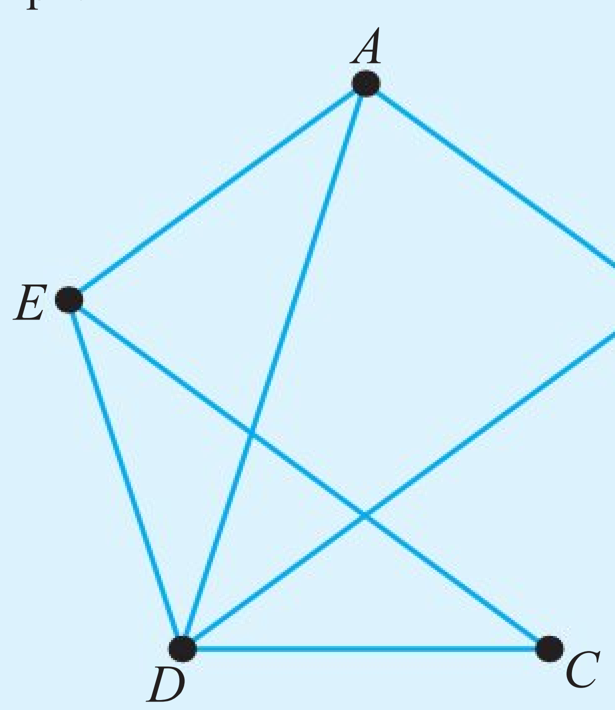
	A	**B**	**C**	**D**
A	–	13	8	10
B	13	–	12	–
C	8	12	–	12
D	10	–	12	–

b

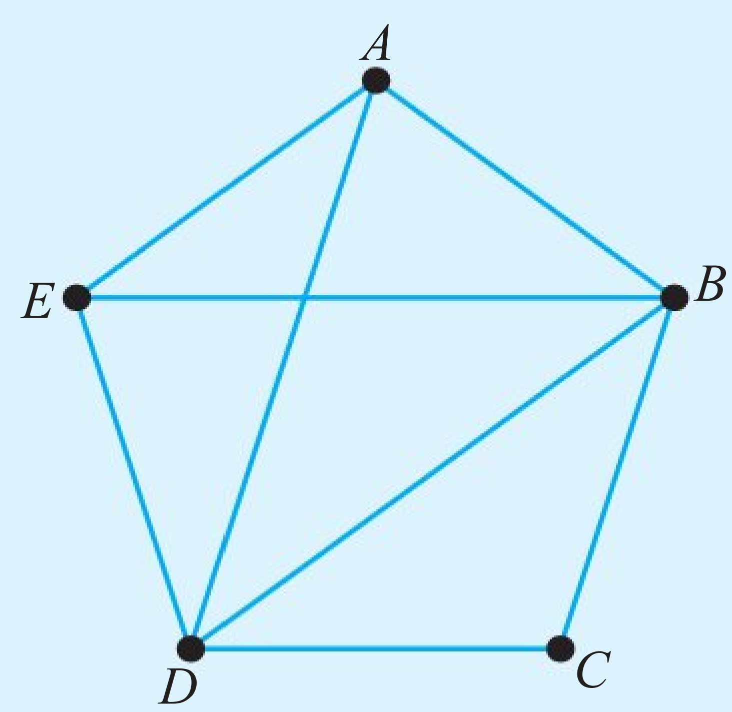
	A	**B**	**C**	**D**
A	–	8	12	9
B	8	–	10	18
C	12	10	–	–
D	9	18	–	–

For questions 5 and 6, use the method demonstrated in Worked Example 7.13 to construct the transition matrix for each graph.

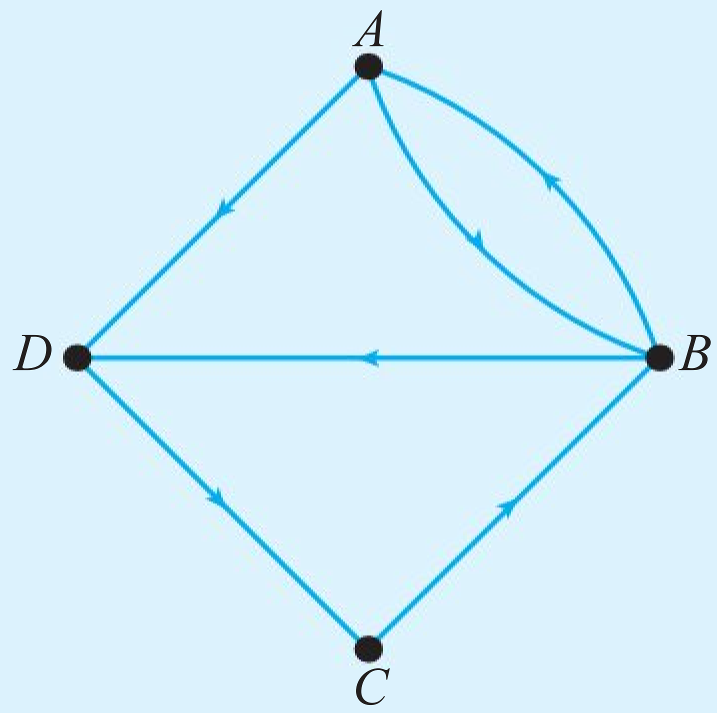
5 a



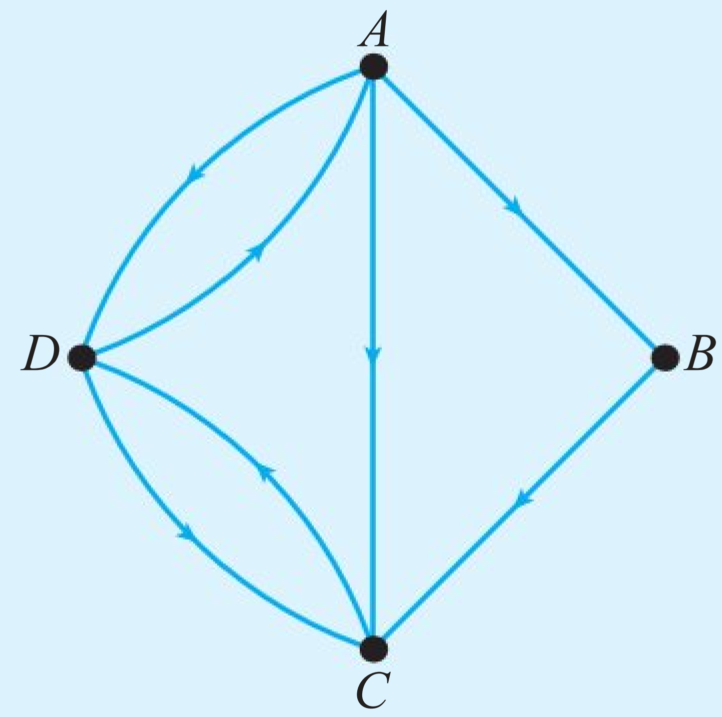
b



6 a



b



Questions 7 and 8 refer to the graphs from questions 5 and 6. Use the method demonstrated in Worked Example 7.14 to find the proportion of the time that a random walk would spend at the given vertex.

7 a Question 5a, vertex B

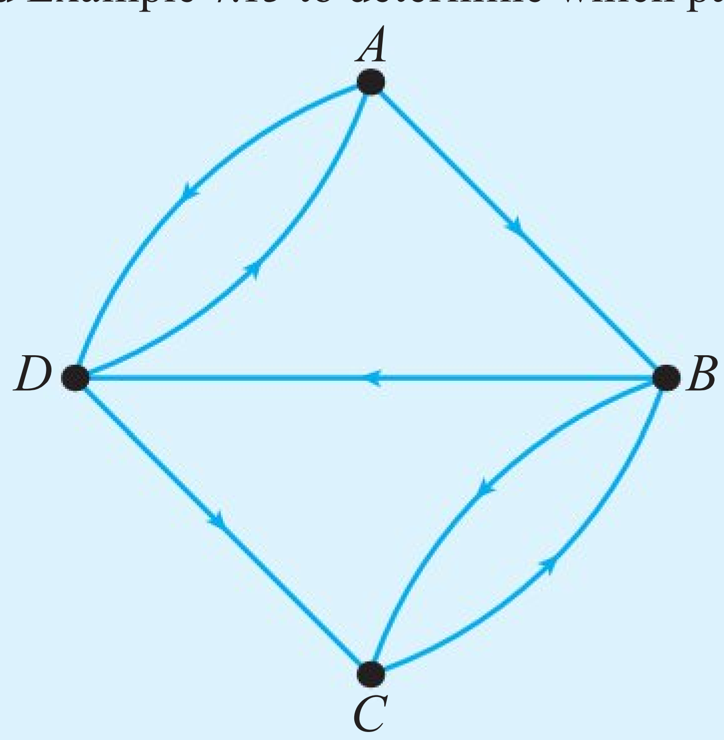
b Question 5b, vertex C

8 a Question 6a, vertex A

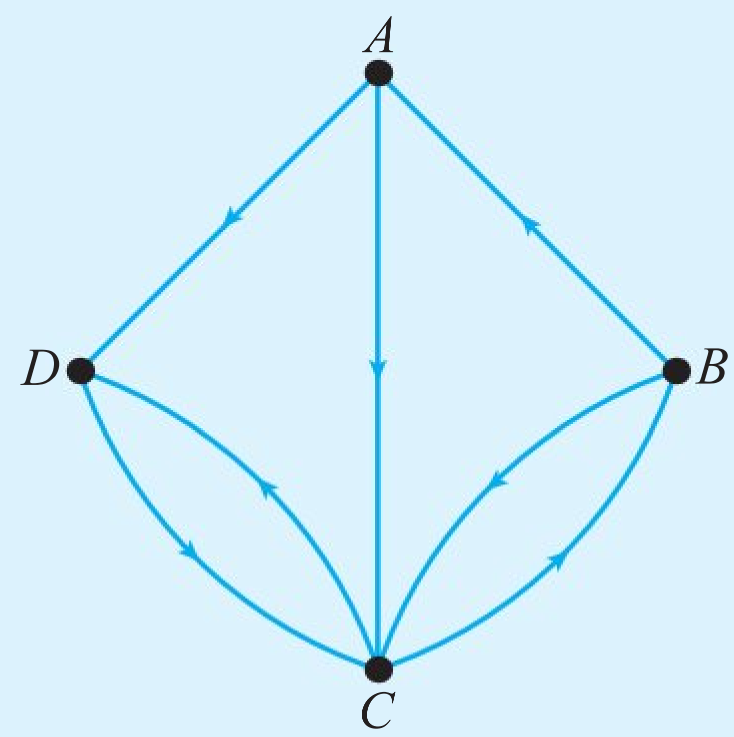
b Question 6b, vertex D

The graphs in questions 9 and 10 show the links between several web pages. Use the method demonstrated in Worked Example 7.15 to determine which page should come top in a search.

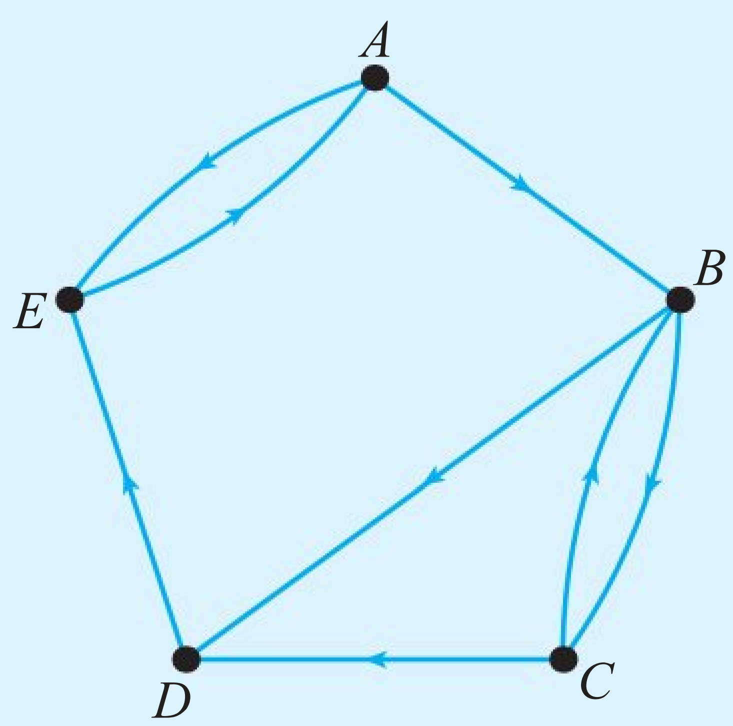
9 a



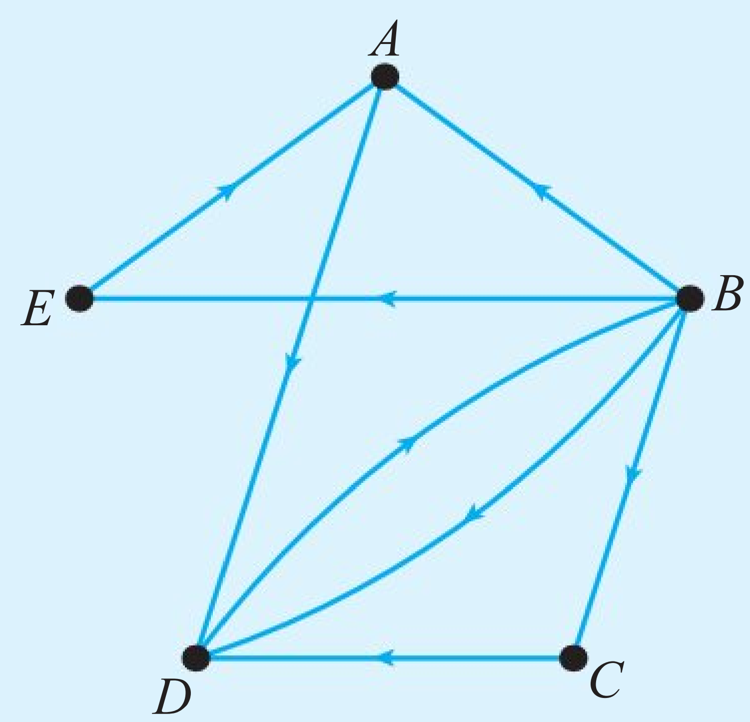
b



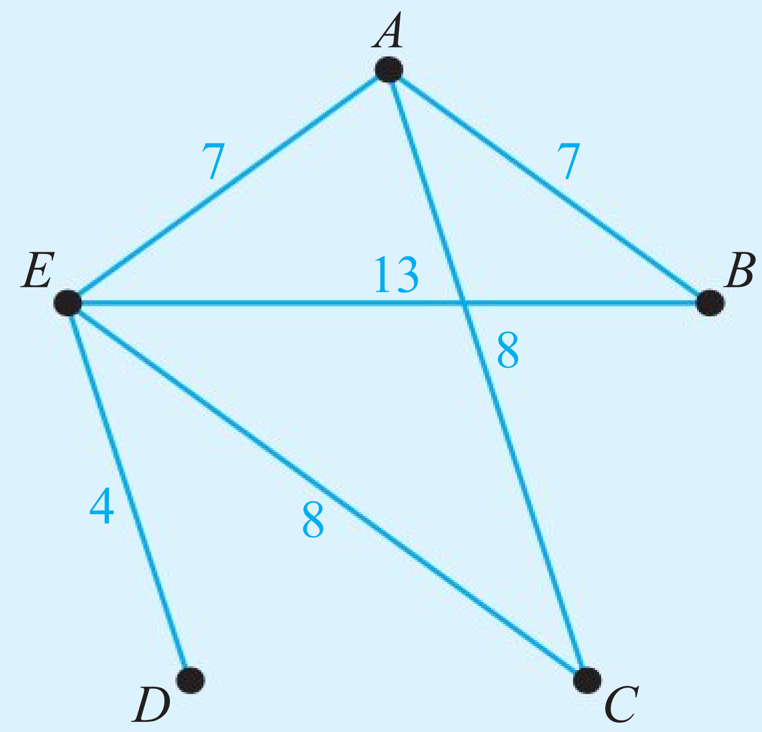
10 a



b



11 The figure shows a graph representing direct flight distances, in thousands of kilometres, between five cities.



a Find the shortest total flight distance between cities B and C.

The weighted adjacency table shows the cost of flights, in hundreds of dollars, between the same five cities.

	A	B	C	D	E
A	–	0.9	2.7	–	0.8
B	0.9	–	–	–	1.2
C	2.7	–	–	–	2.3
D	–	–	–	–	1.2
E	0.8	1.2	2.3	1.2	–

b Find the cheapest route between cities B and C.

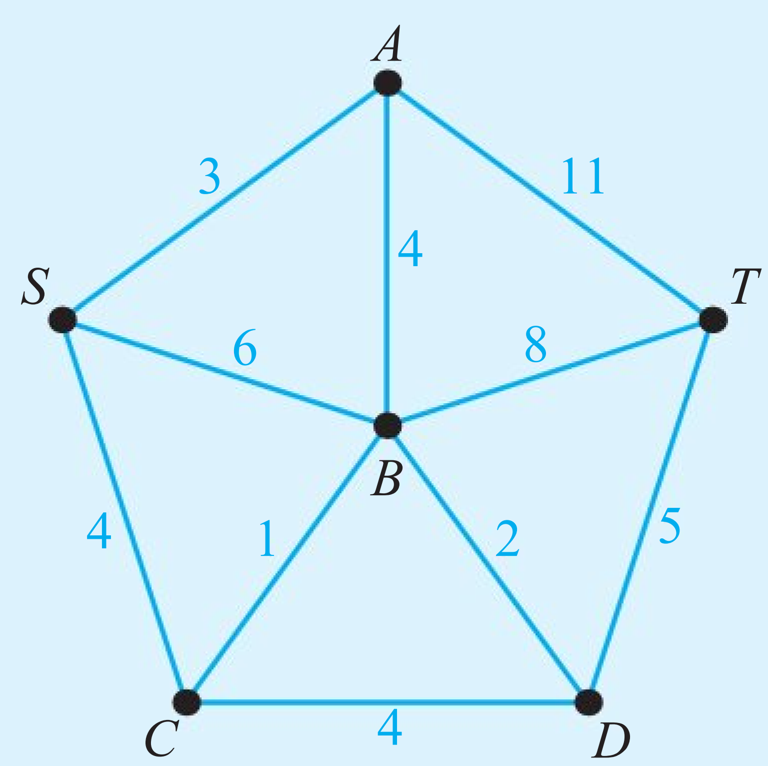
12 Six locations in a park are marked A to F. There are direct paths between some of them. The time, in minutes, taken to walk along those direct paths is given in the following table.

	A	B	C	D	E	F
A	–	3	–	–	5	–
B	3	–	5	5	–	–
C	–	5	–	–	–	4
D	–	5	–	–	4	3
E	5	–	–	4	–	–
F	–	–	4	3	–	–

a Draw a graph to represent the information in the table.

b What is the quickest way to walk from A to D? How long does it take?

13 The graph shows the lengths, in kilometres, of roads between six villages.



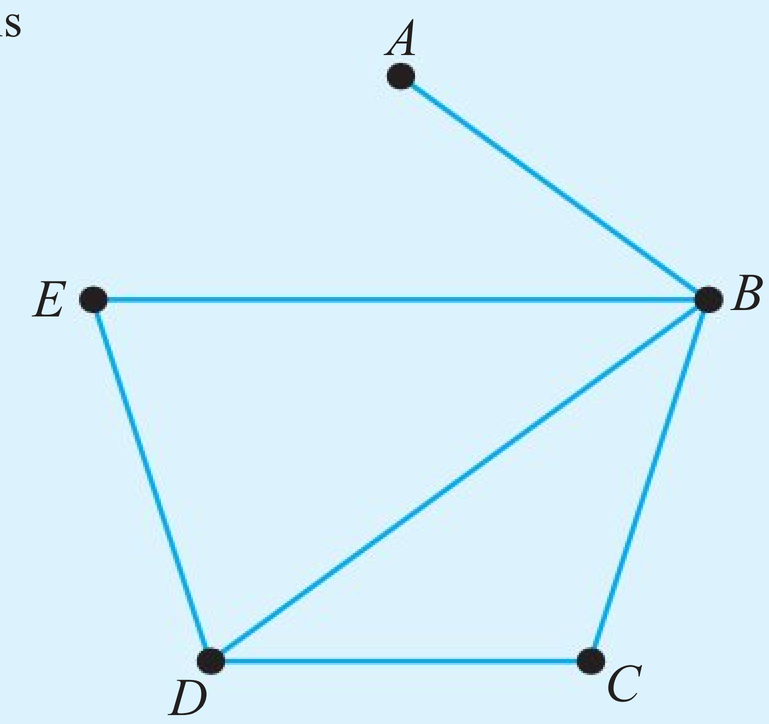
a Find the length of the shortest route from S to T.

b The road between B and D is closed. What is the new shortest route from S to T? State its length.

14 Some of the five ports on an island are connected by roads. The connections are shown in the graph alongside.

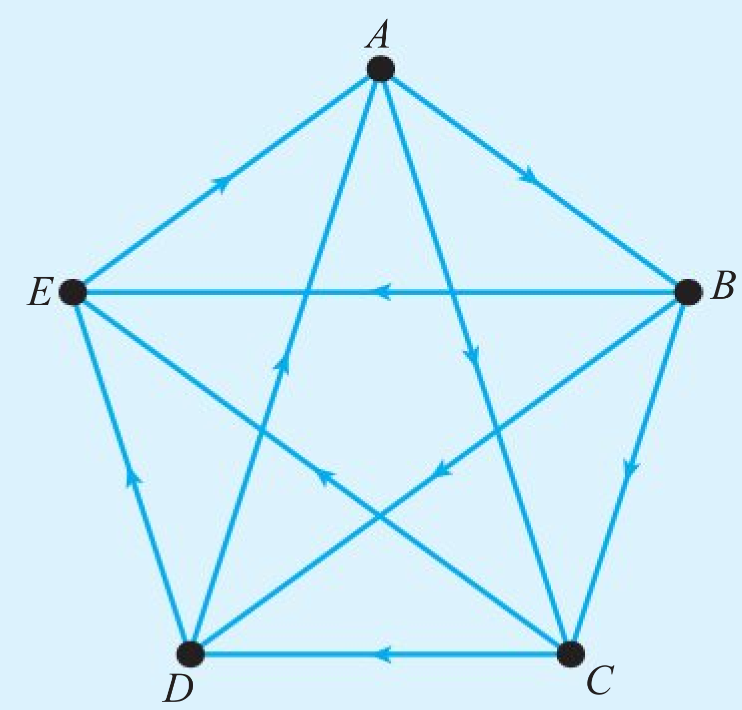
A tourist arrives at port E. Every day (including the first), he selects one of the available roads at random and travels to another port.

- **a** Construct a transition matrix for the graph.
- **b** Find the probability that the tourist travels to port C on the 7th day.
- **c** If the tourist continues travelling around the island for a long time, which port will he visit most often?



15 Links between five websites are represented as a directed graph. For example, website A contains links to websites B and C.

Rank the pages in order of importance, justifying your answer.

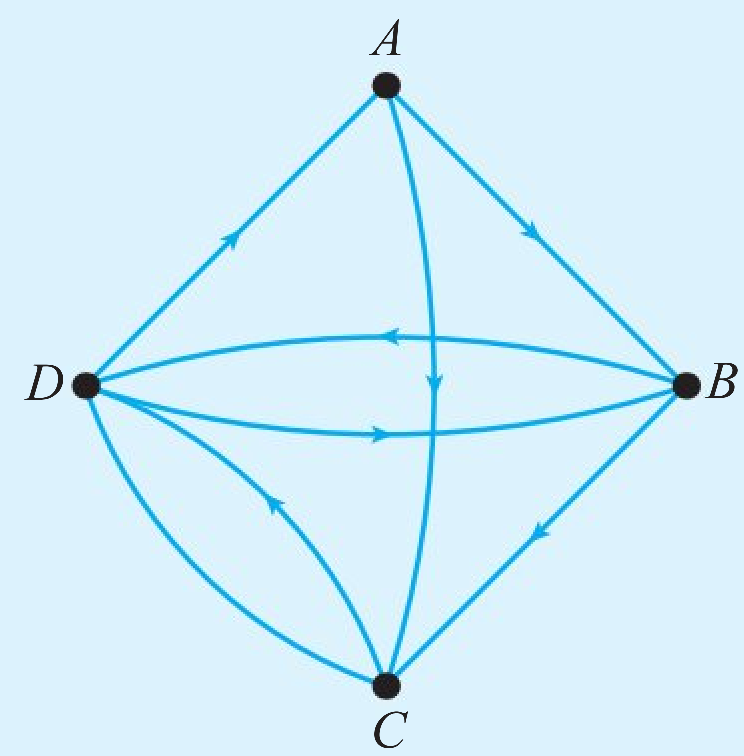


16 The graph shows links between four websites.

- **a** Which website would come top in an internet search?

In an alternative model of an internet search, at each stage the user stops the search with probability 0.1.

- **b** By adding a fifth site corresponding to the end of the search, write down the transition matrix to model this new situation.
- **c** Does the new model predict a different order of importance of the four original websites?



7D Minimum spanning tree algorithms

The table shows the distances (in km) between six villages. Roads are to be built between some of the towns so that it is possible to get from any town to any other town. What is the minimum length of road required?

	A	*B*	*C*	*D*	*E*	*F*
A	–	5	7	–	8	8
B	5	–	6	–	5	–
C	7	6	–	4	4	3
D	–	–	4	–	5	2
E	8	5	4	5	–	–
F	8	–	3	2	–	–

Clearly, if town *A* is connected to town *B*, and town *B* is connected to town *C*, then we don't need a direct road between *A* and *C*. In the language of graph theory, the resulting graph will have no cycles – it will be a tree. The problem is to find the tree of minimum total length (weight) which includes every vertex of the graph. This is called the **minimum spanning tree**.

In this section, you will learn about two different algorithms for finding the minimum spanning tree. Kruskal's algorithm adds edges to the graph, starting with the shortest, until the graph is connected. Prim's algorithm starts with the vertex and then adds the closest possible vertex at each stage.

Tip

The minimum spanning tree is not necessarily unique – there may be more than one tree with the same weight. You are only required to find one answer unless explicitly told otherwise.

Kruskal's algorithm

KEY POINT 7.7

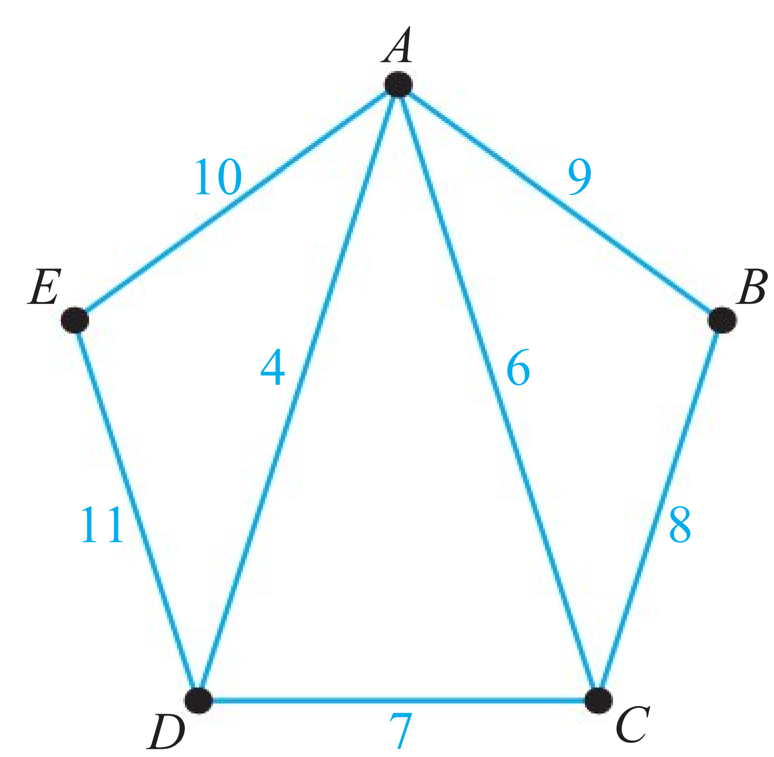
Kruskal's algorithm builds a minimum spanning tree by adding edges one at a time.

- Start by adding the shortest edge.
- At each stage, add the shortest remaining edge as long as it does not create a cycle.
- Keep adding edges until all of the vertices have been connected.

WORKED EXAMPLE 7.16

Use Kruskal's algorithm to find a minimum spanning tree for this graph.

List the edges in the order in which you add them and state the weight of your tree.



Start with the shortest edge, in this case AD

AD (4)

Add edges in order of increasing length, skipping those that would create a cycle

AC (6)

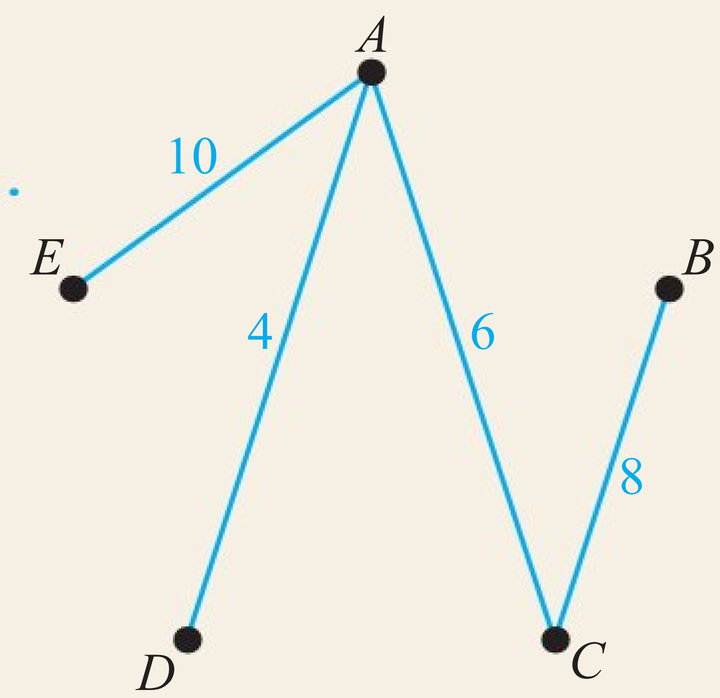
CD skip

BC (8)

AB skip

AE (10)

Draw the graph as you go along, so you can check that there are no cycles and know when all the vertices have been connected



Add up the lengths of all the edges to find the weight of the tree

The weight of the tree is

$$4 + 6 + 8 + 10 = 28$$

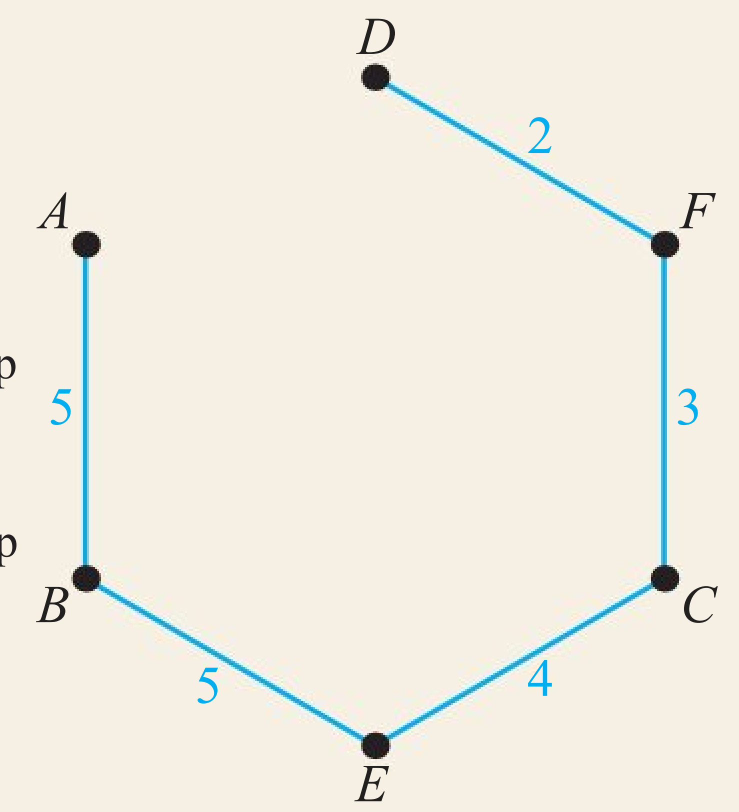
You can also use Kruskal's algorithm when the graph is given by a weighted adjacency table. In this case, it is even more important to draw the graph as you go along, as it is difficult to check for cycles from the table.

WORKED EXAMPLE 7.17

Use Kruskal's algorithm to find a minimum spanning tree for the graph given by this weighted adjacency table.

	A	**B**	**C**	**D**	**E**	**F**
A	–	5	7	–	8	8
B	5	–	6	–	5	–
C	7	6	–	4	4	3
D	–	–	4	–	5	2
E	8	5	4	5	–	–
F	8	–	3	2	–	–

Strat with the shortest edge, which is *DF*	*DF* (2)
Draw the graph as you go along to make sure that you do not create any cycles	*CF* (3)
CD and *CE* both have length 4. *CD* would create a cycle, so add *CE*	*CD* skip *CE* (4)
There are three edges of length 5	*ED* skip
ED would create a cycle (*CEDF*)	*AB* (5)
You can add both *AB* and *EB* without creating a cycle	*EB* (5)
All six vertices have been connected, so the tree is complete	



Prim's algorithm

Kruskal's algorithm is quick and easy to implement on small graphs. However, on a large graph, it is difficult to check whether you are creating a long cycle. Also note that, for an algorithm to be implemented on a computer, you would need an additional algorithm to check for cycles. This can be very time consuming, so for large graphs an alternative algorithm is used.

Tip

Kruskal's and Prim's algorithms will give the same minimum spanning tree. To make it clear which algorithm you have used, you must give the order in which you have added the edges.

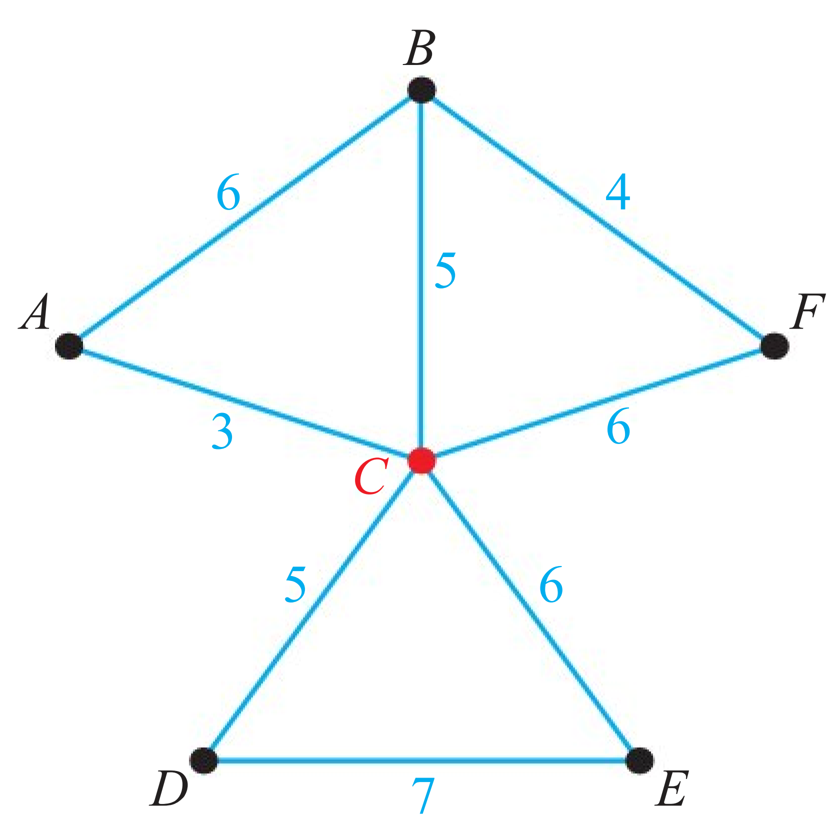
KEY POINT 7.8

Prim's algorithm builds up the tree by adding vertices one at a time.

- Specify a starting vertex.
- At each stage, find the vertex (which has not yet been added) that has the shortest possible distance to any of the vertices that have already been added. Add that vertex and the corresponding edge.
- Stop when all the vertices have been connected.

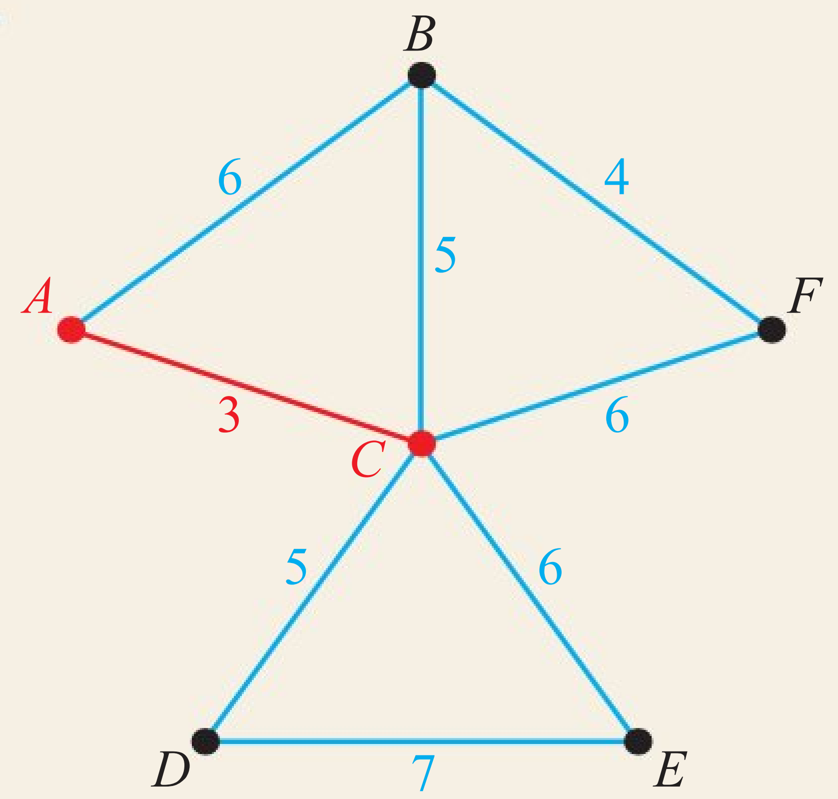
WORKED EXAMPLE 7.18

Use Prim's algorithm starting with vertex *C* to find the minimum spanning tree for this graph. List the edges in order in which they have been added.

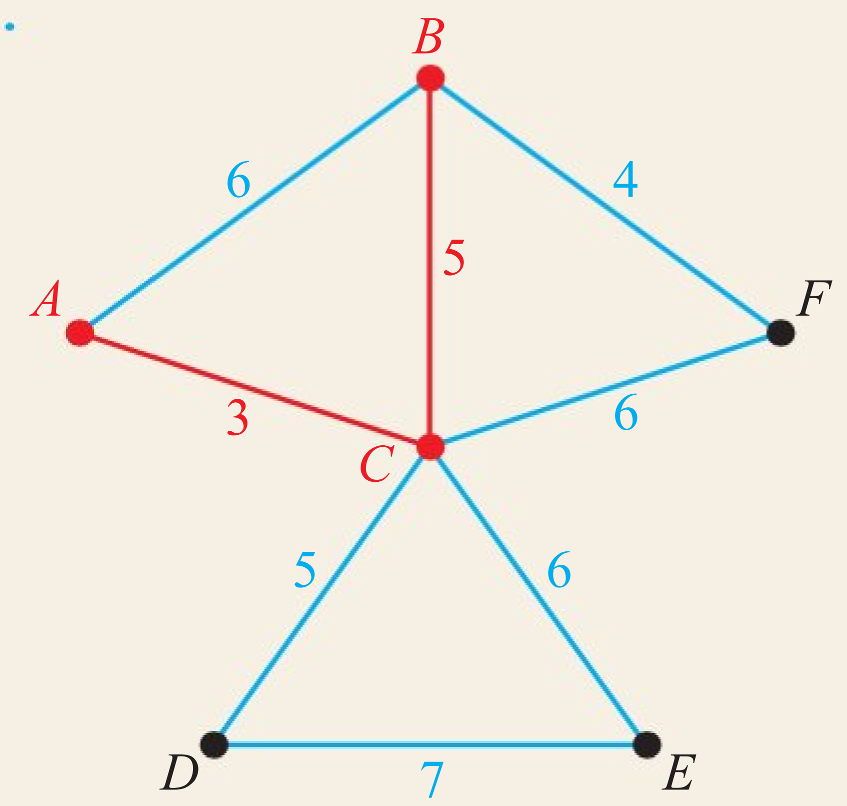


Note: in the diagrams on the right, the vertices and edges that have been added to the graph are marked in red

Find the vertex closest to C – this is A. Add it, together with the edge CA



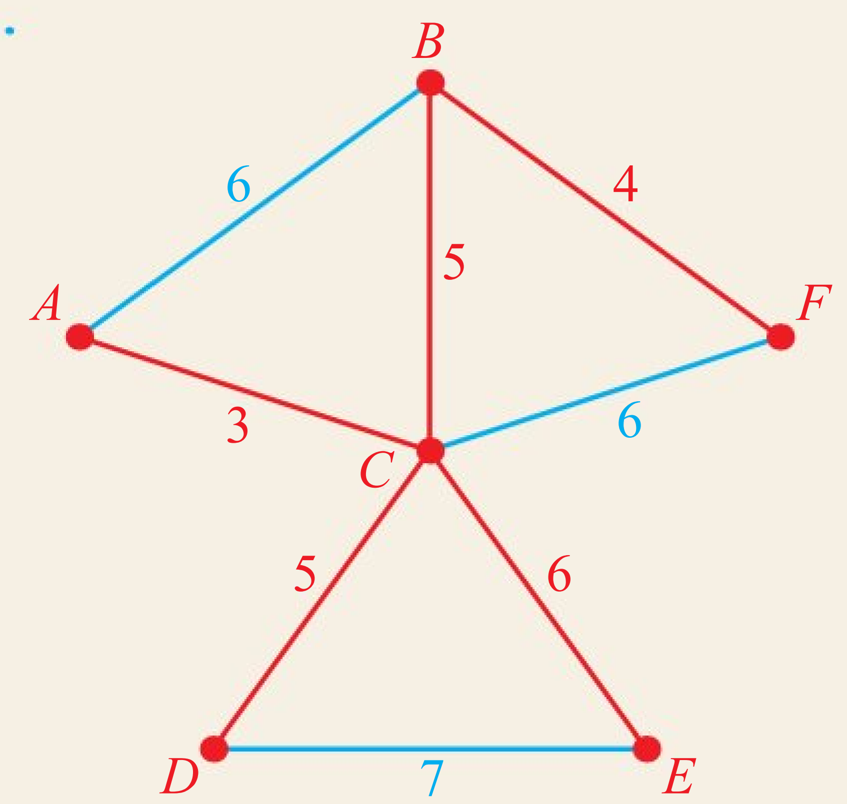
Next look for the vertex closest to *either* A *or* C. This is vertex B, so we add the edge BC



Then look for the vertex closest to *any of* A, B, or C. This is F, so add the edge BF

The vertex closest to *any of* A, B, C or F is D, so add the edge CD

Finally, the only remaining vertex is E and the shortest edge leading to it is CE



The edges have been added in the following order:
AC, *CB*, *EF*, *CD*, *CE*

To implement an algorithm on a computer, the graph needs to be stored in the form of its adjacency table. The next example illustrates the implementation of Prim's algorithm in a table. Once a vertex has been connected, we circle its column label and cross out its row. We also draw a box around the connecting edge length. By crossing out a row, we know that the corresponding vertex has already been connected and we do not need to look at any other edges leading to it.

Tip

You do not have to draw a different table for each stage; your method can be seen from the boxed numbers. You may be asked to write down the order in which the edges have been added.

WORKED EXAMPLE 7.19

Graph *G* has the following weighted adjacency table.

	A	*B*	*C*	*D*	*E*	*F*
A	–	6	3	–	–	5
B	6	–	3	–	3	5
C	3	3	–	–	6	4
D	–	–	–	–	5	2
E	–	3	6	5	–	–
F	5	5	4	2	–	–

Use Prim's algorithm starting at vertex *A* to find the minimum spanning tree for *G*. Draw your tree and state its length.

The first connected vertex is *A*. Circle its column label and cross out its row. Box the shortest edge length in *A*'s column

	A	*B*	*C*	*D*	*E*	*F*
~~*A*~~	~~–~~	~~6~~	~~3~~	~~–~~	~~–~~	~~5~~
B	6	–	3	–	3	5
C	$\boxed{3}$	3	–	–	6	4
D	–	–	–	–	5	2
E	–	3	6	5	–	–
F	5	5	4	2	–	–

The next connected vertex is *C*. Circle the column label and cross out the row

Look at all the numbers in *A* and *C* columns that haven't been crossed out; box the smallest

	A	*B*	*C*	*D*	*E*	*F*
~~*A*~~	~~–~~	~~6~~	~~3~~	~~–~~	~~–~~	~~5~~
B	6	–	$\boxed{3}$	–	3	5
~~*C*~~	~~$\boxed{3}$~~	~~3~~			~~6~~	~~4~~
D	–	–	–	–	5	2
E	–	3	6	5	–	–
F	5	5	4	2	–	–

Vertex *B* has now been connected, so we circle its column label and cross out its row

The smallest uncrossed number in columns *A*, *B* and *C* is 3, corresponding to vertex *E*

	A	*B*	*C*	*D*	*E*	*F*
~~*A*~~	~~–~~	~~6~~	~~3~~	~~–~~	~~–~~	~~5~~
~~*B*~~	~~6~~	~~–~~	~~$\boxed{3}$~~	~~–~~	~~3~~	~~5~~
~~*C*~~	~~$\boxed{3}$~~	~~3~~			~~6~~	~~4~~
D	–	–	–	–	5	2
E	–	$\boxed{3}$	6	5	–	–
F	5	5	4	2	–	–

Vertex *E* has now been connected

We look for the smallest uncrossed number from columns *A*, *B*, *C*, *E*. It is in row *F*

	A	B	C	D	E	F
~~A~~	–	~~6~~	~~3~~	–	–	~~5~~
~~B~~	~~6~~	–	$\boxed{3}$	–	~~3~~	~~5~~
~~C~~	$\boxed{3}$	~~3~~	–	–	~~6~~	~~4~~
D	–	–	–	–	5	2
~~E~~	–	$\boxed{3}$	~~6~~	~~5~~	–	–
F	5	5	$\boxed{4}$	2	–	–

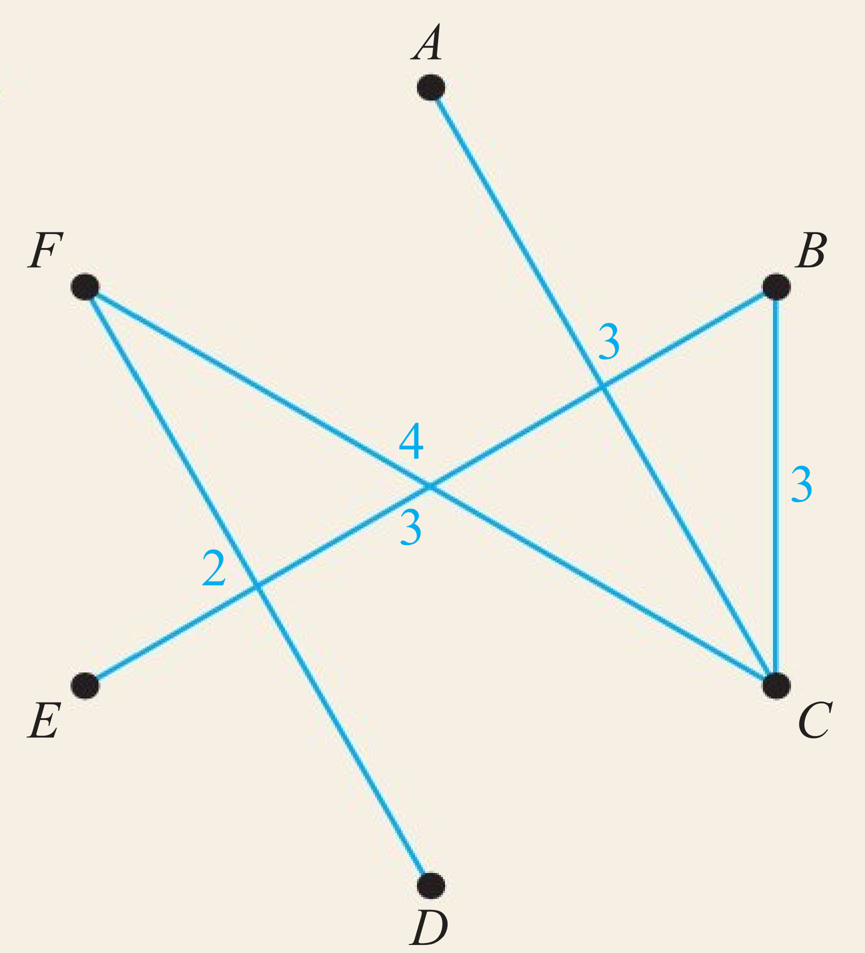
Vertex *F* has now been connected

The smallest uncrossed number is 2, and it corresponds to vertex *D* (the only vertex that has not been connected yet)

	A	B	C	D	E	F
~~A~~	–	~~6~~	~~3~~	–	–	~~5~~
~~B~~	~~6~~	–	$\boxed{3}$	–	~~3~~	~~5~~
~~C~~	$\boxed{3}$	~~3~~	–	–	~~6~~	~~4~~
D	–	–	–	–	5	$\boxed{2}$
~~E~~	–	$\boxed{3}$	~~6~~	~~5~~	–	–
~~F~~	~~5~~	~~5~~	$\boxed{4}$	~~2~~	–	–

All vertices have been connected, so the tree is complete

You can use the boxed numbers to draw the tree and find its weight



Weight = 15

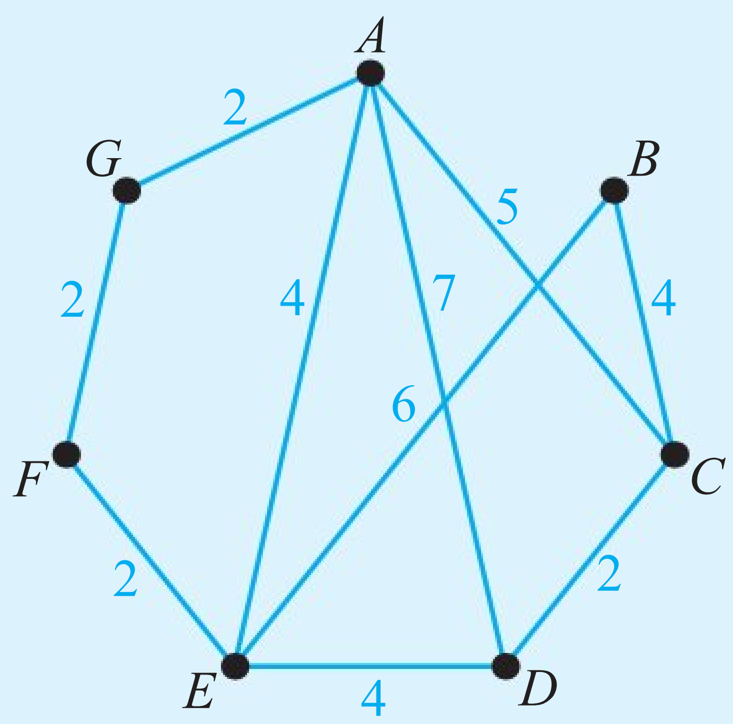
You are the Researcher

Both Kruskal's and Prim's are examples of so-called **greedy algorithms**, where at each stage we pick the best possible option (in this case, the shortest possible edges). This strategy does not work for other types of problems on graphs, for example finding the shortest path between two vertices. One method to solve this type of problem is called Djikstra's algorithm.

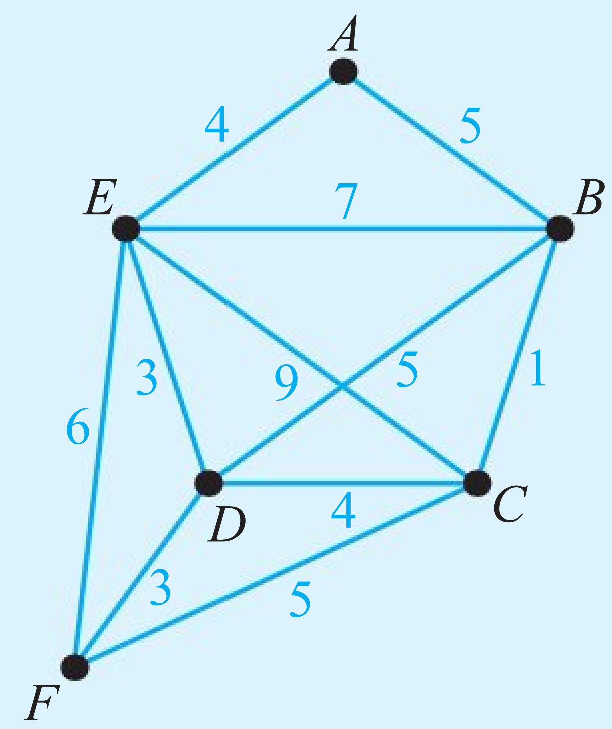
Exercise 7D

For questions 1 to 3, use Kruskal's algorithm, illustrated in Worked Example 7.16, to find the minimum spanning tree for these graphs. Draw your tree and state its weight.

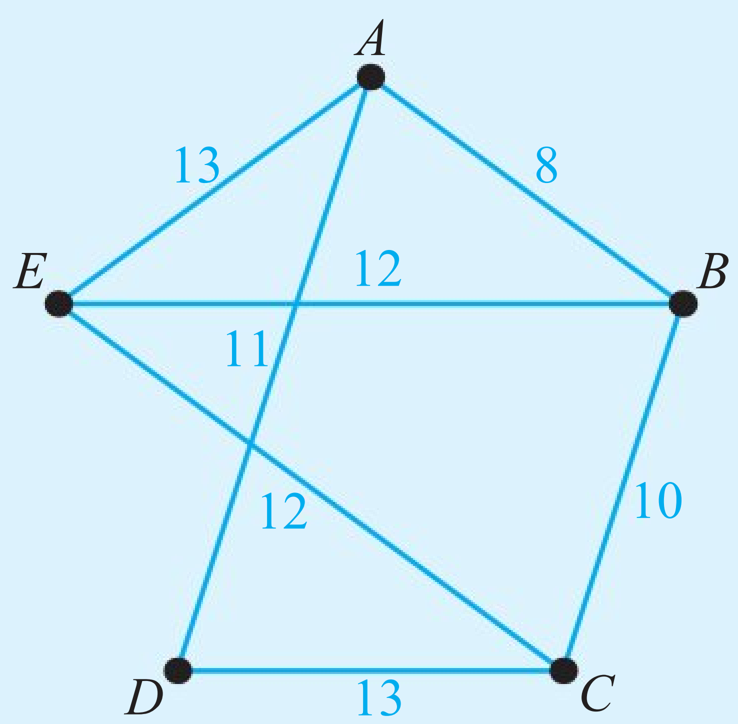
1 a



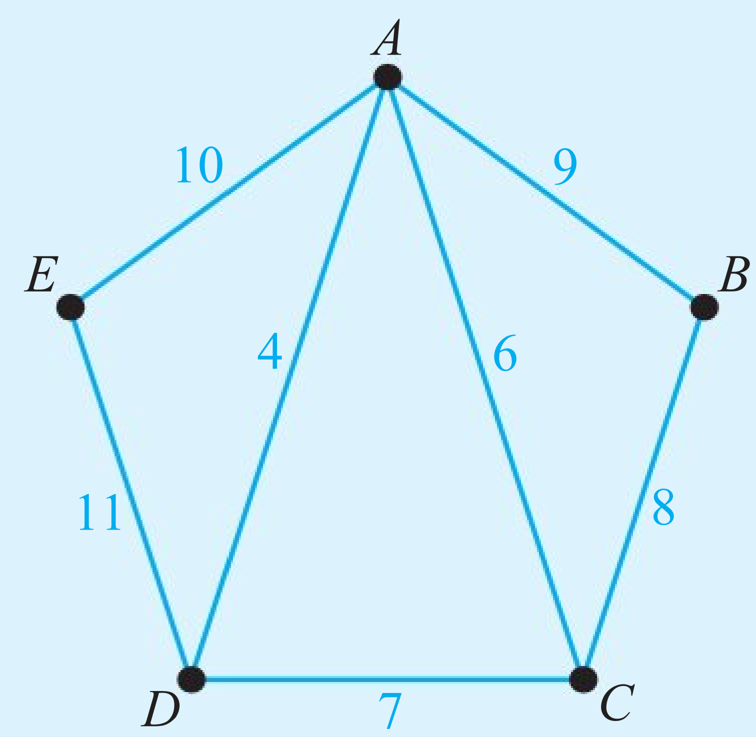
b



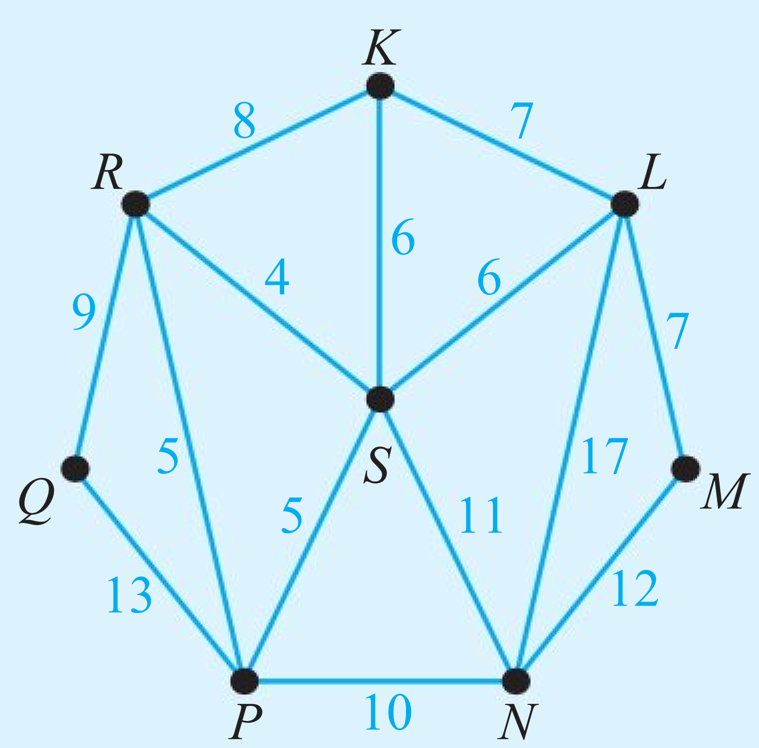
2 a



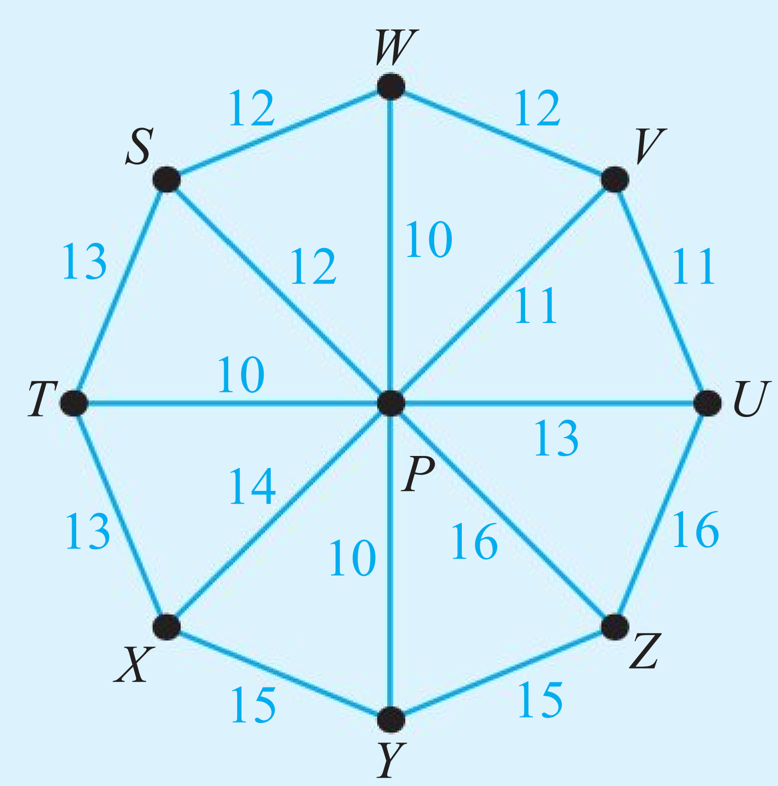
b



3 a



b



For questions 4 to 6, use Kruskal's algorithm, illustrated in Worked Example 7.17, to find the minimum spanning tree for the graphs given by these adjacency tables. Draw your tree and state its weight.

4 a

	A	B	C	D	E
A	–	10	8	7	10
B	10	–	5	4	9
C	8	5	–	7	10
D	7	4	7	–	8
E	10	9	10	8	–

b

	A	B	C	D	E
A	–	12	16	11	16
B	12	–	13	14	19
C	16	13	–	15	18
D	11	14	15	–	18
E	16	19	18	18	–

5 a

	A	B	C	D	E	F	G
A	–	–	30	–	–	50	45
B	–	–	70	35	40	–	–
C	30	70	–	50	–	–	20
D	–	35	50	–	10	–	15
E	–	40	–	10	–	15	–
F	50	–	–	–	15	–	10
G	45	–	20	15	–	10	–

b

	A	B	C	D	E	F	G
A	–	25	35	40	–	–	–
B	25	–	50	35	40	–	–
C	35	50	–	45	45	–	40
D	40	35	45	–	20	35	30
E	–	40	45	20	–	15	25
F	–	–	–	35	15	–	–
G	–	–	40	30	25	–	–

6 a

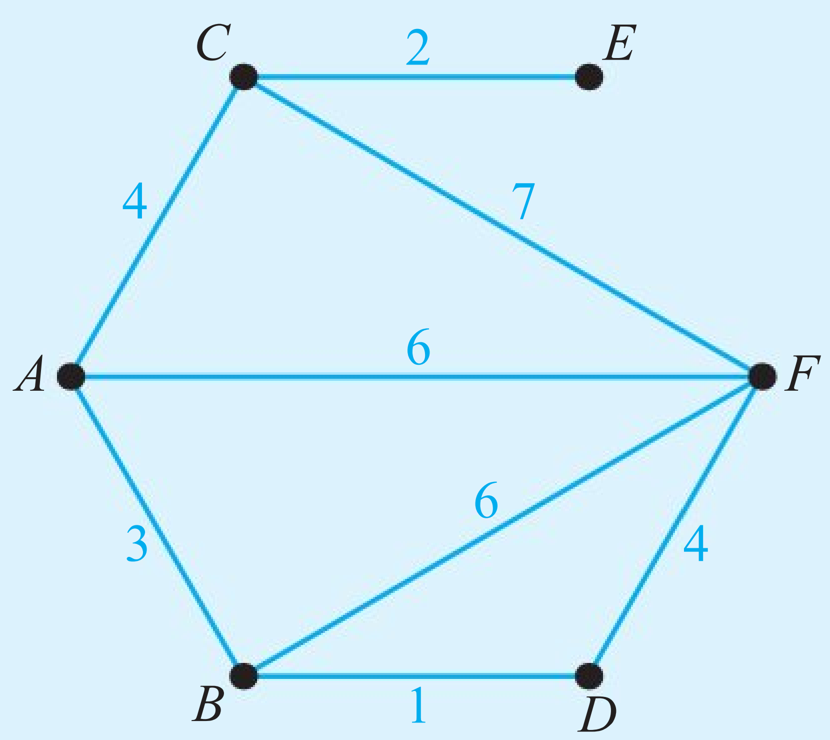
	A	B	C	D	E	F
A	–	4	–	6	–	10
B	4	–	5	–	5	–
C	–	5	–	6	–	4
D	6	–	6	–	7	–
E	–	5	–	7	–	2
F	10	–	4	–	2	–

b

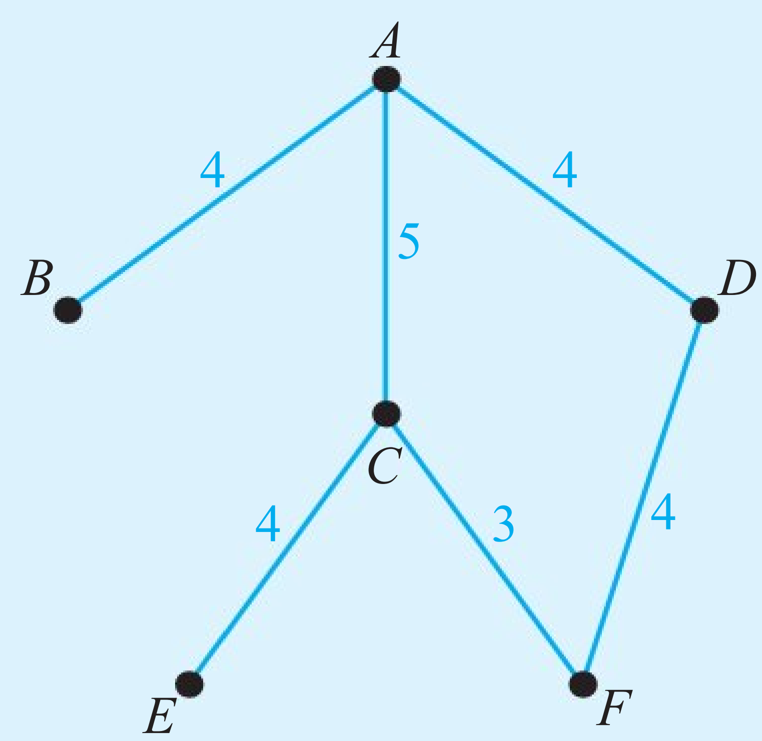
	A	B	C	D	E	F
A	–	5	7	–	8	8
B	5	–	6	–	5	–
C	7	6	–	4	4	3
D	–	–	4	–	5	2
E	8	5	4	5	–	–
F	8	–	3	2	–	–

For questions 7 to 9, use Prim's algorithm starting at vertex A, as illustrated in Worked Example 7.18, to find the minimum spanning tree for each graph. State the edges in the order that you add them.

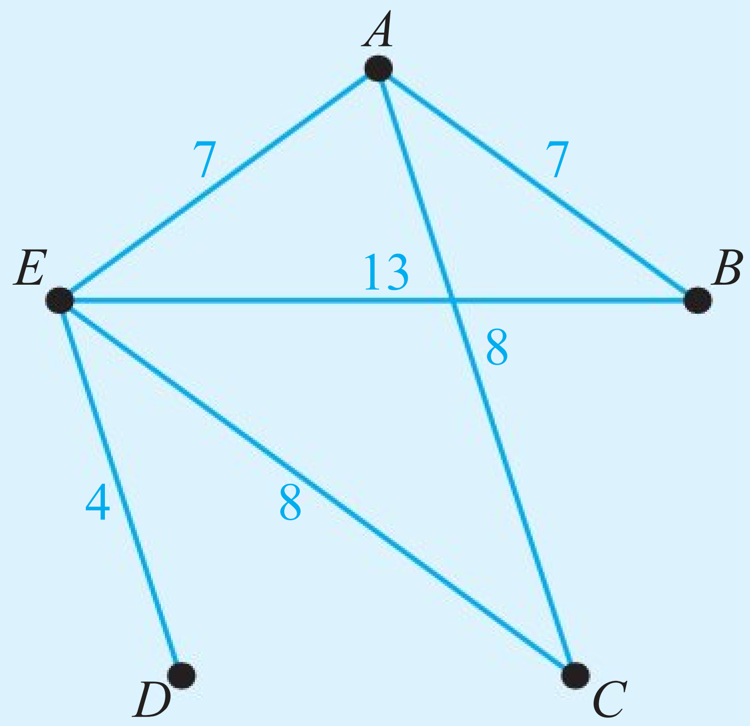
7 a



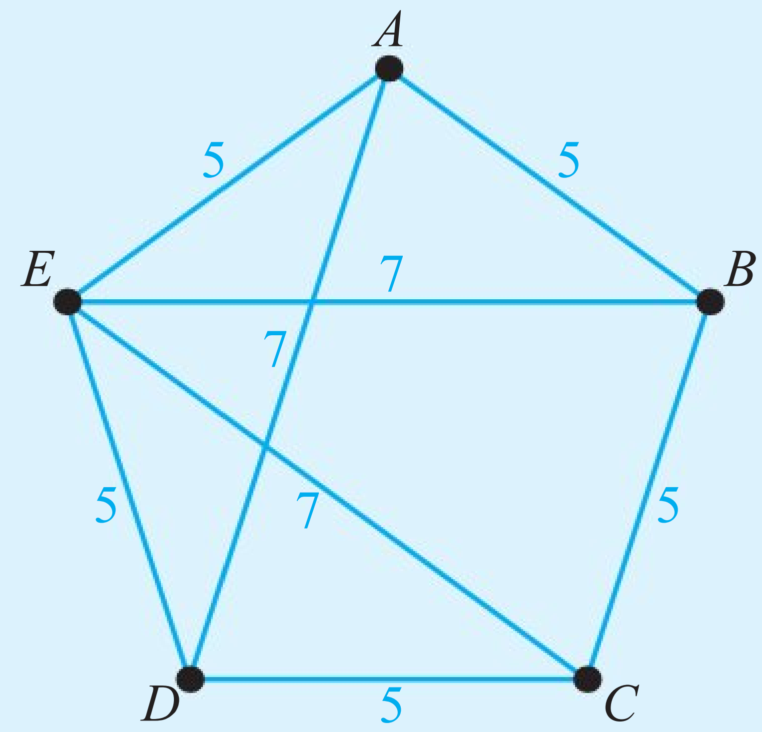
b



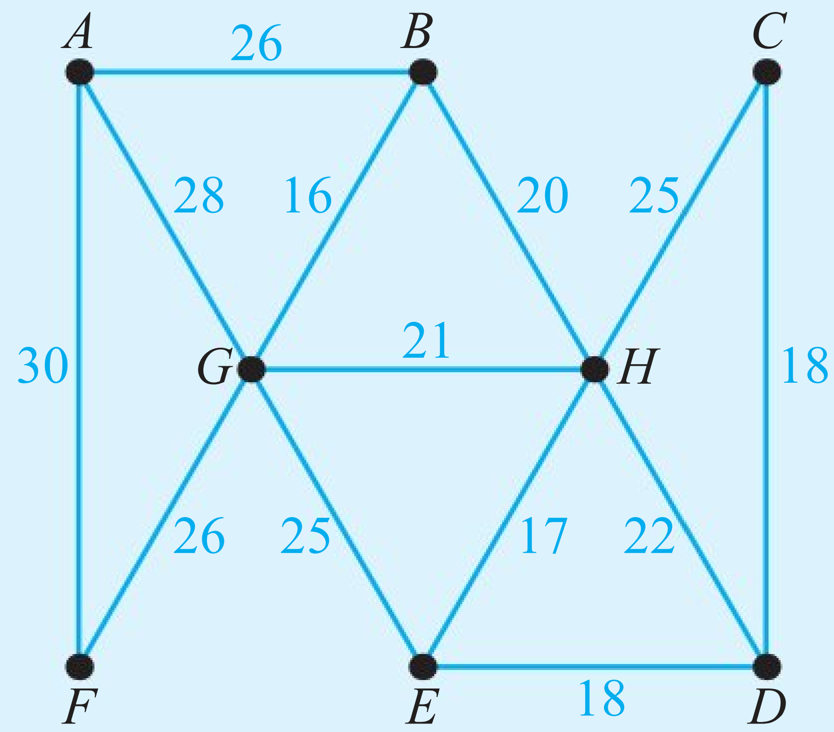
8 **a**



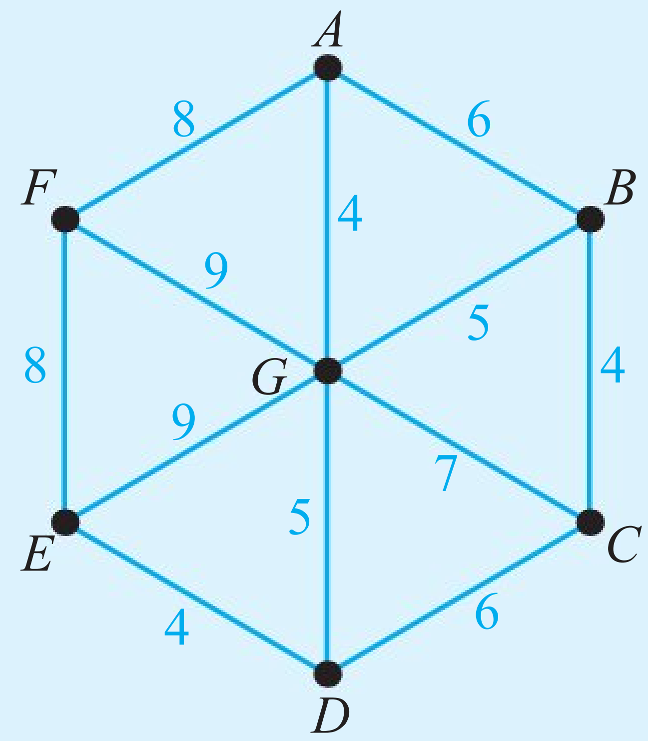
b



9 **a**



b



For questions 10 to 12, use the matrix method for Prim's algorithm, illustrated in Worked Example 7.19, to find the minimum spanning tree for the graphs given by adjacency tables in questions 4 to 6. State the order in which you added the edges. The starting vertex is given in each question.

10 **a** Question 4a, start at A

b Question 4b, start at A

11 **a** Question 5a, start at E

b Question 5b, start at E

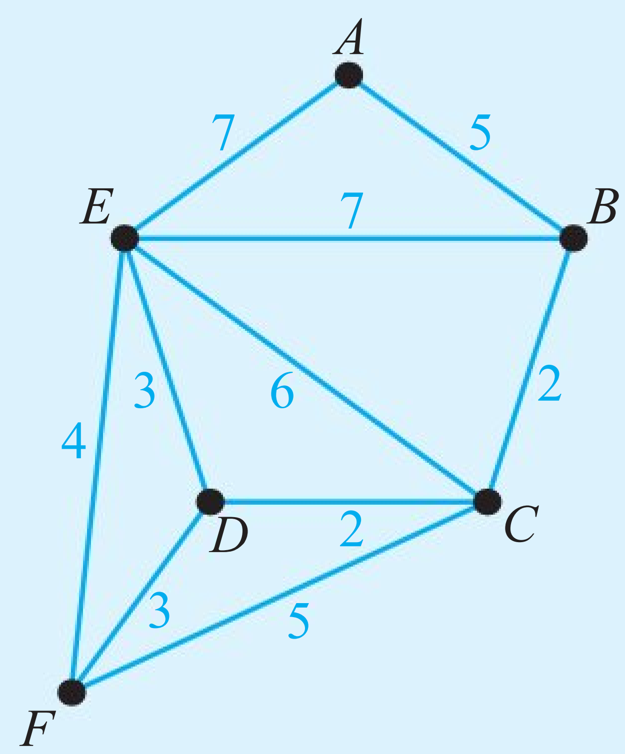
12 **a** Question 6a, start at C

b Question 6b, start at C

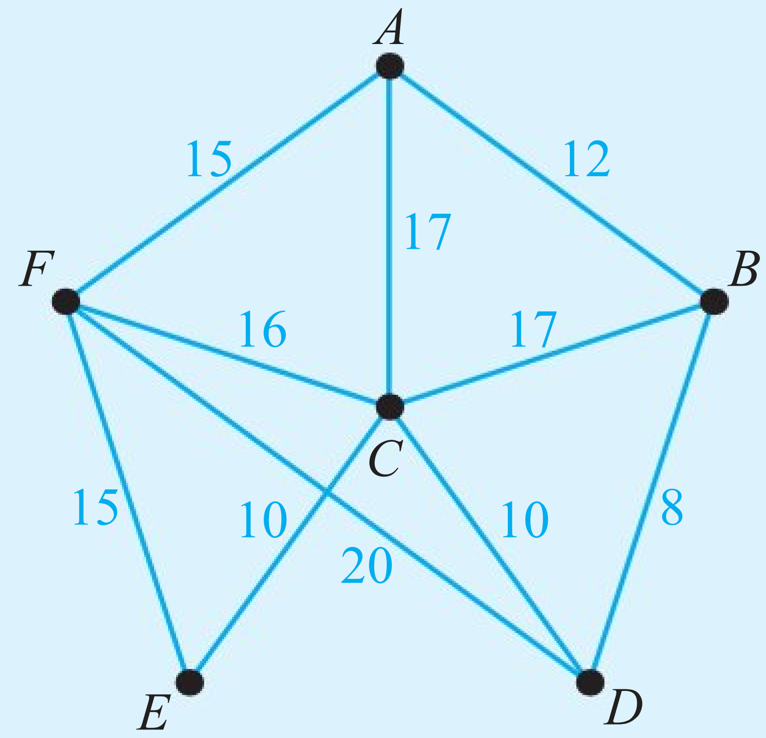
13 Six villages are to be connected by new roads. The graph shows potential roads and their lengths in kilometres.

a Use Prim's algorithm, starting at vertex A, to find the minimum spanning tree for the graph. Draw your tree.

b What is the minimum length of road required to ensure that each village is connected to all the others?



14 The graph alongside shows travel time, in minutes, for bus routes between different locations in a town. Some of the routes are to be closed, but it must still be possible to get from any location to any other.



a Use Kruskal's algorithm to determine which routes should remain open.

It is required that the route CF remains open.

b Describe how you could adapt Kruskal's algorithm to determine which other routes should remain open.

15 **a** Explain briefly the differences between Kruskal's and Prim's algorithms for finding the minimum spanning tree of a graph.

b Use Prim's algorithm to find the minimum spanning tree of the graph given by the weighted adjacency table shown. Perform the algorithm on the matrix and use A as the starting vertex. Draw your tree.

	A	**B**	**C**	**D**	**E**	**F**
A	–	12	8	14	9	10
B	12	–	10	7	11	15
C	8	10	–	12	14	11
D	14	7	12	–	13	10
E	9	11	14	13	–	12
F	10	15	11	10	12	–

The numbers in the table represent distances, in metres, between different workstations in an office. New wiring needs to be laid down to supply electricity to each workstation. An electricity supply is located next to workstation A.

c State the minimum length of wiring required.

d Explain the significance, in this context, of your minimum spanning tree being

i connected

ii a tree.

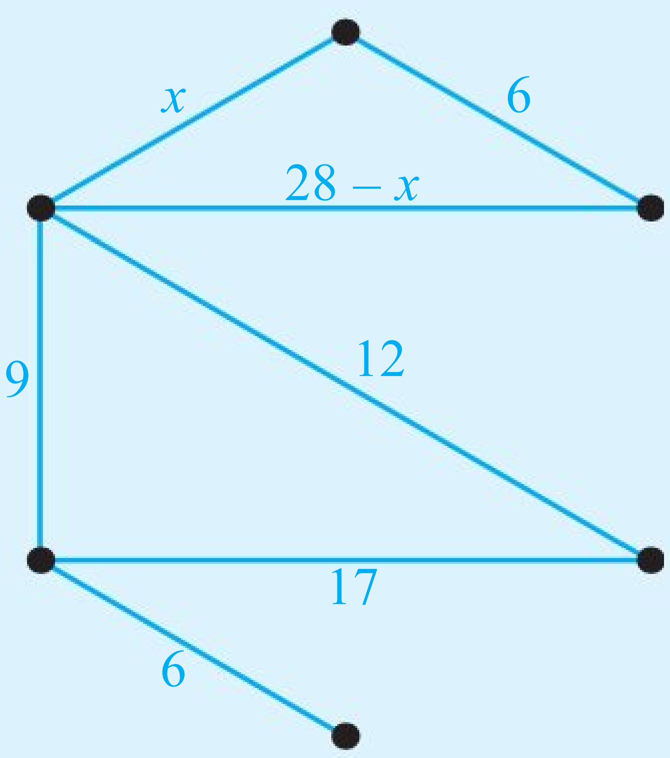
16 A simple connected graph has 5 vertices and 7 edges of lengths 10, 11, 13, 15, 18, 20 and 22. Find the

a minimum possible

b maximum possible

weight of the minimum spanning tree.

17 The minimum spanning tree of the graph shown in the diagram has weight less than 40. Find the range of possible values of x given that all the edges have positive weight.



7E The Chinese postman problem

In the final two sections of this chapter you will study two of the most famous problems of graph theory: The Chinese postman problem and the travelling salesman problem.

The Chinese postman problem is to find the shortest path around the graph which uses each edge at least once and returns to the starting point. Think about a postman who needs to walk along every street. This is sometimes also called the route inspection problem.

You know from Section 7B that a graph is Eulerian if all of its vertices have even degrees.

If the graph is Eulerian, any Eulerian cycle is a solution to the problem; it uses each edge exactly once so the length of the required path is the sum of the lengths of all the edges.

If the graph is not Eulerian, then some edges need to be repeated. This is equivalent to adding extra edges ('doubling up' some of the existing ones) to make the graph Eulerian. The method for doing this depends on the number of odd degree vertices in the graph.

Graphs with two odd vertices

KEY POINT 7.9

Chinese postman algorithm for a graph with two odd vertices:

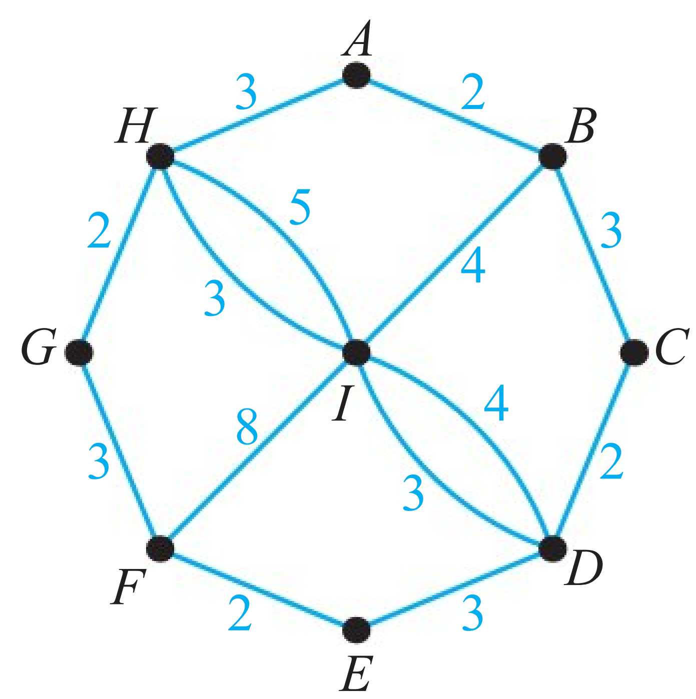
- Identify the two vertices of odd degree.
- Find the shortest path between those two vertices.
- The edges in this shortest path need to be used twice, and all the other edges are used only once.
- The length of the route is the total weight of all the edges plus the length of the edges that are used twice (the shortest path from the previous step).

WORKED EXAMPLE 7.20

a Find the length of the shortest Chinese postman route for the graph on the right.

b State which edges need to be used twice.

c Find one such route starting and finishing at A.



Check the degrees of all the vertices — **a** There are two odd vertices: B and F

Find the shortest path from E to F — Shortest path from B to F: B-I-F (length = 7)

The length is the weight of all the edges plus 7 — Total length = 42 + 7 = 49

The edges in the shortest path from E to F need to be repeated — **b** The edges BI and IF are used twice.

Find one possible route by inspection, remembering that BI and IF can be used twice. The easiest way to achieve this, if possible, is to include $BIFIB$ at some point — **c** A possible route: $ABIFIBCDIDEFGHIHA$

Graphs with four odd vertices

You need to split the four vertices into two pairs and find which particular pairing option produces the smallest possible extra distance.

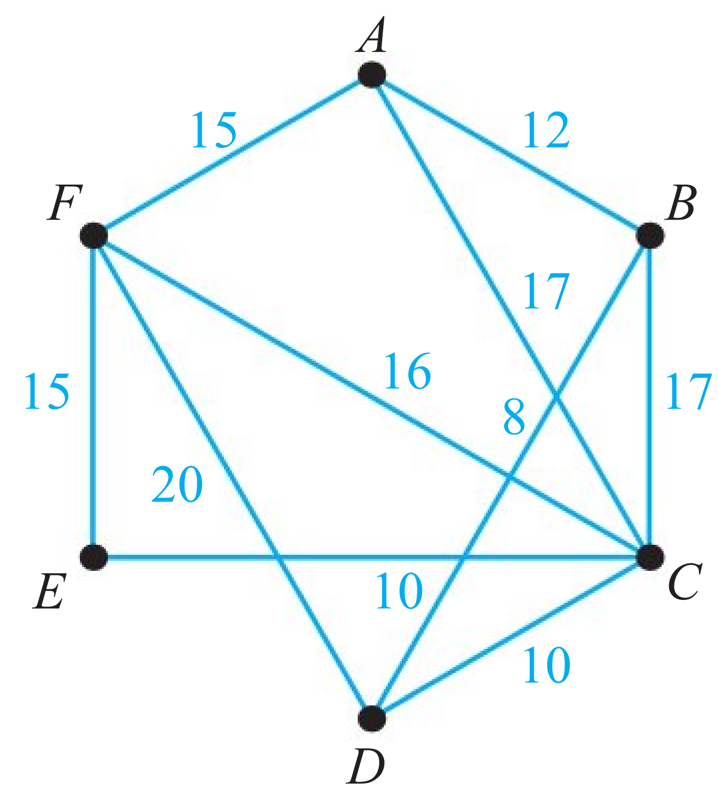
KEY POINT 7.10

Chinese postman algorithm for a graph with four odd vertices:

- Write down all possible pairings between the four odd vertices (there will be three of them).
- For each pairing possibility, find the total of the shortest distances between the two vertices in a pair.
- Select the pairing which gives the smallest total. The corresponding edges need to be repeated.

WORKED EXAMPLE 7.21

Solve Chinese postman problem for the graph below. State which edges need to be repeated, and the length of the shortest route.



Identify the vertices of odd degree

Vertices of odd degree:
$A(3),\ B(3),\ C(5),\ D(3)$

Consider all possible pairings. For each pairing, find the shortest path between the vertices

AB and CD: $12(AB)+10(CD)=22$
AC and BD: $17(AC)+8(BD)=25$
AD and BC: $10(ABD)+17(BC)=27$

Select the pairing with the smallest total

Edges AB and CD need to be repeated.

The sum of all the weights in the original graph is 140

The length of the shortest route
$140+22=162$

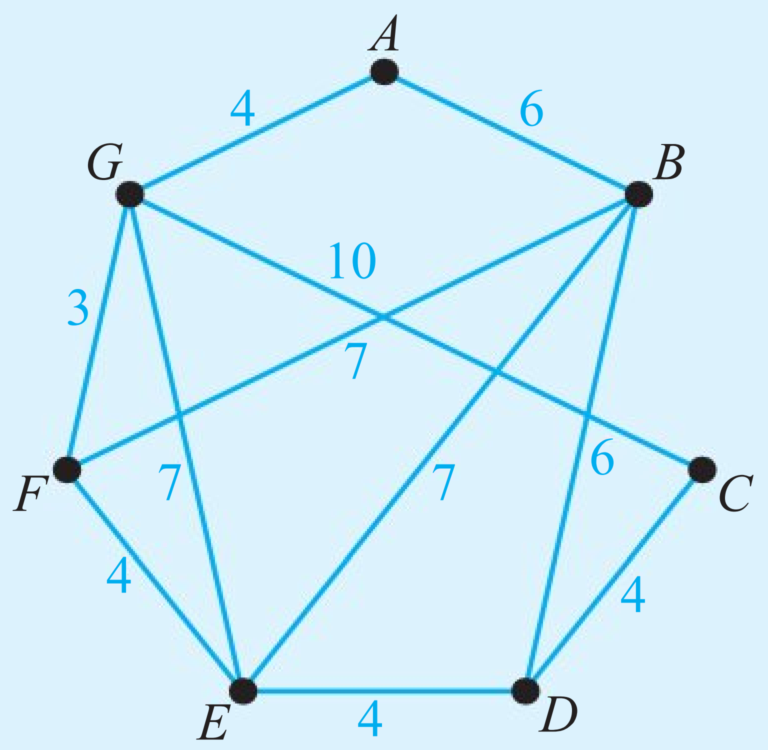
You are the Researcher

You may be wondering why we have not considered graphs with one or three odd vertices. Find out more about the handshaking lemma, the important result mentioned earlier that restricts the possible number of odd vertices in a graph.

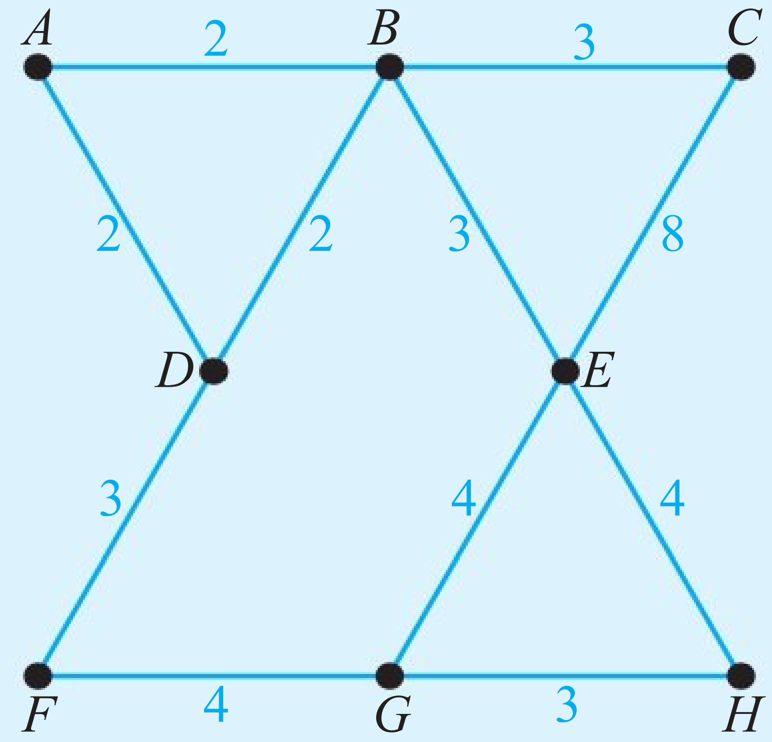
Exercise 7E

For questions 1 and 2, use the method demonstrated in Worked Example 7.20 to solve the Chinese postman problem for each graph. State which edges need to be repeated and the length of the shortest route.

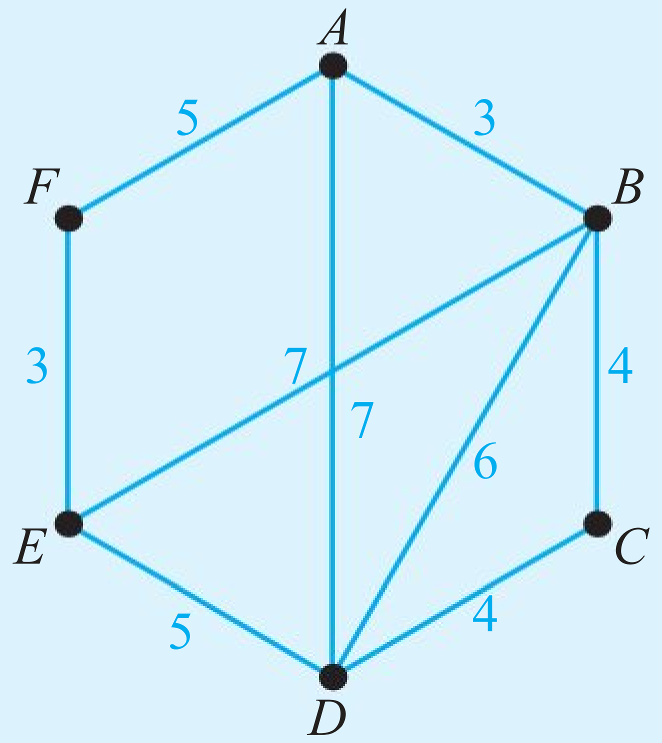
1 a



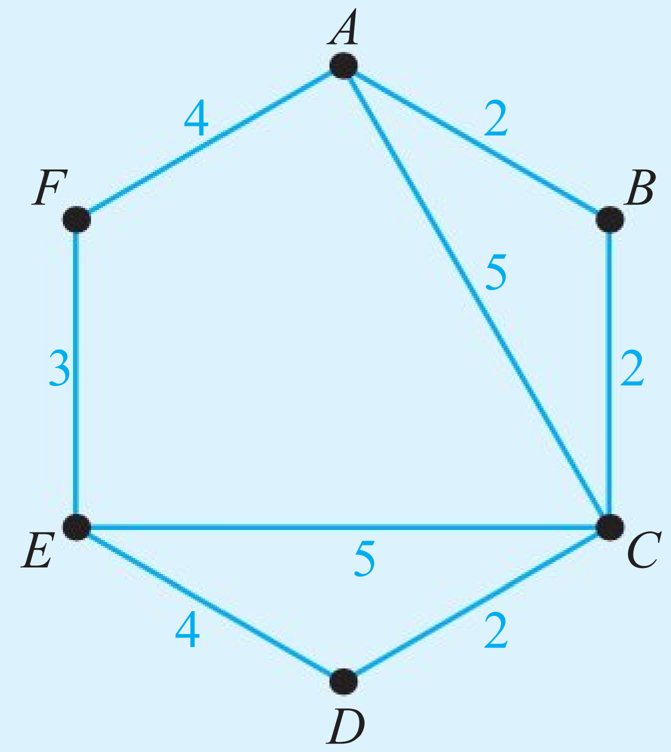
b



2 a

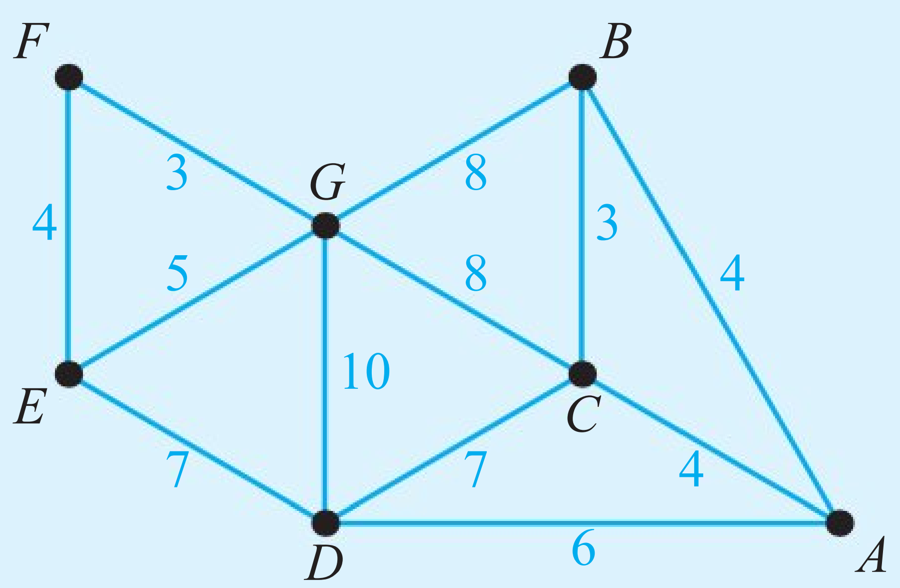


b

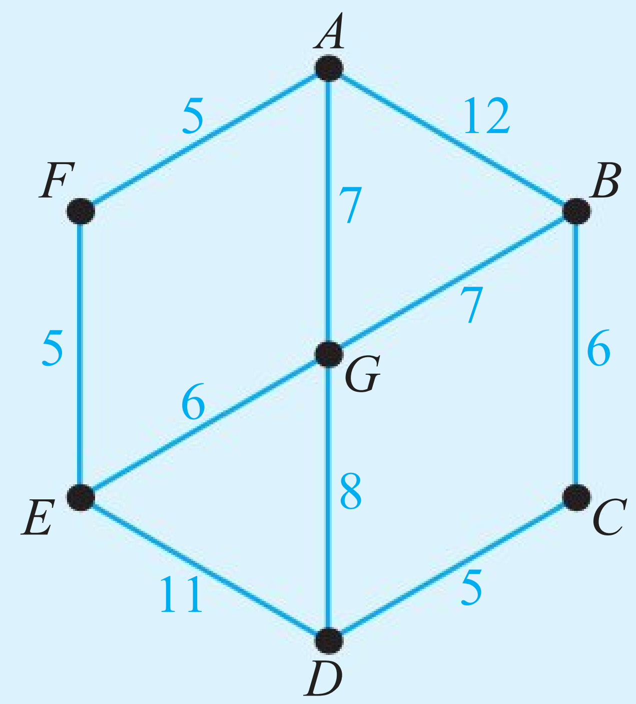


For questions 3 and 4, use the method demonstrated in Worked Example 7.21 to solve the Chinese postman problem for each graph. State which edges need to be repeated and the length of the shortest route.

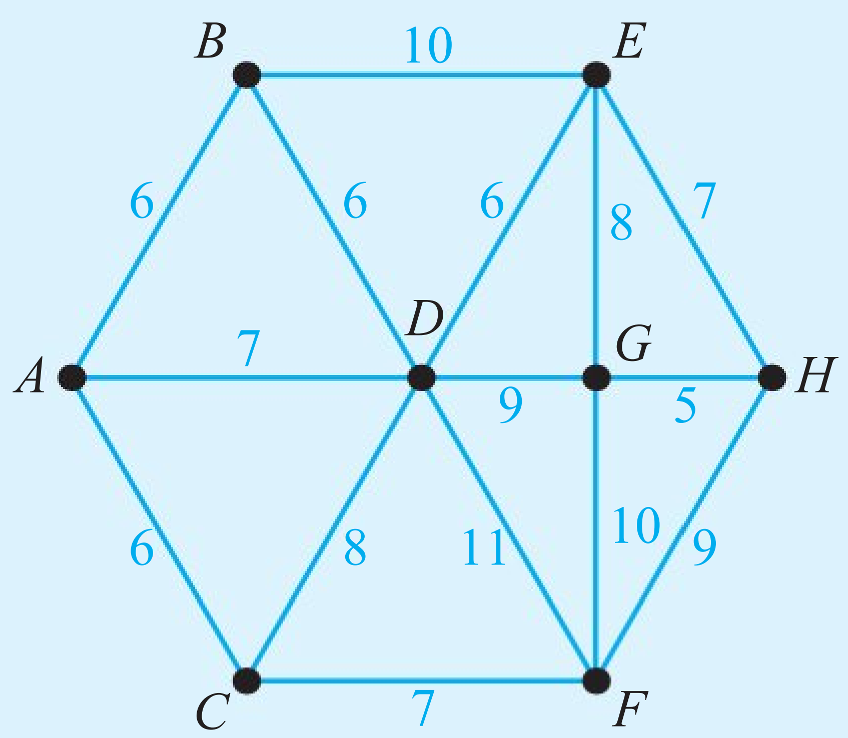
3 a



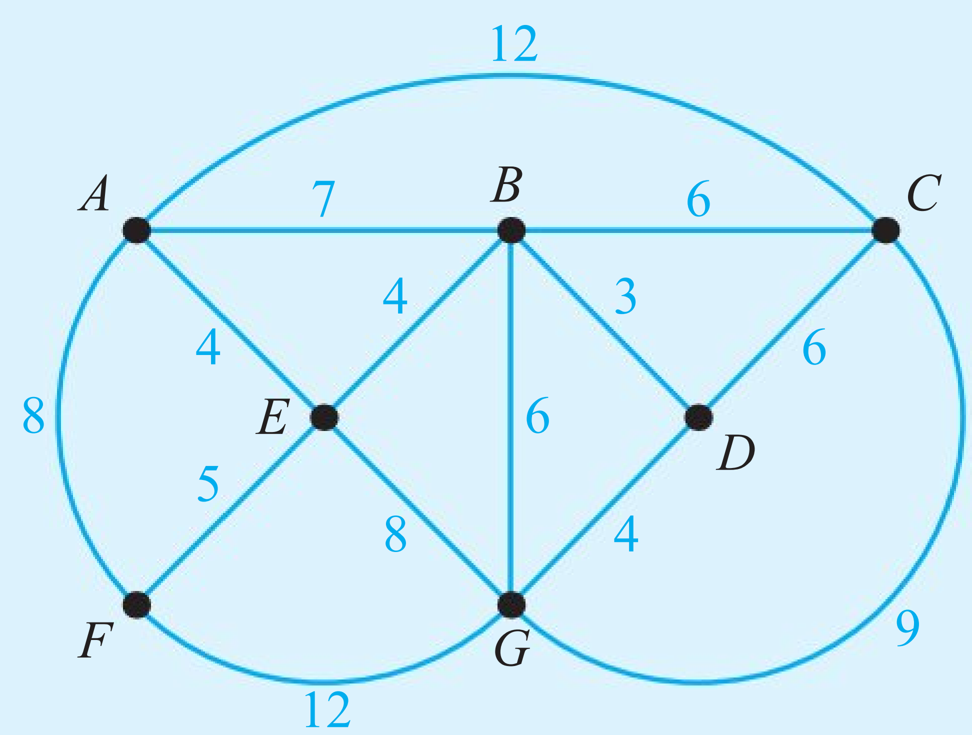
b



4 a



b



5 The graph shows a network of roads connecting seven villages.

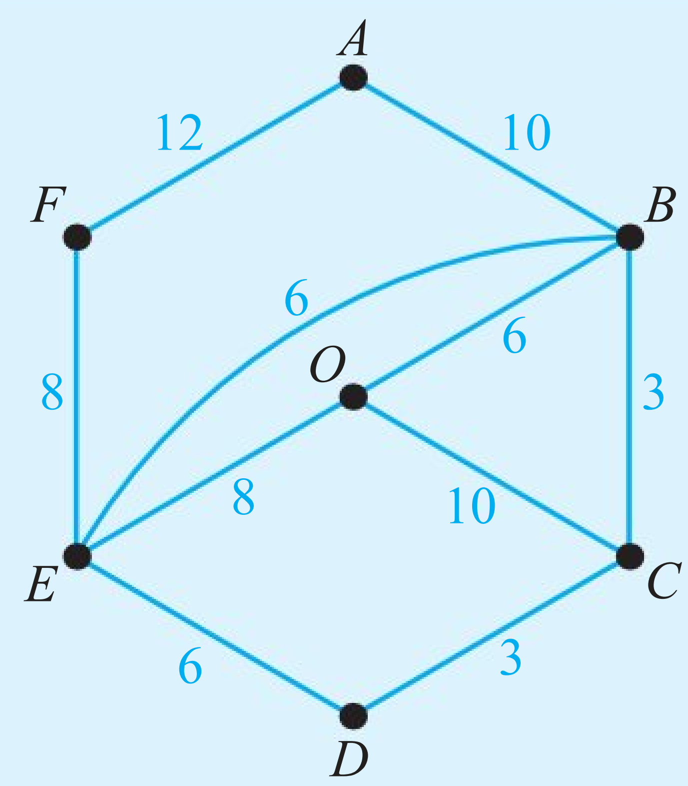
a Write down the two vertices of odd degree.

b State the length of the shortest path between the two vertices of odd degree.

A snow plough is located at village *C*. It needs to clear all the roads and return to *C*.

c Find the length of the shortest route the snowplough can take.

d Which roads does the snowplough need to use twice?



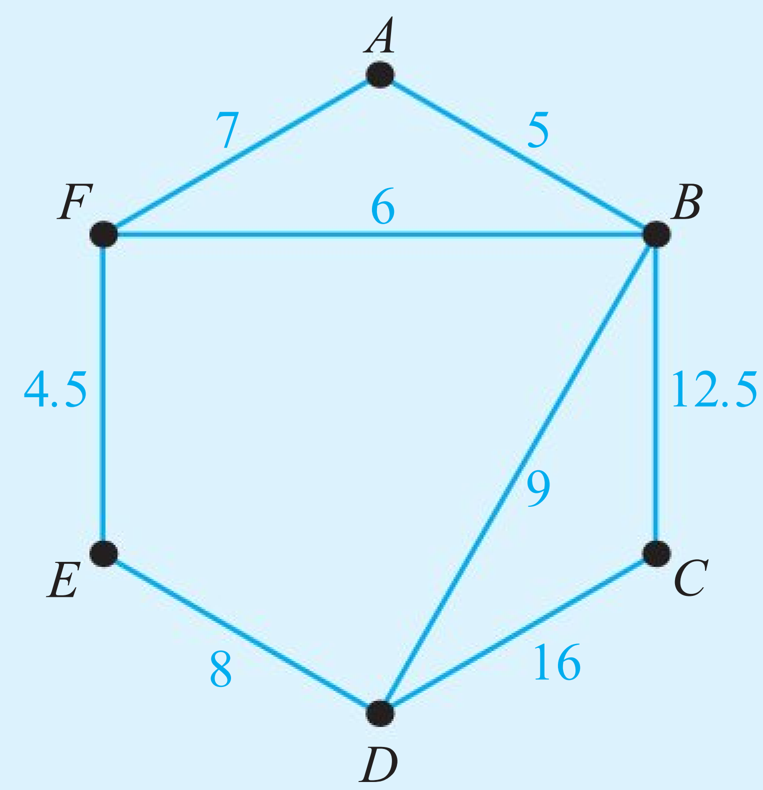
6 A post office is based at the point *A*. The graph shows the length of streets, in hundreds of metres, in a village. A postman needs to walk along each street at least once in order to deliver letters.

a The postman wishes to walk along each street *exactly once*, and can choose the start and end points (which do not need to be the same). State at which points he should start and end, and the length of the shortest possible route he can take.

b Explain why it is not possible for the postman to start at *A*, walk along each street exactly once, and return to *A*.

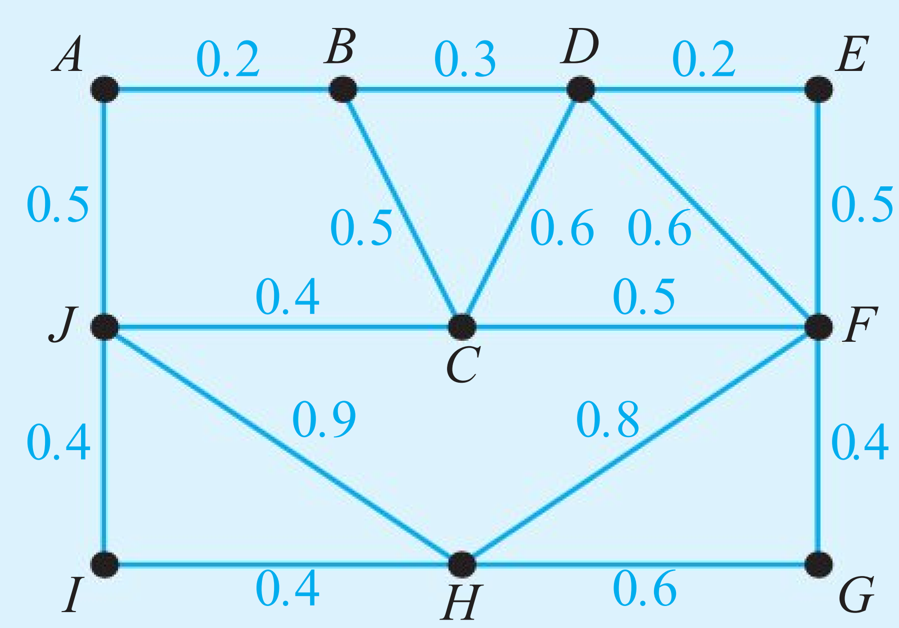
c Find the length of the shortest route which uses each street *at least once*, starting and finishing at the post office.

d In the route from part **c**, which streets are used more than once?

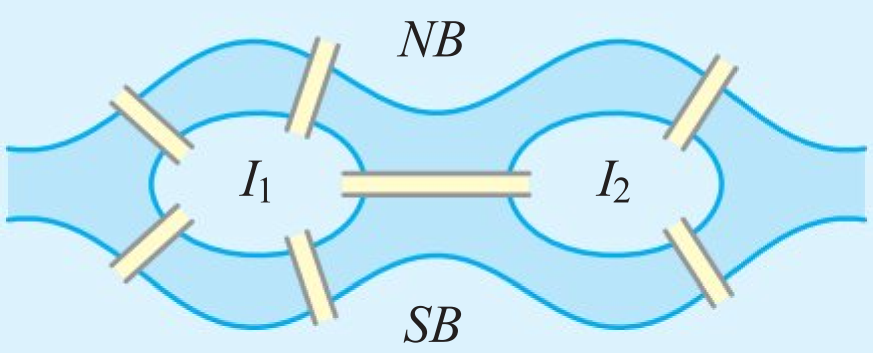


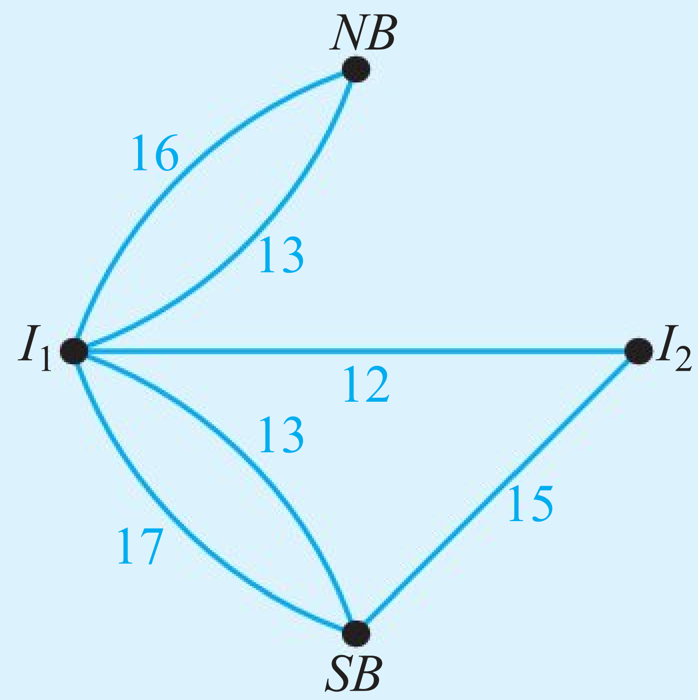
7 The graph represents the network of roads in a small town, with the weights of the edges representing the lengths of the roads, in kilometres. A salesman wants to visit all the houses in the town, so he must walk along each road at least once. He wants to start and finish at his shop, which is located at junction *C*.

Find the shortest possible route for the salesman and state its length.



8 The diagram shows the town of Königsberg with its two islands and seven bridges. The bridge between the north bank (NB) and the second island (I_2) is closed. In the corresponding graph, the weights of the edges represent the time, in minutes, to walk between different places.



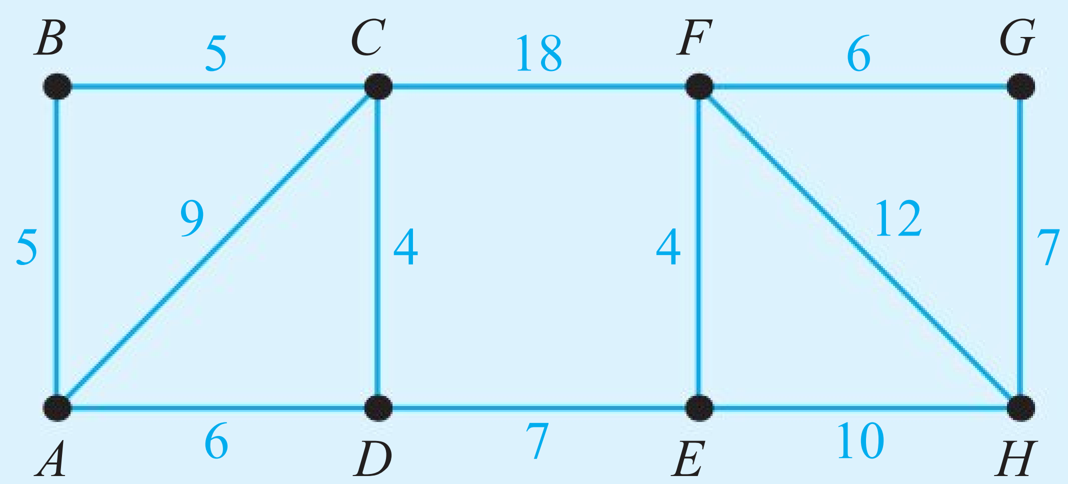


a A tour around the town starts at the north bank, crosses every bridge at least once (except for the closed one), and returns to the north bank. Find the shortest possible time taken by such a tour. You must make your method clear.

b The bridge between NB and I_2 reopens, and it takes 16 minutes to walk between the two places. Find the shortest possible time for a route which crosses every bridge at least once, starting and finishing at the north bank.

c State one factor you have ignored in your calculation.

9 The graph shows a network of paths in a park. The weights of the edges represent the number of minutes it takes to walk between the junctions.



A park ranger needs to inspect all the paths, starting and finishing at her hut, which is located at junction A. She wants to find the route that takes the shortest possible time.

a The ranger will need to walk along some of the paths twice. Determine which ones.

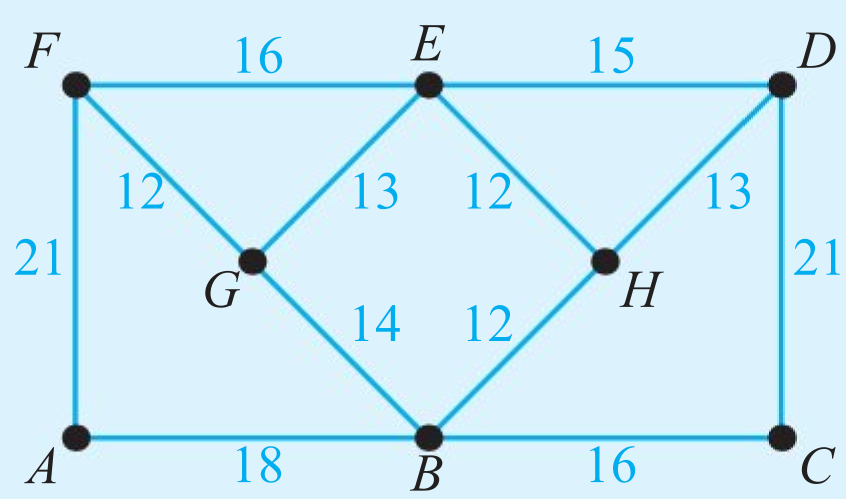
b Find the total time for the shortest possible route.

A tourist wants to walk along all paths in the park, starting at the hut at A, but finishing at the café which is located at H.

c Find the shortest time that the tourist can take.

d Find one possible route for the tourist.

10 The graph shows a network of paths between different sites within a school. The total length of all the paths is 183 m.



a Find the length of the shortest possible route around the school which starts and finishes at A and uses each path at least once.

b State which paths need to be used twice.

c Find the length of the shortest possible route which starts at D and finishes at F.

d A new path of length 16 m is built between sites G and H. Find the new length of the shortest route which starts and finishes at A and uses every path at least once.

11 Graph K is represented by the following weighted adjacency table.

	A	**B**	**C**	**D**	**E**	**F**
A	–	21	18	–	–	–
B	21	–	8	14	42	–
C	18	8	–	21	4	16
D	–	14	21	–	6	4
E	–	42	4	6	–	11
F	–	–	16	4	11	–

a Explain why K is not Eulerian.

b Find the shortest path between vertices C and F.

c Find the shortest route, starting and finishing at D, which uses each edge of graph K at least once. Make your method clear and state the length of your route.

d It is required to find the shortest route which uses each edge of graph K at least once, but does not need to return to the starting vertex.

i Find the least possible length of such a route.

ii What starting and finishing vertices should be used to achieve this shortest route?

7F The travelling salesman problem

The travelling salesman problem is to find the shortest route around a graph which visits each vertex at least once and returns to the starting point. Think about a salesman who wants to visit every town in a region.

You met complete graphs in Section A and Hamiltonian cycles in Section B.

The classical travelling salesman problem

For the simplest version of the travelling salesman problem, assume that the graph is complete. Assume, further, that the shortest route between two vertices is the direct route (we say that the graph satisfies the 'triangle inequality'). Such a graph has several Hamiltonian cycles (which visit each vertex *exactly* once), so the problem becomes finding the shortest one.

One way to solve the classical travelling salesman problem is to list all Hamiltonian cycles and find their weights. This is guaranteed to find the shortest cycle, but is inefficient, or may not be feasible, for large graphs. For example, a complete graph with five vertices has 12 different Hamiltonian cycles, but a complete graph with ten vertices has 181 440.

It turns out that this problem is a lot more difficult than it might at first appear. In fact, there is no known algorithm which guarantees finding the shortest Hamiltonian cycle. The best approach is to find **upper and lower bounds** for the problem. These are two numbers such that we can guarantee that the shortest possible Hamiltonian cycle has a length that is somewhere between them.

Nearest neighbour algorithm for determining an upper bound

Any Hamiltonian cycle will provide an upper bound (UB) – the shortest cycle cannot be any longer. The following algorithm finds a good upper bound by selecting the shortest edge at each stage and then returning to the starting point.

KEY POINT 7.11

Nearest neighbour algorithm for an upper bound:

- Pick a starting vertex.
- Go to the closest vertex not yet visited.
- Repeat until all the vertices have been used.
- Add the final edge to return to the starting vertex.

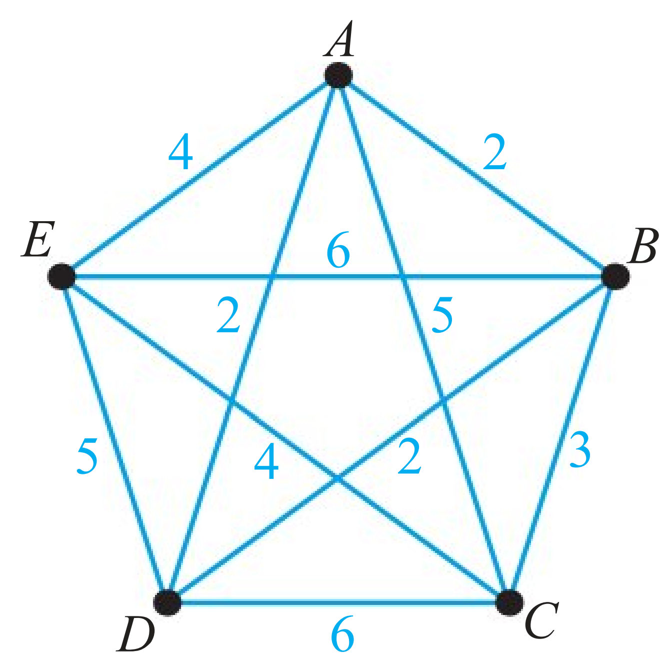
You want the upper bound to be as small as possible. Starting at a different vertex may give a better upper bound.

WORKED EXAMPLE 7.22

Consider this graph.

a Use the nearest neighbour algorithm starting at B to find an upper bound for the travelling salesman problem.

b What does your answer to part **a** tell you about the shortest path that visits each vertex and returns to the starting point?



The vertex closest to B is D

a Edges to include:

$BD(2)$

The vertex closest to D and different from B is A

$DA(2)$

$AE(4)$

Continue until you have visited all the vertices

$EC(3)$

Finally, return to B directly from C

$CB(3)$

The length of the cycle $BDAECB$ gives an upper bound

$\therefore\ UB = 15$

The shortest cycle can't be longer than the upper bound

b It has length $\leqslant 15$.

Deleted vertex algorithm for determining a lower bound

A lower bound (*LB*) is a length such that the shortest Hamiltonian cycle is definitely at least that long.

KEY POINT 7.12

Deleted vertex algorithm for a lower bound:

- Remove one vertex and all associated edges.
- Find the length of the minimum spanning tree for the remaining graph.
- Add the two shortest edges that were originally connected to the removed vertex.

You want the lower bound to be as large as possible. Removing a different vertex may give a better lower bound.

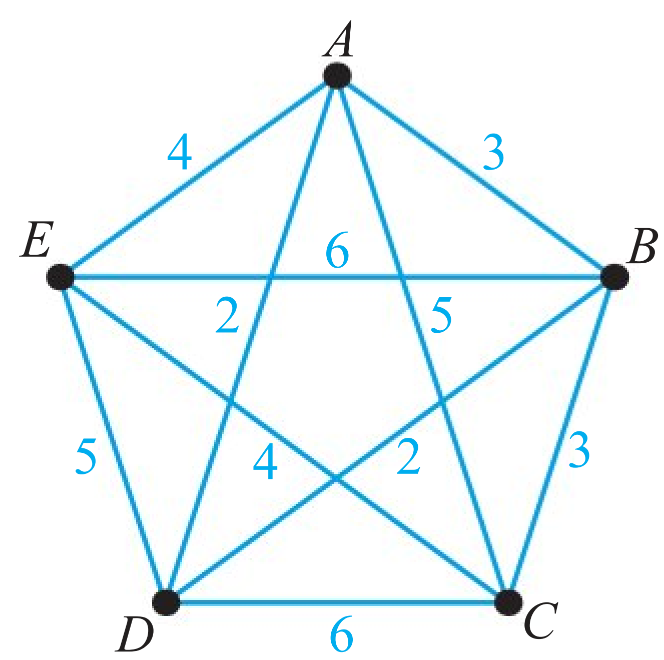
WORKED EXAMPLE 7.23

Consider this graph.

a By deleting vertex A, find a lower bound for the travelling salesman problem.

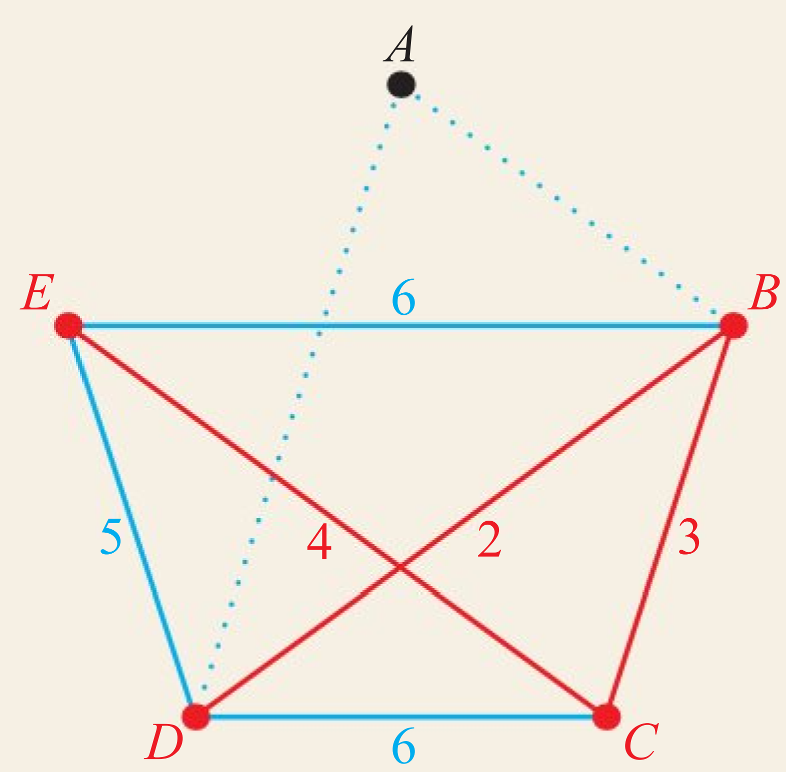
b By deleting vertex C, find a different lower bound.

c Which of the two lower bounds is better?



Use Kruskal's algorithm to find the minimum spanning tree for the graph with vertices B, C, D and E: add edges starting with the shortest

a Remove A:



Minimum spanning tree:
$BD\,(2)$, $BC\,(3)$, $CE\,(4)$

Tip

Notice that these five edges in part **a** do not form a cycle. This will often happen when finding a lower bound.

Add the two shortest edges from A

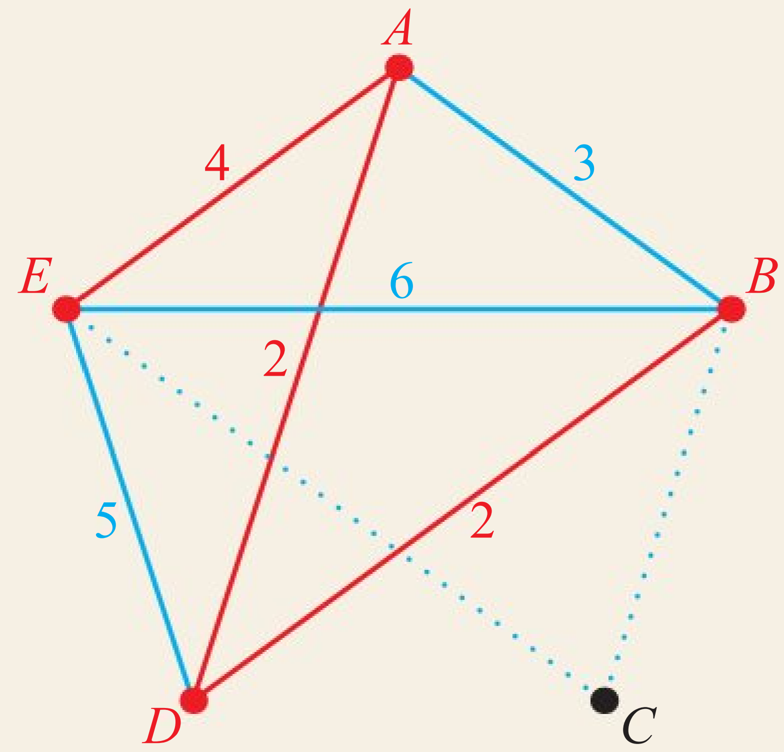
Shortest edges from A:
$AD(2)$, $AB(3)$

The total length gives a lower bound

$$LB = (2+3+4) + (2+3) = 14$$

Now find the minimum spanning tree for the graph with vertices A, B, D E

b Remove C:



Minimum spanning tree:
$AD(2)$, $BD(2)$, $AE(4)$

Add the two shortest edges from C

Shortest edges from C:
$BC(3)$, $BE(4)$

$$LB = (2+2+4) + (3+4) = 15$$

You want the lower bound to be as large as possible

c The lower bound of 15 is better.

In Worked Examples 7.22 and 7.23 you found that both upper and lower bounds are 15. This means that the lower bound is achievable, so the shortest Hamiltonian cycle has length 15. One possible such cycle is $BDAECB$, found in Worked Example 7.22.

KEY POINT 7.13

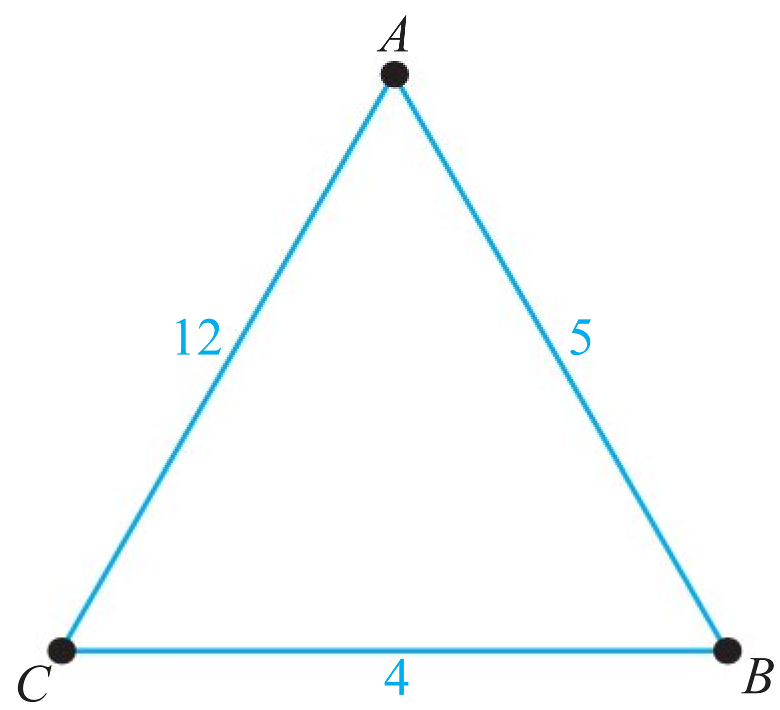
- The solution to the travelling salesman problem lies between the lower and the upper bounds.
- If $LB = UB$, then this is also the length of a shortest cycle.
- If you have found a cycle of the same length as the LB, then this is a shortest cycle.

You are the Researcher

There is no efficient algorithm for solving the travelling salesman problem. We say that the problem is 'NP-hard'. Investigate what this means and find examples of some other problems that fall into this category.

Practical problems

In a practical situation, the graph may not be complete, which means that it may not have a Hamiltonian cycle at all. Furthermore, a graph may not satisfy the triangle inequality, meaning that the shortest route between two vertices is not always the direct one.



In this graph, the shortest route between *A* and *C* is *ABC* (9), not *AC* (12)

In both of these cases, some vertices may need to be visited more than once. You can convert any practical problem into the classical problem by drawing a new graph which is complete and shows the shortest distance between any two vertices.

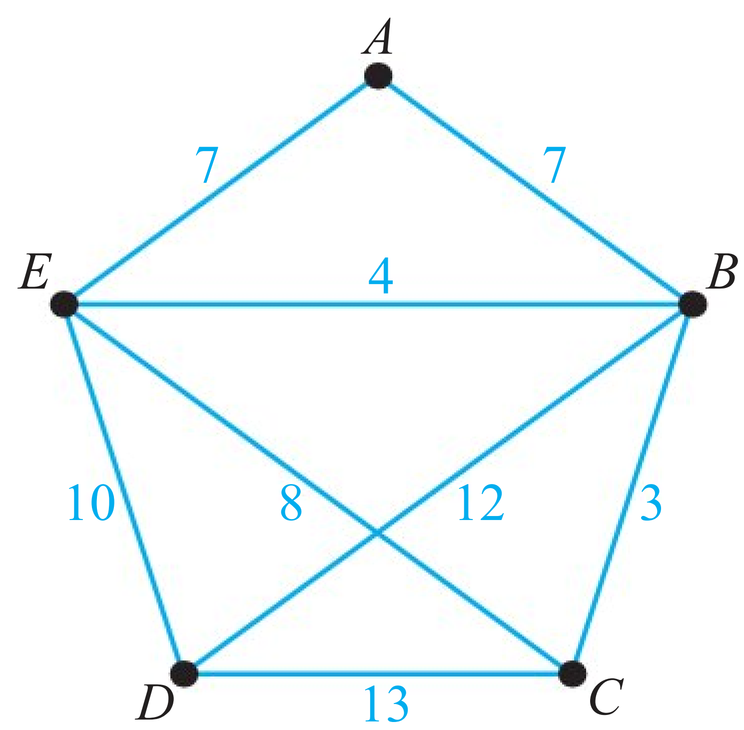
WORKED EXAMPLE 7.24

Look at this graph.

a Construct a table showing the shortest distances between the vertices.

b Use the nearest neighbour algorithm starting at *B* to find an upper bound for the travelling salesman problem on the new graph.

c Find the corresponding route in the original graph and state its length. Which vertices are visited twice?



The graph is not complete – it is missing edges *AC* and *AD*

a The shortest distance between *A* and *C* is 10 (*ABC*).
The shortest distance between *A* and *D* is 17 (*AED*).

The shortest distance between *C* and *E* is not the direct edge

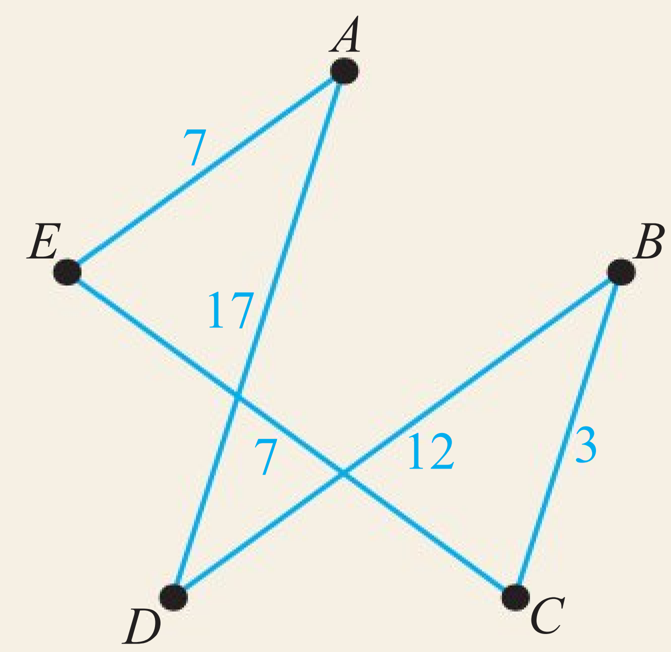
The shortest distance between *C* and *E* is 7 (*CBE*).

The other shortest distances are given by the edges

	A	B	C	D	E
A	–	7	10	17	7
B	7	–	3	12	4
C	10	3	–	13	7
D	17	12	13	–	10
E	7	4	7	10	–

You can perform the nearest neighbour algorithm on the table. Draw the graph as you go along to ensure that you have visited each vertex exactly once

b BC (3)
CE (7)
EA (7)
AD (17)
DB (12)
Hence, UB = 46



The above route uses the edge AD, which did not exist in the original graph, which needs to use A-E-D instead

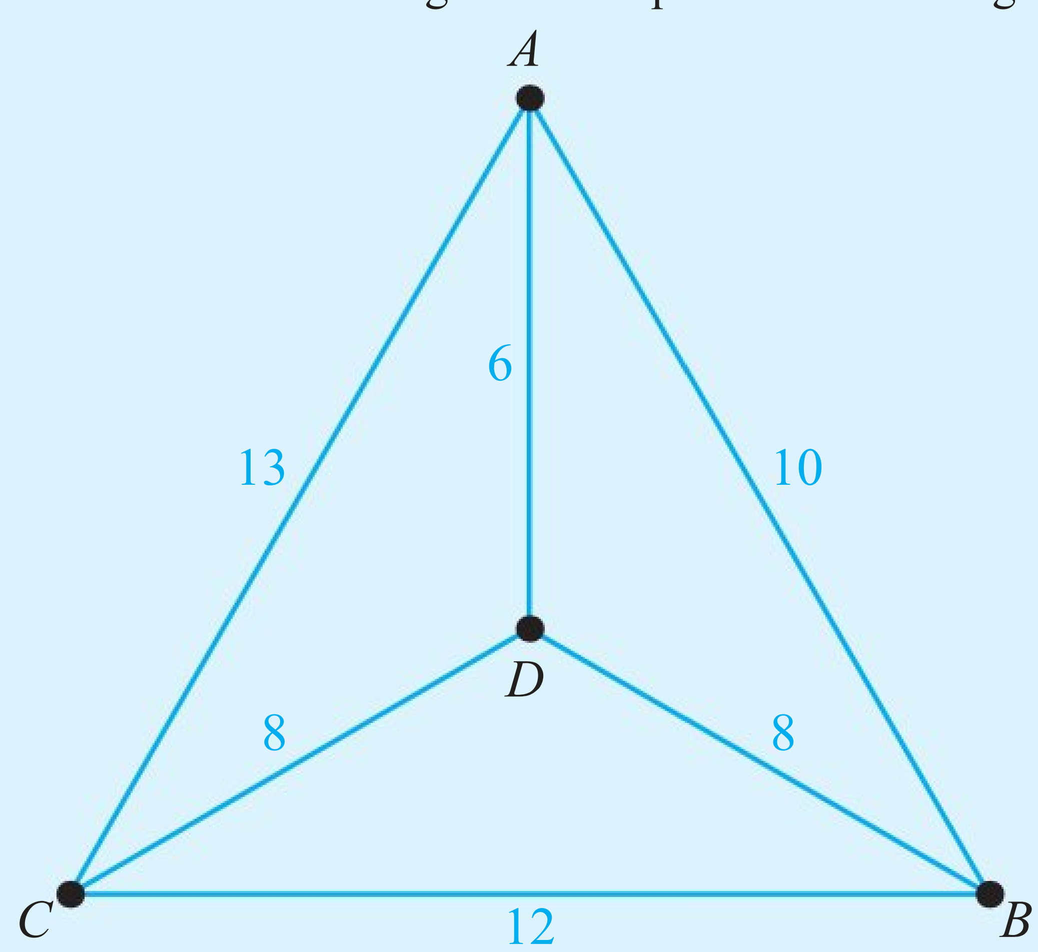
The distance of 7 between C and E was achieved by C-B-E

c The corresponding route, of length 46, in the original graph is:
B-C-B-E-A-E-D-B
Vertices B and E are visited twice.

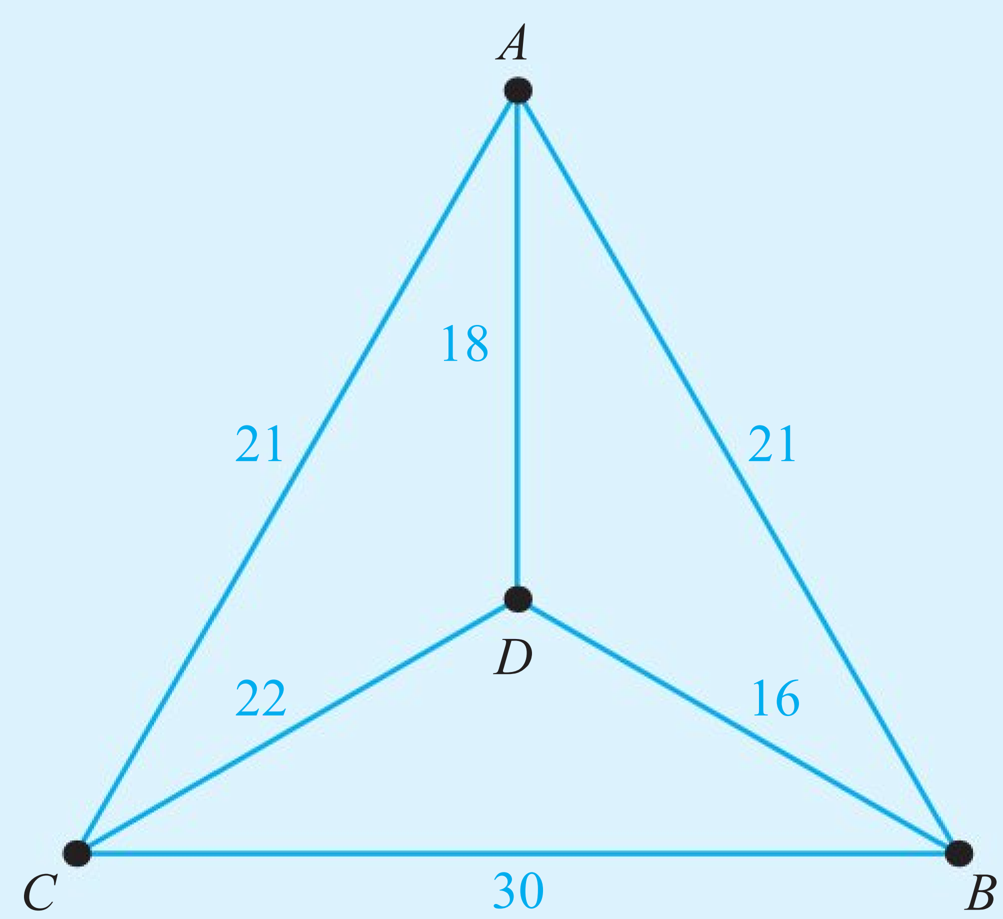
Exercise 7F

For questions 1 to 3, use the nearest neighbour algorithm starting at A, as illustrated in Worked Example 7.22, to find an upper bound for the travelling salesman problem for each graph.

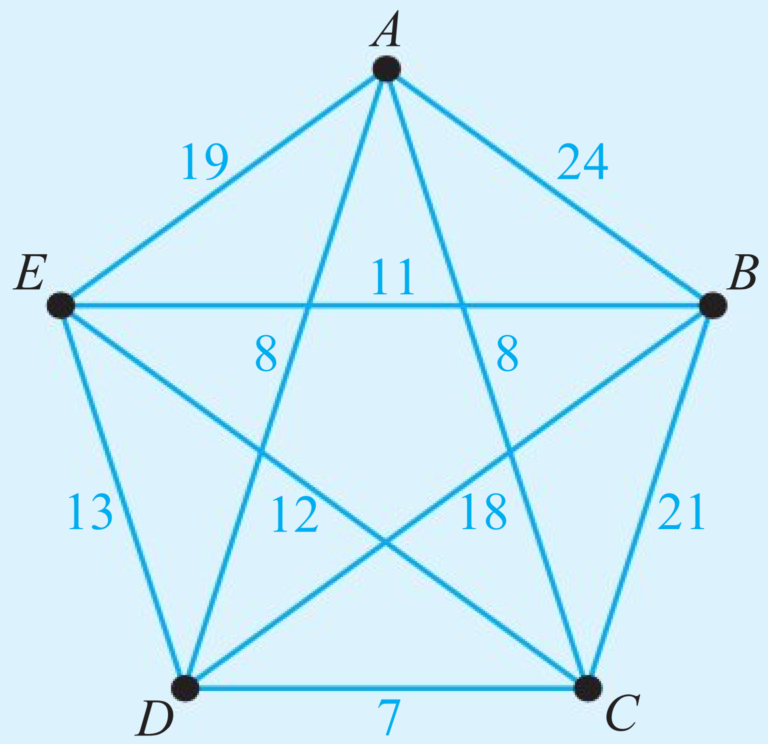
1 a



b



2 a



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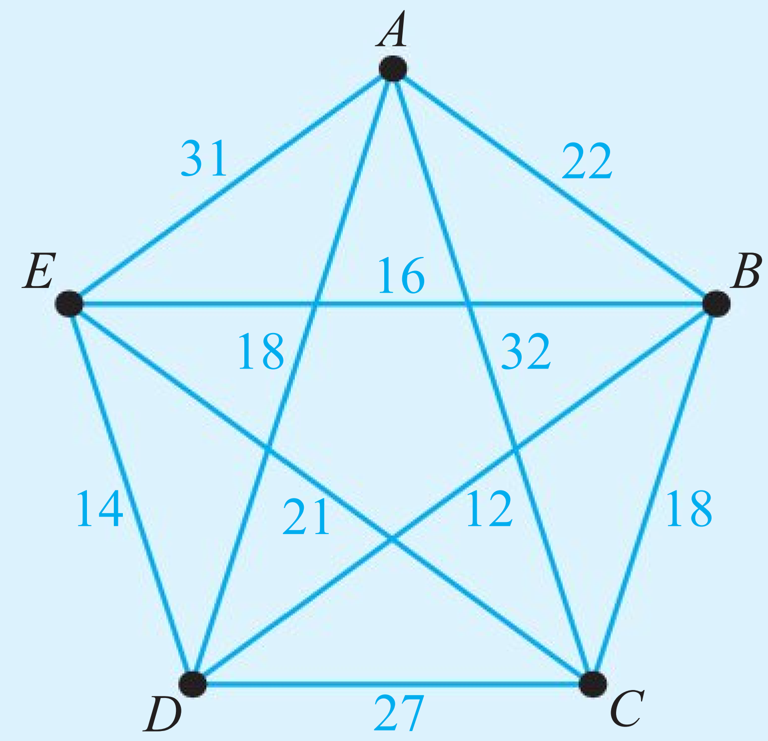


3 a

	A	B	C	D	E
A	–	32	28	26	19
B	32	–	14	21	26
C	28	14	–	18	18
D	26	21	18	–	22
E	19	26	18	22	–

	A	B	C	D	E
A	–	22	21	17	22
B	22	–	22	23	31
C	21	22	–	18	31
D	17	23	18	–	26
E	22	31	31	26	–

For questions 4 to 6, use the deleted vertex algorithm, as illustrated in Worked Example 7.23, to find a lower bound for the travelling salesman problem for each of the graphs from questions 1 to 3. The vertex to be removed is given in each question.

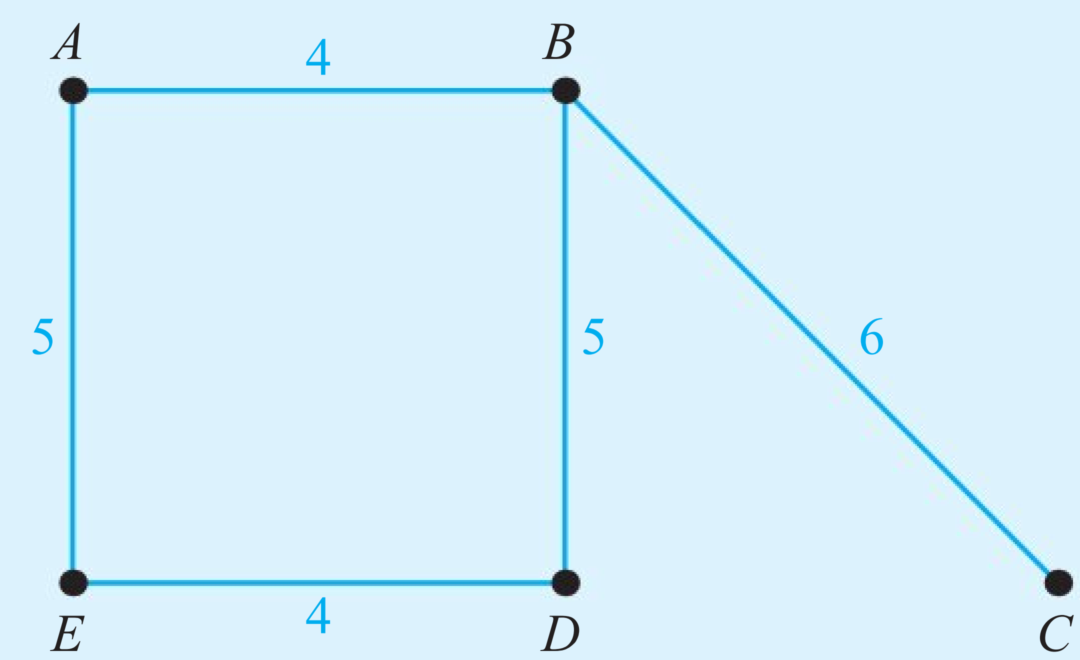
4 a Question 1a, vertex A
b Question 1b, vertex A

5 a Question 2a, vertex B
b Question 2b, vertex B

6 a Question 3a, vertex E
b Question 3b, vertex E

For questions 7 to 9, use the method demonstrated in Worked Example 7.24a to construct the table of shortest distances for each graph.

7 a



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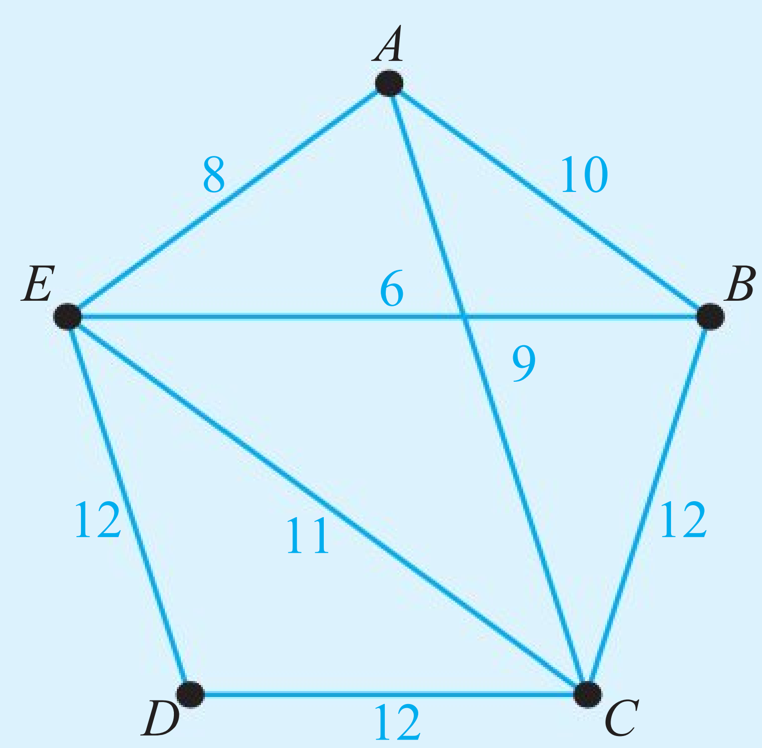
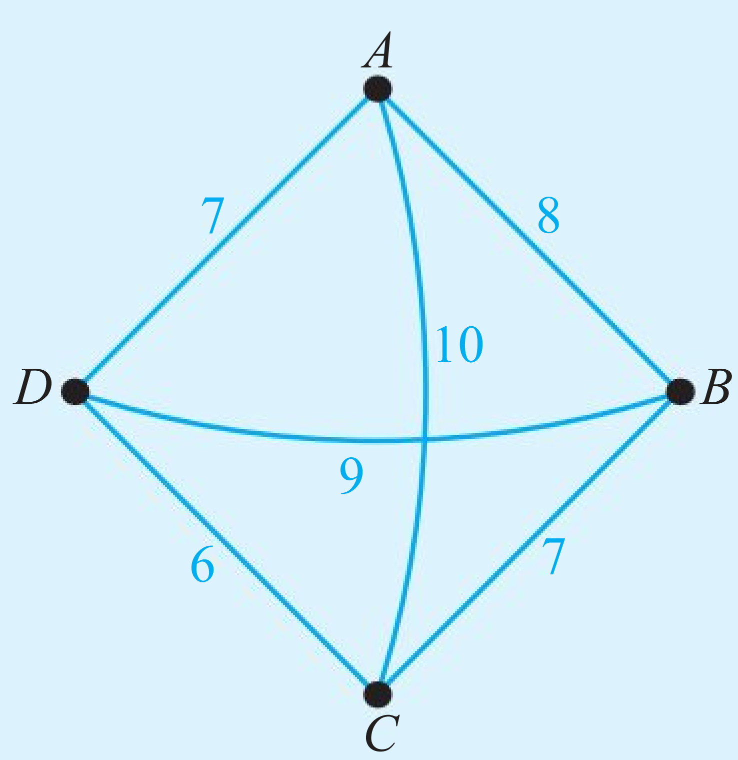


8 a



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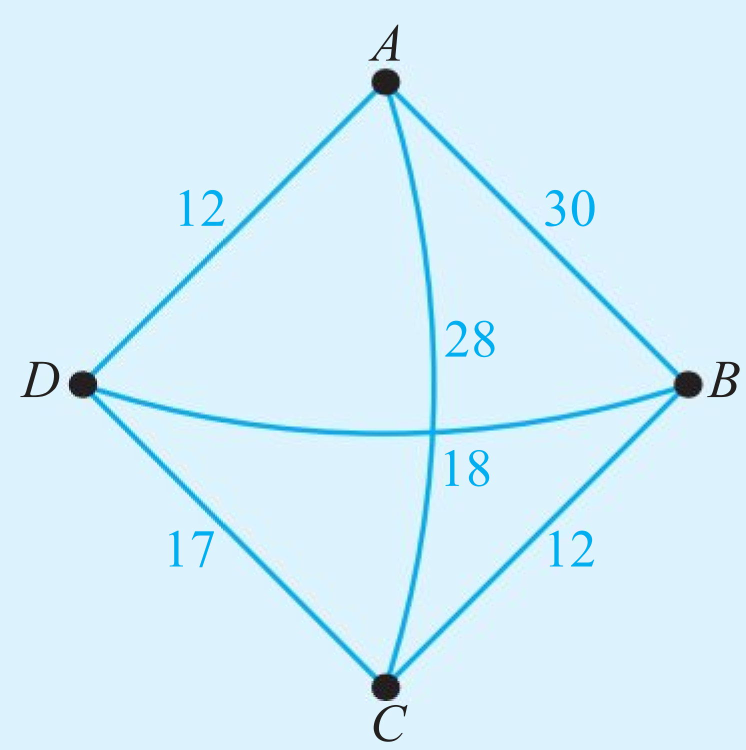
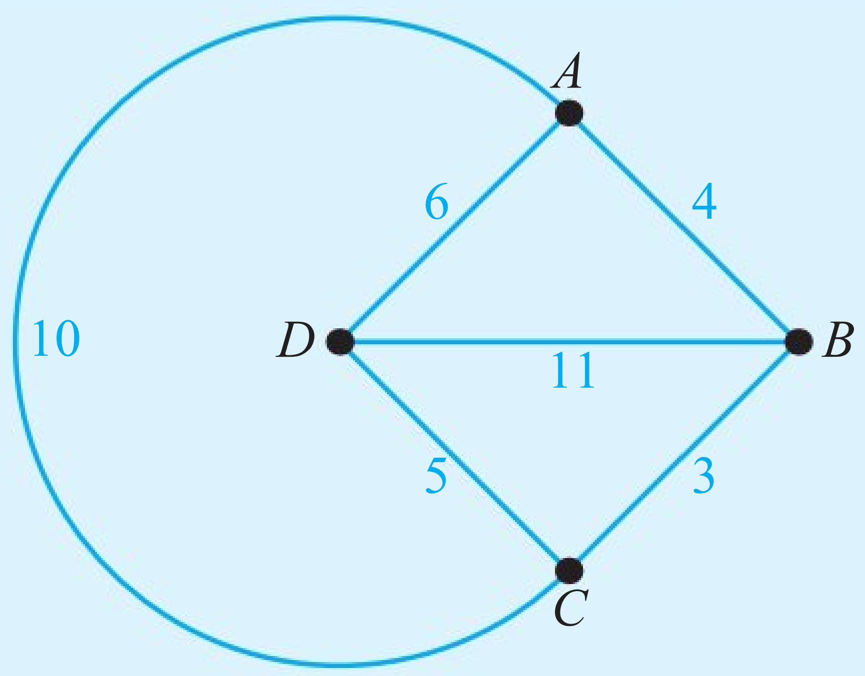
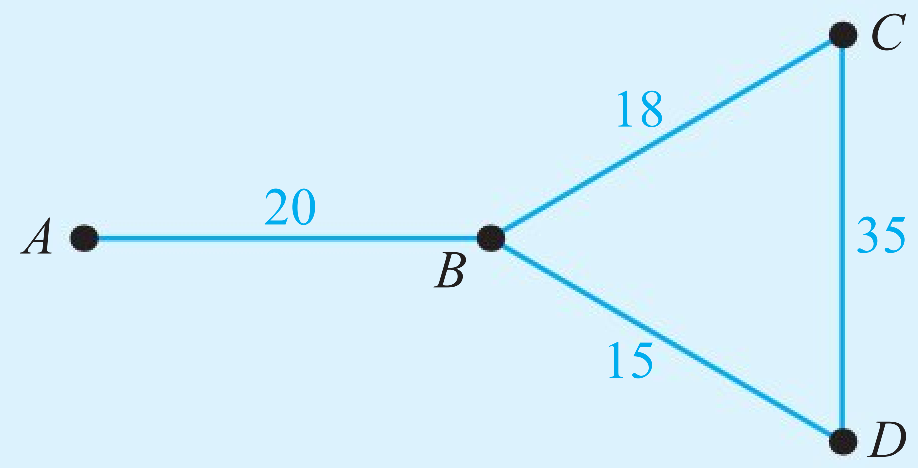


9 **a**



b



10 Graph G is given by the following cost adjacency matrix.

	A	**B**	**C**	**D**
A	–	16	12	8
B	16	–	18	8
C	12	18	–	9
D	8	8	9	–

a List all distinct Hamiltonian cycles starting and finishing at A. You only need to give each cycle in one direction.

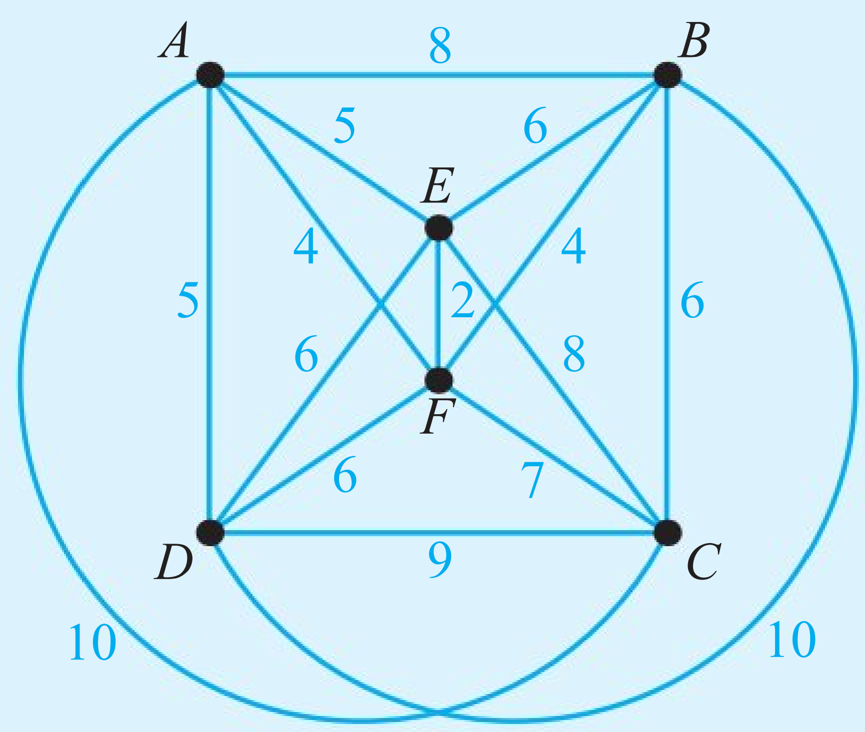
b Hence solve the travelling salesman problem for graph G.

11 The graph alongside shows six places of interest in a nature reserve and a network of paths connecting them. The distances are given in kilometres.

a Use the nearest neighbour algorithm starting at A to find an upper bound for the travelling salesman problem for this graph.

b Use the nearest neighbour algorithm starting at B to find a different upper bound.

c A visitor wishes to visit all six sites and return to the starting point. What can you conclude about the length of the shortest possible route they can use?



12 The table shows the times, in minutes, required to walk between five different stores on a high street. A shopper wants to visit all five stores and return to the starting point.

	A	**B**	**C**	**D**	**E**
A	–	4	3	7	6
B	4	–	3	9	7
C	3	3	–	8	5
D	7	9	8	–	9
E	6	7	5	9	–

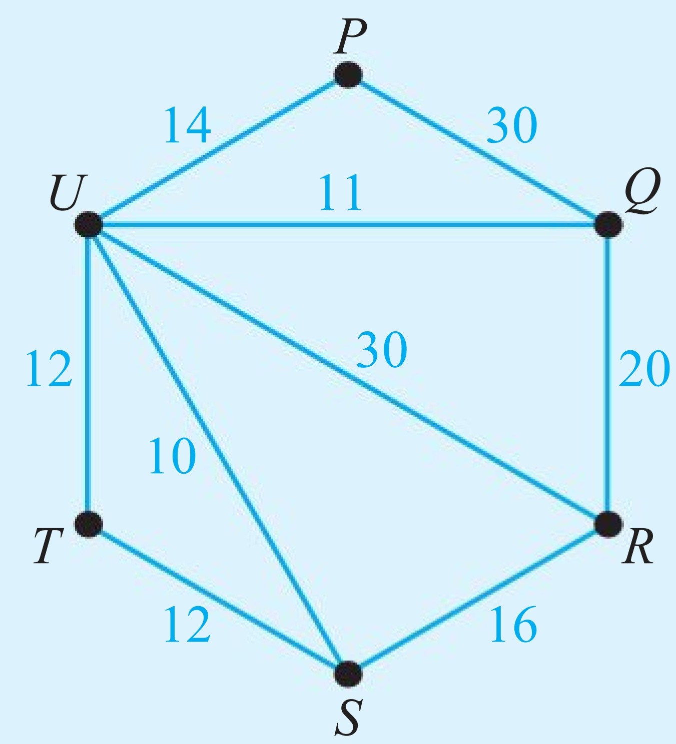
a Explain how you know that the fastest possible route takes at most 29 minutes.

b Use the nearest neighbour algorithm, starting with each vertex in turn, to find an improved upper bound.

c By deleting vertex D from the graph, find a lower bound.

d What can you conclude about the shortest possible time required to visit all five stores and return to the starting point?

13 The diagram shows a network of roads connecting six towns, with distances given in kilometres. A delivery company is based at Q. They want to find the shortest possible route that visits every town and returns to Q.



a Explain why the shortest route between P and R is 40 km long.

b Find the shortest route between P and Q and state its length.

c Copy and complete this table showing the shortest distance between each pair of towns.

	P	**Q**	**R**	**S**	**T**	**U**
P	–	–	40	–	–	14
Q	–	–	20	21	23	11
R	40	20	–	–	–	26
S	–	21	–	–	–	10
T	–	23	–	–	–	12
U	14	11	26	10	12	–

d Use the nearest neighbour algorithm starting at Q to find an upper bound for the shortest possible route.

e By deleting vertex Q, find a lower bound for the length of the shortest possible route.

14 Consider the graph given by the following cost adjacency matrix.

	A	**B**	**C**	**D**	**E**	**F**
A	–	5	10	11	8	7
B	5	–	6	7	8	7
C	10	6	–	7	10	12
D	11	7	7	–	9	10
E	8	8	10	9	–	6
F	7	7	12	10	6	–

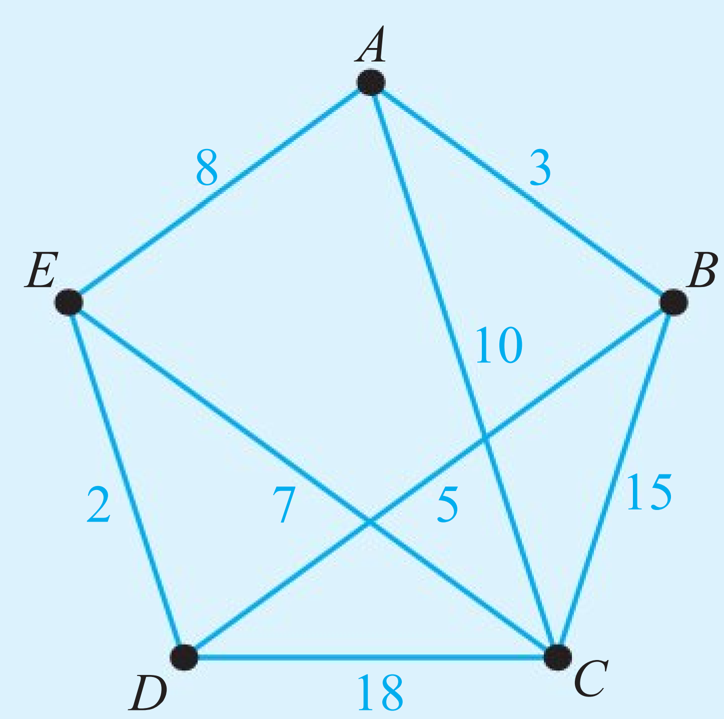
a Use nearest neighbour algorithm starting at A to find an upper bound for the travelling salesman problem.

b By deleting vertex B and using Kruskal's algorithm to find the minimum spanning tree for the resulting graph, find a lower bound for the travelling salesman problem.

c Explain how you know that you have found a solution to the travelling salesman problem, and write down the weight of the optimal route.

d Write down one possible Hamiltonian cycle of shortest length.

15 The graph shows the times, in minutes, to walk between five houses. Some of the houses are not connected by direct paths.



a State the quickest possible routes, and their lengths, between the following pairs of houses.

i B and E

ii B and C

iii C and D

b Draw a complete graph showing the times of the quickest routes between the houses.

A student lives in house C and wants to deliver party invitations to her friends at the other four houses, starting and ending at her own house.

c Use the nearest neighbour algorithm starting at C to find an upper bound for the minimum time she can take.

d By deleting vertex C, find a lower bound.

e By considering the options for the outward and return path connecting to C, explain how you know that this lower bound cannot be achieved for a path beginning and ending at C.

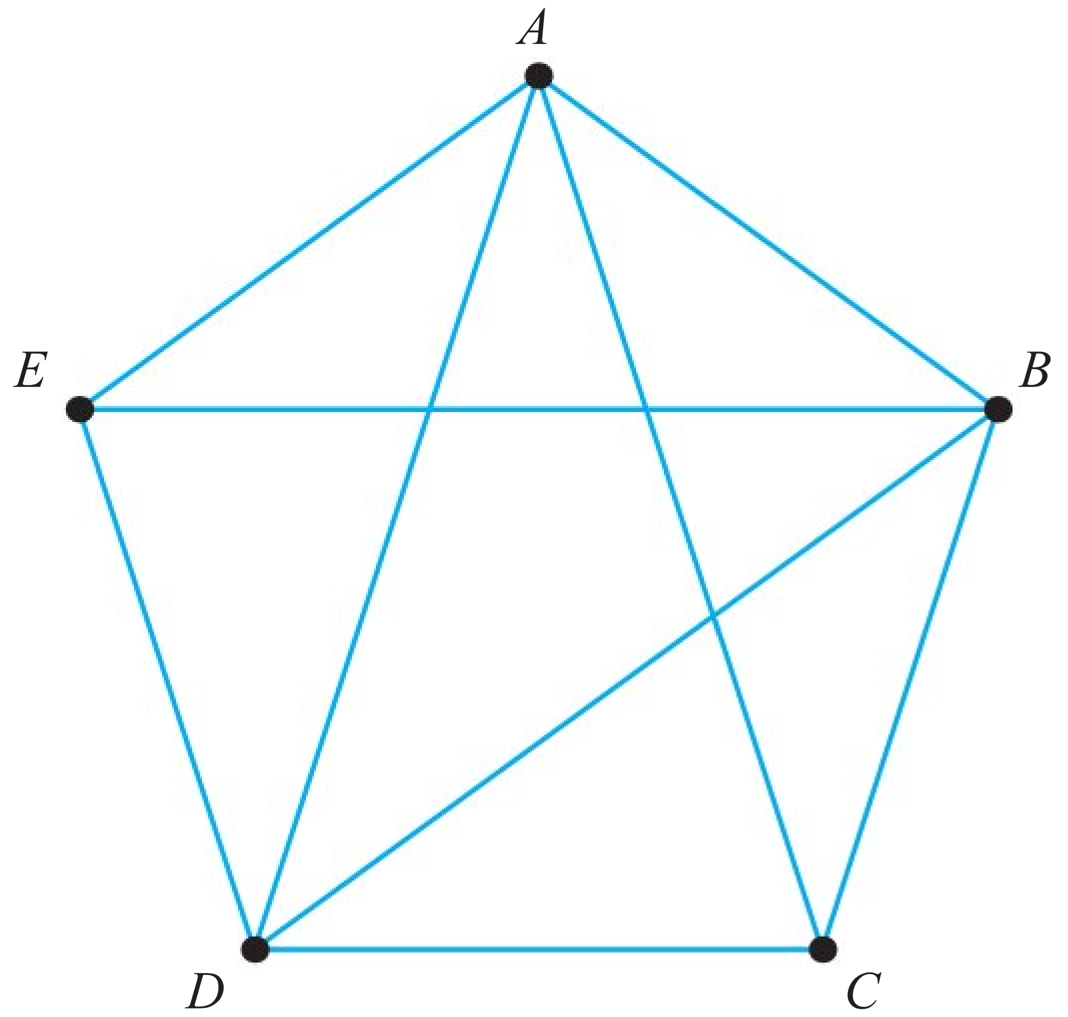
f Find the quickest possible route for the original graph and state its length.

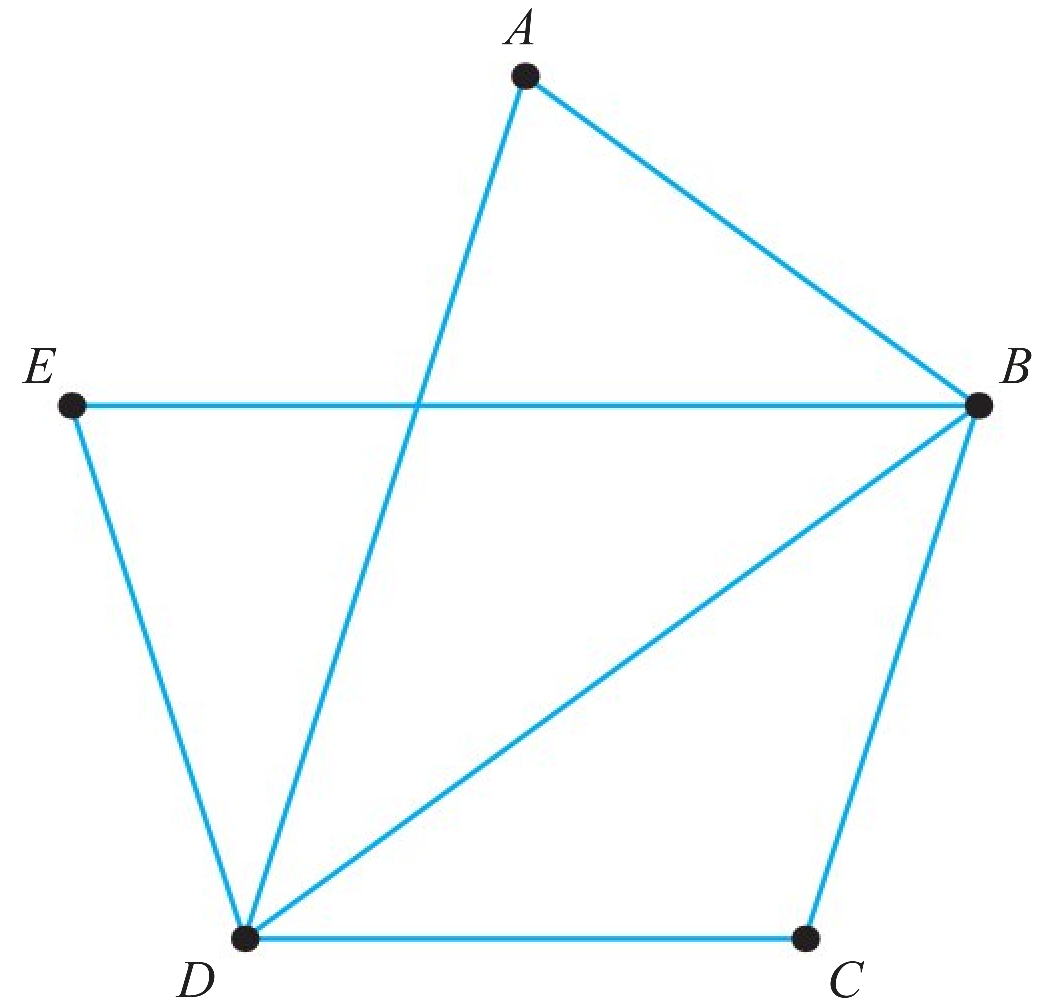
Checklist

- You need to be able to distinguish between undirected and directed, and unweighted and weighted graphs, and be able to classify graphs as simple, connected (or strongly connected), complete or trees.
- You need to be able construct an adjacency matrix for a graph and use it to
  - find the degree of each vertex
  - find the number of walks between two vertices.
- You need to be able to construct a transition matrix for a graph and use it to
  - find the probability of being at a certain vertex after a given number of steps
  - determine long-term probabilities in a random walk
  - use the PageRank algorithm to rank web pages in order of importance.
- You need to be able to identify various types of walks in a graph. In particular:
  - An Eulerian trail uses each edge exactly once, and an Eulerian circuit then returns to the starting point.
  - A graph is Eulerian if all vertices have even degree.
  - A Hamiltonian graph has a Hamiltonian cycle, which visits each vertex exactly once and returns to the starting point.
- You need to be able to find a minimum spanning tree of a weighted graph. This is a subgraph of smallest possible weight which is also a tree.
  - Kruskal's algorithm for finding a minimum spanning tree involves adding edges in order of weight and skipping edges which would form a cycle.
  - Prim's algorithm for finding a minimum spanning tree involves choosing a starting vertex and then adding vertices one at a time so that we always join an unconnected vertex to a connected vertex using the edge of smallest possible weight.
  - When using Prim's algorithm in a table, the labels for each vertex are recorded in a box, with permanent labels in the top row.
- You need to understand the Chinese postman problem for finding the shortest route around a graph which uses each edge at least once and returns to the starting vertex.
  - If the graph is Eulerian, the solution is any Eulerian path.
  - If the graph is not Eulerian, we need to find the shortest path between pairs of vertices odd degree and repeat those edges.
- You need to understand the travelling salesman problem for finding the Hamiltonian cycle of least weight in a complete graph (i.e. the shortest route which visits every vertex and returns to the starting point).
  - If the graph is not complete, you need to complete it by creating a table of shortest distances.
  - The exact solution can only be found by checking all Hamiltonian cycles, which is inefficient.
  - You need to know methods for finding upper and lower bounds (the nearest neighbour algorithm and the deleted vertex algorithm).
  - The required least weight of a Hamiltonian cycle is somewhere between the lower bound and the upper bound, $LB \leqslant L \leqslant UB$. You need to make the lower bound as large as possible and the upper bound as small as possible.

Mixed Practice

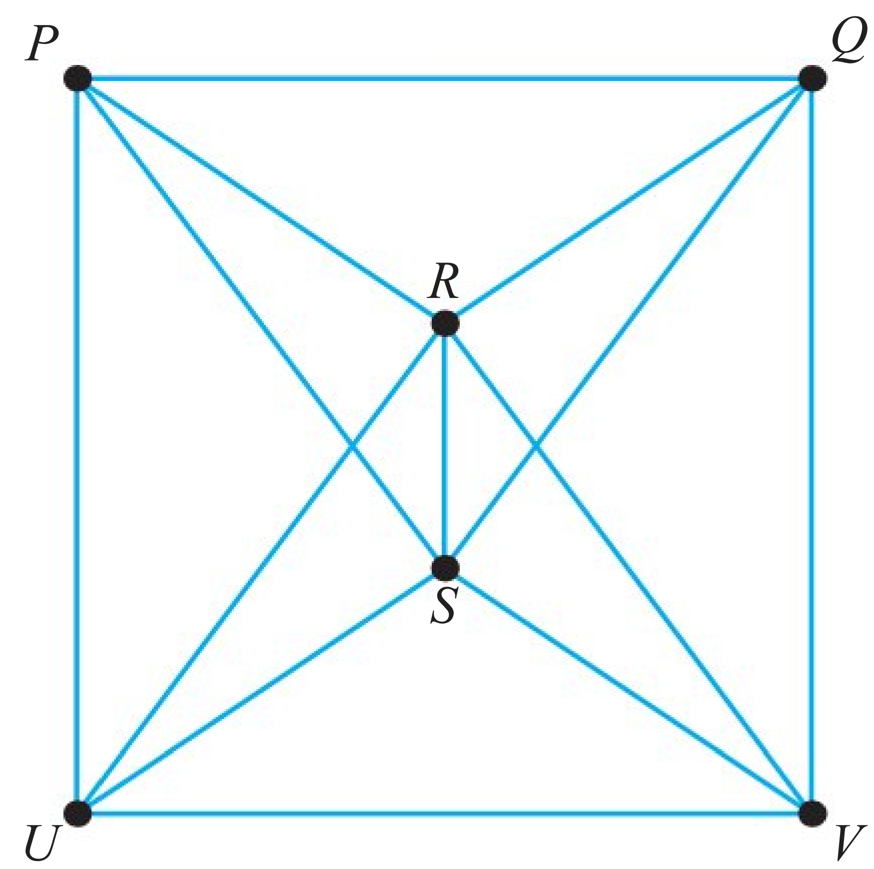
1 The two graphs below show a network of train lines connecting five cities. Sachin wants to use each line exactly once, starting and finishing at the same point.



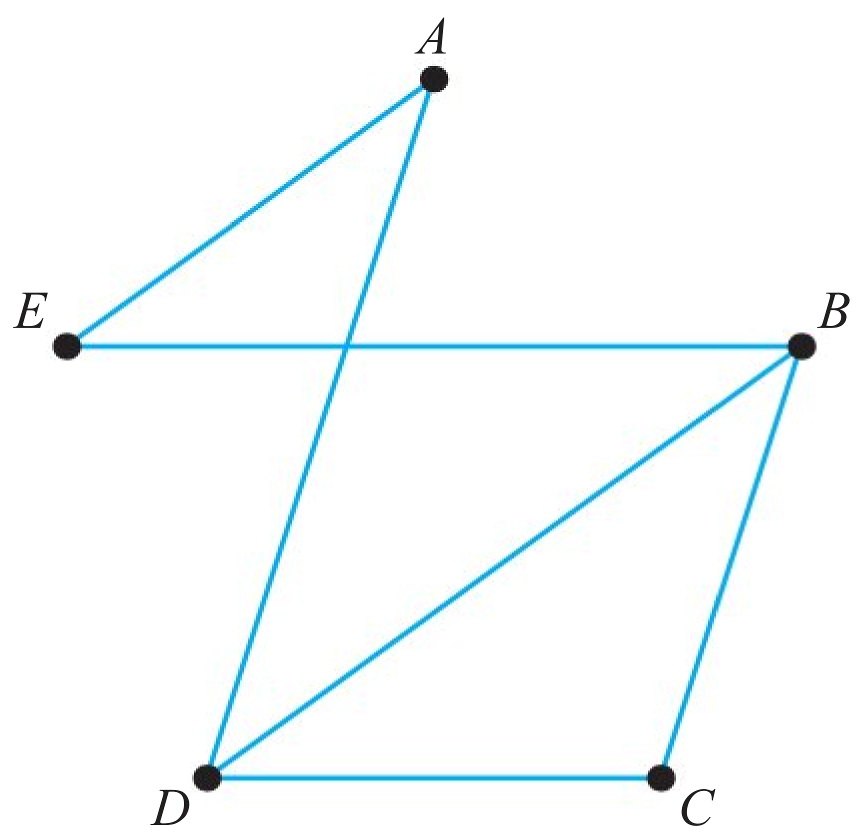


a For which of the two networks is it possible to do this? Justify your answer.
b Find a possible route Sachin could take, starting and finishing at A.
c For the other network, Sachin still wants to use each line exactly once, but can start and finish at different cities. At which city could he start?

2 Graph K is shown in the diagram.
a Which edge should be removed from K so that the resulting graph is Eulerian?
b Find a spanning tree for K.
c Show that K has a Hamiltonian cycle.



3 Consider this graph.

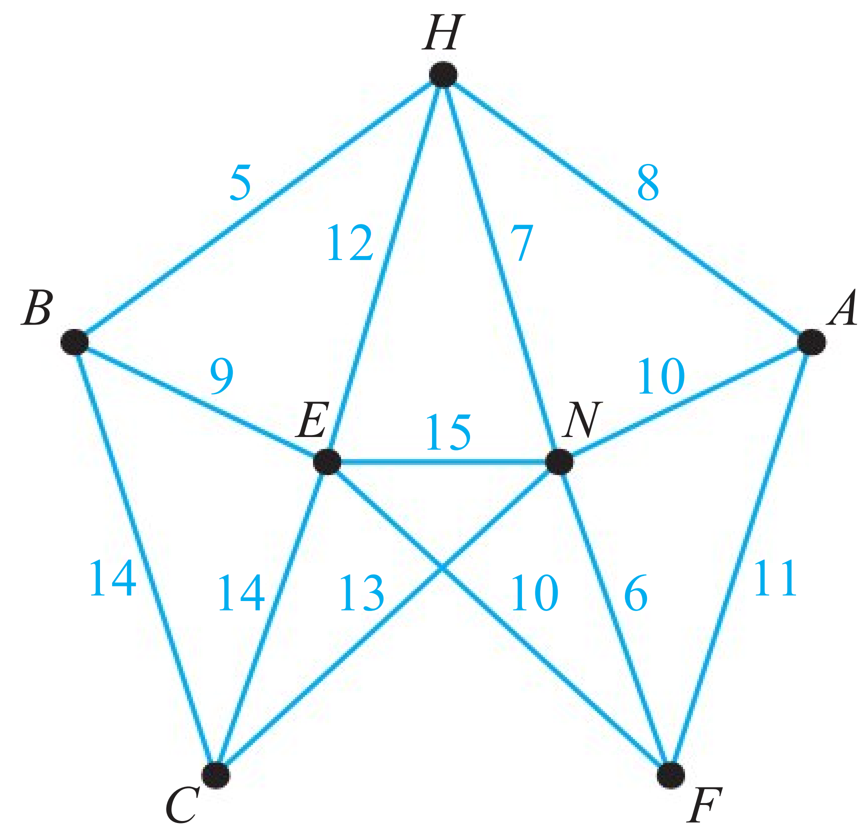


a Construct an adjacency matrix.
b Find the number of walks of length 5 from A to B.
c Find two vertices which are connected by seven different paths of length 4.

4 **a** Explain what is meant by a minimum spanning tree.

b Use Kruskal's algorithm to find the minimum spanning tree for the graph shown. List edges in the order you added them and state the weight of your tree.

c The graph represents a network of roads connecting seven mountain villages, with distances given in kilometres. After a snow storm, some of the roads need to be cleared so that each village can be reached from any other village. State which roads should be selected in order to minimize the total length of roads to clear?

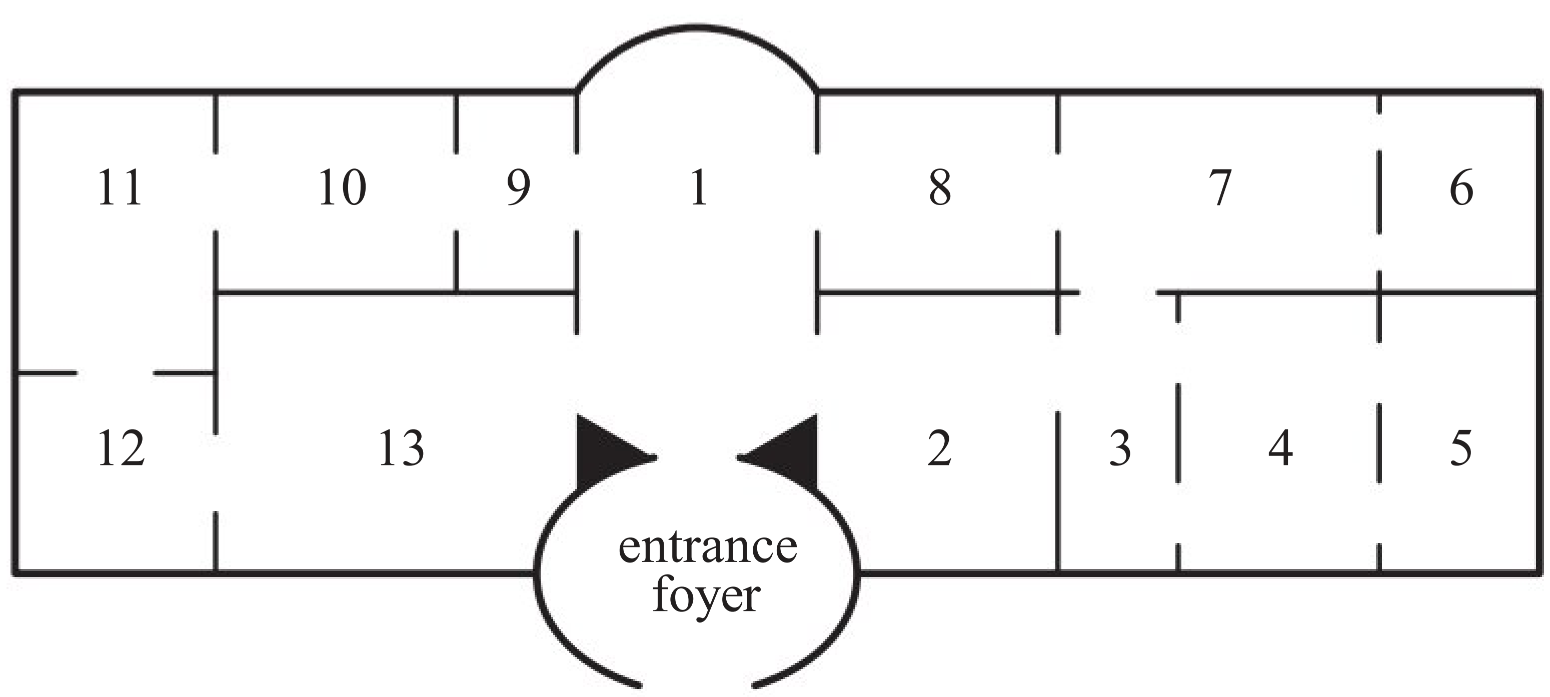


5 **a** Explain briefly the difference between Kruskal's and Prim's algorithms for finding the minimum spanning tree.

b Use Prim's algorithm, starting at A, to find the minimum spanning tree for the graph with the following weighted adjacency table. Draw the tree and state its weight.

	A	*B*	*C*	*D*	*E*	*F*	*G*	*H*	*I*	*J*
A	–	14	–	–	–	–	–	–	15.5	15
B	14	–	8.5	–	–	–	–	–	12	–
C	–	8.5	–	22.5	–	–	–	18.5	13	–
D	–	–	22.5	–	21	10	25	–	–	–
E	–	–	–	21	–	8	16	–	–	–
F	–	–	–	10	8	–	20.5	–	–	–
G	–	–	–	25	16	20.5	–	19	–	–
H	–	–	18.5	–	–	–	19	–	18	–
I	15.5	12	13	–	–	–	–	18	–	11.5
J	15	–	–	–	–	–	–	–	11.5	–

6 The following figure shows the floor plan of a museum.



a **i** Draw a graph G that represents the plan of the museum where each exhibition room is represented by a vertex labelled with the exhibition room number and each door between exhibition rooms is represented by an edge. Do not consider the entrance foyer as a museum exhibition room.

ii Write down the degrees of the vertices that represent each exhibition room.

iii Virginia enters the museum through the entrance foyer. Use your answers to **i** and **ii** to justify why it is possible for her to visit the thirteen exhibition rooms going through each internal doorway exactly once.

b Let G and H be two graphs whose adjacency matrices are represented below.

G

	A	**B**	**C**	**D**	**E**	**F**
A	0	2	0	2	0	0
B	2	0	1	1	0	1
C	0	1	0	1	2	1
D	2	1	1	0	2	0
E	0	0	2	2	0	2
F	0	1	1	0	2	0

H

	A	**B**	**C**	**D**	**E**	**F**
A	0	1	3	0	1	2
B	1	0	1	3	2	0
C	3	1	0	2	1	3
D	0	3	2	0	2	0
E	1	2	1	2	0	1
F	2	0	3	0	1	0

Using the adjacency matrices,

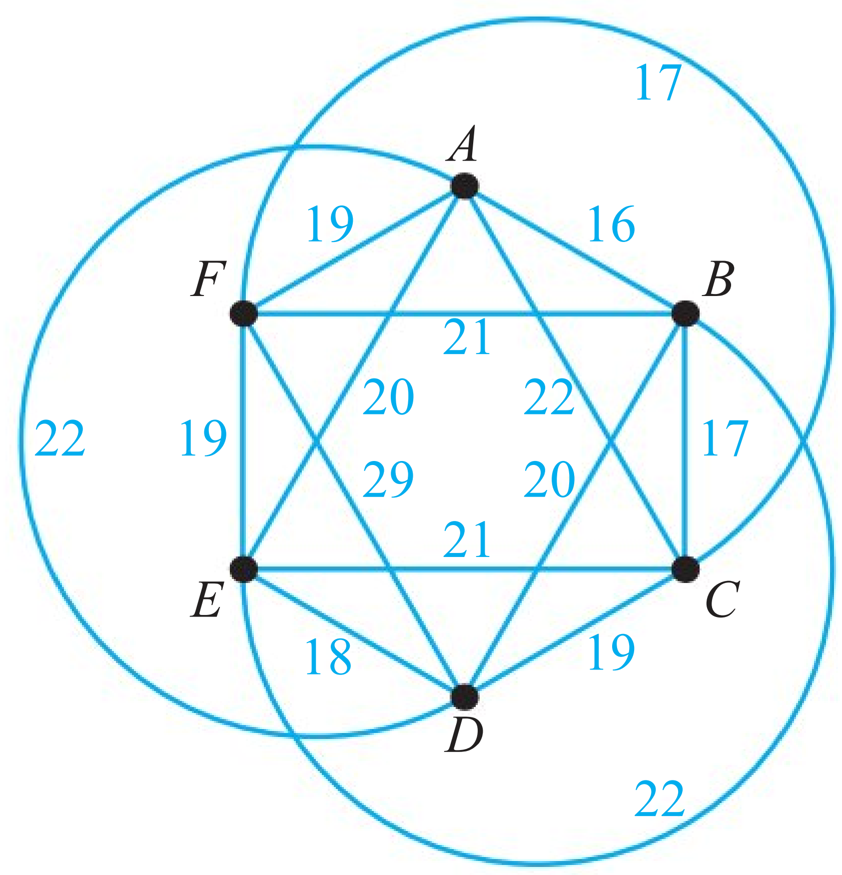
i find the number of edges of each graph

ii show that exactly one of the graphs has a Eulerian trail

iii show that neither of the graphs has a Eulerian circuit.

Mathematics HL November 2013 Paper 3 Q2

7 The graph alongside shows the time required to walk between different classrooms in a school.
A teacher, located in classroom A, wants to take pens to each classroom and return to A.



a Which of the following does the teacher need to find: an Eulerian circuit or a Hamiltonian cycle?

Let T seconds be the minimum possible time the teacher needs to take.

b Use the nearest neighbour algorithm, starting at vertex B, to find an upper bound for T.

c By removing vertex B find a lower bound.

d Write down an inequality satisfied by T.

8 **a** State the number of edges in a tree with n vertices.

b The table shows distances (in metres) between five workstations in an office.

	A	**B**	**C**	**D**	**E**
A	–	3	6	3	8
B	3	–	3	5	5
C	6	3	–	4	6
D	3	5	4	–	3
E	8	5	6	3	–

i Use Prim's algorithm, starting at A, to find the minimum spanning tree for the graph represented by this table.

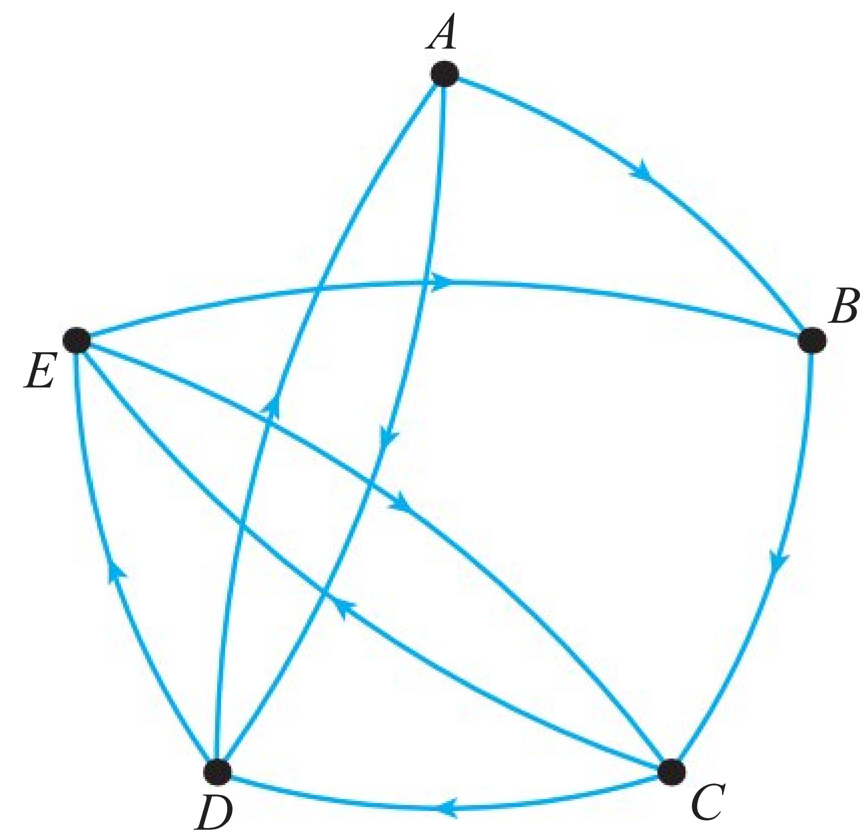
ii State the minimum length of cable required to connect all the workstations.

c **i** Explain how you can tell that the graph is Eulerian.

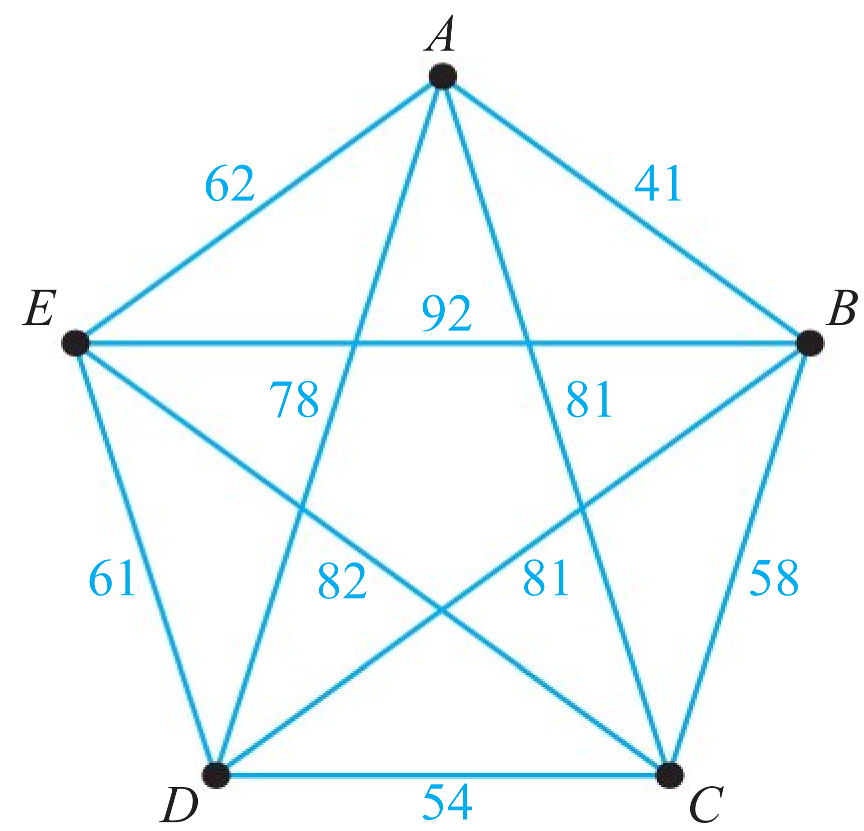
ii Find an Eulerian cycle starting at A and state its length.

9 The graph shows links between five web pages; for example, page A contains links to pages B and D.

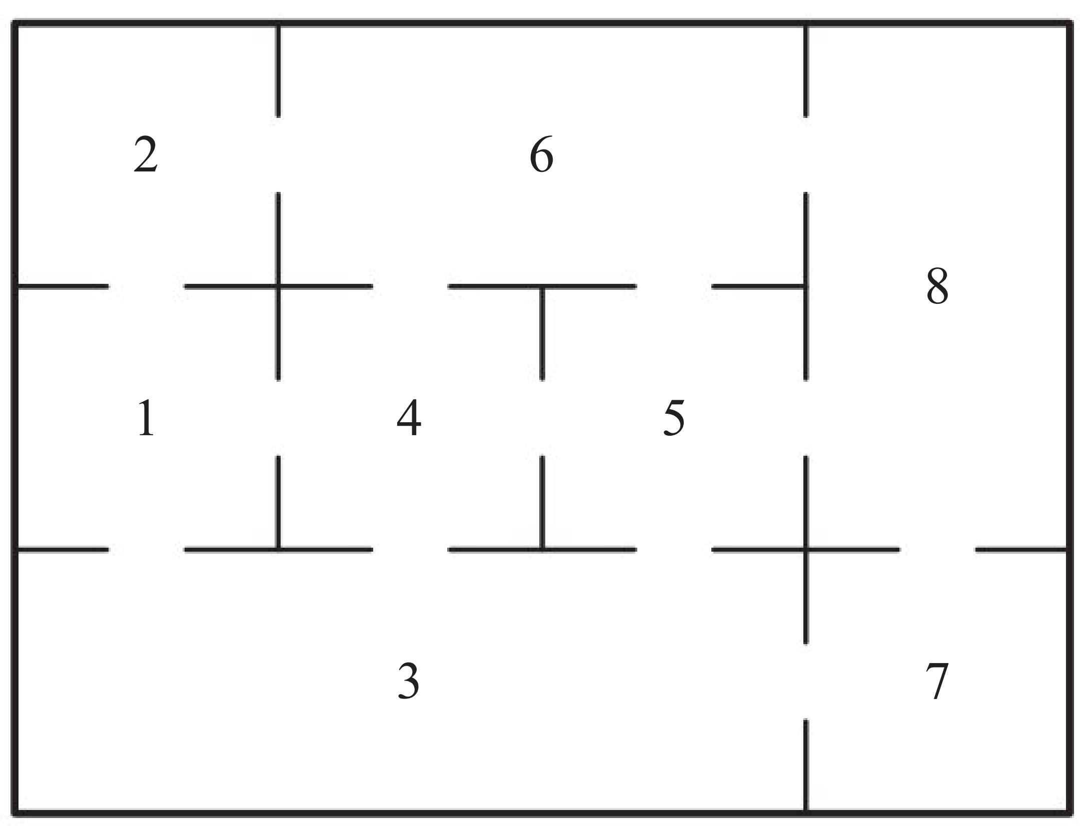
Use the PageRank algorithm to rank the five pages in order of importance.



10 By deleting each vertex in turn, find a lower bound for the travelling salesman problem for the graph alongside.



11 The diagram shows a plan of a house, with doors connecting adjacent rooms.



a Represent the plan on a graph, with vertices representing rooms and edges representing doors.

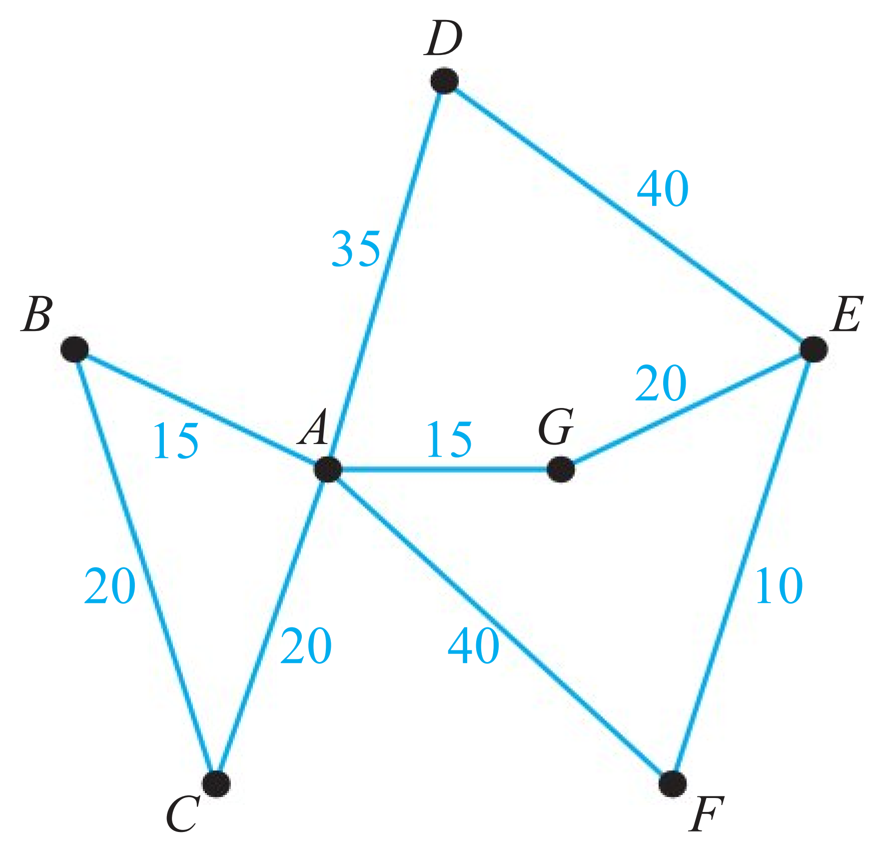
b Explain why it is not possible to walk through each door once and return to the starting room.

c Construct a transition matrix for the graph. A child starts in Room 1 and walks around randomly.

d Find the probability that the third room the child enters is Room 3.

e If the child continues to walk around the house for a long time, what percentage of time will they spend in Room 8?

12 The graph below represents a network of paths connecting seven fountains in a park, their lengths given in metres.

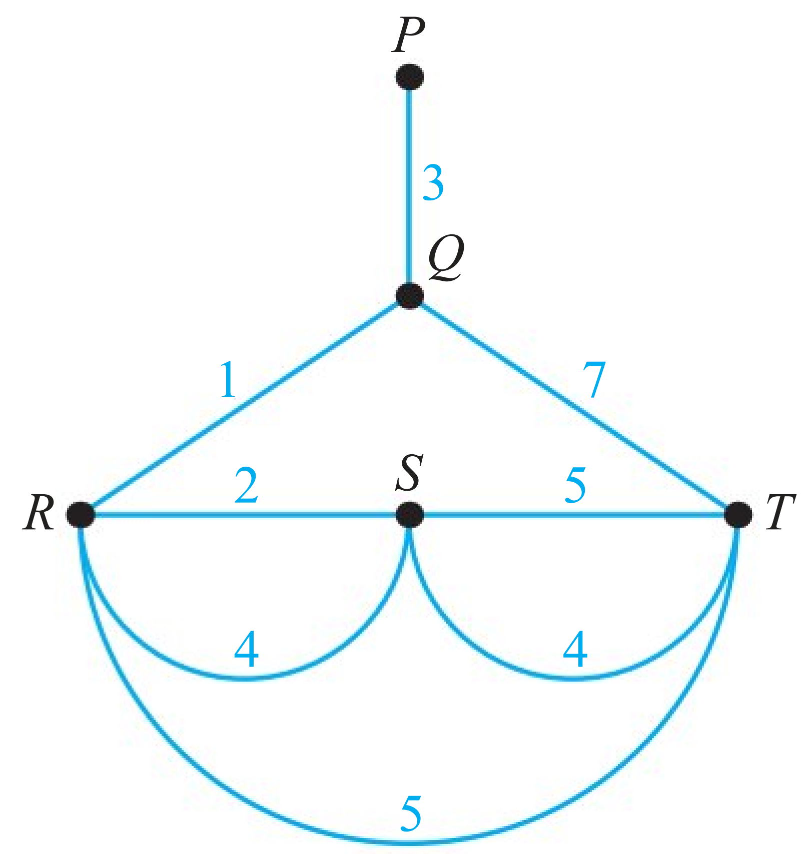


a Explain why it is not possible to start from fountain A, walk along each path exactly once and return to A.

b Find the length of the shortest route which uses each path at least once and returns to A. Show your method clearly.

c A new path is to be built so that it becomes possible to walk along each path exactly once and return to the starting point. Between which two fountains should this new path be built?

13 **a** Consider the following weighted graph.



i Write down a solution to the Chinese postman problem for this graph.

ii Calculate the total weight of the solution.

b **i** State the travelling salesman problem.

ii Explain why there is no solution to the travelling salesman problem for this graph.

Mathematics HL May 2015 Paper 3 Q1 parts b and c

14 The graph G has adjacency matrix $\mathbf{M}$ given below.

$$\begin{array}{c} \\ A \\ B \\ C \\ D \\ E \\ F \end{array} \begin{array}{c} \begin{array}{cccccc} A & B & C & D & E & F \end{array} \\ \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 1 \\ 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 & 0 & 1 \\ 1 & 0 & 0 & 0 & 1 & 0 \end{pmatrix} \end{array}$$

a Draw the graph G.

b What information about G is contained in the diagonal elements of $\mathbf{M}^2$?

c Find the number of walks of length 4 starting at A and ending at C.

d List the trails of length 4 starting at A and ending at C.

Mathematics HL May 2012 Paper 3 Q3

15 The complete graph H has the following cost adjacency matrix.

	A	**B**	**C**	**D**	**E**
A	–	19	17	10	15
B	19	–	11	16	13
C	17	11	–	14	13
D	10	16	14	–	18
E	15	13	13	18	–

Consider the travelling salesman problem for H.

a By first finding a minimum spanning tree on the subgraph of H formed by deleting vertex A and all edges connected to A, find a lower bound for this problem.

b Find the total weight of the cycle $ADCBEA$.

c What do you conclude from your results?

Mathematics HL May 2011 Paper 3 Q2

16 The table shows the lengths (in km) of main roads between eight villages.

	A	**B**	**C**	**D**	**E**	**F**	**G**	**H**
A	–	7	9	–	9	–	–	–
B	7	–	10	12	–	12	8	–
C	9	10	–	–	11	–	–	–
D	–	12	–	–	10	8	8	5
E	9	–	11	10	–	12	–	7
F	–	12	–	8	12	–	11	6
G	–	8	–	8	–	11	–	8
H	–	–	–	5	7	6	8	–

a A road inspector needs to drive along each road to inspect it.

i Explain why he cannot return to the starting village without using some roads more than once.

ii Can he use each road exactly once if he does not have to return to the starting village? Justify your answer.

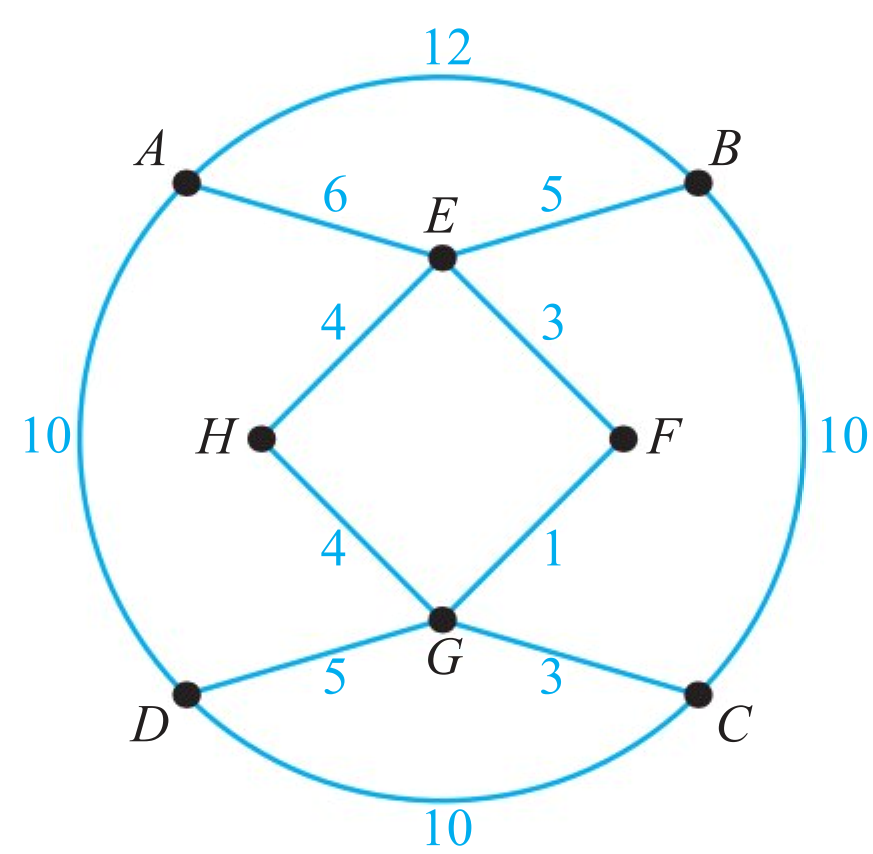
b A snowplough needs to clear all the roads. To do this, it must travel along each road twice (not necessarily in opposite directions).

i Explain why the snowplough can start from village A, travel along each road exactly twice, and return back to A.

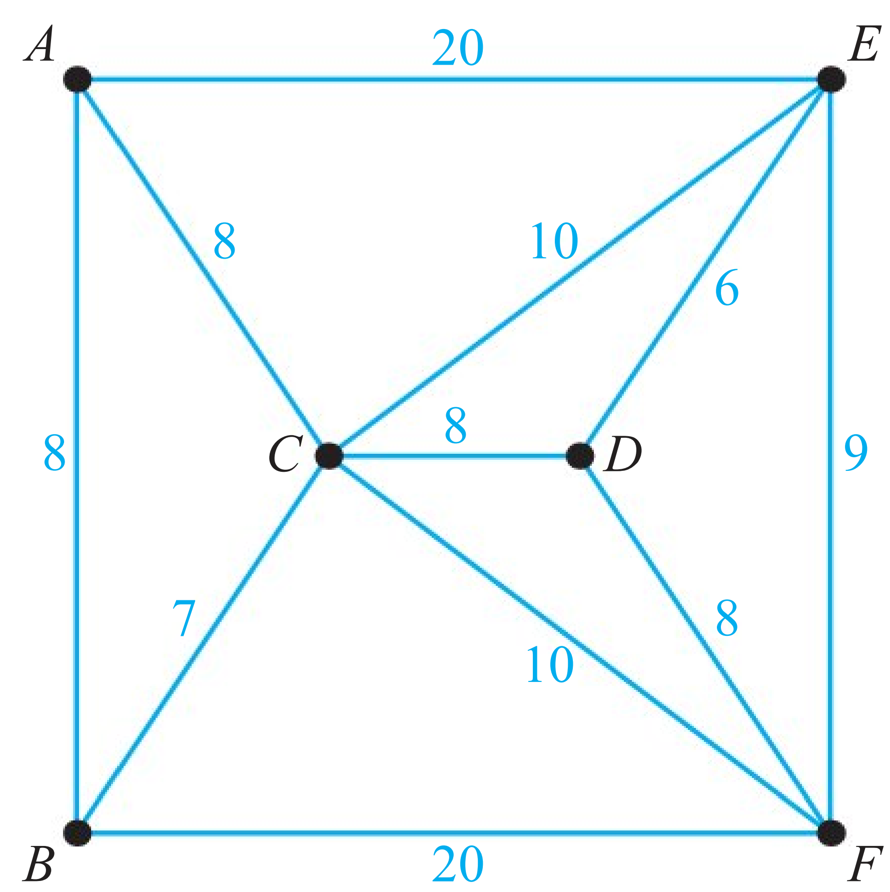
ii Find the distance the snowplough needs to cover.

17 The diagram shows a network of bus routes connecting cities A to H. The weight of the edges represent the cost, in \$, of travel on each route. A passenger wants to travel along each route at least once and return to the starting point.

Find the cheapest possible way to do this and state which routes need to be used more than once.



18 The graph shows boat routes between six islands and the time, in minutes, to sail each route.



A boat starts at port A and needs to deliver food to every island before returning to A.

a Construct a table showing the shortest travel times between each pair of islands.

The shortest time for the boat's tour is T minutes.

b By removing vertex B from the graph, find a lower bound for T.

c Use the nearest neighbour algorithm, starting at C, to find an upper bound for T.

d Write down an inequality satisfied by T.

e Find a tour of the islands which corresponds to your upper bound from part **c**.

19 Consider the graph with the following cost adjacency matrix.

	A	B	C	D	E	F
A	–	36	41	44	50	52
B	36	–	38	42	48	40
C	41	38	–	35	41	52
D	44	42	35	–	44	50
E	50	48	41	44	–	48
F	52	40	52	50	48	–

a Use Kruskal's algorithm to find the minimum spanning tree.

b For the travelling salesman problem on this graph:

i Remove vertex A to find a lower bound.

ii Remove vertex B to find another lower bound.

iii State which of the two lower bounds is better.

c By considering the cycle $ABCDEFA$, find an upper bound, and hence write an inequality satisfied by the solution of the travelling salesman problem.

20 Graph G is represented by the following weighted adjacency matrix.

$$\begin{pmatrix} - & 6 & 10 & 9 & 7 \\ 6 & - & 9 & 8 & 8 \\ 10 & 9 & - & 6 & 10 \\ 9 & 8 & 6 & - & 5 \\ 7 & 8 & 10 & 5 & - \end{pmatrix}$$

a By deleting each vertex in turn, find a lower bound for the travelling salesman problem for G.

b Explain why this lower bound is, in fact, the solution to the travelling salesman problem.

8 Probability

ESSENTIAL UNDERSTANDINGS

- Probability enables us to quantify the likelihood of events occurring and so evaluate risk.
- Both statistics and probability provide important representations which enable us to make predictions, valid comparisons and informed decisions. These fields have power and limitations and should be applied with care and critically questioned in detail to differentiate between the theoretical and the empirical/observed.
- Probability theory allows us to make informed choices, to evaluate risk, and to make predictions about seemingly random events.

In this chapter you will learn...

- how linear transformations of a single random variable affect the expected value and variance
- how to find the expected value and variance of a linear combination of two or more random variables
- how to find the expected value and variance of the mean of a random variable
- about the distribution of linear combinations of normally distributed random variables
- about the distribution of the mean of a normally distributed random variable
- about the distribution of the sum of many observations and the mean of a random variable from any distribution (the central limit theorem)
- about the circumstances under which the Poisson distribution is an appropriate model
- how to find the mean and variance of the Poisson distribution
- about the distribution of the sum of two independent Poisson distributions
- about the role of transition matrices in Markov chains
- about initial state probability vectors
- how to calculate steady state and long-term probabilities of Markov chains.

CONCEPTS

The following concepts will be addressed in this chapter:

- **Modelling** and finding structure in seemingly random events facilitates prediction.
- Different probability distributions provide a **representation** of the **relationship** between the theory and reality, allowing us to make predictions about what might happen.
- **Representation** of probabilities using transition matrices enables us to efficiently predict long-term behaviour and outcomes.
- Statistical literacy involves identifying the **reliability** and the **validity** of samples and whole populations in a closed system.

LEARNER PROFILE - Risk Takers

How do you determine what is a reasonable risk?

Figure 8.1 How do we compare different distributions?

PRIOR KNOWLEDGE

Before starting this chapter, you should already be able to complete the following:

x	1	2	3	4
$\mathrm{P}(X = x)$	0.3	0.1	p	0.2

1 The probability distribution of a random variable X is shown in the table.

a Find the value of p.

b Find $\mathrm{E}(X)$.

2 If $X \sim \mathrm{N}(12, 25)$, find $\mathrm{P}(X < 10)$.

3 If $X \sim \mathrm{B}(8, 0.4)$, find $\mathrm{P}(X > 4)$.

4 a If A and B are independent events with $\mathrm{P}(A) = 0.3$ and $\mathrm{P}(B) = 0.4$, find $\mathrm{P}(A \cap B)$.

b If $\mathrm{P}(C) = 0.6$ and $\mathrm{P}(C \cap D) = 0.5$, find $\mathrm{P}(D \mid C)$.

5 Calculate

$$\begin{pmatrix} 2 & 5 & 1 \\ 1 & 0 & 3 \\ 3 & 6 & 4 \end{pmatrix}^3 \begin{pmatrix} 5 \\ 2 \\ 3 \end{pmatrix}$$

6 Solve the following system of equations.

$$\begin{cases} 3x - y + 4z = 5 \\ x + 4y - z = 8 \\ 2x + 3y + z = 12 \end{cases}$$

7 Consider the matrix $\mathbf{M} = \begin{pmatrix} 0.6 & 0.5 \\ 0.4 & 0.5 \end{pmatrix}$

a Find the eigenvalues and associated eigenvectors of $\mathbf{M}$.

b Hence express $\mathbf{M}$ in the form $\mathbf{M} = \mathbf{PDP}^{-1}$, where $\mathbf{D}$ is a diagonal matrix.

Random variables can be used to model many situations where the outcome depends on chance. You have already seen how the binomial distribution can model these outcomes in certain circumstances, and in this chapter you will meet a new discrete distribution, which is suitable in circumstances where you have events occurring at a constant rate. However, it is another distribution that you are already familiar with, the normal distribution, which turns out to be by far the most significant, even in circumstances when it seems to be unrelated.

Starter Activity

Look at the pictures in Figure 8.1. Discuss which of the two archers is better. What measures could you use to compare them?

Now look at this problem:

Use technology to simulate the outcome of

a the scores of 1000 rolls of a fair dice

b the sum of the scores of two fair dice rolled 1000 times

c the sum of the scores of 20 fair dice rolled 1000 times.

In each case, produce a bar chart of the results. What do you notice?

8A Transforming variables

Linear transformations of a single random variable

In Chapter 8 of the Mathematics: applications and interpretation SL book, you met the idea of the expected value of a discrete random variable, $\mathrm{E}(X)$.

Knowing the average value alone, however, does not tell you much about the values the variable is likely to take. It is also useful to know how spread out those values are, which can be measured by the variance, $\mathrm{Var}(X)$.

In Section 6B of the Mathematics: applications and interpretation SL book, you learnt that the same rules as those in Key Point 8.1 apply to constant changes to data sets.

Although you do not have to be able to calculate the variance of a random variable like you do with expected value, you do need to be able to work with a given value of variance. In particular, you need to know how the expected value and variance change if the random variable is transformed.

KEY POINT 8.1

$$\mathrm{E}(aX+b) = a\mathrm{E}(X)+b$$

$$\mathrm{Var}(aX+b) = a^2\mathrm{Var}(X)$$

CONCEPTS – CHANGE

The results from Key Point 8.1 are used when rescaling quantities or **changing** units. In such situations, it is important to know what changes and what remains the same.

WORKED EXAMPLE 8.1

A random variable X has expected value 12.5 and variance 4.8. A random variable Y is given by $Y = 3X - 2$.

Find the expected value and the variance of Y.

Use $\mathrm{E}(aX+b) = a\mathrm{E}(X)+b$

$$\begin{aligned}\mathrm{E}(Y) &= \mathrm{E}(3X-2)\\ &= 3\mathrm{E}(X)-2\\ &= 3(12.5)-2\\ &= 35.5\end{aligned}$$

Use $\mathrm{Var}(aX+b) = a^2\mathrm{Var}(X)$

$$\begin{aligned}\mathrm{Var}(Y) &= \mathrm{Var}(3X-2)\\ &= 3^2 \times 4.8\\ &= 43.2\end{aligned}$$

Tip

Variance can never be negative.

Be the Examiner 8.1

A random variable X has $\mathrm{E}(X) = 5$ and $\mathrm{Var}(X) = 3.5$.

Find the variance of the random variable $Y = 10 - 4X$.

Which is the correct solution? Identify the errors made in the incorrect solutions.

Solution 1	Solution 2	Solution 3
$\mathrm{Var}(10-4X) = 10 - 4(3.5)$ $= -5$	$\mathrm{Var}(10-4X) = 10 + 16(3.5)$ $= 66$	$\mathrm{Var}(10-4X) = 16(3.5)$ $= 56$

Expected value and variance of linear combinations of n random variables

The results of Key Point 8.1 can be extended to include more than one random variable.

KEY POINT 8.2

- $\mathrm{E}(aX + bY + c) = a\mathrm{E}(X) + b\mathrm{E}(Y) + c$
- $\mathrm{Var}(aX + bY + c) = a^2\mathrm{Var}(X) + b^2\mathrm{Var}(Y)$

The second result is only true if X and Y are independent.

This, in turn, can be extended to a linear combination of any number of random variables.

WORKED EXAMPLE 8.2

The random variable X has mean 8.5 and variance 1.4. The random variable Y has mean 4.7 and variance 0.6. The random variable Z has mean 10.2 and variance 3.8.

The random variable W is given by $W = X + 5Y - Z$.

Find

a $\mathrm{E}(W)$

b $\mathrm{Var}(W)$

$\mathrm{E}(aX + bY + cZ) = a\mathrm{E}(X) + b\mathrm{E}(Y) + c\mathrm{E}(Z)$

a
$$\begin{aligned}\mathrm{E}(W) &= \mathrm{E}(X + 5Y - Z)\\ &= \mathrm{E}(X) + 5\mathrm{E}(Y) - \mathrm{E}(Z)\\ &= 8.5 + 5(4.7) - 10.2\\ &= 21.8\end{aligned}$$

$\mathrm{Var}(aX + bY + cZ) = a^2\mathrm{Var}(X) + b^2\mathrm{Var}(Y) + c^2\mathrm{Var}(Z)$

b
$$\begin{aligned}\mathrm{Var}(W) &= \mathrm{Var}(X + 5Y - Z)\\ &= \mathrm{Var}(X) + 5^2\,\mathrm{Var}(Y) + (-1)^2\,\mathrm{Var}(Z)\\ &= 1.4 + 25(0.6) + 3.8\\ &= 20.2\end{aligned}$$

It is important to be clear whether you are taking, say, two different observations from a population, or taking one observation and doubling it. Although this will not result in a different expected value, it will result in a different variance.

WORKED EXAMPLE 8.3

The mean mass of an orange is 106 g with standard deviation 15 g. The mean mass of a lemon is 64 g with standard deviation 9 g.

A shopkeeper thinks that if five lemons are picked at random their total mass will be more than three times the mass of a randomly selected orange.

Let D be the amount by which five lemons are heavier than three times the mass of a single orange.

Find the mean and variance of D assuming that all the fruit are chosen independently.

Define the variables

Let

X = mass of an orange

Y = mass of a lemon

The five lemons are different but the mass of just one orange is multiplied by 3

Then

$$D = Y_1 + Y_2 + Y_3 + Y_4 + Y_5 - 3X$$

Use the results for the mean and variance of a linear combination of random variables, as before

$$\begin{aligned} E(D) &= E(Y_1) + E(Y_2) + E(Y_3) + E(Y_4) \\ &\quad + E(Y_5) - 3E(X) \\ &= 5(64) - 3(106) \\ &= 2\text{ g} \end{aligned}$$

Standard deviation of X is 15 so $\text{Var}(X) = 15^2$ and standard deviation of Y is 9 so $\text{Var}(Y) = 9^2$

$$\begin{aligned} \text{Var}(D) &= \text{Var}(Y_1) + \text{Var}(Y_2) + \text{Var}(Y_3) \\ &\quad + \text{Var}(Y_4) + \text{Var}(Y_5) + (-3)^2\,\text{Var}(X) \\ &= 5(9^2) + 9(15^2) \\ &= 2430\text{ g}^2 \end{aligned}$$

So, the standard deviation of D = 49.3 g

TOOLKIT: Modelling

Does it make sense that the variance of one observation of X tripled is larger than the variance of three independent observations of X added together? Try simulating this effect by rolling some dice and explain why extreme outcomes are more likely with a single observation.

The calculations above suggest that, on average, the mass of five randomly selected lemons will be larger than three times the mass of a randomly selected orange. However, since the standard deviation is very large compared with the average difference, there is a strong possibility that the mass of the lemons will be smaller.

Expected value and variance of the sample mean

When calculating the mean of a sample of size n of the variable X we add up the n independent observations of X and then divide by n. We give this sample mean the symbol $\bar{X}$ and it is itself a random variable (i.e. it might change each time it is observed):

$$\bar{X} = \frac{X_1 + X_2 + \ldots + X_n}{n}$$

Since $\bar{X}$ is a linear combination of random variables, we can use the results in Key Point 8.2 to find its mean and variance.

KEY POINT 8.3

If $X_1, X_2, \ldots, X_n$ are independent observations of the random variable X, then

- $\mathrm{E}(\bar{X}) = \mathrm{E}(X)$
- $\mathrm{Var}(\bar{X}) = \dfrac{\mathrm{Var}(X)}{n}$

CONCEPTS – RELIABILITY

The second result in Key Point 8.3 says that the standard deviation of a sample mean (which can be thought of as a measure of the error caused by randomness) is smaller than the standard deviation of a single observation. This confirms our intuition that finding an average of several results produces a more **reliable** outcome than just looking at one result.

Tip

The result for $\mathrm{Var}(\bar{X})$ can be proved in a similar way.

Proof 8.1

Prove that $\mathrm{E}(\bar{X}) = \mathrm{E}(X)$.

$$\mathrm{E}(\bar{X}) = \mathrm{E}\left(\frac{X_1 + X_2 + \ldots + X_n}{n}\right)$$

$$= \mathrm{E}\left(\frac{1}{n}X_1 + \frac{1}{n}X_2 + \ldots + \frac{1}{n}X_n\right)$$

$\mathrm{E}(a_1X_1 + a_2X_2 + \ldots + a_nX_n) = a_1\mathrm{E}(X_1) + a_2\mathrm{E}(X_2) + \ldots + a_n\mathrm{E}(X_n)$

$$= \frac{1}{n}\mathrm{E}(X_1) + \frac{1}{n}\mathrm{E}(X_2) + \ldots + \frac{1}{n}\mathrm{E}(X_n)$$

$\mathrm{E}(X_i) = \mathrm{E}(X)$ for each observation

$$= \frac{1}{n}\mathrm{E}(X) + \frac{1}{n}\mathrm{E}(X) + \ldots + \frac{1}{n}\mathrm{E}(X)$$

$$= \frac{1}{n}n\mathrm{E}(X)$$

$$= \mathrm{E}(X)$$

TOOLKIT: Modelling

The result from Worked Example 8.4 provides some mathematical interpretation of why we take means. We expect the sample mean to be quite close to the true population mean, but it has a narrower spread around the population mean than a single observation. The mean is often described as having a smaller error than a single observation. What does the fact that this error is inversely proportional to the square root of n tell us about experimental design?

WORKED EXAMPLE 8.4

The mean mass of a packet of breakfast cereal is 400 g with a variance of 15 g^2.

A sample of 30 packets is taken.

Find the expected value and the variance of the sample mean.

Define the random variable X — Let X = mass of a packet of cereal.

Use $\mathrm{E}(\bar{X}) = \mathrm{E}(X)$ — $\mathrm{E}(\bar{X}) = 400$

Use $\mathrm{Var}(\bar{X}) = \dfrac{\mathrm{Var}(X)}{n}$ — $\mathrm{Var}(\bar{X}) = \dfrac{15}{30} = 0.5$

So, the expected mean of the sample is 400 g with a variance of 0.5 g^2.

Exercise 8A

For questions 1 to 5, use the method demonstrated in Worked Example 8.1 to find $\mathrm{E}(Y)$ and $\mathrm{Var}(Y)$ given that $\mathrm{E}(X)=9$ and $\mathrm{Var}(X)=5$.

1 a $Y=2X+3$
  b $Y=4X-5$

2 a $Y=\frac{1}{3}X-7$
  b $Y=\frac{1}{10}X+4$

3 a $Y=2-3X$
  b $Y=6-X$

4 a $Y=2(X+3)$
  b $Y=3(X-2)$

5 a $Y=\frac{X-4}{5}$
  b $Y=\frac{X+1}{2}$

For questions 6 to 11, use the method demonstrated in Worked Examples 8.2 and 8.3 to find $\mathrm{E}(Z)$ and $\mathrm{Var}(Z)$ given that $\mathrm{E}(X)=0.5$, $\mathrm{Var}(X)=2$, $\mathrm{E}(Y)=-8$ and $\mathrm{Var}(Y)=3$, where X and Y are independent.

6 a $Z=2X-3Y+5$
  b $Z=4X+5Y+1$

7 a $Z=\frac{1}{2}X+4Y-6$
  b $Z=X-\frac{1}{3}Y-2$

8 a $Z=3(X-Y+1)$
  b $Z=4(X+2Y-3)$

9 a $Z=\frac{X-3Y-1}{5}$
  b $Z=\frac{2X-Y+3}{4}$

10 a $Z=X_1+X_2+X_3$
  b $Z=Y_1-Y_2$

11 a $Z=2X-(Y_1+Y_2)$
  b $Z=X_1+X_2+X_3+X_4-3Y$

For questions 12 to 15, use the method demonstrated in Worked Example 8.4 to find $\mathrm{E}(\bar{X})$ and $\mathrm{Var}(\bar{X})$ for a sample of n observations.

12 a $\mathrm{E}(X)=5,\ \mathrm{Var}(X)=1.2,\ n=7$
  b $\mathrm{E}(X)=6,\ \mathrm{Var}(X)=2.5,\ n=12$

13 a $\mathrm{E}(X)=-4.7,\ \mathrm{Var}(X)=0.8,\ n=20$
  b $\mathrm{E}(X)=-15.1,\ \mathrm{Var}(X)=0.7,\ n=15$

14 a $X\sim \mathrm{N}(12,\ 9),\ n=10$
  b $X\sim \mathrm{N}(8,\ 0.25),\ n=12$

15 a $X\sim \mathrm{B}(6,\ 0.5),\ n=10$
  b $X\sim \mathrm{B}(12,\ 0.3),\ n=8$

16 The table shows the probability distribution of a random variable X.

x	1	3	5	7
$\mathrm{P}(X=x)$	0.2	0.3	0.2	0.3

a Find $\mathrm{E}(X)$.

The random variable Y is given by $Y=3X+1$.

b Find $\mathrm{E}(Y)$.

c Given that $\mathrm{Var}(X)=4.96$, find $\mathrm{Var}(Y)$.

17 The random variable W has the probability distribution given by

$$\mathrm{P}(W=w)=\frac{k}{w} \text{ for } w=1,\ 2,\ 4,\ 8$$

a Find the value of k.

The random variable V is given by $V=2W-2$.

b Given that $\mathrm{Var}(W)=3.45$, find $\mathrm{Var}(V)$.

18 The probability distribution of a random variable X is given by

$$P(X=x)=\frac{2x-1}{16} \quad \text{for } x=1, 2, 3, 4$$

a Find $E(X)$.

The random variable Y is given by $Y=10X+3$.

b Find the mean of Y.

c Given that the variance of X is $\frac{55}{64}$, find the variance of Y.

19 Let X and Y be two independent variables with $E(X)=6$, $\text{Var}(X)=3$, $E(Y)=1$ and $\text{Var}(Y)=5$. Find:

a $E(2X-Y+4)$

b $\text{Var}(2X-Y+4)$

20 The mean thickness of the base of a burger bun is 1.4 cm with variance $0.02\,\text{cm}^2$.

The mean thickness of a burger is 3.0 cm with variance $0.14\,\text{cm}^2$.

The mean thickness of the top of the burger bun is 2.2 cm with variance $0.2\,\text{cm}^2$.

Find the mean and standard deviation of the total height of the whole burger in the bun, assuming that the thickness of each part is independent.

21 Eggs are packed in boxes of 12. The mass of the box itself is exactly 50 g. The mass of one egg has mean 12.4 g and standard deviation 1.2 g. Find the mean and the standard deviation of the mass of a box of eggs.

22 A machine produces chocolate bars so that the mean mass of a bar is 102 g and the standard deviation is 8.6 g. As a part of the quality control process, a sample of 20 chocolate bars is taken and the mean mass is calculated. Find the mean and standard deviation of the sample mean of these 20 chocolate bars.

23 The random variable A has mean 3.8 and variance 1.2. The random variable B is such that $A+2B=10$.

Find the mean and the variance of B.

24 The random variable U has mean 25 and variance 16. The random variable V satisfies $2U+5V=20$.

Find the mean and variance of V.

25 The average mass of a man in a lift is 78 kg with standard deviation 12 kg. The average mass of a woman in the lift is 62 kg with standard deviation 8 kg. The empty lift has a mass of 500 kg.

Find the expectation and standard deviation of the total mass of the lift when 3 women and 5 men are inside.

26 A weighted dice has mean outcome 4 with standard deviation 1. Alain rolls the dice once and doubles the outcome. Beatrice rolls the dice twice and adds the results together.

Find the mean and standard deviation of the result when Beatrice's score is subtracted from Alain's.

27 Exam scores at a large school have mean 64 and standard deviation 25. Two students are selected at random.

Find the mean and standard deviation of the result when the score of the second student is subtracted from the score of the first.

28 The standard deviation of the mean mass of a sample of 2 cucumbers is 20 g smaller than the standard deviation in the mass of a single cucumber. Find the standard deviation of the mass of a cucumber.

29 Ayane cycles to school with a mean time of 15 minutes and a standard deviation of 3 minutes. Hiroki walks to school with a mean time of 28 minutes and a standard deviation of 2 minutes. They each calculate the total time it takes them to get to school over a five day week.

Find the mean and standard deviation of how much less time Ayane takes on her journeys over the course of a week, assuming that journey times are independent.

30 A manufacturer believes that their battery life follows a normal distribution with a mean of 4.8 hours and variance 1.7 hours^2. They wish to take a sample to estimate the mean battery life.

a If they want the standard deviation of the sample mean to be less than 0.3 hours, find the minimum sample size needed.

b When the sample size is increased by 80, the standard deviation of the sample mean decreases to a third of its original size. Find the original sample size.

8B Distribution of $\bar{X}$

Linear combination of n independent normal variables

Although the proof is beyond the scope of this course, it turns out that any linear combination of normal variables will also follow a normal distribution. The parameters of the resulting normal distribution can be found using the rules from Key Point 8.2.

KEY POINT 8.4

If the random variables X and Y are independent with $X \sim \mathrm{N}\left(\mu_X, \sigma_X^2\right)$ and $Y \sim \mathrm{N}\left(\mu_Y, \sigma_Y^2\right)$ and if $W = aX + bY + c$, then

$$W \sim \mathrm{N}\left(a\mu_X + b\mu_Y + c,\ a^2\sigma_X^2 + b^2\sigma_Y^2\right)$$

Again, this result can be extended to a linear combination of any number of independent normal random variables.

TOOLKIT: Modelling

You might like to verify this result by using technology to generate some random numbers from a normal distribution and then creating a histogram for W. However, you should be careful – not every bell-shaped curve is a normal distribution. You could research the various methods available for verifying that data could plausibly be taken from a normal distribution.

WORKED EXAMPLE 8.5

$X \sim \mathrm{N}(12, 5)$ and $Y \sim \mathrm{N}(4, 6)$, where X and Y are independent.

If $W = 2X - 3Y + 1$, find $\mathrm{P}(W < 15)$.

Use $W \sim \mathrm{N}\left(a\mu_X + b\mu_Y + c,\ a^2\sigma_X^2 + b^2\sigma_Y^2\right)$

$$W \sim \mathrm{N}(2 \times 12 - 3 \times 4 + 1,\ 2^2 \times 5 + (-3)^2 \times 6)$$

So, $W \sim \mathrm{N}(13, 74)$

Find the required probability from the GDC

$$\mathrm{P}(W < 15) = 0.592$$

We saw in Section 8A that $\mathrm{E}(\bar{X}) = \mathrm{E}(X)$ and $\mathrm{Var}(\bar{X}) = \frac{\mathrm{Var}(X)}{n}$. Following on from the result that a linear combination of normal random variables itself follows a normal distribution gives us the following result for $\bar{X}$.

KEY POINT 8.5

If $X_1, X_2, \ldots, X_n$ are independent observations of the random variable $X \sim \mathrm{N}\left(\mu, \sigma^2\right)$, then

$$\bar{X} \sim \mathrm{N}\left(\mu, \frac{\sigma^2}{n}\right)$$

WORKED EXAMPLE 8.6

The height of sunflowers follows a normal distribution with a mean of 2.4 m and a standard deviation of 0.5 m.

A random sample of 5 sunflowers was taken and their heights measured.

Find the probability that the mean height of the sample was greater than 2.5 m.

Define the random variable X and state its distribution

Let X be the height of sunflowers.

Then $X \sim \mathrm{N}(2.4, 0.5^2)$

Use the fact that if $X \sim \mathrm{N}(\mu, \sigma^2)$, then $\overline{X} \sim \mathrm{N}\left(\mu, \frac{\sigma^2}{n}\right)$

So, $\overline{X} \sim \mathrm{N}\left(2.4, \frac{0.5^2}{5}\right)$

Find the required probability from the GDC

$\mathrm{P}(\overline{X} > 2.5) = 0.327$

Be the Examiner 8.2

The distance Heidi can long jump is normally distributed with mean 4.8 m and standard deviation 0.6 m.

Find the probability that in two jumps her total distance covered is greater than 10 m.

Which is the correct solution? Identify the errors made in the incorrect solutions.

Solution 1	Solution 2	Solution 3
$X \sim \mathrm{N}(4.8, 0.6^2)$ 10 m in two jumps is equivalent to 5 m in each jump. $2\mathrm{P}(X > 5) = 0.359$	$X \sim \mathrm{N}(4.8, 0.6^2)$ $2X \sim \mathrm{N}(2 \times 4.8, 2^2 \times 0.6^2)$ $2X \sim \mathrm{N}(9.6, 1.44)$ $\mathrm{P}(2X > 10) = 0.369$	$X \sim \mathrm{N}(4.8, 0.6^2)$ $X_1 + X_2 \sim \mathrm{N}(2 \times 4.8, 2 \times 0.6^2)$ $X_1 + X_2 \sim \mathrm{N}(9.6, 0.72)$ $\mathrm{P}(X_1 + X_2 > 10) = 0.319$

Central limit theorem

As you may have noticed from the problem at the beginning of this chapter, summing a large number of independent outcomes from a uniform distribution leads to a normal distribution. In fact, if we sum enough independent observations from any distribution with a finite variance, the result will follow an approximately normal distribution.

KEY POINT 8.6

Central limit theorem (CLT):

If $X_1, X_2, \ldots, X_n$ are independent observations from *any* distribution you will meet, and if $n > 30$, then, approximately:

- $\sum_{i=1}^{n} X_n \sim \mathrm{N}\left(n\mu, n\sigma^2\right)$
- $\bar{X} \sim \mathrm{N}\left(\mu, \frac{\sigma^2}{n}\right)$

You are the Researcher

Although the guideline given here is $n > 30$, there are certain distributions for which n needs to be much larger before the CLT is valid. Moreover, there are distributions to which the CLT does not apply at all. One example is the Cauchy distribution, which does not have a finite variance. How might this form the basis of a sophisticated and rigorous exploration?

WORKED EXAMPLE 8.7

Pierre sleeps an average of 7.8 hours each night with a standard deviation of 1.2 hours.

Find the probability that in a 31 day month he sleeps more that 8 hours per night on average.

Let X be the number of hours Pierre sleeps each night.

We do not know the distribution of X but since $n > 30$, we can use the CLT

By the CLT,

$$\bar{X} \sim \mathrm{N}\left(7.8, \frac{1.2^2}{31}\right)$$

Find the required probability from the GDC

$$\mathrm{P}\left(\bar{X} > 8\right) = 0.177$$

Exercise 8B

For questions 1 to 5, use the methods demonstrated in Worked Examples 8.5 and 8.6 to find the distribution of W, given that $X \sim \mathrm{N}(10, 4)$ and $Y \sim \mathrm{N}(5, 3)$.

1. a $W = X + 2Y + 1$
   b $W = 4X + Y - 3$
2. a $W = 3X - 5Y + 2$
   b $W = X - Y - 6$
3. a $W = 0.5X + 0.4Y$
   b $W = 0.2X - 1.5Y$
4. a $W = X_1 - X_2$
   b $W = Y_1 + Y_2 + Y_3$
5. a $W = \bar{X}$, where $n = 8$
   b $W = \bar{Y}$, where $n = 10$

For questions 6 to 8, use the method demonstrated in Worked Example 8.7 to find, where possible, the approximate distribution of the stated random variable, given that X has mean 30 and standard deviation 10.

6. a $\bar{X}$, where $n = 80$
   b $\bar{X}$, where $n = 200$
7. a $\bar{X}$, where $n = 15$
   b $\bar{X}$, where $n = 10$
8. a $\sum_{i=1}^{50} X_i$
   b $\sum_{i=1}^{40} X_i$

9 Given that $X \sim \mathrm{N}(12,16)$ and $Y \sim \mathrm{N}(8,25)$, find:

a $\mathrm{P}(3X+1<5Y)$

b $\mathrm{P}(X_1+X_2>Y_1-Y_2)$

10 The random variable X has mean 100 and standard deviation 25. For a sample of 30 observations, find:

a $\mathrm{P}(\overline{X}<98)$

b $\mathrm{P}(|\overline{X}-95|>10)$

11 An airline has found that the mass of their passengers follows a normal distribution with mean 82.2 kg and variance 10.7 kg^2. The mass of their hand luggage follows a normal distribution with mean 9.1 kg and variance 5.6 kg^2.

a State the distribution of the total mass of a passenger and their hand luggage and find any necessary parameters.

b Find the probability that the total mass of a passenger and their luggage exceeds 100 kg.

12 Evidence suggests that the time Alan takes to run 100 m is normally distributed with mean 13.1 seconds and standard deviation 0.4 seconds. The time Bryan takes to run 100 m is normally distributed with mean 12.8 seconds and standard deviation 0.6 seconds.

a Find the mean and standard deviation of the difference (Alan – Bryan) between Alan's and Bryan's times.

b Find the probability that Alan finishes a 100 m race before Bryan.

c Find the probability that Bryan beats Alan by more than one second.

13 A machine produces metal rods so that their length follows normal distribution with mean 65 cm and variance 0.03 cm^2. The rods are checked in batches of six, and a batch is rejected if the average length is less than 64.8 cm or more than 65.3 cm.

a Find the mean and the variance of the average of a random sample of six rods.

b Hence find the probability that a batch is rejected.

14 The distribution of lengths of pipes produced by a machine is normal with mean 40 cm and standard deviation 3 cm.

a Find the probability that a randomly chosen pipe has a length of 42 cm or more.

b Find the probability that the average length of a randomly chosen set of 10 pipes of this type is 42 cm or more.

15 The random variable X has mean 12 and standard deviation 3.5. A sample of 40 independent observations of X is taken.

Use the central limit theorem to calculate the probability that the mean of the sample is between 13 and 14.

16 The mean mass of a pineapple is 145 g with variance 96 g^2. A crate is filled with 70 pineapples.

Find the probability that the total mass of the pineapples in the crate is less than 10 kg.

17 Winnie eats an average of 1900 kcal each day with a standard deviation of 400 kcal.

Find the probability that in a 31-day month she eats more than 2000 kcal per day on average.

18 The masses, X kg, of male birds of a certain species are normally distributed with mean 4.6 kg and standard deviation 0.25 kg. The masses, Y kg, of female birds of this species are normally distributed with mean 2.5 kg and standard deviation 0.2 kg.

a Find the mean and variance of $2Y-X$.

b Find the probability that the mass of a randomly chosen male bird is more than twice the mass of a randomly chosen female bird.

c Find the probability that the total mass of three male birds and 4 female birds (chosen independently) exceeds 25 kg.

19 A shop sells apples and pears. The masses, in grams, of the apples may be assumed to have a $\mathrm{N}(180, 12^2)$ distribution and the masses of the pears, in grams, may be assumed to have a $\mathrm{N}(100, 10^2)$ distribution.

a Find the probability that the mass of a randomly chosen apple is more than double the mass of a randomly chosen pear.

b A shopper buys 2 apples and a pear. Find the probability that the total mass is greater than 500 grams.

20 The length of a grass snake is normally distributed with mean 1.2 m. The probability of a randomly selected sample of 5 grass snakes having a mean length greater than 1.4 m is 5%.

Find the standard deviation of the length of a grass snake.

21 Given that $X \sim \mathrm{B}(10, 0.6)$, find the probability that the mean of 35 independent observations of X is greater than 6.5.

22 The average mass of a sheet of A4 paper is 5 g, and the standard deviation of the masses is 0.08 g.

a Find the mean and standard deviation of the mass of a ream of 500 sheets of A4 paper.

b Find the probability that the mass of a ream of 500 sheets is within 5 g of the expected mass.

c Explain how you have used the central limit theorem in your answer.

23 Company A and company B specialize in roadside repairs of vehicles that have broken down. The mean time for A to attend a breakdown is 45 minutes with a standard deviation of 22 minutes; for B the mean time is 51 minutes with a standard deviation of 25 minutes.

A sample of 40 breakdowns for A and 50 breakdowns for B are taken.

a Find the probability that, from these samples, the mean time for B to attend is within 4 minutes of the mean time for A to attend.

b Explain why you needed to use the central limit theorem in part a.

24 Boys' scores in a test follow the distribution $\mathrm{N}(50,25)$. Girls' scores follow $\mathrm{N}(60,16)$.

a Find the probability that a randomly chosen boy and a randomly chosen girl differ in score by less than five.

b Find the probability that a randomly chosen boy scores less than three quarters of the mark of a randomly chosen girl.

25 The daily rainfall in Statsville follows a normal distribution with mean μ mm and standard deviation σ mm. The rainfall each day is independent of the rainfall on other days.

On a randomly chosen day, there is a probability of 0.1 that the rainfall is greater than 8 mm.

In a randomly chosen 7-day week, there is a probability of 0.05 that the mean daily rainfall is less than 7 mm.

Find the value of μ and of σ.

26 Tambara goes to school by train. The time she waits each morning is normally distributed with a mean of 12 minutes and a standard deviation of 4 minutes.

a On a specific morning, find the probability that Tambara waits more than 20 minutes.

b During a particular week (Monday to Friday), find the probability that

i her total morning waiting time does not exceed 70 minutes

ii she waits less than 10 minutes on exactly two mornings of the week

iii her average morning waiting time is more than 10 minutes.

c Given that the total morning waiting time for the first four days is 50 minutes, find the probability that the average for the week is over 12 minutes.

d Given that Tambara's average morning waiting time in a week is over 14 minutes, find the probability that it is less than 15 minutes.

27 The times Johannes takes to answer a multiple choice question are normally distributed with mean 1.5 minutes and standard deviation 0.6 minutes. He has one hour to complete a test consisting of 35 questions.

a Assuming that the questions are independent, find the probability that Johannes does not complete the test in time.

b Explain why you did not need to use the central limit theorem in your answer to part a.

28 A random variable has mean 15 and standard deviation 4. A large number of independent observations of the random variable is taken.

Find the minimum sample size so that the probability that the sample mean is more than 16 is less than 0.05.

8C Poisson distribution

You have already met two 'standard' probability distributions that arise as the result of commonly occurring circumstances: one discrete (the binomial distribution) and one continuous (the normal distribution).

Another of these 'standard' distributions is the **Poisson distribution**, which models the number of events in a fixed period given the average rate at which they occur. For example, if you know the average number of visits per hour to a website, then you could use the Poisson distribution to predict the probability of getting a certain number of visits in the next hour.

As with the binomial distribution, you need certain conditions to be satisfied for the Poisson distribution to be appropriate.

Tip

If the average rate of success, or events occurring at a constant rate, is mentioned, you should use the Poisson distribution.

If you can identify a fixed number of trials, then the binomial distribution is usually required.

KEY POINT 8.7

The Poisson distribution occurs when the following conditions are satisfied:

- events are independent of each other
- events occur at a constant average rate
- events occur singly (one at a time).

CONCEPTS – MODELLING

It is often the case in the real world that the conditions for a Poisson distribution won't be perfectly met. In the case of the example of visits to a website, it is unlikely that the average rate of visits is constant throughout the day. Nevertheless, it may still be that **modelling** this situation with a Poisson distribution provides useful information.

If a random variable X has the Poisson distribution with constant average rate λ, we write $X \sim \text{Po}(\lambda)$.

WORKED EXAMPLE 8.8

Jorge receives emails independently of each other at a constant rate of 6 per hour.

The random variable X is the time between emails arriving.

Is this situation suitable for X to be modelled by a Poisson distribution?

The conditions are met, but for X to follow a Poisson distribution it would need to measure the number of events, not the time between events

No, since X is not the number of emails arriving in a given time period.

You are the Researcher

In Worked Example 8.8, X is actually modelled by a distribution closely related to the Poisson distribution called the **exponential distribution**.

As with the binomial and normal distributions, you can calculate probabilities from the Poisson distribution with your GDC.

WORKED EXAMPLE 8.9

A water pipe has on average three leaks in each 5 km section, distributed independently of each other. An inspector from the water company examines a 5 km section of the pipe.

Find the probability that he

a finds exactly 2 leaks

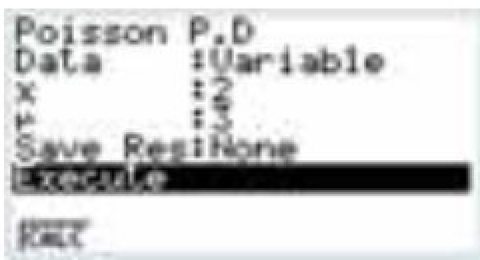
b finds at least 2 leaks.

Let X be the number of leaks in a 5 km section of pipe.

X is Poisson with $\lambda = 3$

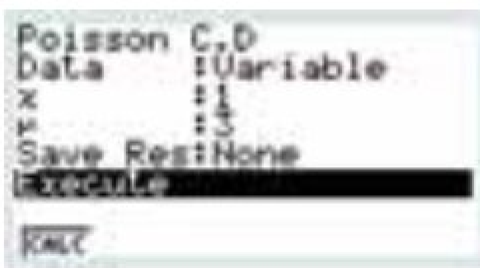
$X \sim \mathrm{Po}(3)$

Use the GDC in Poisson P.D. mode as you want the probability of X taking a single value



a $\mathrm{P}(X = 2) = 0.224$

Use the GDC in C.D. mode as you want the probability of X taking a range of values



b $\mathrm{P}(X \geqslant 2) = 1 - \mathrm{P}(X \leqslant 1)$

$= 1 - 0.199$

$= 0.801$

Tip

Some calculators can find this probability directly without the preliminary calculation.

Mean and variance of the Poisson distribution

As for the binomial distribution, there are formulae for the expected value and variance of a Poisson distribution. Since λ is the constant average rate at which events occur, it should be no surprise that λ is the expected value.

KEY POINT 8.8

If $X \sim \mathrm{Po}(\lambda)$, then

- $\mathrm{E}(X) = \lambda$
- $\mathrm{Var}(X) = \lambda$

CONCEPTS – MODELLING

Notice that the mean is equal to the variance for the Poisson distribution. This is something we look out for when determining whether data is likely to fit a Poisson **model**, whether although that in itself is not sufficient to decide – there are other distributions with this feature.

WORKED EXAMPLE 8.10

A radioactive substance emits beta particles at a average constant rate of 20 per hour. If X is the number of beta particles emitted per hour, find

a the mean of X

b the standard deviation of X.

X is Poisson with $\lambda = 20$ ⋯ $X \sim \mathrm{Po}(20)$

Use $\mathrm{E}(X) = \lambda$ ⋯ **a** $\mathrm{E}(X) = 20$

Use $\mathrm{Var}(X) = \lambda$ ⋯ **b** $\mathrm{Var}(X) = 20$

Standard deviation is the square root of the variance ⋯ So, standard deviation = 4.47

The Poisson distribution is scalable. For example, if the number of birds seen on the branch of a tree in 10 minutes follows a Poisson distribution with mean λ, then the number of birds seen on the branch in 20 minutes follows a Poisson distribution with mean 2λ, and the number of birds seen on the branch in 5 minutes follows a Poisson distribution with mean $\frac{\lambda}{2}$.

WORKED EXAMPLE 8.11

Assuming that the number of buses arriving at a bus stop in a one-hour period follows a Poisson distribution, with mean 15, find the probability that there are fewer than 8 buses in a 20 minute period.

Let X be the number of buses arriving in 20 mins.

Buses arrive at a rate of 15 per hour so $\lambda = 5$ per 20 minutes ⋯ $X \sim \mathrm{Po}(5)$

Find the required probability from the GDC. Remember that the GDC will give probabilities of the form $\mathrm{P}(X \leqslant k)$ ⋯ $\mathrm{P}(X < 8) = \mathrm{P}(X \leqslant 7)$
$= 0.867$

Sum of two independent Poisson distributions

The scalability of the Poisson distribution is a consequence of a more general result about the Poisson distribution. If two independent variables both follow a Poisson distribution, then so does their sum.

KEY POINT 8.9

If $X \sim \mathrm{Po}(\lambda_X)$ and $Y \sim \mathrm{Po}(\lambda_Y)$ are two independent Poisson distributions and $Z = X + Y$, then

$$Z \sim \mathrm{Po}(\lambda_X + \lambda_Y)$$

WORKED EXAMPLE 8.12

Zhuo runs a website that provides video tutorials for IB maths and IB physics topics. The website receives an average of 7.8 hits an hour for maths and 6.5 hits an hour for physics.

a Assuming that the hits for maths and physics are from an independent Poisson distribution, find the probability that his website gets more than 15 hits an hour.

b Explain why the assumption that the hits for maths and physics form independent Poisson distributions is unlikely to be true.

a Let Z be the number of hits per hour.

Use $Z \sim \text{Po}(\lambda_X + \lambda_Y) = \text{Po}(7.8 + 6.5)$

$Z \sim \text{Po}(14.3)$

Find the required probability from the GDC

$$\begin{aligned} \text{P}(Z > 15) &= 1 - \text{P}(Z \leqslant 15) \\ &= 1 - 0.639 \\ &= 0.361 \end{aligned}$$

b The rate of hits to the website is unlikely to be constant – there will probably be more at some times of the day than others.
The two distributions are probably not independent of each other, as times when more maths hits occur are likely to be similar to times when more physics hits occur (possibly even by the same users).

Be the Examiner 8.3

The number of errors in a Maths text book is believed to follow a Poisson distribution with a mean of 1.5 errors per 10 pages.

Find the probability that there are more than 5 errors in 50 pages.

Which is the correct solution? Identify the errors made in the incorrect solutions.

Solution 1	Solution 2	Solution 3
$X \sim \text{Po}(1.5)$ 5 errors in 50 pages is equivalent to 1 error in each 10 pages. $\text{P}(X > 1) = 1 - \text{P}(X \leqslant 1)$ $= 0.442$	$X \sim \text{Po}(7.5)$ $\text{P}(X > 5) = 1 - \text{P}(X \leqslant 5)$ $= 0.759$	$X \sim \text{Po}(7.5)$ $\text{P}(X > 5) = 1 - \text{P}(X \leqslant 4)$ $= 0.868$

Exercise 8C

For questions 1 to 4, use the method demonstrated in Worked Example 8.8 to decide whether the random variable X could be modelled by a Poisson distribution. If so, state how well the conditions are likely to be met.

1 a A radioactive element emits beta particles at a mean rate of 3 per minute and X is the number of minutes in the next hour in which at least 2 particles emitted.

b Fish occur at a mean rate of 4 per m^3 in a certain $1000\,m^3$ volume of the sea and X is the number of $1\,m^3$ volumes of water, out of 5 surveyed, that contain exactly 4 fish.

2 a A radioactive element emits beta particles at a mean rate of 3 per minute and X is the number emitted each minute.

b Fish occur at a mean rate of 4 per m^3 in a certain $1000\,m^3$ volume of the sea and X is the number of fish per m^3.

3 a A radioactive element emits beta particles at a mean rate of 3 per minute and X is length of time until the next emission.

b Fish occur at a mean rate of 4 per m^3 in a certain $1000\,m^3$ volume of the sea and X is the volume of water than needs to be searched before the next fish is found.

4 a A radioactive element emits beta particles at a mean rate of 3 per minute and X is the number of minutes that pass until the first emission is observed.

b Fish occur at a mean rate of 4 per m^3 in a certain $1000\,m^3$ volume of the sea and X is the number of $1\,m^3$ volumes of water that need to be searched until the first fish is found.

For questions 5 to 12, use the method demonstrated in Worked Example 8.9 to find the required probabilities given that $X \sim \mathrm{Po}(8.4)$.

5 a $\mathrm{P}(X = 7)$
b $\mathrm{P}(X = 10)$

6 a $\mathrm{P}(X \leqslant 4)$
b $\mathrm{P}(X \leqslant 9)$

7 a $\mathrm{P}(X < 11)$
b $\mathrm{P}(X < 8)$

8 a $\mathrm{P}(X \geqslant 6)$
b $\mathrm{P}(X \geqslant 3)$

9 a $\mathrm{P}(X > 12)$
b $\mathrm{P}(X > 7)$

10 a $\mathrm{P}(7 \leqslant X \leqslant 10)$
b $\mathrm{P}(5 \leqslant X \leqslant 8)$

11 a $\mathrm{P}(7 < X < 9)$
b $\mathrm{P}(6 < X < 12)$

12 a $\mathrm{P}(10 \leqslant X < 13)$
b $\mathrm{P}(10 < X \leqslant 15)$

For questions 13 and 14, use the method demonstrated in Worked Example 8.10 to find the required values.

13 a Find $\mathrm{E}(X)$ if $X \sim \mathrm{Po}(3.1)$

b Find $\mathrm{Var}(X)$ if $X \sim \mathrm{Po}(3.1)$

14 a Find the standard deviation of X if $X \sim \mathrm{Po}(5.3)$

b Find the mean of X if $X \sim \mathrm{Po}(5.3)$

For questions 15 to 17, use the method demonstrated in Worked Example 8.11 to find the distribution of Y given that X is the number of cars passing a checkpoint in a 20 second period and $X \sim \mathrm{Po}(12)$.

15 a Y is the number of cars passing in 1 minute

b Y is the number of cars passing in 40 seconds

16 a Y is the number of cars passing in 4 seconds

b Y is the number of cars passing in 5 seconds

17 a Y is the number of cars passing in 1.5 minutes

b Y is the number of cars passing in 16 seconds

For questions 18 and 19, use the method demonstrated in Worked Example 8.12 to find the distribution of Z, where $Z = X + Y$, and X and Y are independent.

18 a $X \sim \mathrm{Po}(2)$ and $Y \sim \mathrm{Po}(4.5)$

b $X \sim \mathrm{Po}(5.3)$ and $Y \sim \mathrm{Po}(3.6)$

19 a X follows a Poisson distribution with mean 6.2 and Y follows a Poisson distribution with mean 7.8

b X follows a Poisson distribution with mean 4 and Y follows a Poisson distribution with mean 3

20 From a particular observatory, shooting stars are observed in the night sky at an average rate of one every five minutes.

Assuming that this rate is constant and that shooting stars occur (and are observed) independently of each other, find the probability that more than 20 are seen over a period of one hour.

21 When examining blood from a healthy individual under a microscope, a haematologist knows that she should see on average four white blood cells in each high power field.

Find the probability that blood from a healthy individual will show

a seven white blood cells in a single high power field

b a total of 28 white blood cells in six high power fields, selected independently.

22 A wire manufacturer is looking for flaws. Experience suggests that there are on average 1.8 flaws per metre in the wire.

a Determine the probability that there is exactly one flaw in one metre of the wire.

b Determine the probability that there is at least one flaw in 2 metres of the wire.

23 Mario receives an average of 5.4 emails and 2.1 texts each hour. These are the only types of messages he receives.

a Assuming that the emails and texts each form independent Poisson distributions, find the probability that he receives more than 4 messages in an hour.

b Explain why the assumption that the emails and texts form independent Poisson distributions is unlikely to be true.

24 The number of road traffic accidents at a particular intersection follows a Poisson distribution. If the probability of there being at least one accident in a given week is 0.6, find

a the mean of the distribution

b the probability of there being more than two accidents in a given week.

25 The number of outbreaks of mumps in a certain district is modelled by a Poisson distribution.

a If the probability of there being more than two outbreaks in a given month is 0.3, find the probability of there being fewer than two outbreaks in a given month.

b Explain whether the Poisson distribution is likely to be a good model for this situation.

26 The number of telephone calls per minute to a call centre follows a Poisson distribution with a mean of 6. Let X be the number of calls received in one minute and let Y be the number of calls received in 10 minutes.

a Calculate:

i $\mathrm{P}(X = 6)$

ii $\mathrm{P}(Y = 60)$

b Find the probability that the call centre receives exactly 6 calls in at least 5 minutes of a 10 minutes period.

27 The number of eagles observed in a forest in one day follows a Poisson distribution with mean 1.4.

a Find the probability that more than three eagles will be observed on a given day.

b Given that at least one eagle is observed on a particular day, find the probability that exactly two eagles are seen that day.

28 The number of mistakes a teacher makes while marking homework has a Poisson distribution with a mean of 1.6 errors per piece of homework.

a Find the probability that there are at least two marking errors in a randomly chosen piece of homework.

b Find the most likely number of marking errors occurring in a piece of homework. Justify your answer.

c Find the probability that, in a class of 12 pupils, fewer than half of them have errors in their marking.

29 A car company has two limousines that it hires out by the day. The number of requests per day has a Poisson distribution with a mean of 1.3 requests per day.

a Find the probability that neither limousine is hired.

b Find the probability that some requests have to be denied.

c If each limousine is to be equally used, on how many days in a period of 365 days would you expect a particular limousine to be in use?

30 The daily sales of cupcakes from a bakery can be modelled by a Poisson distribution with mean 21.5. A fresh batch is baked every three days, and any cupcakes older than three days cannot be sold.

a Find the probability of selling more than 15 cupcakes per day for three consecutive days.

b Find the number of cupcakes the bakery would have to produce to be at least 95% certain of not selling out.

8D Markov chains

Often the probability of a certain event occurring depends on what had happened previously. For example, it is more likely to be sunny today if it was sunny yesterday. We might represent this type of situation in a tree diagram, but with several outcomes possible at each stage, and perhaps several stages too, this can quickly become complicated.

A more efficient way of representing this type of situation is as a **Markov chain**. A Markov chain is a system consisting of two or more states, for example rainy and sunny, in which the probability of being in any given state depends only on the previous event.

The conditional probabilities of changing from one state to another, or staying in the same state, are represented in a **transition matrix**.

You met this idea of a transition matrix in Section 7C in the context of moving from one vertex to another in a graph.

Transition matrices

KEY POINT 8.10

A transition matrix is a matrix in which:

- the probability of going from state A to state B is given by the entry in column A and row B
- the entries in each column must sum to 1.

WORKED EXAMPLE 8.13

Any given day is either deemed to be sunny or rainy. If it is sunny today, then the probability of it being sunny tomorrow is 0.8. If it is rainy today, then the probability of it being sunny tomorrow is 0.4.

Write down the transition matrix for this system.

We are given that probability Sunny → Sunny = 0.8 and probability Rainy → Sunny = 0.4

$$\begin{array}{cc} & \begin{array}{cc} S & R \end{array} \\ \begin{array}{c} S \\ R \end{array} & \begin{pmatrix} 0.8 & 0.4 \\ & \end{pmatrix} \end{array}$$

Complete the matrix by noting that the columns must sum to 1

$$\begin{array}{cc} & \begin{array}{cc} S & R \end{array} \\ \begin{array}{c} S \\ R \end{array} & \begin{pmatrix} 0.8 & 0.4 \\ 0.2 & 0.6 \end{pmatrix} \end{array}$$

The information about a system could also be given in a transition diagram, which is a directed graph with the various states as vertices.

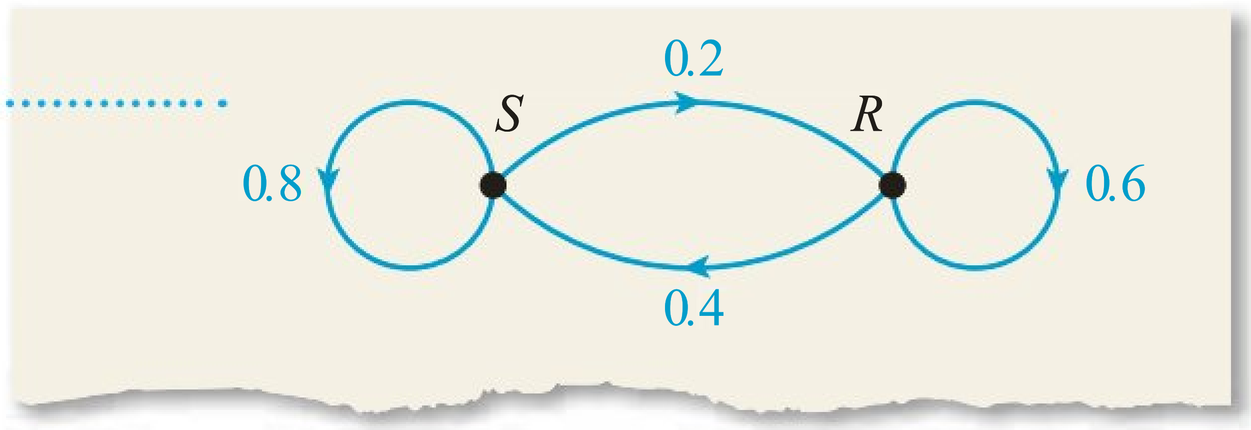
WORKED EXAMPLE 8.14

Represent the information in the transition matrix in a transition diagram.

$$\mathbf{T} = \begin{array}{c} \\ S \\ R \end{array} \begin{array}{c} \begin{array}{cc} S & R \end{array} \\ \begin{pmatrix} 0.8 & 0.4 \\ 0.2 & 0.6 \end{pmatrix} \end{array}$$

The graph has a loop from S→S with probability 0.8, and a loop from R→R with probability 0.6

The directed edge from S→R has probability 0.2, and the directed edge from R→S has probability 0.4



Powers of transition matrices

From a transition matrix, $\mathbf{T}$, you can read off the probability of being in any particular state at $t = 1$ given that you know the state initially ($t = 0$). To find these probabilities at $t = n$, find the matrix $\mathbf{T}^n$.

WORKED EXAMPLE 8.15

For the system in Worked Example 8.14, find the probability that it is sunny three days later given that it is rainy today.

Use the GDC to find $\mathbf{T}^3$

$$\mathbf{T}^3 = \begin{pmatrix} 0.8 & 0.4 \\ 0.2 & 0.6 \end{pmatrix}^3$$

$$= \begin{pmatrix} 0.688 & 0.624 \\ 0.312 & 0.376 \end{pmatrix}$$

You want the probability in the R column and the S row (going from Rainy to Sunny)

The probability of it being sunny three days later given that it is rainy today is 0.624.

Tip

A transition matrix, $\mathbf{T}$, is said to be regular if for some integer, n, all entries of $\mathbf{T}^n$ are positive (none are 0). The corresponding Markov chain is said to be a regular Markov chain. In this course, all Markov chains will be regular.

CONCEPTS – REPRESENTATION

Try finding the probability from Worked Example 8.15 by **representing** the system as a tree diagram instead of a Markov chain. Although you could already do this question by using tree diagrams, the Markov chain method is far more efficient.

Initial state probability vectors

Another way of finding the required probability in Worked Example 8.15 would have been to multiply $\mathbf{T}^3$ by a vector giving the probability of being in the initial state (a rainy day). Since it is known to be a rainy day initially, this would be the vector $\begin{array}{c} S \\ R \end{array}\begin{pmatrix} 0 \\ 1 \end{pmatrix}$:

$$\begin{pmatrix} 0.8 & 0.4 \\ 0.2 & 0.6 \end{pmatrix}^3 \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 0.624 \\ 0.367 \end{pmatrix} \begin{array}{c} S \\ R \end{array}$$

The resulting vector gives the probabilities for being sunny or rainy after three days, given that it was rainy initially.

KEY POINT 8.11

For a transition matrix $\mathbf{T}$ and initial state vector $\mathbf{s}_0$, the state after n time periods, $\mathbf{s}_n$, is given by $\mathbf{T}^n\mathbf{s}_0 = \mathbf{s}_n$

WORKED EXAMPLE 8.16

Two taxi firms, Alpha Cabs and Beta Cars are well established in the market and have market shares of 50% and 40%, respectively. A new company Gamma Carriages has recently entered the market and has market share of 10%.

Given the monthly transition matrix,

$$\begin{array}{c|c} & \begin{array}{ccc} A & B & G \end{array} \\ \begin{array}{c} A \\ B \\ G \end{array} & \begin{pmatrix} 0.85 & 0.05 & 0.1 \\ 0.05 & 0.8 & 0.1 \\ 0.1 & 0.15 & 0.8 \end{pmatrix} \end{array}$$

find the projected market share of each firm after five months.

Use $\mathbf{s}_n = \mathbf{T}^n\mathbf{s}_0$ with

$$\mathbf{s}_0 = \begin{pmatrix} 0.5 \\ 0.4 \\ 0.1 \end{pmatrix} \begin{array}{c} A \\ B \\ G \end{array}$$

$$\mathbf{s}_5 = \begin{pmatrix} 0.85 & 0.05 & 0.1 \\ 0.05 & 0.8 & 0.1 \\ 0.1 & 0.15 & 0.8 \end{pmatrix}^5 \begin{pmatrix} 0.5 \\ 0.4 \\ 0.1 \end{pmatrix}$$

Evaluate using the GDC

$$= \begin{pmatrix} 0.375 \\ 0.287 \\ 0.338 \end{pmatrix}$$

So, after 5 months, the market shares for Alpha, Beta and Gamma are 37.5%, 28.7% and 33.8%, respectively.

Calculation of steady state and long term probabilities

As well as finding the state of a system after a set number of time periods, we might also want to see whether in the long term the system settles into a unchanging state or a **steady state**.

WORKED EXAMPLE 8.17

For the system in Worked Example 8.16, find the likely steady-state market shares of the three taxi operators.

We can find the long-term behaviour of the system by applying large powers of the transition matrix to the initial state probabilities.

$$\mathbf{T}^{50}\begin{pmatrix}0.5\\0.4\\0.1\end{pmatrix}=\begin{pmatrix}0.345\\0.276\\0.379\end{pmatrix}$$

There is no change between the effect of $\mathbf{T}^{50}$ and $\mathbf{T}^{100}$ so it seems that the system has reached a steady state

$$\mathbf{T}^{100}\begin{pmatrix}0.5\\0.4\\0.1\end{pmatrix}=\begin{pmatrix}0.345\\0.276\\0.379\end{pmatrix}$$

So, the steady-state market shares of Alpha, Beta and Gamma are 34.5%, 27.6%, and 37.9%, respectively.

Notice that this means **s** is an eigenvector of the matrix **T** with corresponding eigenvector 1. This fact is used in questions 27 and 28 of Exercise 8D.

Since the steady-state vector is one that does not any longer change under application of **T**, it can be found without ever knowing the initial probability vector.

KEY POINT 8.12

For a transition matrix **T**, the steady-state vector **s** satisfies $\mathbf{Ts}=\mathbf{s}$.

Tip

Since the long-term probabilities are independent of the initial probabilities, you could have found the long-term market share in Worked Example 8.17 by just finding a high power of the transition matrix itself. The entry in each row will converge to the long-term probability of being in that state:

$$\mathbf{T}^{50}=\begin{pmatrix}0.345&0.345&0.345\\0.276&0.276&0.276\\0.379&0.379&0.379\end{pmatrix}$$

You may be asked to find the exact steady-state vector by setting up and solving a system of equations.

WORKED EXAMPLE 8.18

For the system in Worked Example 8.16,

a set up a system of linear equations for the steady-state market shares of the three taxi operators

b solve this system to find the exact values of these market shares.

When the steady state, **s**, is reached, $\mathbf{Ts} = \mathbf{s}$

$$\mathbf{Ts} = \mathbf{s}$$

Let $s = \begin{pmatrix} s_1 \\ s_2 \\ s_3 \end{pmatrix}$

a $$\begin{pmatrix} 0.85 & 0.05 & 0.1 \\ 0.05 & 0.8 & 0.1 \\ 0.1 & 0.15 & 0.8 \end{pmatrix}\begin{pmatrix} s_1 \\ s_2 \\ s_3 \end{pmatrix} = \begin{pmatrix} s_1 \\ s_2 \\ s_3 \end{pmatrix}$$

Multiply out to get three simultaneous equations

$$\begin{cases} 0.85s_1 + 0.05s_2 + 0.1s_3 = s_1 \\ 0.05s_1 + 0.8s_2 + 0.1s_3 = s_2 \\ 0.1s_1 + 0.15s_2 + 0.8s_3 = s_3 \end{cases}$$

Rearrange into the standard format with a constant on the RHS

There is also a fourth equation, which arises because the probabilities s_1, s_2, s_3 must sum to 1

$$\begin{cases} -0.15s_1 + 0.05s_2 + 0.1s_3 = 0 \\ 0.05s_1 - 0.2s_2 + 0.1s_3 = 0 \\ 0.1s_1 + 0.15s_2 - 0.2s_3 = 0 \\ s_1 + s_2 + s_3 = 1 \end{cases}$$

Solve using the GDC

b $s_1 = \frac{10}{29},\ s_2 = \frac{8}{29},\ s_3 = \frac{11}{29}$

So, the exact steady-state market shares of Alpha, Beta and Gamma are $\frac{10}{29}, \frac{8}{29}$ and $\frac{11}{29}$, respectively.

Exercise 8D

For questions 1 to 3, use the method demonstrated in Worked Example 8.13 to write down the transition matrix for the given information.

1 **a** $P(A \to B) = 0.75$
$P(B \to B) = 0.3$

b $P(A \to A) = 0.45$
$P(B \to A) = 0.9$

2 **a** $P(A \to A) = 0.82$, $P(A \to B) = 0.06$
$P(B \to B) = 0.73$, $P(B \to C) = 0.19$
$P(C \to A) = 0.22$, $P(C \to C) = 0.67$

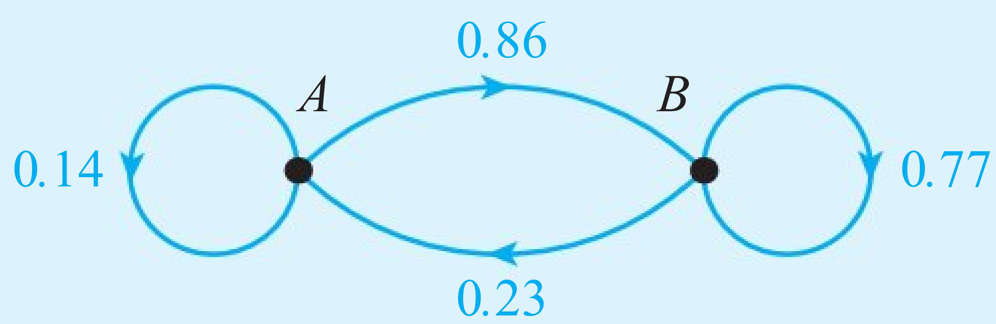
b $P(A \to B) = 0.02$, $P(A \to C) = 0.34$
$P(B \to B) = 0.57$, $P(B \to C) = 0.43$
$P(C \to A) = 0.45$, $P(C \to B) = 0$

3 **a** $\mathrm{P}(A \to A) = 0.71$, $\mathrm{P}(A \to B) = 0.29$, $\mathrm{P}(A \to C) = 0$
$\mathrm{P}(B \to B) = 0.88$, $\mathrm{P}(B \to C) = 0.05$, $\mathrm{P}(B \to D) = 0.07$
$\mathrm{P}(C \to A) = 0.24$, $\mathrm{P}(C \to B) = 0$, $\mathrm{P}(C \to C) = 0.57$
$\mathrm{P}(D \to A) = 0.45$, $\mathrm{P}(D \to C) = 0.07$, $\mathrm{P}(D \to D) = 0.48$

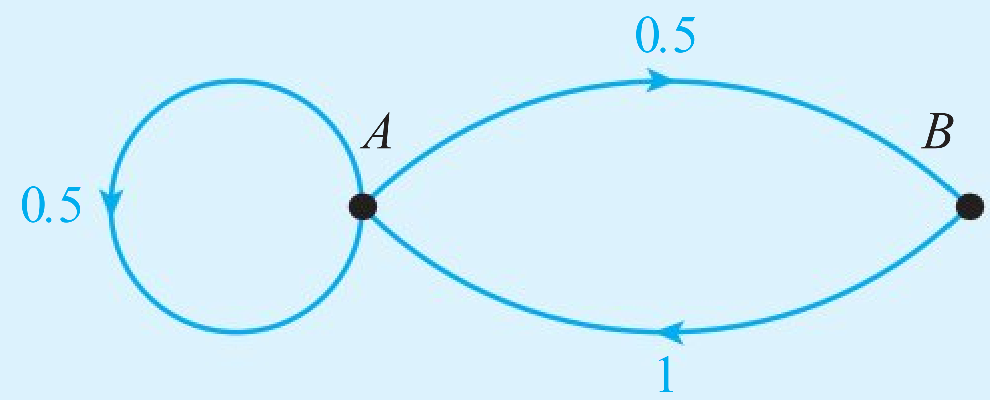
b $\mathrm{P}(A \to B) = 0.1$, $\mathrm{P}(A \to C) = 0.25$, $\mathrm{P}(A \to D) = 0.3$
$\mathrm{P}(B \to A) = 0.15$, $\mathrm{P}(B \to C) = 0.2$, $\mathrm{P}(B \to D) = 0.25$
$\mathrm{P}(C \to A) = 0.1$, $\mathrm{P}(C \to B) = 0.05$, $\mathrm{P}(C \to D) = 0.35$
$\mathrm{P}(D \to A) = 0.2$, $\mathrm{P}(D \to B) = 0.2$, $\mathrm{P}(D \to C) = 0.05$

For questions 4 to 6, use the method demonstrated in Worked Example 8.14 to write down the transition matrix for each given transition diagram.

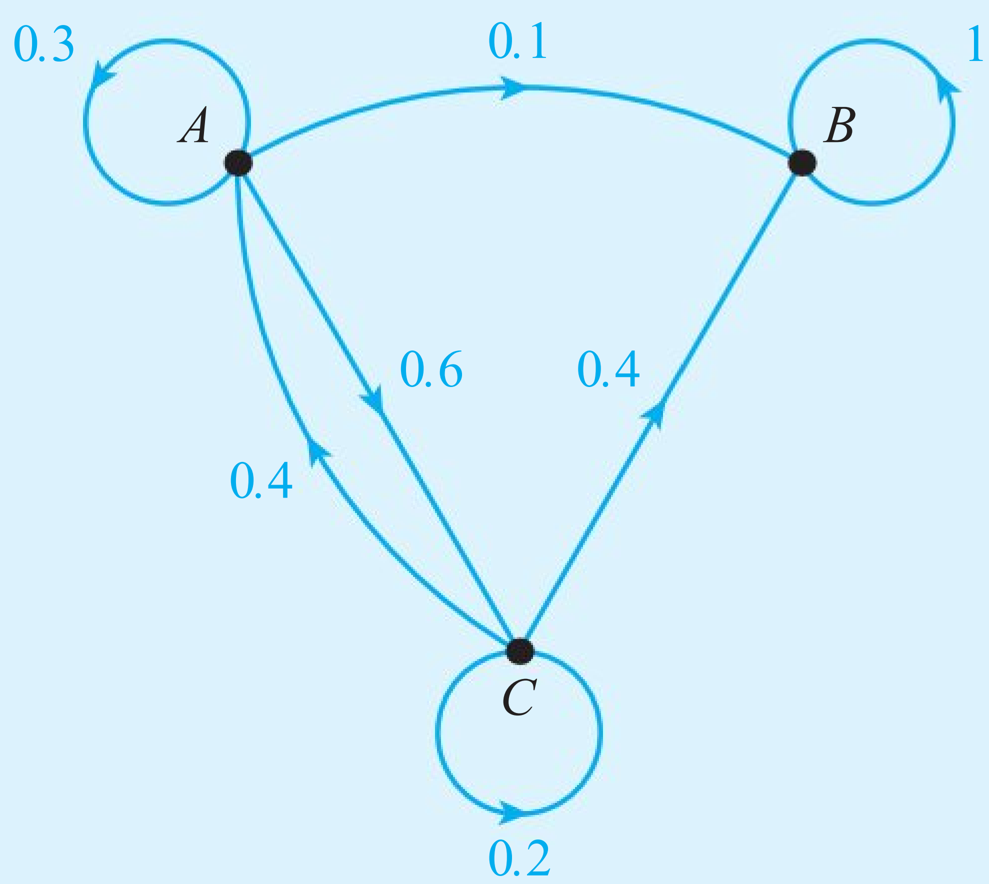
4 **a**



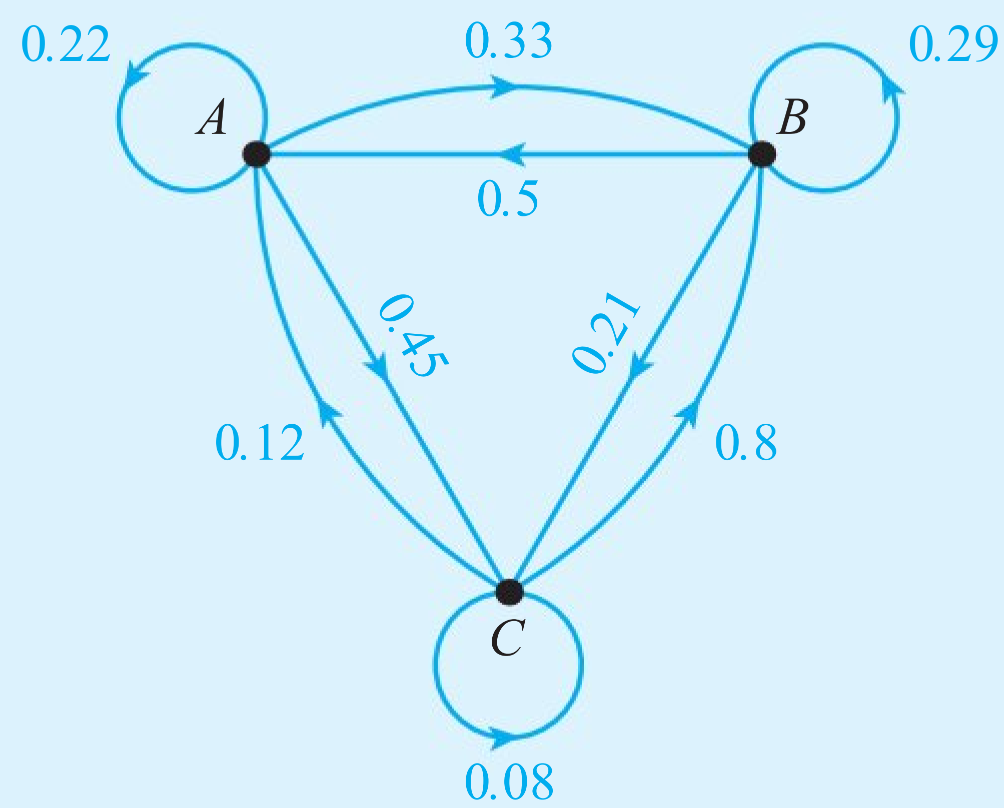
b



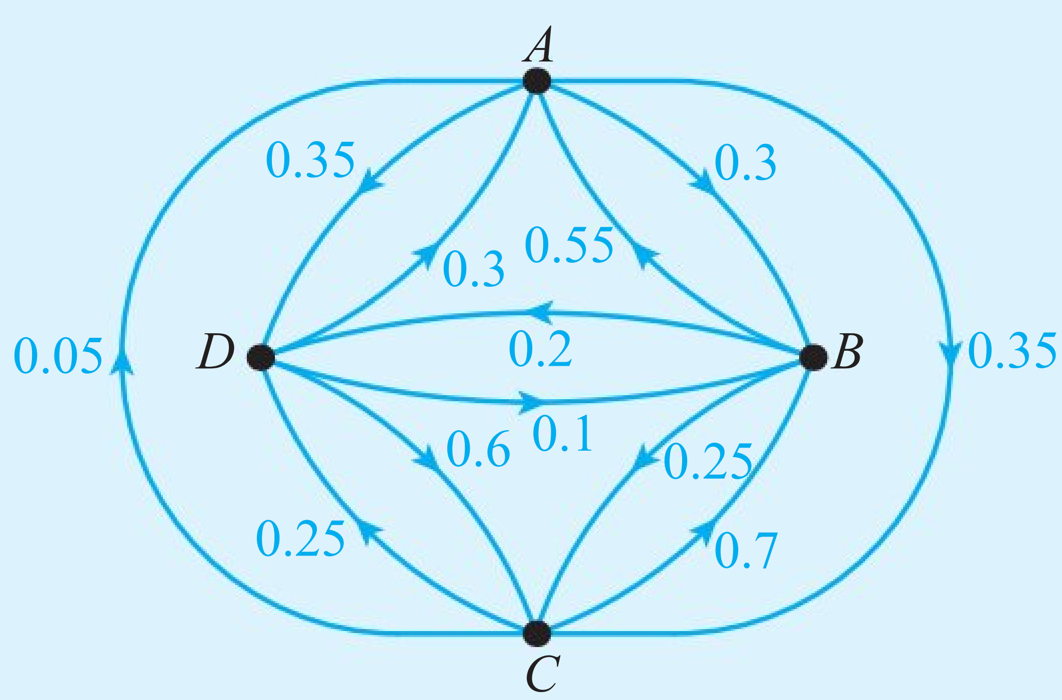
5 **a**



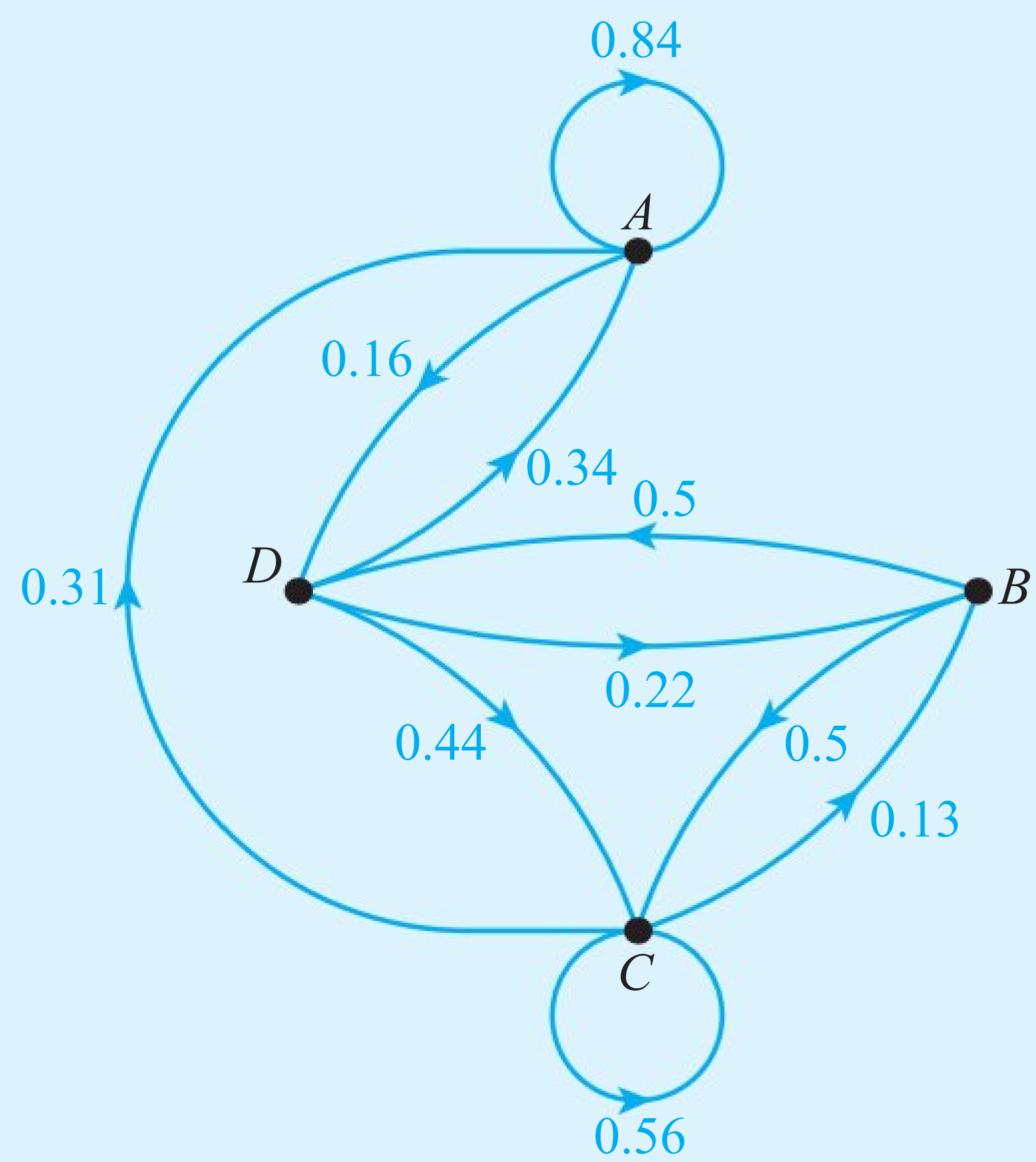
b



6 **a**



b



For questions 7 to 9, use the method demonstrated in Worked Example 8.15 to find the required probability from the given transition matrix.

7 **a** $\mathbf{T} = \begin{array}{c} \begin{array}{cc} A & B \end{array} \\ \begin{pmatrix} 0.2 & 0.4 \\ 0.8 & 0.6 \end{pmatrix} \end{array} \begin{array}{c} \\ A \\ B \end{array}$

Probability of being in state B four days from now, given that in state A now.

b $\mathbf{T} = \begin{array}{c} \begin{array}{cc} A & B \end{array} \\ \begin{pmatrix} 0.65 & 0.5 \\ 0.35 & 0.5 \end{pmatrix} \end{array} \begin{array}{c} \\ A \\ B \end{array}$

Probability of being in state A three days from now, given that in state A now.

8 **a** $\mathbf{T} = \begin{array}{c} \begin{array}{ccc} A & B & C \end{array} \\ \begin{pmatrix} 0.25 & 0.45 & 0.35 \\ 0.5 & 0.15 & 0.35 \\ 0.25 & 0.4 & 0.3 \end{pmatrix} \end{array} \begin{array}{c} \\ A \\ B \\ C \end{array}$

Probability of being in state C five days from now, given that in state C now.

b $\mathbf{T} = \begin{array}{c} \begin{array}{ccc} A & B & C \end{array} \\ \begin{pmatrix} 0.9 & 0.1 & 0 \\ 0.1 & 0.7 & 0.2 \\ 0 & 0.2 & 0.8 \end{pmatrix} \end{array} \begin{array}{c} \\ A \\ B \\ C \end{array}$

Probability of being in state C six days from now, given that in state B now.

9 **a** $\mathbf{T} = \begin{array}{c} \begin{array}{cccc} A & B & C & D \end{array} \\ \begin{pmatrix} 0.54 & 0.25 & 0.1 & 0.3 \\ 0.16 & 0 & 0.73 & 0.08 \\ 0.2 & 0.61 & 0.17 & 0 \\ 0.1 & 0.14 & 0 & 0.62 \end{pmatrix} \end{array} \begin{array}{c} \\ A \\ B \\ C \\ D \end{array}$

Probability of being in state D two days from now, given that in state B now.

b $\mathbf{T} = \begin{array}{c} \begin{array}{cccc} A & B & C & D \end{array} \\ \begin{pmatrix} 1 & 0 & 0.5 & 0.2 \\ 0 & 0.4 & 0.1 & 0 \\ 0 & 0.3 & 0.4 & 0.6 \\ 0 & 0.3 & 0 & 0.2 \end{pmatrix} \end{array} \begin{array}{c} \\ A \\ B \\ C \\ D \end{array}$

Probability of being in state B four days from now, given that in state D now.

For questions 10 to 12, use the method demonstrated in Worked Example 8.16 to find the probability vector $\mathbf{s}_n$ for each given transition matrix $\mathbf{T}$ and initial state probability vector $\mathbf{s}_0$.

10 **a** $\mathbf{T} = \begin{pmatrix} 0.82 & 0.26 \\ 0.18 & 0.74 \end{pmatrix}$, $\mathbf{s}_0 = \begin{pmatrix} 0.7 \\ 0.3 \end{pmatrix}$

Find $\mathbf{s}_3$.

b $\mathbf{T} = \begin{pmatrix} 0.43 & 0.61 \\ 0.57 & 0.39 \end{pmatrix}$, $\mathbf{s}_0 = \begin{pmatrix} 0.1 \\ 0.9 \end{pmatrix}$

Find $\mathbf{s}_2$.

11 **a** $\mathbf{T} = \begin{pmatrix} 0.8 & 0.05 & 0.06 \\ 0.12 & 0.92 & 0.1 \\ 0.08 & 0.03 & 0.84 \end{pmatrix}$, $\mathbf{s}_0 = \begin{pmatrix} 0.4 \\ 0 \\ 0.6 \end{pmatrix}$

Find $\mathbf{s}_4$.

b $\mathbf{T} = \begin{pmatrix} 0.9 & 0 & 0.1 \\ 0 & 0.8 & 0.2 \\ 0.1 & 0.2 & 0.7 \end{pmatrix}$, $\mathbf{s}_0 = \begin{pmatrix} 0.2 \\ 0.3 \\ 0.5 \end{pmatrix}$

Find $\mathbf{s}_3$.

12 **a** $\mathbf{T} = \begin{pmatrix} 0 & 0.7 & 0.1 & 0.2 \\ 0.2 & 0.2 & 0.6 & 0 \\ 0.3 & 0.1 & 0 & 0.7 \\ 0.5 & 0 & 0.3 & 0.1 \end{pmatrix}$, $\mathbf{s}_0 = \begin{pmatrix} 0.1 \\ 0.2 \\ 0.3 \\ 0.4 \end{pmatrix}$

Find $\mathbf{s}_5$

b $\mathbf{T} = \begin{pmatrix} 0.05 & 0.2 & 0.2 & 0.05 \\ 0.1 & 0.6 & 0.2 & 0.7 \\ 0.45 & 0.15 & 0.4 & 0.05 \\ 0.4 & 0.05 & 0.2 & 0.2 \end{pmatrix}$, $\mathbf{s}_0 = \begin{pmatrix} 0.7 \\ 0 \\ 0.1 \\ 0.2 \end{pmatrix}$

Find $\mathbf{s}_4$

For questions 13 to 15, use the method demonstrated in Worked Example 8.17 to find the steady-state probability vector for each of the transition matrices from questions 10 to 12. As noted in the tip below Worked Example 8.17, you do not need an initial state vector to calculate this for these matrices.

13 **a** $\mathbf{T} = \begin{pmatrix} 0.82 & 0.26 \\ 0.18 & 0.74 \end{pmatrix}$

b $\mathbf{T} = \begin{pmatrix} 0.43 & 0.61 \\ 0.57 & 0.39 \end{pmatrix}$

14 **a** $\mathbf{T} = \begin{pmatrix} 0.8 & 0.05 & 0.06 \\ 0.12 & 0.92 & 0.1 \\ 0.08 & 0.03 & 0.84 \end{pmatrix}$

b $\mathbf{T} = \begin{pmatrix} 0.9 & 0 & 0.1 \\ 0 & 0.8 & 0.2 \\ 0.1 & 0.2 & 0.7 \end{pmatrix}$

15 **a** $\mathbf{T} = \begin{pmatrix} 0 & 0.7 & 0.1 & 0.2 \\ 0.2 & 0.2 & 0.6 & 0 \\ 0.3 & 0.1 & 0 & 0.7 \\ 0.5 & 0 & 0.3 & 0.1 \end{pmatrix}$

b $\mathbf{T} = \begin{pmatrix} 0.05 & 0.2 & 0.2 & 0.05 \\ 0.1 & 0.6 & 0.2 & 0.7 \\ 0.45 & 0.15 & 0.4 & 0.05 \\ 0.4 & 0.05 & 0.2 & 0.2 \end{pmatrix}$

For questions 16 to 18, use the method demonstrated in Worked Example 8.18 to find the exact steady-state probability vector for each of the transition matrices from questions 13 to 15 by setting up and solving a system of equations.

16 **a** $\mathbf{T} = \begin{pmatrix} 0.82 & 0.26 \\ 0.18 & 0.74 \end{pmatrix}$

b $\mathbf{T} = \begin{pmatrix} 0.43 & 0.61 \\ 0.57 & 0.39 \end{pmatrix}$

17 **a** $\mathbf{T} = \begin{pmatrix} 0.8 & 0.05 & 0.06 \\ 0.12 & 0.92 & 0.1 \\ 0.08 & 0.03 & 0.84 \end{pmatrix}$

b $\mathbf{T} = \begin{pmatrix} 0.9 & 0 & 0.1 \\ 0 & 0.8 & 0.2 \\ 0.1 & 0.2 & 0.7 \end{pmatrix}$

18 **a** $\mathbf{T} = \begin{pmatrix} 0 & 0.7 & 0.1 & 0.2 \\ 0.2 & 0.2 & 0.6 & 0 \\ 0.3 & 0.1 & 0 & 0.7 \\ 0.5 & 0 & 0.3 & 0.1 \end{pmatrix}$

b $\mathbf{T} = \begin{pmatrix} 0.05 & 0.2 & 0.2 & 0.05 \\ 0.1 & 0.6 & 0.2 & 0.7 \\ 0.45 & 0.15 & 0.4 & 0.05 \\ 0.4 & 0.05 & 0.2 & 0.2 \end{pmatrix}$

19 The transition matrix $\mathbf{T}$ gives the probabilities of people's next purchase of car being an automatic A or a manual (M).

$$\mathbf{T} = \begin{array}{c} \begin{array}{cc} A & M \end{array} \\ \begin{pmatrix} 0.95 & 0.35 \\ 0.05 & 0.65 \end{pmatrix} \end{array} \begin{array}{c} \\ A \\ M \end{array}$$

a Write down the probability of someone who currently owns a manual buying an automatic as their next car.

b Find the long-term probability matrix.

c Hence state the long-term percentage of manual cars.

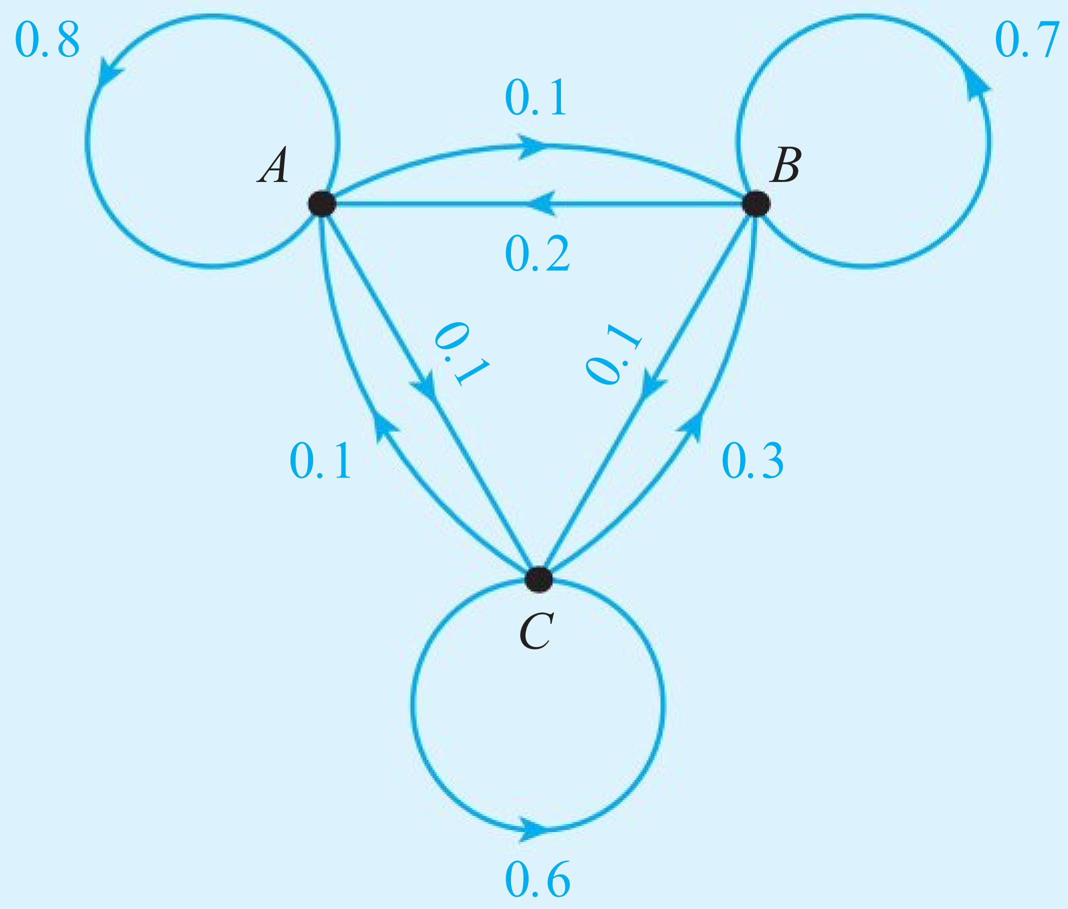
20 If Cristiano scored his previous penalty, the probability of him scoring his next penalty is 0.95. If he missed his previous penalty, his probability of scoring next time is 0.8.

a Write down the transition matrix for this situation.

b Given that he missed his last penalty, find the probability that he scores with his third penalty after the one he missed.

c Find his steady-state probability vector for scoring and missing.

21 There are three convenience stores in a small town. The transition diagram gives the probabilities of people swapping between stores for their weekly shop.



a Write down the transition matrix for this information.

In a particular week, 400 people use Store A, 240 people use Store B and 360 people use Store C.

b Find the number of customers using each store four weeks later.

c Find the steady-state number of customers using each store.

22 The population of otters in a particular area of wetland was surveyed over the course of many years. Each year, the area is designated as either empty (E), lightly populated (L) or heavily populated (H).

The probabilities of changing between these states each year are given in the transition matrix $\mathbf{T}$:

$$\begin{array}{c} \\ \mathbf{T} = \end{array} \begin{array}{c} \begin{array}{ccc} E & L & H \end{array} \\ \begin{pmatrix} 0.1 & 0.1 & 0.6 \\ 0.4 & 0.6 & 0.4 \\ 0.5 & 0.3 & 0 \end{pmatrix} \end{array} \begin{array}{c} \\ E \\ L \\ H \end{array}$$

a Draw a transition diagram for this matrix.

b If the area is initially empty, find the probability of it being heavily populated three years later.

c Find the long-term probability of the area being lightly populated.

23 Financial markets are categorized as bullish (values generally rising), bearish (values generally falling) or stagnant (values generally neither rising nor falling).

If a market has had a bullish week, the probability of the next week also being bullish is 0.85 and the probability of the next week being bearish is 0.05.

If a market is bearish, the probability of the next week being bullish is 0.1 and the probability of the next week being bearish is 0.75.

If a market is stagnant, then the probability of the next week being bullish is 0.25 and the probability of the next week being bearish is 0.25.

a Write down the transition matrix for this system.

b If the current week is bearish, find the probability that three weeks from now will be a bullish week.

c Find the long-term probabilities for this system.

24 A gambler takes \$20 into a casino to play roulette. He places \$10 on red each time. If the ball lands on red, he wins \$10; if the ball lands on black, he loses \$10. He continues to play until he either loses all his money or reaches \$30.

a Assuming that the probabilities of getting red and black are both 0.5, set up a transition matrix between the states \$0, \$10, \$20 and \$30.

b Find the probability that he ends up leaving with \$30.

25 At the end of the summer season, the entire colony of a particular species of bird migrates to one of two locations, A or B.

If a bird migrates to A this year, the probability that it migrates to A next year is 0.7.

If a bird migrates to B this year, the probability that it migrates to A next year is 0.4.

a Write down the transition matrix for this system.

Initially, 45% of birds migrate to location A and 55% to location B.

b Find the proportion of birds migrating to A two years later.

c Set up a system of equations for the long-term proportions migrating to A and B.

d Solve these equations to find the exact steady state proportions.

26 Pairs of genes can be dominant (D), hybrid (H) or recessive (R).

If one of the parents is known to be hybrid, then the probabilities of any offspring being dominant, hybrid or recessive depends on the second parent as given by the matrix

$$\begin{array}{c} \textbf{Parent} \\ \begin{array}{ccc} D & H & R \end{array} \end{array}$$

$$\mathbf{T} = \begin{pmatrix} 0.5 & 0.25 & 0 \\ 0.5 & 0.5 & 0.5 \\ 0 & 0.25 & 0.5 \end{pmatrix} \begin{array}{l} D \\ H \\ R \end{array} \quad \textbf{Offspring}$$

A rabbit of unknown characteristic is mated with a hybrid rabbit, and then the offspring are mated with a hybrid, and so on through the generations.

a Find the probability that that the second generation offspring are recessive given that the parent was dominant.

b Set up a system of equations for the long-term probabilities of offspring being dominant, hybrid or recessive.

c Solve these equations to find the long-term probabilities.

27 In a model of population movement, people either live in an urban (U) or a rural (R) area. The proportion of those moving from urban to rural areas each year is 0.1 and the proportion of those moving from rural to urban areas is 0.15.

a Write down the transition matrix, $\mathbf{T}$, for population movement each year.

b Find the eigenvalues and corresponding eigenvectors of $\mathbf{T}$.

c Hence write down matrices $\mathbf{P}$ and $\mathbf{D}$ such that $\mathbf{T} = \mathbf{PDP}^{-1}$.

Initially, 65% of people live in urban areas and 35% live in rural areas.

d Find an expression for the proportion of people in urban areas after n years.

e Hence find the proportion of people living in urban areas in the long term.

28 In any given year, a field of crops is either diseased or healthy. If the crop is healthy in a particular year, then the probability of it being diseased the next year is 0.2. If the crop is diseased in a particular year, then the probability of it being healthy the next year is 0.6.

a Write down the transition matrix, $\mathbf{T}$, for the state of the field.

b Find the eigenvalues and corresponding eigenvectors of $\mathbf{T}$.

c Hence write down matrices $\mathbf{P}$ and $\mathbf{D}$ such that $\mathbf{T} = \mathbf{PDP}^{-1}$.

In year zero the field is healthy.

d Find an expression for the probability of it being diseased in year n.

e Hence find the long term probability of the field being diseased.

Checklist

- You should know how linear transformations of a single random variable affect the expected value and variance:
  - $\mathrm{E}(aX + b) = a\mathrm{E}(X) + b$
  - $\mathrm{Var}(aX + b) = a^2\mathrm{Var}(X)$
- You should be able to find the expected value and variance of a linear combination of two or more random variables:
  - $\mathrm{E}(aX + bY + c) = a\mathrm{E}(X) + b\mathrm{E}(Y)$
  - $\mathrm{Var}(aX + bY + c) = a^2\mathrm{Var}(X) + b^2\mathrm{Var}(Y) + c$

  The second result is only true if X and Y are independent.
- You should be able to find the expected value and variance of the mean of a random variable.
  If $X_1, X_2, \ldots, X_n$ are independent observations of the random variable X, then:
  - $\mathrm{E}(\bar{X}) = \mathrm{E}(X)$
  - $\mathrm{Var}(\bar{X}) = \dfrac{\mathrm{Var}(X)}{n}$
- You should know about the distribution of linear combinations of normally distributed random variables.

  If the random variables X and Y are independent with $X \sim \mathrm{N}(\mu_X, \sigma_X^2)$ and $Y \sim \mathrm{N}(\mu_Y, \sigma_Y^2)$ and if $W = aX + bY + c$, then:

  $W \sim \mathrm{N}(a\mu_X + b\mu_Y + c,\ a^2\sigma_X^2 + b^2\sigma_Y^2)$
- You should know about the distribution of the mean of a normally distributed random variable.

  If $X_1, X_2, \ldots, X_n$ are independent observations of the random variable $X \sim \mathrm{N}(\mu, \sigma^2)$, then:

  $\bar{X} \sim \mathrm{N}\left(\mu, \dfrac{\sigma^2}{n}\right)$
- You should know about the distribution of the sum or mean of many observations of a random variable from any distribution (the central limit theorem).

  If $X_1, X_2, \ldots, X_n$ with $n > 30$ are independent observations from any distribution you will meet, then approximately:
  - $\displaystyle\sum_{i=1}^{n} X_n \sim \mathrm{N}(n\mu, n\sigma^2)$
  - $\bar{X} \sim \mathrm{N}\left(\mu, \dfrac{\sigma^2}{n}\right)$
- You should know about the circumstances under which the Poisson distribution is an appropriate model.

  The Poisson distribution occurs when the following conditions are satisfied:
  - events are independent of each other
  - events occur at a constant average rate
  - events occur singly (one at a time).
- You should know how to find the mean and variance of the Poisson distribution.

  If $X \sim \mathrm{Po}(\lambda)$, then:
  - $\mathrm{E}(X) = \lambda$
  - $\mathrm{Var}(X) = \lambda$

- You should know about the distribution of the sum of two independent Poisson distributions.

  If $X \sim \text{Po}(\lambda_X)$ and $\sim \text{Po}(\lambda_Y)$ are two independent Poisson distributions and $Z = X + Y$, then:

  $Z \sim \text{Po}(\lambda_X + \lambda_Y)$
- You should know about the role of transition matrices in Markov chains.

  A transition matrix is a matrix in which:
  - the probability of going from state A to state B is given by the entry in column A and row B
  - the entries in each column must sum to 1.
- You should know about initial state probability vectors.

  For a transition matrix $\mathbf{T}$ and initial state vector $\mathbf{s}_0$, the state after n time periods, $\mathbf{s}_n$, is given by $\mathbf{T}^n\mathbf{s}_0 = \mathbf{s}_n$.
- You should know how to calculate steady-state and long-term probabilities of Markov chains.

  For a transition matrix $\mathbf{T}$, the steady-state vector $\mathbf{s}$ satisfies $\mathbf{Ts} = \mathbf{s}$.

Mixed Practice

1 The probability distribution of a random variable X is given in the following table.

$\boldsymbol{x}$	1	2	3	4	5
$\mathbf{P}(\boldsymbol{X} = \boldsymbol{x})$	0.1	0.2	0.3	0.2	0.2

a Find $\text{E}(X)$.

The random variable Y is given by $Y = 2 - 3X$.

b Find $\text{E}(Y)$.

c Given that $\text{Var}(X) = 1.56$, find $\text{Var}(Y)$.

2 Random variable V has the probability distribution given in the table.

$\boldsymbol{v}$	2	3	5	7
$\mathbf{P}(\boldsymbol{V} = \boldsymbol{v})$	0.4	p	$2p$	$3p$

a Find the value of p.

b Find the mean of V.

The random variable W is given by $W = 9 - 2V$.

c Find the mean of W.

d Given that the standard deviation of V is 2.14, find the standard deviation of W.

3 The random variable X has the probability distribution given by

$$\text{P}(X = x) = \frac{3x - 1}{26} \quad \text{for } x = 1, 2, 3, 4$$

a Show this probability distribution in a table.

b Find the exact value of $\text{E}(X)$.

c Given that $\text{Var}(X) = 0.917$ to three significant figures, find $\text{Var}(20 - 5X)$ correct to three significant figures.

4 A case of wine contains 5 bottles. The mean mass of a bottle of wine is 1.2 kg with a variance of 0.1 kg^2.

An empty case has a mass of 0.4 kg with a variance of 0.02 kg^2.

a Find
 i the mean mass of a full case of wine
 ii the variance of a full case.

b State an assumption you needed to make in part **a ii**.

5 The heights of trees in a forest have a mean of 20 m and a variance of 55 m^2. A sample of 35 trees is measured.

a Find the mean and variance of the average height of the trees in the sample.

b Use the central limit theorem to find the probability that the average height of the trees in the sample is less than 18 m.

6 The number of cars arriving at a car park in a five-minute interval follows a Poisson distribution with mean 8, and the number of motorbikes follows a Poisson distribution with mean 1.4.

Find the probability that exactly 10 vehicles arrive at the car park in a particular five-minute interval.

7 The number of tweets per day from a school Twitter account follows a distribution with mean 9 and standard deviation 3.

a Find the mean and standard deviation of the total number of tweets put out in a five-day week.

b State any assumptions made in part **a**.

8 X is the random variable 'number of pizzas ordered per hour in a restaurant'. It is thought that $X \sim \text{Po}(7.3)$.

a Write down two conditions required for the Poisson distribution to model data.

b Find $P(4 < X \leqslant 10)$.

9 Based on long experience, a gardener knows that birds tend to arrive at his garden at an average rate of 12 per hour.

a State two assumptions required to model the birds' arrival using a Poisson distribution. Are these reasonable assumptions?

b If these assumptions do hold, find the probability of the gardener observing more than 15 birds in an hour.

10 Consumers have three options for their broadband provider: Pacey Play, Rapid Rate or Super Speedy.

The transition matrix, **T**, for changing between these companies each year is given by

$$\mathbf{T} = \begin{array}{c} \\ \\ \\ \\ \end{array}\begin{array}{c} \begin{array}{ccc} P & R & S \end{array} \\ \begin{pmatrix} 0.71 & 0.08 & 0.04 \\ 0.22 & 0.82 & 0.05 \\ 0.07 & 0.10 & 0.91 \end{pmatrix} \end{array} \begin{array}{c} \\ P \\ R \\ S \end{array}$$

a State the probability of a customer who currently uses Super Speedy changing to Pacey Play next year.

b Given that the current market share is Pacey Play 42%, Rapid Rate 38%, Super Speedy 20%, find the market share of each company in three years time.

c Find the steady-state market share of each company.

11 The n independent random variables $X_1, X_2, \ldots, X_n$ all have the distribution $\mathrm{N}(\mu, \sigma^2)$.

a Find the mean and the variance of

i $X_1 + X_2$

ii $3X_1$

iii $X_1 + X_2 - X_3$

iv $\bar{X} = \dfrac{(X_1 + X_2 + \ldots + X_n)}{n}$

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12 The mass of men in an office block is known to be normally distributed with mean 78 kg and standard deviation 8 kg. An elevator in the office block has a maximum recommended load of 600 kg.

With 7 men in the elevator, calculate the probability that their combined mass exceeds the maximum recommended load.

13 The volume of soda, X ml, in a can follows a normal distribution with mean 331 ml and standard deviation 2 ml. These cans are sold in packs of 6.

The total amount of soda in a pack is Y ml.

a Find $\mathrm{P}(Y < 1980)$.

b Find $\mathrm{E}(Y - 6X)$ and $\mathrm{Var}(Y - 6X)$.

c Find the probability that the volume of soda in a randomly chosen pack is more than 5 ml greater than 6 times the volume in a randomly chosen can.

14 The error a machine makes in cutting 10 m lengths of rope has a mean of 0 cm and a standard deviation of 0.5 cm.

A sample of 35 pieces of rope is taken.

Find the probability that the mean error of the sample is less than 0.1 cm.

15 A receptionist at a hotel answers on average 45 phone calls a day.

a State a possible model for the number of calls received per day and any assumptions you are making.

b Use your model to find the probability that, on a particular day, she will answer more than 50 phone calls.

c Find the probability that she will answer more than 45 phone calls every day during a five-day week.

16 In a particular town, rainstorms occur at an average rate of two per week and can be modelled using a Poisson distribution.

a Find the probability of at least eight rainstorms occurring during a particular four-week period.

b Given that the probability of at least one rainstorm occurring in a period of n complete weeks is greater than 0.99, find the least possible value of n.

17 A geyser erupts randomly. The eruptions at any given time are independent of one another and can be modelled using a Poisson distribution with mean 20 per day.

- **a** Determine the probability that there will be exactly one eruption between 9 am and 10 am.
- **b** Determine the probability that there are more than 22 eruptions during one day.
- **c** Determine the probability that there are no eruptions in the 30 minutes Dale spends watching the geyser.
- **d** Find the probability that the first eruption of a day occurs between 3 am and 4 am.
- **e** Determine the probability that there will be at least one eruption in at least six out of the eight hours the geyser is open for public viewing.
- **f** Given that there is at least one eruption in an hour, find the probability that there is exactly one eruption.

18 Patients arrive at random at an emergency room in a hospital at the rate of 14 per hour throughout the day.

- **a** Find the probability that exactly four patients will arrive at the emergency room between 12:00 and 12:15.
- **b** Given that fewer than 15 patients arrive in one hour, find the probability that more than 12 arrive.
- **c** Dr Chris works a 10 hour shift. Find the probability that, in at least 5 of those 10 hours, more than 15 patients arrive at the emergency room.

19 Compared to its value at the start of trading, the value of a share at the end of the day can have risen (R), fallen (F) or stayed the same (S). Over a prolonged period, a certain share was observed to perform in the following way. If its price:

- rises on a given day, the probability of it rising again the next day is 0.6, while the probability of it falling is 0.1
- falls on a given day, the probability of it rising the next day is 0.4, while the probability of it falling again is 0.4
- stays the same on a given day, the probability of it rising the next day is 0.5, while the probability of it falling is 0.2.

- **a** Write down the transition matrix for the share price movement.
- **b** If the price fell today, find the probability that it will rise in 3 days time.
- **c** **i** Set up a system of equations satisfied by the steady-state probabilities of the share price movement.
  - **ii** Hence find these exact steady-state probabilities.

20 Engine oil is sold in cans of two capacities, large and small. The amount, in millilitres, in each can, is normally distributed according to Large $\sim \mathrm{N}(5000, 40)$ and Small $\sim \mathrm{N}(1000, 25)$.

- **a** A large can is selected at random. Find the probability that the can contains at least 4995 millilitres of oil.
- **b** A large can and a small can are selected at random. Find the probability that the large can contains at least 30 millilitres more than five times the amount contained in the small can.
- **c** A large can and five small cans are selected at random. Find the probability that the large can contains at least 30 millilitres less than the total amount contained in the small cans.

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21 The number of cats visiting Helena's garden each week follows a Poisson distribution with mean $\lambda = 0.6$.
Find the probability that
i in a particular week no cats will visit Helena's garden
ii in a particular week at least three cats will visit Helena's garden
iii over a four-week period no more than five cats in total will visit Helena's garden
iv over a twelve-week period there will be exactly four weeks in which at least one cat will visit Helena's garden.

Mathematics HL November 2013 Paper 2 Q11 part **a**

22 The random variable X has probability distribution Po(8).
a **i** Find $P(X = 6)$.
ii Find $P(X = 6 | 5 \leqslant X \leqslant 8)$.
b $\bar{X}$ denotes the sample mean of $n > 1$ independent observations from X.
i Write down $E(\bar{X})$ and $\mathrm{Var}(\bar{X})$.
ii Hence, give a reason why $\bar{X}$ is not a Poisson distribution.
c A random sample of 40 observations is taken from the distribution for X.
i Find $P(7.1 < \bar{X} < 8.5)$.
ii Given that $P(|\bar{X} - 8| \leqslant k) = 0.95$, find the value of k.

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23 A marine biologist is investigating the lengths of a particular type of fish. It is known that the lengths have standard deviation 4.6 cm. She wishes to take a sample to estimate the mean length. She requires that the standard deviation of the sample mean is smaller than 0.5.

What sample size should she take? Justify any assumptions that you make.

24 Jars of jam have mean mass of 498 g and standard deviation σ g. The probability that 50 jars of jam weigh more than 25 kg is 4.23%.

Find the value of σ.

25 The marks students scored in a maths test follow a normal distribution with mean 63 and variance 64. The marks of the same group of students in an English test follow a normal distribution with mean 61 and variance 71.
a Find the probability that a randomly chosen student scored a higher mark in English than in maths.
b Find the probability that the average English mark of a class of 12 students is higher than their average maths mark.

26 The number of worms in each square metre of woodland is modelled by a Poisson distribution with mean 1.2.
a Find the probability that in a 2 m^2 area of woodland there are exactly two worms.
b Find the probability that in each of two 1 m^2 areas of woodland there is exactly one worm.
A scientist searches many different 1m^2 areas of the woodland. She only records the number of worms in areas where she finds some.
c Find the mean of her observations.

27 A shop has a delivery of 50 pairs of sunglasses every week during the summer season. Weekly demand for sunglasses during this period can be modelled by a Poisson distribution with mean 42.5.

a Assuming that there are no sunglasses in stock when a fresh delivery arrives, find the probability that the store then sells out of sunglasses that week.

b Find the most likely number of sunglasses sold in a given week.

c Find the minimum number of sunglasses the store should order to be 99% sure of meeting demand.

28 Each year, a particular plant either flowers or grows. If the plant flowers in a given year, the probability that it flowers the next year is 0.1. If the plant grows in a particular year, the probability that it flowers the next year is 0.6.

a Write down the transition matrix, $\mathbf{T}$, for the plant's activity.

b Find the eigenvalues and corresponding eigenvectors of $\mathbf{T}$.

c Hence write down matrices $\mathbf{P}$ and $\mathbf{D}$ such that $\mathbf{T} = \mathbf{PDP}^{-1}$.

In year zero a plant flowers.

d Find an expression for the probability of it flowering in year n.

e Hence find the long-term probability of the plant flowering.

9 Statistics

ESSENTIAL UNDERSTANDINGS

- Statistics is concerned with the collection, analysis and interpretation of quantitative data and uses the theory of probability to estimate parameters, discover empirical laws, test hypotheses and predict the occurrence of events.
- Statistical representations and measures allow us to represent data in many different forms to aid interpretation.
- Both statistics and probability provide important representations which enable us to make predictions, valid comparisons and informed decisions. These fields have power and limitations and should be applied with care and critically questioned in detail to differentiate between the theoretical and the empirical/observed.

In this chapter you will learn...

- how to conduct a statistical investigation
- about validity and reliability
- how to find unbiased estimates of the population mean and variance
- how and when to combine data in a χ^2 table
- how to choose an appropriate number of degrees of freedom in a χ^2 goodness of fit test when estimating parameters
- how to find non-linear regression models
- how to calculate the sum of square residuals for a regression model and use it to measure the fit for a model
- how to find the coefficient of determination and use it to measure the fit for a model
- how to find and interpret confidence intervals for the population mean
- how to conduct a hypothesis test for the population mean for the normal distribution when the population variance is known and when it is unknown
- how to conduct a hypothesis test for the difference between two population means when the population variance is known and when it is unknown
- how to conduct a hypothesis test for the difference in population mean for paired samples
- how to conduct a hypothesis test for the population proportion using the binomial distribution
- how to conduct a hypothesis test for the population mean rate using the Poisson distribution
- how to conduct a hypothesis test for the Pearson's population product-moment correlation coefficient
- how to find the critical region in hypothesis tests for normal tests, binomial tests and Poisson tests
- how to find the probability of Type II and Type II errors in hypothesis tests.

Figure 9.1 What level of certainty can we expect in the results?

CONCEPTS

The following concepts will be addressed in this chapter:

- Different statistical techniques require justification and identification of their limitations and **validity**.
- Correlation and regression are powerful tools for identifying **patterns** and equivalence of **systems**.
- **Modelling** and finding structure in seemingly random events facilitates prediction.
- Statistical literacy involves identifying reliability and **validity** of samples and whole populations in a closed **system**.
- A systematic approach to hypothesis testing allows statistical inferences to be tested for **validity**.

PRIOR KNOWLEDGE

Before starting this chapter, you should already be able to complete the following:

1 Find the mean and standard deviation for this data set.

2, 3, 4, 5, 5, 6, 7, 8, 9, 9

2 Conduct a χ^2 goodness of fit test on the data alongside where the critical value is 12.8 and H_0: Data come from $B(5, 0.7)$.

Data value	**0**	**1**	**2**	**3**	**4**	**5**
Observed frequency	7	28	103	188	132	42

3 For the contingency table alongside, test at the 10% significance level whether the two variables are independent.

18	13	8
13	16	21
8	18	22

4 For the data alongside,

a find Pearson's product-moment correlation coefficient and interpret this value

b find the equation of the regression line.

x	1	2	3	4	4	5	6	6	7	9
y	1	3	5	5	6	7	8	11	9	10

5 For the data in the table alongside, conduct a t-test at the 10% significance level of the hypotheses $H_0: \mu_1 = \mu_2$, $H_1: \mu_1 < \mu_2$

	Sample 1	**Sample 2**
Mean	5.4	5.8
Variance	4.3	3.9
Size	48	35

6 If $X \sim N(12, 4^2)$, find $P(X < 9)$.

7 If $X \sim B(10, 0.6)$, find $P(X \geqslant 5)$.

8 If $X \sim Po(5.2)$, find $P(X > 6)$.

This chapter extends a number of ideas met in the statistics chapters of the Mathematics: applications and interpretation SL book. As well as the possibility of fitting a linear regression model to data, we now look at several other regression models that might be more suitable, from quadratic and cubic to exponential and sinusoidal. We also revisit χ^2-tests, t-tests and tests for the correlation coefficient, as well as introducing hypothesis tests using the normal, binomial and Poisson distributions.

Of course, hypothesis tests do not offer certainty – it is always possible that the conclusion arrived at was incorrect. Being able to find the probability that the conclusion was in error is an important feature of setting up and evaluating any hypothesis test.

Starter Activity

Look at the pictures in Figure 9.1. Discuss what the possible errors are in any conclusion reached? Which potential error is the more serious to make in each case?

Now look at this problem:

A coin is flipped 20 times.

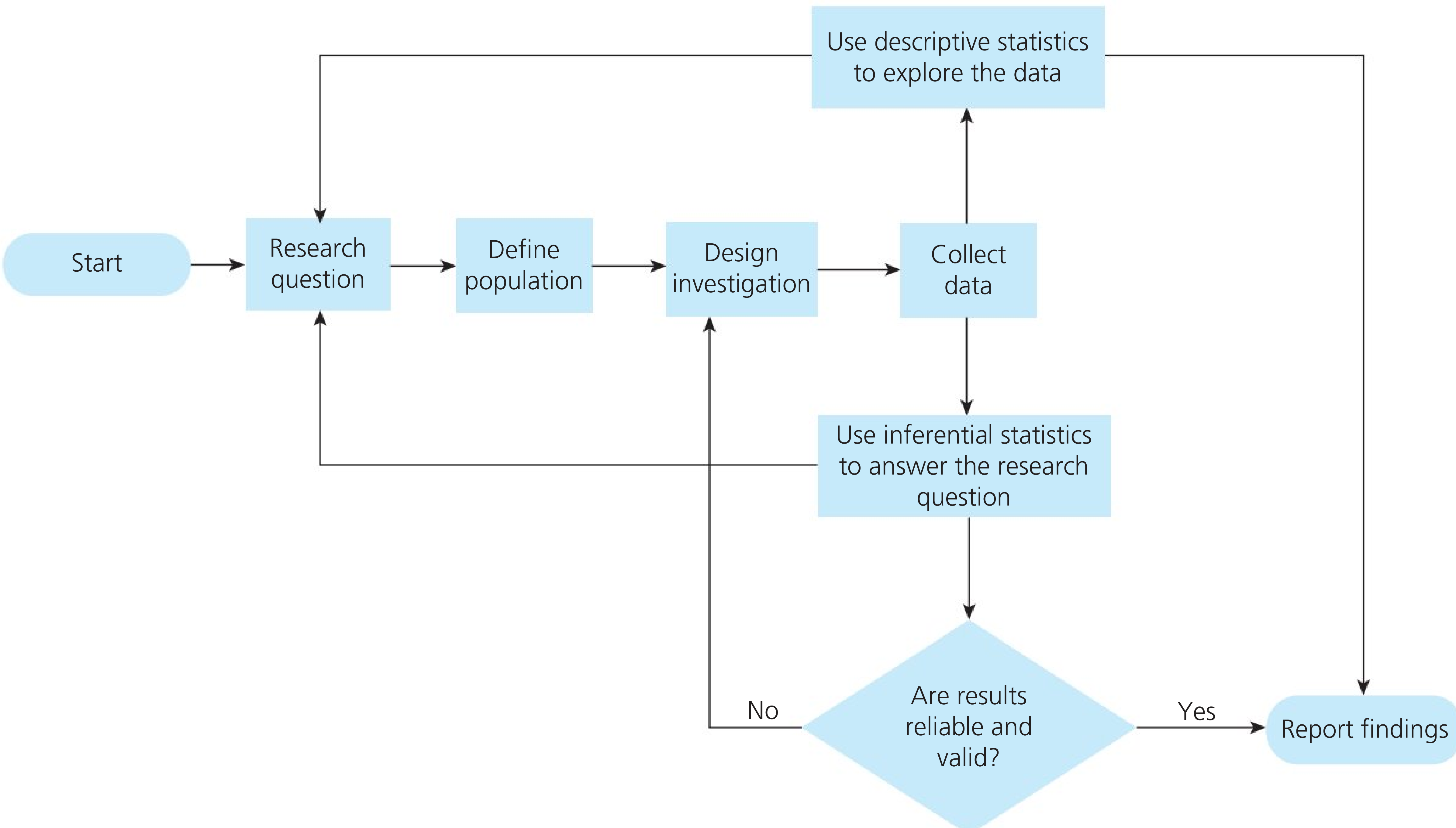
a What is the minimum number of heads would you need to see to conclude that the coin was biased in favour of heads?

b If you did see that number of heads, how confident would you be in your conclusion that the coin was biased in favour of heads?

9A Statistical design and unbiased estimators

The statistical investigation cycle

In the SL book, you met some very advanced ways to analyse data, such as regression and statistical tests. However, these analytical tools are only one part of the statistical process.



Research questions

The starting point for any investigation is to be clear about what you want to know. This might start off as a broad question such as 'does healthy body equal healthy mind?' You often then have to narrow this down to something which is clearer and easier to work with – for example 'Do people who are good at athletics do better in their academic studies?'

Define populations

A key idea in all of statistics is that you are inferring something about a population from a sample. Even if you had data for everybody in your school, that is just one school, in a particular year. You must be clear what the population of interest is – perhaps all 18-year-old IB students in your school – so that the sampling method you need to use becomes clear.

LEARNER PROFILE – Principled

Does 'fair' mean the same thing as 'equal'? Would it be fair for everybody to get equal results in an exam? How can mathematics be used to define 'fairness'?

Design investigation

This comprises several parts.

1 Selecting relevant variables from many variables

Most interesting research questions relate to ideas which are hard to measure. You have to choose proxy variables which are related to, but not quite the same as, the idea you are interested in. In the example above, we might choose time in a 100 m race as a proxy for athletic ability and score in their maths examination as a proxy for academic ability. Both of these have the advantage of being relatively easy to measure, but they are not identical to the original idea – people might be very good at the javelin, or very bad at examinations.

TOK Links

Are things which can be measured numerically more useful than intangible quantities? Can everything important be assigned a numerical value?

Equally, there might be a lot more data available than we want. We could create very complex models by taking into account every exam a student has ever taken, but that suffers from increasing technical difficulty and it is harder to communicate results clearly.

2 Designing data collection method

A **survey** is any method for collecting data for analysis. This might include questionnaires, interviews, direct observations or measurements and collecting secondary data from other sources. In the example above, we might collect academic data from the school's examination database and observe individuals' 100 m times at a sports day. However, this might have practical issues – we might not have access to the school's database and not everybody might take part in the 100 m in the school sports day. It might be easier to use a questionnaire to ask each pupil to self-report their exam results and 100 m times, although this still has issues with respondents not answering honestly and many people not responding at all.

A questionnaire is a list of questions, which sounds simple; however, it can be hard to design good questions.

- They should be unbiased – not revealing the personal opinions of the person setting the questionnaire, for example 'What is your opinion about burgers for lunch?' is better than 'Do you agree that burgers are unhealthy, disgusting and cruel to animals?'
- Questionnaires can be unstructured ('Describe your pain') or structured ('Rank your pain on a scale from 1 to 5, circling your answer'). Both have advantages – it is generally easier to analyse responses to structured questions, but unstructured questions can reveal more insight into a situation.

- Questionnaires may be personal, requiring people to be self-aware. This often causes issues. For example, asking people to rank their own happiness is notoriously difficult as one person's 7 out 10 happiness might be comparable to another person's 9 out of 10.
- Questionnaires should also be precise, so that respondents are clear about what is required. For example, the question 'How many people are in your family?' might be interpreted by some people as including aunts, cousins and grandparents. Some people might have step-parents or half-brothers and they may not be sure about whether these should be included. Another common imprecise question might be 'Would you like sausages or chips to be available in the canteen?' Some people might answer 'chips' and some people might just say 'yes'.

In Chapter 6 of the Mathematics: applications and interpretation SL book, you saw a variety of different sampling methods.

3 Choosing relevant and appropriate data

If you want to know about the salaries of IB graduates there is no point looking at the salaries of all the people in a country. Your sample should be relevant to the research question. It should also be appropriate for the analysis you are choosing to do. For example, a huge model linking hundreds of variables will require a very large data set. You should choose a sampling method to try to get a representative sample from the population of interest.

4 Choosing an appropriate statistical process

For exploratory work, you might want to use descriptive statistics – such as calculating averages or examining scatter graphs. This might be the final process, or it might inform further research questions.

To answer more sophisticated questions we tend to use inferential statistics such as *t*-tests and chi-squared tests. The types of questions you can ask are guided by the tests you have available – for example, you cannot really test causality (if one thing causes another), just correlation between variables.

Test for reliability and validity

The conclusions of a test are **reliable** if similar conclusions would be reached on each occasion the test is conducted in similar circumstances. There are two procedures you should be aware of to check on reliability.

- Test-retest is when you conduct the test and repeat it on new data from the same group sometime later. If the test is reliable you would expect a strong correlation between the results on the two occasions. There will be intervening factors and natural variation so the correlation does not have to be perfect.
- Parallel forms is when the same concept is measured in slightly different but comparable ways – for example, measuring 100 m and 200 m times of athletes. If the test is reliable, then there should be a strong correlation between the results when either measurement is used. However it can be quite difficult to find two different but comparable things to measure.

You are the Researcher

You might want to see how Cronbach alpha measures parallel forms.

The process is **valid** if it is measuring what you really want to measure. There are multiple threats to validity. The variables you are working with might not be entirely representative of the concept. For example, you might want to measure how healthy people are in a country and use life expectancy as a proxy, but this might really be measuring how good the doctors are at keeping people alive, with people having long but unhealthy lives. There might also be issues with the statistical processes being used. For example, you might use a t-test on data which are not drawn from a normal distribution. There are two procedures which can be used to check validity.

- Content validity checks are when experts assess whether the test is relevant to the content required. For example, whether a test on trigonometry is really about trigonometry or whether it also brings in tests of reading, using calculators, understanding contexts and other such confounding variables. Within statistics, this would also include checking whether the assumptions of statistical tests are satisfied.
- Criterion validity checks whether the test agrees with an external standard which is considered an authentic measure of the quality being investigated. For example, if a test for entering a school has criterion validity, then students who do well on the test should do well in the school.

If the tests for reliability and validity fail, then you might have to go back to adapt your research question, redesign your data collection (for example, use a larger sample) or use a different test.

You are the Researcher

There are a suite of tests called non-parametric tests which have fewer assumptions than the tests you have met so far. For example, instead of the t-test you could use the Mann-Whitney test.

Statistical investigations are very popular choices for mathematical explorations. Having a formal approach to testing reliability and validity would be a good way to demonstrate the sophistication and rigour aspects of Criterion E – Use of Mathematics.

WORKED EXAMPLE 9.1

A school surveys its staff by asking them to rank how tired they are, on a scale of 1 to 10.

Suggest

a whether this question is valid

b whether the results are reliable.

Validity will depend on how well the criteria are explained and what exactly is being investigated. Measuring work-related fatigue could be confounded by illness, self-awareness and wanting to exaggerate

a This is unlikely to be valid as the number is subjective, with the same number meaning different things to different people.

Reliability is about whether the results will be repeatable

b This is unlikely to be reliable as on a different day the answers may change – the last day of term may have a different answer to the first day of term.

Unbiased estimators

Descriptive statistics, such as the mean, range or variance, can be calculated for a sample. However, we are very rarely only interested in the sample. We want to infer something about the population. One way of doing this is to use an **unbiased estimator**. This is a statistic that, if we were to calculate it for many samples, would average to the true population value.

For the mean, it turns out that the sample mean $\bar{x}$ is an unbiased estimate of the population mean, μ.

KEY POINT 9.1

$\bar{x}$ is an unbiased estimate of μ.

Proof 2.1

Key Point 9.1 might seem intuitive, but the proof shows you how unbiased estimates work.

The sample mean is a random variable formed by adding up n independent observations of X and dividing by n

$$\bar{X} = \frac{1}{n}(X_1 + X_2 + \ldots + X_n)$$

Taking an average over many samples is mathematically done by finding an expectation

$$\mathrm{E}(\bar{X}) = \mathrm{E}\left(\frac{1}{n}(X_1 + X_2 + \ldots + X_n)\right)$$

This can be simplified by using linear transformations of random variables

$$= \frac{1}{n}(\mathrm{E}(X_1) + \mathrm{E}(X_2) + \ldots + \mathrm{E}(X_n))$$

Each expected value is the true population mean, μ

$$= \frac{1}{n}(\mu + \mu + \ldots + \mu)$$

There are n terms in the sum, each equal to μ so this simplifies

$$= \frac{1}{n}(n\mu)$$

$$= \mu$$

The variance of a sample s_n^2 will tend to slightly underestimate the true variance, σ^2, because a sample does not usually explore all the extremes of a population. The unbiased estimate of the population variance is s_{n-1}^2, and it can be found using the following formula.

KEY POINT 9.2

An unbiased estimate of the population variance is

$$s_{n-1}^2 = \frac{n}{n-1}s_n^2$$

This will usually be found directly from your calculator. Different calculators might use slightly different notation so you must make sure that you know how your calculator describes the unbiased estimate.

Tip

A common error is to think that this means that s_{n-1} is an unbiased estimate of the population standard deviation. It turns out that this is not the case.

You are the Researcher

You might like to prove the formula in Key Point 9.2. To do so you will need to research a general formula for variance of a random variable.

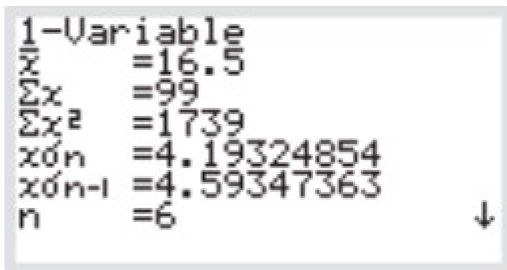
WORKED EXAMPLE 9.2

For the sample

11, 16, 14, 14, 21, 23

evaluate unbiased estimates of the population mean and variance.

Put the data into the calculator



The unbiased estimate for μ is $\bar{x} = 16.5$.

The unbiased estimate for $\sigma^2 \approx 4.59^2 = 21.1$.

Exercise 9A

For questions 1 to 3, use the ideas in Worked Example 9.1 to suggest **i** whether this question is valid and **ii** whether the results are reliable. Justify your answer.

1 **a** 'Have you ever committed a crime?'
b 'What is the best football team?'

2 **a** 'Do you like pizza?'
b 'Do you have any brothers?'

3 **a** 'Do you like maths or physics?'
b 'Have you enjoyed the session today?'

For questions 4 to 6, use the technique of Worked Example 9.2 to evaluate unbiased estimates of the population mean and variance for the given sample.

4 **a** 14, 19, 20, 22, 25, 25
b 7, 14, 21, 28, 35, 42

5 **a** 6, −8, 12, 0, −9, 4
b 9, −19, 14, 20, −6, −7

6 **a** 0.1, 0.2, 0.4, −0.4, 0.2, 0.1
b 0.1, 0.2, 3.9, 4.1, −0.6, 0.1

7 The standard deviation, s_n, of a sample of size 10 is 12.4. Find s_{n-1}.

8 The best estimate of the population standard deviation, s_{n-1}, of a sample is 1.118 (to 4 s.f.) times bigger than the sample standard deviation, s_n. Find the value of n.

9 Claudia collects a sample of the number of eggs laid by 6 snakes of a particular breed. Here are her results.

4, 3, 3, 8, 5, 4

a Find an unbiased estimate for the population mean of the number of eggs of snakes from this breed.
b Find an unbiased estimate for the population variance of the number of eggs of snakes from this breed.
c Claudia only sampled snakes from her local breeder. Explain why the processes used in parts **a** and **b** might not be valid.
d How could Claudia test the reliability of her result in part **a**?

10 A social scientist wants to look at people's experience of crime. He conducts a survey using a questionnaire. One of the questions asked is:

'How many criminal activities affected you in the last year?'

He leaves his questionnaire in a police station, asking people to post their completed questionnaires in a box. One week later, he has received the following responses to the question:

1, 2, 4, 2, 1, 3498

- **a** Explain why it is reasonable to remove the data item 3498 from his sample.
- **b** For the remaining data, calculate unbiased estimates of the population mean and population variance.
- **c** Comment on the content validity of the question.
- **d** Comment on the validity of the sampling method.
- **e** Is the social scientist's estimate of the population mean more likely to be an overestimate or an underestimate? Justify your answer.
- **f** How could the reliability of the survey be improved?

11
- **a** Define reliability.
- **b** A psychology test tries to assess people's personality types on four different categories. A test uses four questions to assess each category spread throughout a questionnaire. The psychologist wants to check whether the results of each question testing the same category correlate. Name the reliability test she is using.
- **c** Suggest an alternative way to assess the reliability of the test.

12
- **a** Define validity.
- **b** A career coach wants to assess his impact on people's business success. He does this by asking the following question at the end of a course he has run:

  'Do you think that this course will increase your salary?'

  Give two reasons why this question does not have content validity.
- **c** Explain why making the questionnaire anonymous might increase the validity of the question.
- **d** The career coach followed up on these responses four years later and checked to see whether the salary increases were greater for the people who answered yes to his question. What aspect of validity is he testing by doing this?

9B Further χ^2 tests

Categorizing numerical data in a χ^2 table

In the Mathematics: applications and interpretation SL book, you met the idea that the chi-squared distribution only works if all expected values are greater than 5. We can combine groups to make this happen.

WORKED EXAMPLE 9.3

In a survey at a sports club, members were asked to name their favourite sport out of baseball, basketball and ice-hockey.

		Age		
		13–14	**15–16**	**17–19**
Sport	**Baseball**	3	12	21
	Basketball	0	12	16
	Ice-hockey	8	16	11

a Find the expected values in a test for independence.

b Combine the columns to form an appropriate data set to test whether age and sport are independent.

c Hence test at 5% significance to see whether these data provide evidence for a link between age and sport.

You can use your calculator to find a matrix of expected values

a The expected values are

		Age		
		13–14	**15–16**	**17–19**
Sport	**Baseball**	4	14.5	17.5
	Basketball	3.11	11.3	13.6
	Ice-hockey	3.89	14.1	17.0

The 13–14 age group contains expected frequencies less than 5, so it needs to be combined with another column. The only column it makes sense to combine it with is the 15–16 age group

b Combining 13–14 and 15–16:

		Age	
		13–16	**17–19**
Sport	**Baseball**	15	21
	Basketball	12	16
	Ice-hockey	24	11

H_0: Sport and Age categories are independent.
H_1: Sport and Age categories are dependent.

The calculator can be used to conduct the chi-squared test. You should quote the degrees of freedom, chi-squared value and the p-value

c $\chi^2 = 6.31$, 2 degrees of freedom,
$p = 0.0425 < 0.05$
Therefore, there is significant evidence that age and sport preference are dependent.

You might have wondered why the default letter used to represent unknowns is x. One theory starts from the fact that, historically, maths was not written in equations using letters, but rather words.

$x - 3 = 2$ would have been written as 'three less than the unknown equals two'. However, at the time when such equations were being studied, Arabia was at the forefront of mathematics and 'unknown' would have been written as the Arabic word 'shalan'. Arabian influence in Europe was particularly strong in Spain and there was no Spanish equivalent for the sound 'sh' so they borrowed the Greek letter χ to start the word. Gradually this word was abbreviated to just χ then x.

χ^2 tests with estimated parameters

In the Mathematics: applications and interpretation SL book, you met the idea that you can test to see if data might have been drawn from a particular distribution – for example, $\text{Po}(1.2)$. However, you might not know in advance the parameter of the distribution. If you estimate it using the data, then that adds an extra constraint on the expected frequencies. This leads to a slight change in the formula for the chi-squared distribution.

KEY POINT 9.3

Degrees of freedom $= N - 1 - k$

where N is the number of groups and k is the number of parameters estimated.

WORKED EXAMPLE 9.4

a Estimate the mean of the population from which the following data are drawn.

X	0	1	2	3	4
Frequency	32	49	41	20	12

b Hence test at the 5% significance level to see if it has been drawn from a Poisson distribution, stating your null and alternative hypotheses.

You need to be able to input frequency distributions into a calculator

a From the GDC, $\bar{x} = 1.55$

In this case, the mean is not included in the hypotheses because there was no prior belief about its value. It must be estimated from the data. You can find the expected frequencies using probabilities from the $\text{Po}(1.55)$ distribution multiplied by 154, the total frequency in the data. Notice that even though the highest observed value was 4, the expected values have to add up to the same total frequency. This means that the final group has to be treated as greater than or equal to 4

b H_0: The data is drawn from a Poisson distribution.
H_1: The data is not drawn from a Poisson distribution.
Using the Po(1.55) distribution, the expected frequencies are

X	0	1	2	3	$\geqslant 4$
Exp frequency	32.6	50.6	39.3	20.3	11.1

There are 5 categories, the total frequency is fixed and one parameter (the mean) has been estimated from the data

There are $5 - 1 - 1 = 3$ degrees of freedom.

The calculator can then be used to find the p-value

$\chi^2 = 0.211$

$p = 0.976 > 0.05$

Therefore, there is no significant evidence to suggest that the data was not drawn from a Poisson distribution.

Tip

Be careful – the conclusion should not simply claim that the data does come from a Poisson distribution. The wording is quite tricky, but it is important to be precise.

This idea can also be applied to continuous data.

Tip

The notation [10,20[means $10 \leqslant x < 20$.

WORKED EXAMPLE 9.5

For the following data, find unbiased estimates of the population mean and variance.

Hence determine at the 5% significance level whether it could have been drawn from a normal distribution.

x	[10, 20[	[20, 30[	[30, 40[	[40, 50[	[50, 60[	[60, 70[
Frequency	12	40	48	52	44	16

In this case, the mean and variance are not included in the hypotheses because there was no prior belief about their values. They have to be estimated from the data.

H_0: The data is drawn from a normal distribution.
H_1: The data is not drawn from a normal distribution.

Find the mean and variance from the table using mid-interval values (the midpoints of each group). Write down those values that you use in the calculation that are not given in the question

The mid-interval values are 15, 25, 35, 45, 55 and 65.
From the GDC,
$x = 40.8$
$s_{n-1}^2 = 183$

To find the expected frequencies, find the probabilities using the cumulative normal distribution function on the calculator, then multiply by the total frequency (212). Even though the observed data were between 10 and 70, the expected frequencies for a normal distribution need to cover all real values

The expected frequencies are

x	Frequency
$]-\infty, 20[$	13.1
$[20, 30[$	31.7
$[30, 40[$	55.8
$[40, 50[$	58.3
$[50, 60[$	36.2
$[60, \infty[$	16.7

There are 6 groups, the constraints are the total being the same, and the mean and variance being the same as those estimated from the data

The degrees of freedom are $6 - 1 - 2 = 3$.

The GDC can then do a goodness of fit test

From the GDC, $\chi^2 = 5.73$,
p-value $= 0.126 > 0.05$.
Therefore, there is no significant evidence that this sample was not drawn from a normal distribution.

TOK Links

Notice that we cannot say that the data are drawn from a normal distribution. We would see this type of data only about 12.6% of the time when the real distribution is normal which is not hugely supportive of it being normally distributed. However, there is insufficient evidence to be statistically confident that it was not drawn from a normal distribution. Are we really testing for a good fit, or for a bad fit? Does the name we give to tests influence how we interpret them? Is it easier to prove or disprove a statement using statistics?

You are the Researcher

In Worked Example 9.5, we used unbiased estimators for the mean and variance. Strictly, when doing chi-squared tests we should use a different type of estimator called a maximum likelihood estimator. Sometimes these give the same answer as unbiased estimators but sometimes they give different answers. Explore what is meant by a maximum likelihood estimator and find out when it differs from unbiased estimators.

Exercise 9B

For questions 1 to 3, use the method of Worked Example 9.3 to test variables A and B for independence at the 5% significance level. You will need to combine rows or columns first.

1 a

		A		
		13–14	**15–16**	**17–19**
B	**6**	9	12	20
	7	4	12	15
	8	3	20	10

b

		A		
		[10,20[	**[20,30[**	**[30,40]**
B	**[0,1]**	12	9	5
	]1,2[	16	10	4
	[2,3]	18	15	3

2 a

		A		
		[10,20[	**[20,30[**	**[30,40]**
B	**[0,1]**	6	5	4
	]1,2[	16	15	20
	[2,3]	4	12	19

b

		A		
		1.3	**1.4**	**1.5**
B	**2**	20	30	40
	3	10	10	10
	4	4	4	5

3 a

		A			
		1	**2**	**3**	**4**
B	**2**	0	5	5	10
	3	0	10	10	0
	4	0	10	10	0
	5	10	5	5	0

b

		A			
		1	**2**	**3**	**4**
B	**2**	5	6	4	1
	3	10	12	15	16
	4	20	30	40	50
	5	25	35	45	55

For questions 4 to 6, use the method of Worked Example 9.4 to test at the 5% significance level whether a Poisson distribution is an appropriate model for the given data.

4 a

X	**0**	**1**	**2**	**3**	**4**
Frequency	20	20	20	20	20

b

X	**0**	**1**	**2**	**3**	**4**
Frequency	10	10	10	10	10

5 a

X	**0**	**1**	**2**	**3**	**4**
Frequency	10	20	30	20	10

b

X	**0**	**1**	**2**	**3**	**4**
Frequency	100	200	300	200	100

6 a

X	**0**	**1**	**2**	**3**	**4**	**5**
Frequency	8	14	20	25	16	9

b

X	**0**	**1**	**2**	**3**	**4**	**5**
Frequency	16	28	35	40	16	9

For questions 7 to 9, use the method of Worked Example 9.5 to test at the 5% significance level whether a normal distribution is an appropriate model for the given data.

7 a

x	**[10, 20[**	**[20, 30[**	**[30, 40[**	**[40, 50[**	**[50, 60[**	**[60, 70[**
Frequency	15	20	25	25	20	15

b

x	**[10, 20[**	**[20, 30[**	**[30, 40[**	**[40, 50[**	**[50, 60[**	**[60, 70[**
Frequency	150	200	250	250	200	150

8 **a**

x	$[20,40[$	$[40,60[$	$[60,80[$	$[80,100[$	$[100,120[$
Frequency	10	10	10	10	10

b

x	$[20,40[$	$[40,60[$	$[60,80[$	$[80,100[$	$[100,120[$
Frequency	10	20	30	20	10

9 **a**

x	$[-100,0[$	$[0,50[$	$[50,100[$	$[100,200[$
Frequency	10	50	30	20

b

x	$[-100,0[$	$[0,50[$	$[50,100[$	$[100,200[$
Frequency	20	50	40	20

10 Test Mendel's prediction at the 5% significance level.

Links to: Biology

In the 1850s, based on ideas of heredity, Austrian monk and scientist Gregor Mendel predicted that if tall plants were allowed to self-fertilize, they would produce tall plants and short plants in the ratio 3:1. He actually observed 787 tall plants and 277 short plants.

11 The times taken by eight-year-old children to solve a puzzle can be modelled by a normal distribution with mean 12 minutes and standard deviation 2.5 minutes. The times taken to solve the same puzzle by a random sample of 40 ten-year-old children are as follows.

Time (minutes)	$t \leqslant 9$	$9 < t \leqslant 11$	$11 < t \leqslant 13$	$13 < t \leqslant 15$	$t > 15$
Frequency	6	8	15	7	4

A psychologist wants to test, using a 10% significance level, whether the times of the ten-year-old children come from the same distribution.

a Write down suitable hypotheses for this test.

b Find the expected frequencies.

c Explain why the first two groups and the last two groups need to be combined.

d State the number of degrees of freedom after combining the groups.

e Carry out the test and state the conclusion.

12 Katya wants to find out whether diet choices are dependent on age. She collects data from students at her school and records them in the contingency table.

	Vegetarian	**Vegan**	**Eats meat**
11–13	8	20	14
14–15	15	10	20
16–17	8	8	6
17–18	7	6	3

a State suitable hypotheses for a χ^2 test for independence.

b Explain why the last two rows of the table need to be combined.

c Conduct a χ^2 test for independence, using a 5% significance level. State your conclusion in context.

d **i** Katya's friend says that he is vegetarian on most days but will eat fish at family celebrations if offered, so he did not know which response was required. How could Katya improve her questionnaire to take into account his feedback?

ii Explain why adding too many categories could be problematic.

13 Hermann is investigating whether the number of cars going past his house can be modelled by a Poisson distribution with mean 3.5 per minute. He observes the cars over a period of 60 minutes and records the number of cars in each minute.

Number of cars in a minute	**0**	**1**	**2**	**3**	**4**	**5**	**$\geqslant 6$**
Frequency	3	8	10	13	12	9	5

a State suitable hypotheses for a χ^2 test.

b State which two groups need to be combined.

c State the number of degrees of freedom.

d Find the p-value and state the conclusion of the test.

14 A teacher suggests that exam grades at her college can be modelled by the distribution.

$$\mathrm{P}(G = g) = \frac{g(11-g)}{140} \text{ for } g = 3, 4, 5, 6, 7$$

A random sample of 26 students had the following grades.

Grade	**3**	**4**	**5**	**6**	**7**
Frequency	10	6	4	3	3

a Assuming that the teacher's suggestion is correct, calculate the expected frequencies.

b Test, using a 10% significance level, whether the teacher's model is appropriate for these data.

15 A six-sided dice is rolled 27 times, with the following results.

Outcome	**1**	**2**	**3**	**4**	**5**	**6**
Frequency	6	2	5	7	2	5

Is there evidence, at the 10% significance level, that the dice is not fair?

16 The table shows information about the mode of transport that students use to get to school in four different cities.

	Amsterdam	**Athens**	**Houston**	**Johannesburg**
Car	12	25	48	24
Bus	18	33	12	18
Bicycle	46	12	7	53
Walk	13	3	0	5

Use a χ^2 test to find out whether there is evidence, at the 5% significance level, that there is a relationship between the mode of transport and the city. State the number of degrees of freedom and the p-value.

17 Consider the following data.

x	**[0, 10[**	**[10, 20[**	**[20, 30[**	**[30, 40[**	**[40, 50[**
Frequency	200	500	820	500	200

a Use a chi-squared test to determine whether the following distributions are plausible models for the data at the 5% significance level.

i $\mathrm{N}(\mu, \sigma^2)$

ii $\mathrm{N}(25, \sigma^2)$

iii $\mathrm{N}(25, 110)$

b When would you use a model of the form $\mathrm{N}(25, \sigma^2)$ rather than $\mathrm{N}(\mu, \sigma^2)$?

18 Rajesh is practising tennis serves. He takes three serves at a time and records the number of successful ones. He believes that this number can be modelled by the binomial distribution $B(3, 0.6)$.

Number of successful serves out of 3	0	1	2	3
Frequency	7	28	95	70

a State the hypotheses for a χ^2 goodness of fit test.

b Find the expected frequencies and write down the number of degrees of freedom.

c By finding the p-value, show that there is evidence, at the 5% significance level, that $B(3, 0.6)$ is not a good model.

Rajesh still thinks that the number of successful serves can be modelled by a binomial distribution, but with a different probability of success.

d By finding the mean of the data in the table, estimate the probability of success.

e Hence test, using a 5% significance level, whether the number of successful serves can be modelled by a binomial distribution.

19 A publisher wants to test whether the number of typos per page can be modelled by a Poisson distribution. She collects the following data from a random sample of 100 pages.

Number of typos	0	1	2	3	4	$\geqslant 5$
Number of pages	12	23	29	24	12	0

a Find the mean number of typos per page.

b State suitable hypotheses for a χ^2 test.

c Find the expected frequencies and the number of degrees of freedom.

d Test, using a 5% significance level, whether the number of typos per page can be modelled by a Poisson distribution.

20 A train company claims that times for a particular journey are distributed normally, with mean 23 minutes. Sumaya takes this train to school and wants to test the company's claim. She decides to conduct a χ^2 test and records the lengths of 50 randomly selected journeys.

Time (minutes)	20–21.5	21.5–22.5	22.5–23.5	23.5–24.5	24.5–26
Frequency	3	8	14	17	8

a Estimate the population standard deviation of train times.

b Find the expected frequencies. Do any groups need to be combined?

c Write down the number of degrees of freedom.

d Use a p-value to complete the test at the 5% significance level, stating your conclusion clearly.

21 An athlete believes that her long jump distances follow a normal distribution. In order to test her belief, she recorded the distances from a random sample of 100 jumps, and obtained the following results.

Distance (m)	4.5 to 5	5 to 5.5	5.5 to 6	6 to 7	7 to 7.2
Frequency	9	18	32	33	8

a Assuming that her belief is correct, copy the table and complete the expected frequencies, correct to three decimal places.

Distance (m)	<5	5 to 5.5	5.5 to 6	6 to 7	>7
Frequency	8.148	17.914			

b State the number of degrees of freedom for a χ^2 goodness of fit test.

c State suitable hypotheses.

d Conduct the test at the 10% significance level.

22 Consider the following data.

X	0	1	2	3	4
Frequency	10	20	30	40	50

a Find an unbiased estimate of the mean of the data.

b Find an unbiased estimate for the variance of the data.

c Amelia wants to check whether the data could plausibly come from a Poisson distribution. Show how she could do this

i by comparing her answers to a and b

ii by using a goodness of fit test with the last category being $X \geqslant 4$

iii by using a goodness of fit test with the last category being $X > 4$.

d Compare the validity of Amelia's three methods.

9C Non-linear regression

In Chapter 6 of Mathematics: applications and interpretation SL, you met the idea of linear regression, and used your GDC to find the equation of the regression line for a given data set.

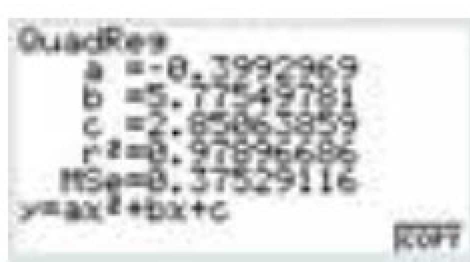
You now need to able to do the same for a number of non-linear possibilities as well: quadratic, cubic, exponential, power and sine regressions.

WORKED EXAMPLE 9.6

For the data below, find the equation of a regression model of the form $y = ax^2 + bx + c$.

x	2	3	3	4	5	6	7	7	9	10
y	13	16	17	19	22	24	23	24	22	21

The GDC will give you the values of the coefficients a, b and c in the quadratic $y = ax^2 + bx + c$



The quadratic model is

$$y = -0.399x^2 + 5.78x + 2.85$$

WORKED EXAMPLE 9.7

For the data below, find the equation of a regression model of the form $y = a\sin(bx+c)+d$.

x	1	2	2	3	4	5	7	8
y	1.4	8.1	8.7	5.0	0.9	7.6	0.7	6.9

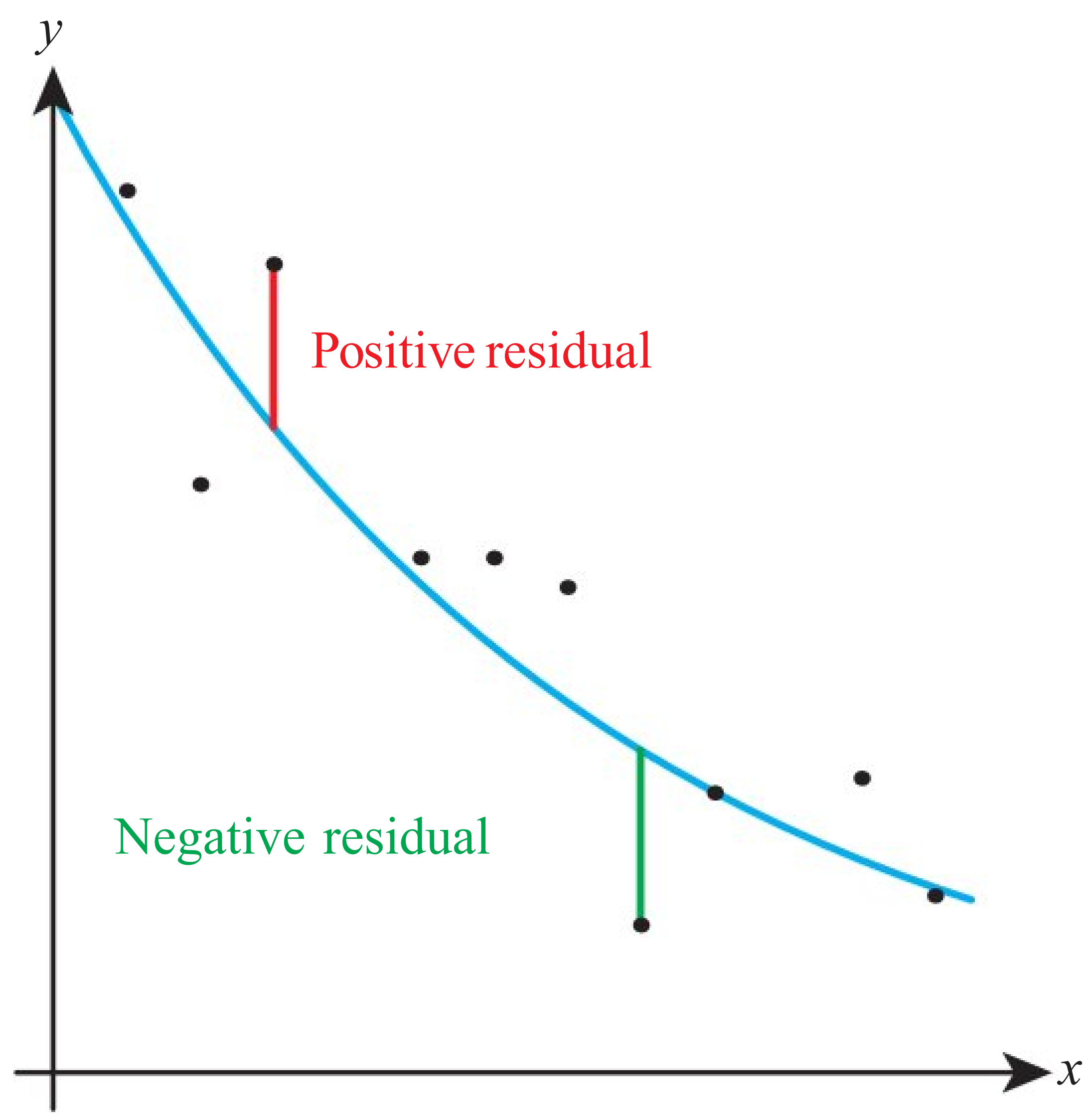
The GDC will give you the values of the coefficients a, b, c and d in the sine model $y = a\sin(bx+c)+d$

The sine model is

$$y = 4.12\sin(2.01x - 2.94) + 4.75$$

Sum of square residuals as a measure of fit for a model

Having formed a model, it is important to know how well it fits the data. To do this we need a measure of the difference between the y-values a model predicts and the actual y-values of the data. These differences are called residuals.



Since residuals can be positive or negative, to get an overall measure we first square the residuals before summing them. This gives a quantity called the **sum of square residuals**, SS_{res}.

KEY POINT 9.4

- $SS_{res} = \sum \left(y_i - \hat{y}_i\right)^2$, where $\hat{y}_i$ are the values the model predicts.
- The smaller the value of SS_{res} the better the model fits the data.

Note that since each component of SS_{res} is positive (as each residual is squared), the more data values that are used to form a model the larger the value of SS_{res} is likely to be for that model.

Therefore, you should not use SS_{res} to compare the goodness fit of two models that have been derived from a different number of data points.

The coefficient of determination (R^2)

The sum of square residuals leads to another measure of the goodness of fit of a model – the coefficient of determination, R^2. This measure does allow for comparison between models derived from a different number of data points.

For a linear model, R^2 is just the square of the Pearson's product-moment correlation coefficient, r.

Tip

A value of R^2 of around 0.7 or more is usually considered to be an indication of a good fit.

KEY POINT 9.5

$0 \leqslant R^2 \leqslant 1$, where a value of 1 indicates that the model perfectly predicts the data values.

In many circumstances, R^2 gives the proportion of variability in the dependent variable accounted for by the chosen model.

You are the Researcher

Find out about the different types of non-linear regression and when R^2 has this interpretation and when not.

WORKED EXAMPLE 9.8

A cubic model is suggested for this data.

x	0	0.5	1.0	1.5	2.0	2.5	3.0
y	3.4	4.9	1.9	5.8	5.5	16.2	16.5

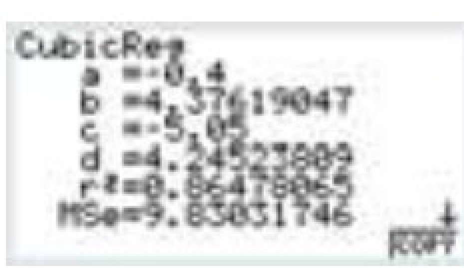
a Find the equation of the regression curve.

b State the value of the coefficient of determination for this model.

The GDC will give you the values of the coefficients a, b, c and d in the cubic $y = ax^3 + bx^2 + cx + d$

a The cubic model is

$$y = -0.400x^3 + 4.38x^2 - 5.05x + 4.25$$



The coefficient of determination is r^2 from this output screen

b $R^2 = 0.865$

Exercise 9C

For questions 1 to 5, use the technique of Worked Examples 9.6 and 9.7 to find a regression model of the given form for each set of data.

1 a $y = ax^2 + bx + c$

x	0	1	2	3	4	5
y	4.1	0.9	1.9	4.4	10.8	22.3

b $y = ax^2 + bx + c$

x	0.5	1.0	1.5	2.0	2.5	3.0	3.5
y	10.8	15.7	17.4	15.8	14.1	9.5	5.1

2 a $y = ax^3 + bx^2 + cx + d$

x	0.2	0.4	0.6	0.8	0.8	1.0	1.2	1.2	1.4	1.6
y	3.8	4.1	6.1	6.7	7.8	8.4	12.9	12.1	12.6	15.0

b $y = ax^3 + bx^2 + cx + d$

x	1	2	2	3	4	5	6	7	8
y	6	8	7	5	12	23	45	60	116

3 a $y = ae^{bx}$

x	5	10	15	20	25
y	4	20	40	155	330

b $y = ae^{bx}$

x	1.4	2.5	3.2	4.6	5.8	6.0
y	14	5.6	3.1	0.4	0.2	0.1

4 a $y = ax^b$

x	10	20	30	40	50	60	70	80	90
y	10	7.5	4.6	4.3	4.2	2.7	2.5	2.4	2.4

b $y = ax^b$

x	1	2	3	4	5
y	1	45	190	780	1000

5 a $y = a\sin(bx + c) + d$

x	10	11	12	13	13	14
y	0.9	1.5	7.3	1.0	1.3	3.5

b $y = a\sin(bx + c) + d$

x	1	1	2	3	4	4	5	6
y	2.4	1.7	3.7	4.8	4.7	4.6	2.5	0.6

For questions 6 to 8, use the technique of Worked Example 9.8 to the find the coefficient of determination for the given model for each set of data.

6 **a** Linear model

x	50	60	70	80	80	90	90	100	110	120
y	86	49	48	36	29	31	22	4	−29	−36

b Linear model

x	10	15	20	25	30
y	2	12	10	13	20

7 **a** Quadratic model

x	0	1	2	2	3	4	5
y	4.3	1.1	−2.9	−2.5	−0.8	5.6	7.1

b Quadratic model

x	100	103	105	106	109	110	112	115
y	12	16	31	29	35	21	15	10

8 **a** Cubic model

x	0.4	0.9	1.3	1.8	2.3	2.5	2.8	3.1
y	5.3	5.5	2.3	0.4	4.6	4.3	4.9	6.8

b Cubic model

x	5	12	18	26	31
y	1.3	9.8	7.6	6.7	−1.1

9 Zoe suspects that a linear model may be appropriate for data she has collected and calculates the Pearson's product-moment correlation coefficient to be −0.879.

Find the value of the coefficient of determination for Zoe's data set.

10 The distance from the Sun, x, in astronomical units (AU), and the orbital period, T, in Earth years, of each of the eight planets in the Solar System are given below.

Planet	Distance from sun (AU)	Orbital period (years)
Mercury	0.387	0.241
Venus	0.723	0.615
Earth	1.00	1.00
Mars	1.52	1.88
Jupiter	5.20	11.9
Saturn	9.54	29.5
Uranus	19.2	84.0
Neptune	30.1	165

a Find a regression model of the form $T = ax^3 + bx^2 + cx + d$, giving the values of a, b, c, d to three significant figures.

b The dwarf planet Pluto is 39.5 AU from the sun.

i Use the model to predict its orbital period.

ii Comment on the reliability of your prediction.

11 The depth of water, h metres, in a harbour t hours after midnight is recorded as follows.

t	d
2	7.7
6	6.5
10	4.1
14	7.5
18	6.7
22	4.5

a Find a regression model of the form $h = a\sin(bt + c) + d$, giving the values of a, b, c, d to two significant figures.

b Use your model to find the

i maximum predicted depth of water in the harbour

ii minimum predicted depth of water in the harbour.

12 A company is considering two functions to model how demand for a particular product D varies with the price charged, p.

The company has gathered the following sales figures at varying prices.

p ($)	D
2	40
4	37
6	23
8	20
10	17
12	12
14	11
16	11
18	10
20	6

Model A: $D = ap + b$

Model B: $D = ap^b$

a Determine the value of R^2 for model A and model B.

b On the basis of these values of R^2, suggest which model is a better fit for the data.

c Write down the equation for your chosen model, giving the values of a and b to three significant figures.

d Use your model to predict the demand at a price of $15.

e Comment on the suitability of using your model to predict demand at prices higher than $20.

13 A biologist is attempting to develop a model for population growth of bacteria. She records the following data.

Time (minutes)	1	2	3	4
Population (thousands)	3.2	5.8	7.4	10.2

She proposes two models:

Model A: $P = 2.5e^{0.3t}$

Model B: $P = 3.5e^{0.2t}$

a Find the sum of square residuals for each model.

b On the basis of the values found in part **a**, suggest which model better fits the data.

14 A maths teacher is attempting to form a model that predicts students' scores, s, in a maths exam from the length of time they spend revising beforehand, t.

He wants to form a separate model for girls and boys based on the following data.

GIRLS		BOYS	
Time spent revising (hours)	**Score in exam (%)**	**Time spent revising (hours)**	**Score in exam (%)**
2	54	1	34
3	60	2	36
3	51	2	52
5	69	4	61
6	83	5	94
6	77	7	78
8	84	8	84
9	91	10	80
10	86		
12	79		

a Form a quadratic regression model for the girls and a separate quadratic model for the boys.

b Why would using the sum of square residuals for each model not be a good way of determining which model best fits the data set from which it is derived?

c Use the coefficient of determination for each model to suggest which model is a better fit.

15 The temperature, T°C, of a kettle t minutes after boiling is as follows.

t	2.5	5	7.5	10	12.5	15
T	84	67	59	49	41	38

a Fit a quadratic regression model to the data.

b Find the value of R^2.

c Explain whether your value of R^2 indicates that a quadratic model is a good fit for the data.

16 Alessandra and Zoe collect data on the number of visits to their business's website every hour in a 24-hour period.

Alessandra fits a quadratic regression model to the data and gets a value of $R^2 = 0.81$.

Zoe fits a cubic regression model and gets a value of $R^2 = 0.90$.

Zoe claims that her model is a better fit to the data.

Explain why Zoe's claim is not necessarily true.

9D Confidence intervals for the mean

The concept of a confidence interval

In Section 9A, you learnt how to use sample data to find unbiased estimates of the population mean and variance. These are called **point estimates** – each one is a single number. They are very unlikely to be the real population mean and variance because of natural variation when choosing samples – a different sample will give different estimates.

Instead of finding a single value to estimate the population mean, it may be better to have an interval of values which is very likely to include the true mean. This is called a **confidence interval**. In this section, you will learn how to use the GDC to find confidence intervals in various situations.

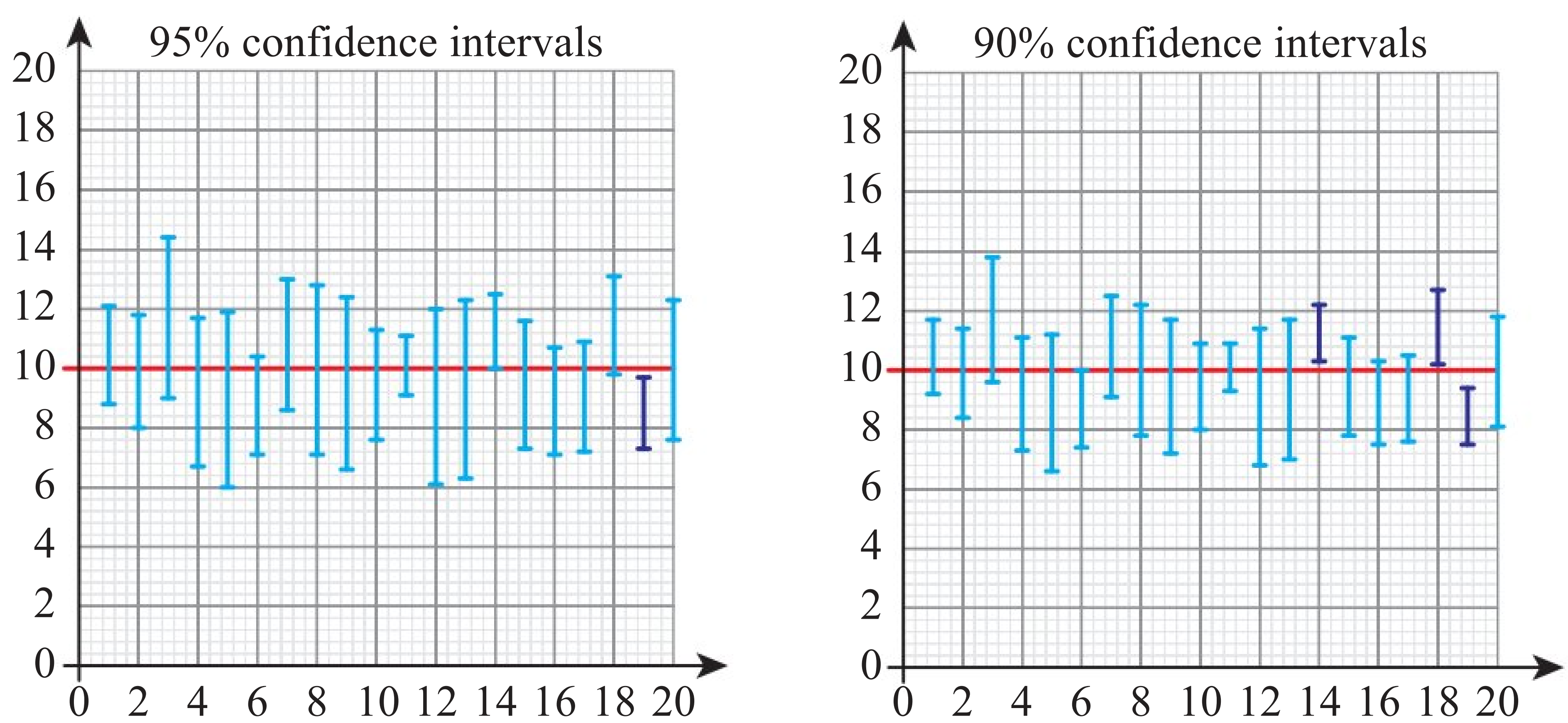
For example, suppose that you measure a sample of six leaves from a certain tree and get the following results (in cm):

6.2, 5.1, 7.3, 5.3, 8.1, 6.5

Based on this sample, the 95% confidence interval for the mean is found to be $5.21 < \mu < 7.63$. If you take a different sample, you will get a different confidence interval. '95% confidence' means that if lots of samples are taken and a confidence interval calculated for each one, then 95% of those confidence intervals will contain the true population mean.

You can choose a different confidence level. For the sample of lengths above, the 90% confidence interval is $5.47 < \mu < 7.37$. You can see that both intervals are centred at 6.42, which is the sample mean, but that the second interval is smaller. The second interval gives a more precise estimate for the population mean, but you can be less confident that the true mean is contained within it.

The diagram below shows a large number of 95% and 90% confidence intervals calculated from samples of size 6 taken from a normal distribution with mean 10. The intervals highlighted are the only ones that don't contain the population mean.



In this course, we only consider symmetric confidence intervals for the population mean, which are centred on the sample mean. The width of the interval depends on the size of the sample, the variance of the data, the chosen confidence level and the distribution of the underlying population. The latter may or may not be known; your GDC calculates confidence intervals assuming that the underlying population distribution is normal.

You are the Researcher

You already know how to use a sample to estimate the population variance – one good estimate is s_{n-1}^2. It turns out that finding a confidence interval for the variance requires the use of the χ^2 distribution.

Confidence interval for the mean of a normal population with unknown variance

In this case, you need to construct a ***t*-interval**.

KEY POINT 9.6

Use a t-interval when the population variance is unknown.

You need to assume that the underlying population follows a normal distribution.

In Section 15B of the Mathematics: applications and interpretation SL book, you learnt about t-tests, which you will revisit in the next section. They are based on the same distribution, called the student's t-distribution, as the calculation of t-intervals.

WORKED EXAMPLE 9.9

A random sample of eight lightbulbs from a particular manufacturer was tested to determine their lifetime. The results, in thousands of hours, were

12.3, 21.7, 18.2, 31.5, 22.8, 16.0, 28.8, 14.5

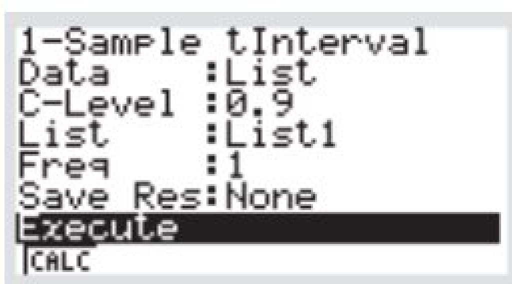
It can be assumed that the population of lifetimes is normally distributed. Find a 90% confidence interval for the mean lifetime of lightbulbs produced by this manufacturer.

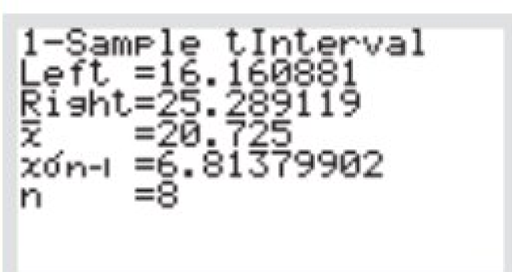
State which interval you are going to find. Since you have not been told the population variance, use the t-interval

t-interval on GDC:

You need to enter the data into a list and select the confidence level

$16200 \text{ hours} < \mu < 25300 \text{ hours}$





Tip

This can also be written using set notation as $\mu \in (16200, 25300)$.

Sometimes the data has been summarized for you, and you can enter those summary statistics into the GDC to find a confidence interval.

WORKED EXAMPLE 9.10

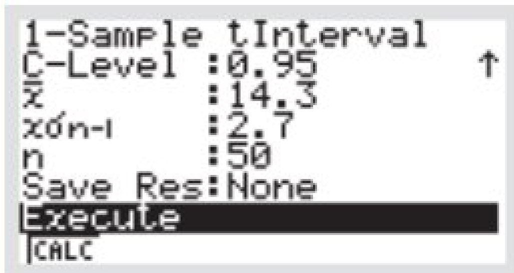
The times taken for 17-year old boys to run 100 m can be assumed to follow a normal distribution. A random sample of 50 times (x seconds) is taken and the results are summarized as follows:

$\bar{x} = 14.3,\ s_{n-1} = 2.7$

Find a 95% confidence interval for the population mean time.

The population variance is unknown, so use the t-interval .... t-interval on GDC:

You need to enter the sample size as well as the summary statistics .... $13.5 < \mu < 15.1$



Confidence interval for the mean of a normal population with known variance

If you happen to know the population variance, it is possible to construct a better confidence interval by using the **z-interval** instead of the t-interval. (It is 'better' in the sense that you can achieve the same level of confidence with a smaller interval, giving you a more accurate estimate for the population mean.)

KEY POINT 9.7

Use a z-interval when the population variance is known.

You need to assume that the underlying population follows a normal distribution.

WORKED EXAMPLE 9.11

The lengths of the leaves of a certain kind of plant are known to follow a normal distribution with standard deviation 2.8 cm. A random sample of 20 leaves is measured and the sample mean length was 16.3 cm. Find a 95% confidence interval for the population mean length of the leaves.

The population variance is known, so use the z-interval .... z-interval on GDC:

.... $\mu \in (15.1, 17.5)$

You are the Researcher

The formula for the upper and lower end-points of a 95% confidence interval for the mean of a normal distribution with variance σ^2, calculated from a sample of size n with sample mean $\bar{x}$, is $\bar{x} \pm 1.96\frac{\sigma}{\sqrt{n}}$. Investigate how this formula follows from the distribution of the sample mean for a normal distribution.

Tip

If the sample size is large and the population variance is known, it is possible to obtain an approximate z-interval even if the underlying distribution is not known to be normal. This is because the central limit theorem ensures that the distribution of sample means is approximately normal.

TOOLKIT: Problem Solving

The following is a random sample of data from a normal distribution with standard deviation 1.2:

3.5, 4.7, 5.2, 5.2, 5.6, 6.1, 6.8, 7.2

a Find a 95% z-interval for the population mean.

b Find a 95% t-interval for the population mean (ignoring the known standard deviation).

Which interval is smaller? Can you explain why?

Use a spreadsheet to generate samples of size 8 from the normal distribution $\mathrm{N}(5.5, 1.2^2)$. For each sample, calculate the two 95% confidence intervals as above. Is the z-interval always smaller than the t-interval? How does the difference change with the sample size?

Exercise 9D

For questions 1 to 3, use the technique demonstrated in Worked Example 9.9 to find the confidence interval for the population mean (with the given confidence level) of a normal population with unknown variance, using the given sample data.

1 a 12.7, 4.8, 5.3, 5.8, 9.7, 11.0, 6.5, 8.2; 90% confidence

b 3.62, 1.05, 3.21, 7.70, 4.06, 5.52, 3.25, 4.01; 90% confidence

2 a 135, 111, 218, 206, 172, 182, 165; 95% confidence

b 317, 425, 515, 627, 515, 388, 517; 95% confidence

3 a 5.3, 8.7, 9.4, 9.4, 3.8, 4.5, 7.3, 6.5, 8.1, 7.5; 90% confidence

b 5.3, 8.7, 9.4, 9.4, 3.8, 4.5, 7.3, 6.5, 8.1, 7.5; 95% confidence

For questions 4 to 6, use the technique demonstrated in Worked Example 9.10 to find the confidence interval for the population mean (with the given confidence level) of a normal population with unknown variance, using the given summary statistics.

	$\bar{x}$	s_{n-1}	Sample size	Confidence level
4 a	12.3	1.7	12	90%
b	14.2	3.8	12	90%
5 a	186	25	8	95%
b	227	21	8	95%
6 a	8.2	0.62	25	80%
b	7.7	0.85	25	80%

For questions 7 to 9, use the technique demonstrated in Worked Example 9.11 to find the confidence interval for the population mean (with the given confidence level) of a normal population with known variance, using the given summary statistics.

		σ	$\bar{x}$	Sample size	Confidence level
7	a	6.3	18.9	20	90%
	b	9.2	18.9	20	90%
8	a	9.2	18.9	6	95%
	b	9.2	36.5	6	95%
9	a	46	173	12	80%
	b	82	355	12	80%

For questions 10 to 12, find the confidence interval for the population mean (with the given confidence level) of a normal population with known variance, using the given sample data.

		σ	Sample data	Confidence level
10	a	1.6	12.7, 4.8, 5.3, 5.8, 9.7, 11.0, 6.5, 8.2	90%
	b	0.87	3.62, 1.05, 3.21, 7.70, 4.06, 5.52, 3.25, 4.01	90%
11	a	12	135, 111, 218, 306, 172, 182, 165	95%
	b	23	317, 425, 515, 627, 515, 388, 517	95%
12	a	0.45	5.3, 8.7, 9.4, 9.4, 3.8, 4.5, 7.3, 6.5, 8.1, 7.5	80%
	b	1.02	5.3, 8.7, 9.4, 9.4, 3.8, 4.5, 7.3, 6.5, 8.1, 7.5	90%

13 A random sample of eight adult cats were weighed, with the following results (in kilograms):

4.6, 2.8, 5.1, 4.2, 3.9, 4.7, 3.6, 4.2

It can be assumed that the masses of cats are normally distributed in the population. Find a 90% confidence interval for the population mean mass of a cat.

14 The scores on a test can be assumed to be normally distributed. For a random sample of 20 people who took the test, the mean score was 63.8 and the unbiased estimate of the population variance was 18.6. Find a 90% confidence interval for the population mean score.

15 The mass of fish in a pond is known to be normally distributed with standard deviation 124 g. The mean mass of 85 fish from the pond is found to be 806 g. Find a 95% confidence interval for the mean mass of all the fish in the pond.

16 The times to complete a task are known to be normally distributed with standard deviation 5.8 minutes. A group of ten children completed the task in the following times (in minutes):

20.5, 31.2, 18.7, 22.5, 39.8, 28.7, 22.1, 18.6, 36.4, 29.2

a Find a 90% confidence interval for the mean time required to complete the task.

b Justify your choice of interval.

17 A randomly selected group of 80 students were asked how long they spent on their last assignment. The mean of the sample was 8.6 hours and the unbiased estimate of the population variance was 1.5 hours^2.

a Find a 95% confidence interval for the population mean time.

b State any assumptions you made about the underlying distribution.

18 The temperature at noon is measured in 15 randomly selected locations in a country. The mean of the sample was 18.6 °C and the standard deviation of the sample was 3.8 °C.

a Find an unbiased estimate of the population variance.

b Assuming that the temperatures are normally distributed, find a 90% confidence interval for the mean temperature in the country.

19 A chemistry student makes many attempts to prepare a 20% solution. The teacher measures the concentration in eight of the attempts and obtains the following results (rounded to the nearest %):

24, 19, 18, 22, 25, 26, 20, 22

a State an assumption you need to make in order to create a confidence interval for the mean.

b Find a 90% confidence interval for the mean concentration of solutions prepared by the student.

c Based on your confidence interval, do you think that the student is preparing the solutions correctly? Justify your answer.

20 A teacher believes that his students can complete a certain task in less than 3 minutes on average. He asks a random sample of ten students to complete the task. He calculates their mean time to be 3.2 minutes and the unbiased estimate of the population standard deviation to be 0.6 minutes.

a Find a 90% confidence interval for the population mean time. Justify your choice of interval and state any assumptions you have made.

b Hence comment on the teacher's belief.

21 A phone manufacturer wishes to estimate the average battery life of their phones. In a random sample of 20 phones, the mean battery life is 26.5 hours and the variance of the sample is 11.56 hours^2.

a Find the unbiased estimate of the population variance.

b Find a 90% confidence interval for the mean battery life of the manufacturer's phones, stating any assumptions you make.

c The manufacturer claims that their battery life is longer than 28 hours on average. Comment on this claim in the light of your confidence interval.

22 A catering manager wants to check the amount of coffee dispensed by the drinks machine in the student common room. A sample of 20 coffee cups taken from the machine has mean 149 ml and standard deviation 4.6 ml.

a Calculate s_{n-1} for this sample.

b Find a 95% confidence interval for the mean amount of coffee dispensed by the machine.

c The catering manager decides that the machine needs to be adjusted if it dispenses more than 153 ml per cup. Does the machine need adjusting?

d A student complains that the machine dispenses less than the stated 150 ml per cup. Does your confidence interval support this claim?

23 The heights, rounded to the nearest centimetre, of a random sample of 60 adults are summarized in the table.

***h* (cm)**	**145–155**	**156–160**	**161–165**	**166–170**	**171–180**	**181–190**
Frequency	4	7	12	14	14	9

It can be assumed that the heights are normally distributed in the population.

a Find an approximate 95% confidence interval for the mean population height.

b Explain why the confidence interval is approximate.

24 The maximum daily temperature, measured in °C, in a certain location is known to follow a normal distribution with variance 8.6. A random sample of 12 days was selected and the maximum temperature on each of the days recorded. The mean of this sample was 17.8°C.

a Find the 80% confidence interval for the mean maximum daily temperature.

b Does this confidence interval provide evidence that the mean maximum daily temperature is more than 16.5°C?.

c Would the answer to part b change if you used the 90% interval instead?

25 A call centre manager wants to estimate the average length of calls. She knows from past experience that the population standard deviation of the length of calls is 2.6 minutes. She takes a random sample of 80 calls and finds that their mean length is 3.7 minutes.

a Find a 95% confidence interval for the mean call length.

An assistant manager points out that it is not known whether the call lengths are normally distributed.

b State, with a reason, whether the calculation of the confidence interval is part a is still valid.

26 A study is conducted to investigate whether drinking coffee raises blood pressure. Twelve subjects have their systolic blood pressure measured, in millimetres of mercury, before and after drinking coffee. The results are recorded in a table.

Subject	**1**	**2**	**3**	**4**	**5**	**6**	**7**	**8**	**9**	**10**	**11**	**12**
Before	92	107	121	85	109	135	141	98	112	126	135	108
After	98	121	123	102	132	126	135	98	127	137	122	128
Difference	6						−6					

a Copy the table and complete the final row showing the increase in blood pressure.

It can be assumed that the blood pressure is normally distributed in the population.

b Calculate the 90% confidence interval for the mean increase in blood pressure.

c Explain whether your confidence interval suggests that drinking coffee increases blood pressure.

9E Hypothesis tests for the mean

Choosing between the *t*-test and the *z*-test

As with confidence intervals, a different test is appropriate depending on whether the population variance is known or needs to be estimated from a sample.

KEY POINT 9.8

Use a *z*-test when the population variance is known.

Use a *t*-test when the population variance has been estimated from the sample.

Both tests require the underlying population distribution to be normal.

WORKED EXAMPLE 9.12

A machine produces screws which have a mean length of 6 cm and a standard deviation of 0.2 cm. The distribution of lengths is assumed to be normal. After the machine was moved, it is believed that the mean length may have changed while the standard deviation stays the same. A sample of 20 screws is measured and found to have the mean length of 5.92 cm. Test, at a 5% significance level, whether there is evidence that the mean length has changed. State the hypotheses and justify your choice of test.

The null hypothesis reflects the default position (no change)

$H_0: \mu = 6$

You are testing for change in either direction, so use a two-tailed test

$H_0: \mu \neq 6$

Choose between a *t*- and a *z*-test depending on whether the population variance is known

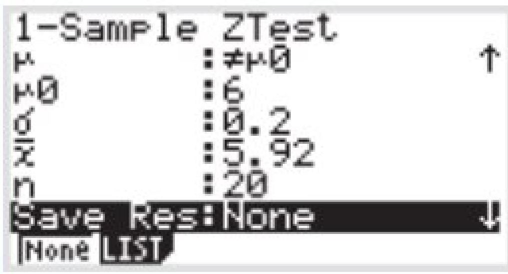
Use a *z*-test because the population variance is known and the population distribution is normal.

Select the appropriate option on your GDC

From the GDC:
$p = 0.0736 > 0.05$

There is insufficient evidence to reject H_0.

Input the population standard deviation, the sample mean, the sample size, the hypotheses and the significance level



The GDC gives you the *p*-value to compare with the significance level

Write a conclusion in context, making it clear that there is some uncertainty

There is insufficient evidence, at the 5% significance level, that the mean length of screws has changed.

Critical values and critical regions for the *z*-test

In Worked Example 9.12, how small would the mean length of the 20 screws need to be in order to provide sufficient evidence against H_0?

Tip

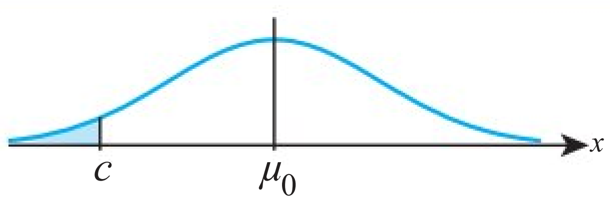
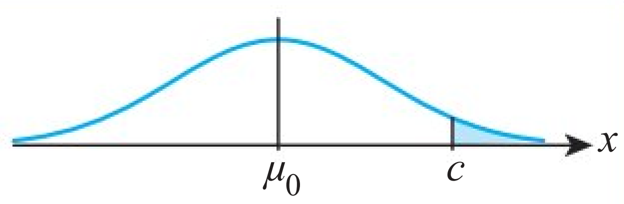
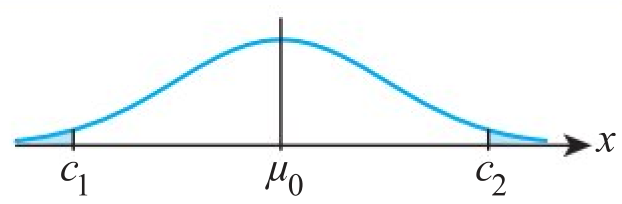
You will not be asked to find critical regions for *t*-tests.

KEY POINT 9.9

The set of all values which would lead us to reject the null hypothesis is called the **critical region**. The value (or values) on the boundary of the critical region is (are) called the **critical value(s)**.

For the *z*-test, the probability of the sample mean being in the critical region equals the significance level of the test.

The form of the critical region depends on the alternative hypothesis.

H_1	$\mu < \mu_0$	$\mu > \mu_0$	$\mu \neq \mu_0$
Critical region	$\bar{X} < c$	$\bar{X} > c$	$\bar{X} < c_1$ and $\bar{X} > c_2$
Probability	$P(\bar{X} < c) =$ significance level	$P(\bar{X} > c) =$ significance level	$P(\bar{X} < c_1) = P(\bar{X} > c_2) = \frac{1}{2}$ significance level
			

You can find the critical values using an inverse normal distribution, because you know that the distribution of the sample mean is $\bar{X} \sim N\left(\mu, \frac{\sigma^2}{n}\right)$.

WORKED EXAMPLE 9.13

A machine is supposed to produce bolts of length 6 cm and standard deviation 0.2 cm. The lengths follow a normal distribution. After the machine was moved, the mean of a sample of 20 bolts is calculated to check whether the mean length has changed. Find the critical region to test, at the 5% significance level,

a whether the mean length has decreased

b whether the mean length has changed.

The null hypothesis and the distribution of $\bar{X}$ will be the same in both cases

$H_0: \mu = 6$

$\bar{X} \sim N\left(6, \frac{0.2^2}{20}\right)$

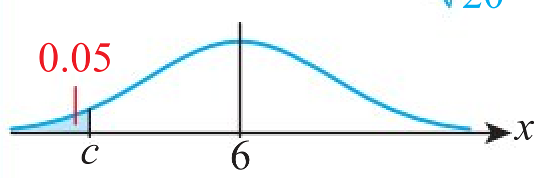
To test for a decrease, you need the critical region of the form $\bar{X} < c$

a $H_1: \mu < 6$

$P(\bar{X} < c) = 0.05 \Rightarrow c = 5.93$

So, the critical region is $\bar{X} < 5.93$.

Use inverse normal to find c. Remember that the standard deviation is $\frac{0.2}{\sqrt{20}}$



To test for a change, you need to split the 0.05 probability between the two tails

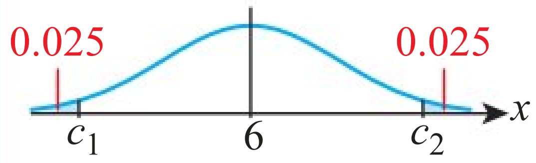
b $H_1: \mu \neq 6$

$P(\bar{X} < c_1) = 0.025 \Rightarrow c_1 = 5.91$

$P(\bar{X} > c_2) = 0.025 \Rightarrow c_2 = 6.09$

So, the critical region is

$\bar{X} < 5.91$ and $\bar{X} > 6.09$



TOOLKIT: Problem Solving

A normal distribution has a known standard deviation 3.8.

a A sample of 20 values from this distribution is found to have a mean of 14.2. Find the 95% confidence interval for the population mean.

b Another sample of 20 values is to be used to test whether the population mean is different from 14.2. Find the critical region for the test at the 5% significance level.

Find out how to derive the formula for a confidence interval. This will help you to explain why the two answers are the same.

The critical value for a test, at the 5% significance level, of whether the population mean is *less* than 14.2 corresponds to the lower limit of another confidence interval for the mean. What is the confidence level of this interval?

Testing for difference: unpaired samples

In Section 15B of the Mathematics: applications and interpretation SL book, you learnt how to test whether two samples come from populations with equal means. The population variances were unknown and you needed to assume that they were equal (so you used the 'pooled variance' option).

When the variances of the two populations are known, you can use a z-test. It does not matter whether the population variances are equal or not.

WORKED EXAMPLE 9.14

A scientist wants to test whether two different species of mice have tails of the same length. She has reason to believe that the tail lengths for both species are distributed normally, the first with standard deviation 6.3 cm and the second with standard deviation 4.8 cm.

She finds that the mean tail length for a random sample of 12 mice from species A is 17.3 cm, and the mean tail length for a random sample of 8 mice from species B is 20.6 cm. Test, using a 10% significance level, whether the two species have different tail lengths on average.

State the hypotheses.
It is a two tail test

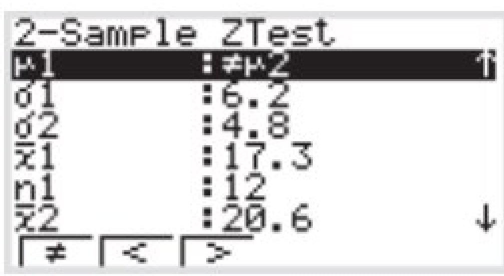
$H_0: \mu_1 = \mu_2$

$H_1: \mu_1 \neq \mu_2$

Select the appropriate option on yotur GDC

From the GDC,

$p = 0.181 > 0.05$



There is insufficient evidence to reject H_0.

There is insufficient evidence, at the 10% significance level, that the tail lengths are different on average.

Testing for difference

Paired samples

In some situations, samples in a two-sample test are paired. For example, you may have measurements from each individual pre- and post-treatment, or results from each student in two different subjects. You can then test whether the population mean has changed by finding the difference between the two measurements for each individual and treating that as a single sample. This helps eliminate differences between individual participants and focuses on the change between the two measurements.

WORKED EXAMPLE 9.15

A tennis coach wants to determine whether a new racquet improves the speed of his students' serves (faster serves are considered better). She randomly selects a group of nine children and measures their service speed with their current racquet and with the new racquet, in metres per second. The results are shown in the table below.

Child	A	B	C	D	E	F	G	H	I
Speed with current racquet	58	68	49	71	80	57	46	57	66
Speed with new racquet	72	81	52	59	75	72	48	62	70

a State, in this context, one advantage of a paired test over an unpaired test.

b Is there evidence, at the 5% significance level, that the new racquet increases the service speed?

c What distributional assumption is needed to make your test valid?

a A paired test eliminates differences between children and only detects any change due to a new racquet.

The data are paired and the population variances are unknown, so use a paired t-test

b Let d be the difference in speeds (new – old).

Child	A	B	C	D	E	F	G	H	I
$\boldsymbol{d}$	14	13	3	−11	−5	15	2	5	4

You need to form a new sample by finding differences

You are testing for an increase in speed. The null hypothesis is that there is no change, so the mean difference would be zero

H_0: $\mu = 0$

H_1: $\mu > 0$

Enter the data from the table into your GDC, using a one-sample t-test

From the GDC, using a t-test:

$$p = 0.0830 > 0.05$$

There is insufficient evidence to reject H_0.

There is insufficient evidence, at the 5% significance level, that the new racquet improves serve speed.

You have used a t-test on the values of d, so you need these values to be normally distributed

c The population of differences needs to be normally distributed.

Tip

Note that you need to assume for this type of test that the differences are normally distributed, not that the original populations were both normally distrubuted.

Exercise 9E

For questions 1 to 20, the underlying populations have a normal distribution.

For questions 1 to 4, you are given the hypotheses, some information about the population and either raw or summarized data. Use the technique demonstrated in Worked Example 9.12 to find the p-value and hence state the conclusion of each test. You will need to choose between a z-test and a t-test.

1 **a** $H_0: \mu = 12$, $H_1: \mu \neq 12$; significance level 5%; population $\sigma = 2.7$; sample size $n = 15$, sample $\bar{x} = 13.2$

b $H_0: \mu = 132$, $H_1: \mu \neq 132$; significance level 5%; population $\sigma = 14.1$; sample size $n = 12$, sample $\bar{x} = 122.5$

2 **a** $H_0: \mu = 4.5$ $H_1: \mu < 4.5$; significance level 2%; sample size $n = 15$, sample $\bar{x} = 3.7$, sample $s_{n-1} = 1.26$

b $H_0: \mu = 72$ $H_1: \mu < 72$; significance level 2%; sample size $n = 8$, sample $\bar{x} = 68.7$, sample $s_{n-1} = 3.84$

3 **a** $H_0: \mu = 16$, $H_1: \mu \neq 16$; significance level 5%; sample data: 14, 18, 22, 12, 21, 17

b $H_0: \mu = 2.1$, $H_1: \mu \neq 2.1$; significance level 5%; sample data: 4.2, 1.7, 3.5, 2.1, 4.2, 1.4

4 **a** $H_0: \mu = 120$, $H_1: \mu > 120$; significance level 10%; population $\sigma = 4.8$; sample data: 132, 127, 122, 119, 121, 117, 131, 128, 122

b $H_0: \mu = 11.5$, $H_1: \mu > 11.5$; significance level 10%; population $\sigma = 1.25$; sample data: 13.2, 12.7, 12.2, 11.9, 12.1, 11.7, 13.1, 12.8, 12.2

For questions 5 to 7, use the technique demonstrated in Worked Example 9.13 to find the critical region for each hypothesis test.

5 **a** $H_0: \mu = 80.4$; $H_1: \mu < 80.4$; 10% significance; $n = 120$; population $\sigma = 20$

b $H_0: \mu = 93$; $H_1: \mu < 93$; 5% significance; $n = 400$; population $\sigma = 20$

6 **a** $H_0: \mu = 80$; $H_1: \mu > 80$; 1% significance; $n = 18$; population $\sigma = 10$

b $H_0: \mu = 750$; $H_1: \mu > 750$; 2% significance; $n = 45$; population $\sigma = 10$

7 **a** $H_0: \mu = 60$; $H_1: \mu \neq 60$; 5% significance; $n = 16$; population $\sigma = 10$

b $H_0: \mu = 120$; $H_1: \mu \neq 120$; 10% significance; $n = 30$, population $\sigma = 10$

For questions 8 to 11, you are given information about two sets of data. Use the technique demonstrated in Worked Example 9.14 to find the p-value for each test. For t-tests, assume that the samples come from populations with equal variances.

	Population σ	H_0	H_1	Sample 1	Sample 2
8 a	unknown	$\mu_1 = \mu_2$	$\mu_1 \neq \mu_2$	$\bar{x} = 11.3$, $s_{n-1} = 2.3$, $n = 9$	$\bar{x} = 13.5$, $s_{n-1} = 1.7$, $n = 10$
b	unknown	$\mu_1 = \mu_2$	$\mu_1 \neq \mu_2$	$\bar{x} = 6.8$, $s_{n-1} = 1.03$, $n = 14$	$\bar{x} = 5.2$, $s_{n-1} = 2.11$, $n = 12$
9 a	$\sigma_1 = 4.5$, $\sigma_2 = 3.2$	$\mu_1 = \mu_2$	$\mu_1 < \mu_2$	$\bar{x} = 23.5$, $n = 20$	$\bar{x} = 24.1$, $n = 30$
b	$\sigma_1 = 11.2$, $\sigma_2 = 13.1$	$\mu_1 = \mu_2$	$\mu_1 < \mu_2$	$\bar{x} = 173$, $n = 15$	$\bar{x} = 180$, $n = 12$
10 a	$\sigma_1 = \sigma_2 = 0.83$	$\mu_1 = \mu_2$	$\mu_1 > \mu_2$	1.3, 2.8, 3.7, 3.8, 3.8, 4.1	0.9, 1.2, 1.7, 2.5, 2.6, 3.1
b	$\sigma_1 = \sigma_2 = 1.26$	$\mu_1 = \mu_2$	$\mu_1 > \mu_2$	13.2, 11.5, 12.6, 11.2, 12.5, 11.2, 12.1, 13.0	12.3, 11.2, 11.8, 12.5, 10.7, 11.7, 11.4, 12.1
11 a	unknown	$\mu_1 = \mu_2$	$\mu_1 < \mu_2$	142, 153, 121, 135, 131	151, 145, 142, 161, 140, 139
b	unknown	$\mu_1 = \mu_2$	$\mu_1 < \mu_2$	1.8, 2.1, 2.6, 2.3, 1.7, 2.5, 1.8, 1.5	2.0, 2.1, 3.2, 2.8, 3.1, 2.2

For questions 12 to 14, use the technique demonstrated in Worked Example 9.15 to conduct a paired t-test.

12 **a** $H_0: \mu_1 = \mu_2$, $H_1: \mu_1 < \mu_2$, 10% significance level

6.2	3.5	2.4	5.3	6.1	3.5
7.1	4.6	2.8	5.7	6.8	4.1

b $H_0: \mu_1 = \mu_2$, $H_1: \mu_1 < \mu_2$, 5% significance level

32	29	42	52	51	38
35	22	47	61	52	38

13 **a** $H_0: \mu_1 = \mu_2$, $H_1: \mu_1 > \mu_2$, 2% significance level

121	135	147	152	165	158	162	138
119	118	141	150	159	147	162	131

b $H_0: \mu_1 = \mu_2$, $H_1: \mu_1 > \mu_2$, 1% significance level

36	42	47	62	47	63	55	72
35	38	45	55	41	62	50	66

14 a H_0: $\mu_1 = \mu_2$, H_1: $\mu_1 \neq \mu_2$, 5% significance level

4.7	5.3	5.1	3.8	4.1	5.2	4.3
4.5	4.6	5.2	3.7	4.3	4.8	4.1

b H_0: $\mu_1 = \mu_2$, H_1: $\mu_1 \neq \mu_2$, 10% significance level

148	172	197	191	153	168	151
155	180	204	193	153	178	142

15 The wingspan of a species of butterfly is known to be normally distributed with mean 10 cm and standard deviation 1 cm. A zoologist suspects he may have found a new sub-species with a larger wingspan. The mean of a sample of 6 of these butterflies is 11.2 cm. Test the scientist's suspicion at the 5% significance level assuming that the wingspans are still normally distributed with standard deviation 1 cm.

16 The scores on a national mathematics test are distributed normally with mean 65.7. Juan thinks that the scores in his school are lower than the national average. A random sample of 20 students gives a mean score of 62.9 and variance 12.635.

a Use these sample values to find an unbiased estimate of the population variance.

b Explain why a t-test is appropriate in this situation.

c Test Juan's belief at the 10% significance level.

17 A baker knows that the time needed for his oven to heat up to 220 °C follows a normal distribution with mean 7.8 minutes and standard deviation 0.8 minutes. After cleaning the oven, he plans to take ten independent measurements to find out whether the time has decreased.

a Write down suitable null and alternative hypotheses for this test.

b Let X be the time taken for the oven to heat up after cleaning. Assuming that the null hypothesis is true, and that the standard deviation is unchanged, write down the distribution of $\bar{X}$.

c Find the critical region for the test at the 5% level of significance.

18 A manufacturer produces two different types of phone, KX03 and KX04. Data collected over a long term indicate that the battery life of KX04 is on average five hours longer than for KX03. For both types, the battery life is distributed normally with variance 16.8 hours^2. A supplier believes that they have evidence that the difference in battery life is not five hours. They provide the following sample of lifetimes, in hours.

KX03	38	36	35	26	22	35		
KX04	42	34	27	35	38	31	32	41

a Explain why a z-test is appropriate in this situation.

b Test the supplier's belief using a 5% significance level.

19 It is believed that the second harvest of apples from trees is smaller than the first. Ten trees are sampled and the total mass of apples (in kg) in the first and second harvests are recorded.

Tree	A	B	C	D	E	F	G	H	I	J
Apples in first harvest	8.0	7.2	4.5	7.3	6.8	5.3	6.4	4.8	8.1	7.0
Apples in second harvest	7.5	7.4	4.0	6.7	6.0	5.5	5.8	3.6	8.9	6.0

a Explain why a paired test is appropriate in this situation.

b It can be assumed that the total mass of apples per tree for each harvest can be modelled by a normal distribution. Carry out an appropriate test at the 5% significance level to decide whether the claim is justified. State your hypotheses and conclusion clearly.

20 The temperature in classrooms in a large school is controlled centrally and is supposed to be kept constant throughout the day. Previous measurements suggest that the temperature in each classroom is distributed normally with standard deviation 0.4°C. A caretaker believes that the classrooms are warmer in the afternoon. To test his belief, he measures the temperature in the morning (M°C) and in the afternoon (A°C) in ten randomly selected classrooms, and obtains the following results.

Classroom	A	B	C	D	E	F	G	H	I	J
M	19.3	19.4	18.3	19.1	19.0	18.6	18.4	19.2	18.7	20.1
A	19.5	19.7	18.3	19.2	19.4	18.7	18.5	19.5	18.9	20.1

a Calculate the difference in temperature, $D = A - M$, for each classroom.

b Find the standard deviation of D in this sample.

c Stating appropriate hypotheses in terms of D, test the caretaker's belief at the 10% significance level.

21 It is believed that, in men, the normal level of testosterone in blood is normally distributed with mean 24 nmol per litre and standard deviation 6 nmol per litre. After a race, a sprinter gives two samples with an average of 34 nmol per litre. Is this sufficiently different (at 1% significance) to suggest that sprinter's testosterone level is above average?

22 The temperature of a water bath is normally distributed with a mean of 60°C and a standard deviation of 1°C. After being serviced it is assumed that the standard deviation is the same. The temperature is measured on five independent occasions and a test is performed at the 5% significance level to see if the temperature has changed from 60°C. What range of mean temperatures would result in accepting that the temperature has changed?

23 Celine believes that, on average, her weekly maths homework takes at least 20 minutes longer to complete than her English homework. For a random sample of eight maths homeworks and six English homeworks, she finds the following.

	Sample mean $\bar{x}$ (minutes)	**Sample variance s_n^2 (minutes^2)**
Maths	135	126
English	98	102

a Find the unbiased estimates for the population variance for each homework.

b Test Celine's belief using a 10% significance level.

c State two assumptions you have used in your test.

24 a Standard light bulbs have an average lifetime of 800 hours and a standard deviation of 100 hours. A low energy light bulb manufacturer claims that their lifetimes have the same standard deviations but that they last longer. A sample of 50 low energy light bulbs has an average lifetime of 829.4 hours. Test the manufacturer's claim at the 5% significance level.

b An advisor is worried that the test is not valid because it has not been checked whether the lifetimes have a normal distribution. Is his worry justified?

25 Rashid is investigating whether a certain drug causes mice to gain mass. He weighs a sample of six mice before taking the drug and after taking the drug for a week. The results (mass in g) are given in the table.

Mouse	1	2	3	4	5	6
Before	86	52	124	96	106	125
After	92	68	124	99	112	126

Rashid decides to use a pooled two-sample t-test.

a State two assumptions necessary for this test to be valid.

b Find the p-value for this test.

He then realizes that the data are paired.

c Construct a new table showing the change in mass.

d Do any assumptions from part **a** still need to hold? What are the required assumptions now?

e State two advantages of using a paired test.

f Find the p-value for the paired test.

g Which test is more appropriate in this situation? Hence state what conclusion Rashid should make at the 10% significance level.

26 It is known that the marks on a trigonometry test follow a normal distribution with standard deviation 8, and marks on a calculus test follow a normal distribution with standard deviation 5. Johannes thinks that students at his school are better at calculus than at trigonometry. To test his belief, he records the test results for a random sample of eight students. (Both tests are marked out of the same total.)

Student	1	2	3	4	5	6	7	8
Trigonometry mark	63	71	73	98	65	88	78	62
Calculus mark	65	82	78	93	72	93	79	61

Let T be the trigonometry mark, let C be the calculus mark, and let $D = C - T$.

a Find the variance of D.

b Assuming that the average mark on the two tests is the same, write down the distribution of $\overline{D}$.

c Use an appropriate test to determine whether there is evidence, at the 10% significance level, that the mean of D is positive.

d Hence comment on Johannes's belief.

27 Times taken for a computer to complete startup are modelled by a normal distribution with standard deviation 0.86 seconds. A new cleaning software claims to decrease the startup time. A random sample of 12 computers is to be tested and the startup times recorded before (B) and after (A) the installation of the software. The random variable D is defined as $A - B$, and the mean of D is denoted by μ_D.

a Write down suitable hypotheses to test whether the startup time has decreased.

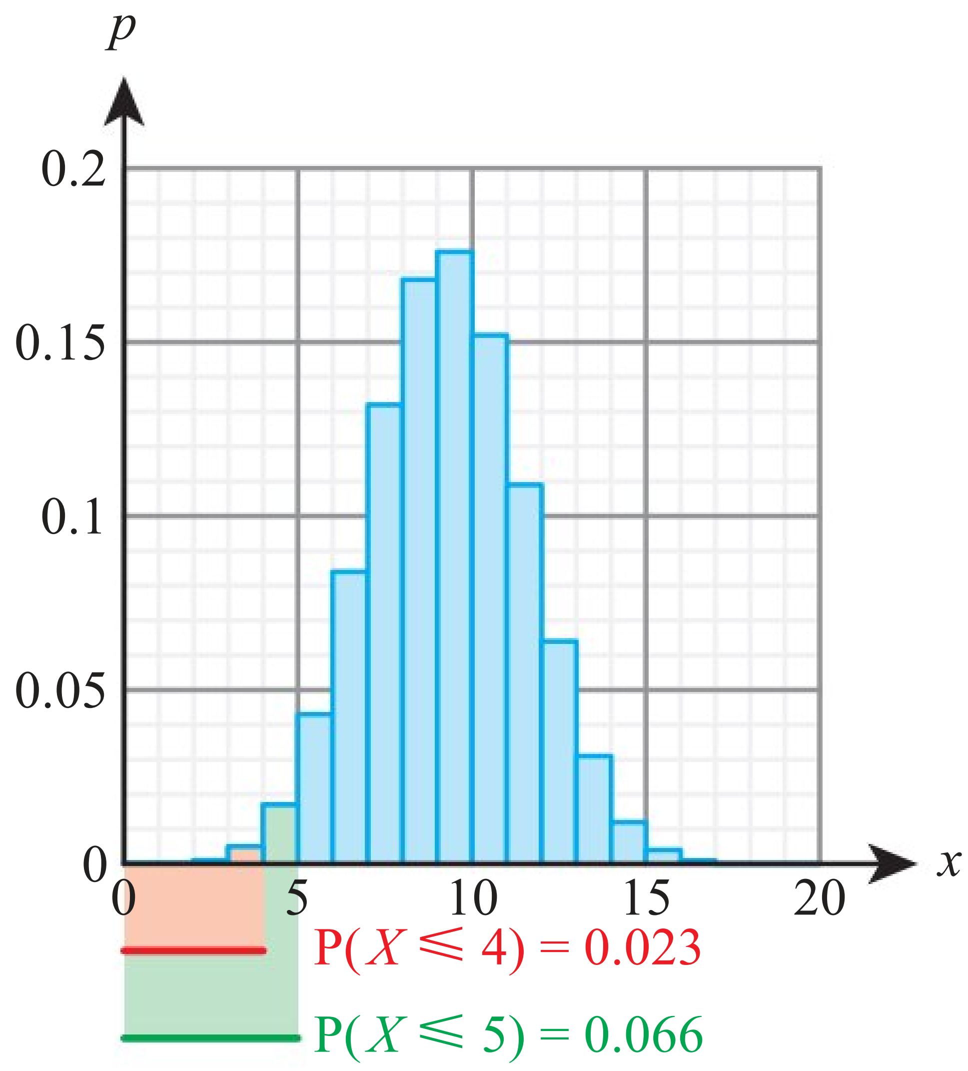
b Assuming that H_0 is true, find the distribution of D and hence write down the distribution of $\overline{D}$. Give the variances correct to four decimal places.

c Find the critical region for the test at the 5% significance level.

9F Hypothesis tests using the binomial and Poisson distributions

Discrete vs continuous variables

In the last section you saw that, when testing for the mean of a normal distribution, you can find the critical region for the test such that P(critical region) = significance level. When working with discrete distributions, such as binomial or Poisson, this is not always possible because the probabilities 'jump' instead of changing continuously.



Tip

In this course, discrete distributions will only require one-tailed tests.

KEY POINT 9.10

For a hypothesis test involving a discrete variable,

- the critical region is of the form $X \leqslant c$ or $X \geqslant c$
- $\mathrm{P}(\text{critical region}) \leqslant$ significance level

Test for the population proportion using the binomial distribution

The binomial distribution measures the number of successful outcomes from n independent trials. You can test whether the probability of success has increased or decreased by conducting n independent trials and counting the number of successes, X.

WORKED EXAMPLE 9.16

A six-sided dice is rolled 30 times, obtaining 8 sixes. Is there evidence, at the 10% significance level, that the probability of rolling a six is more than $\frac{1}{6}$?

The hypothesis involves the probability of success (in this case, rolling a six)

H_0: $p = \frac{1}{6}$, H_1: $p < \frac{1}{6}$

Define the random variable you are going to use, and state its distribution assuming H_0 is true

Let X = number of sixes from 30 rolls

Assuming that H_0 is true, $X \sim B\left(30, \frac{1}{6}\right)$

Find the probability of observing the given outcome or more extreme

$$P(X \geqslant 8) = 1 - P(X \leqslant 7)$$
$$= 0.114 > 0.10$$

There is insufficient evidence, at the 10% significance level, that the probability of rolling a six is more than $\frac{1}{6}$.

As mentioned before, it is usually not possible to find a critical region with the probability exactly equal to the significance level of the test. You need to look for the largest probability which is still below the significance level.

WORKED EXAMPLE 9.17

It is known that 77% of UK households own at least one car. Ganesh lives in an urban area and suspects that car ownership in his neighbourhood is lower than this. To test his belief, he intends to take a random sample of 50 households and find out how many own a car. Write down suitable hypotheses and the find the critical region for the test at the 5% significance level.

The number of households with a car is a binomial random variable

Let X = the number of households out of 50 which own a car. Then $X \sim B(50, p)$.

The hypotheses refer to the value of p

H_0: $p = 0.77$, H_1: $p < 0.77$

You are looking for the largest value of c such that $P(X \leqslant c) \leqslant 0.05$. You may be able to generate a list of probabilities on your GDC. Write down at least two adjacent values to show your working

Assuming $X \sim B(50, 0.77)$:

$$P(X \leqslant 34) = 0.0925 > 0.05$$
$$P(X \leqslant 33) = 0.0508 > 0.05$$
$$P(X \leqslant 32) = 0.0258 < 0.05$$

The critical region is $X \leqslant 32$.

Tip

Since you are looking for the critical region in the left tail, start your search below the mean of X, which is $50 \times 0.77 = 38.5$.

Tip

Some GDCs have an 'inverse binomial' function. Be careful – this sometimes gives the critical value which is too large or too small by one, so you should always test adjacent values.

TOK Links

Some textbooks will advise you to choose the critical region so that its probability is as close as possible to the significance level, rather than below it. So in the Worked Example 9.17 the critical region would be $X \leqslant 33$ rather than $X \leqslant 32$. Is there a 'right' or 'wrong' definition of a critical region? Who should decide which definition to use?

Test for the population mean using the Poisson distribution

The Poisson distribution measures the number of events occurring in a fixed interval. You can test whether the average rate of events has increased or decreased by counting the number of events in a given interval.

WORKED EXAMPLE 9.18

Ingrid runs an IB revision website which gets an average of 18 visits per day. Following the launch of a rival website, she wants to test whether this average rate has decreased. On a randomly selected day, there were 10 visits to her website. Test, using a 5% significance level, whether the average rate of visiting has decreased.

The hypotheses involve the average rate per day

$H_0: \lambda = 18,\ H_1: \lambda < 18$

Define the random variable you are going to use and state its distribution assuming H_0 is true

X = the number of visits in one day

Assuming that H_0 is true, $X \sim \text{Po}(18)$

Find the probability of the observed value or more extreme

$P(X \leqslant 10) = 0.0304 < 0.05$

There is sufficient evidence, at the 5% significance level, that the average rate of visits has decreased.

Critical regions are found in the same way as for the binomial distribution.

WORKED EXAMPLE 9.19

The number of outbreaks of a certain disease in a country is modelled by a Poisson distribution. Over a long period, the average rate of outbreaks has been constant at 2.4 per year. A doctor suspects that the average rate has increased, and wants to test his suspicion by looking at the number of outbreaks in the past year. Find the critical region for this test at the 2% significance level.

The hypotheses refer to the average rate per year

Let X = the number of outbreaks in a year.
Then $X \sim \text{Po}(\lambda)$.

$H_0: \lambda = 2.4,\ H_1: \lambda > 2.4$

You are looking for the smallest value of c such that $P(X \geqslant c) \leqslant 0.02$. Some calculators only give probabilities of the form $P(X \leqslant k)$ so you might have to find the required probabilities using this cumulative function.

Assuming that $X \sim \text{Po}(2.4)$:

$$P(X \geqslant 5) = 1 - P(X \leqslant 4) = 0.096 > 0.02$$
$$P(X \geqslant 6) = 1 - P(X \leqslant 5) = 0.036 > 0.02$$
$$P(X \geqslant 7) = 1 - P(X \leqslant 6) = 0.012 < 0.02$$

The critical region is $X \geqslant 7$.

Be the Examiner 9.1

The number of errors in a maths textbook is believed to follow a Poisson distribution with mean 1.6 per 10 pages. Kelvin worries that the error rate in a new edition is higher than this. He checks a random sample of 10 pages and finds 4 errors. Does this provide evidence, at the 5% significance level, that the average error rate is higher in the new edition?

Which is the correct solution? Identify the errors made in the incorrect solutions.

Solution 1	Solution 2	Solution 3
$P(X = 4) = 0.0551 > 0.05$	$P(X \geqslant 4) = 1 - P(X \leqslant 3)$ $= 0.0788 > 0.05$	$P(X > 4) = 1 - P(X \leqslant 4)$ $= 0.0237 < 0.05$
Insufficient evidence that the average error rate is higher.	Insufficient evidence that the average error rate is higher.	Sufficient evidence that the average error rate is higher.

Test for correlation

In Section 6D of the Mathematics: applications and interpretation SL book, you learnt how to you use your GDC to calculate the Pearson's product-moment correlation coefficient (r) for a set of data. You also learnt how to use a given critical value to decide whether it indicates significant linear correlation. This decision-making procedure can be expressed in the language of hypothesis testing.

The value of r you calculate from data is the sample correlation coefficient. In the underlying population, there may or may not be any correlation between the two variables. The population correlation coefficient is denoted by ρ. When you ask whether the calculated value of r is significant, the question you are really asking is: Is there significant evidence that ρ is not zero?

KEY POINT 9.11

In a hypothesis test for correlation, the null hypothesis is always $\rho = 0$.

The alternative hypothesis can take one of three forms:

Testing for…	**Positive correlation**	**Negative correlation**	**Any correlation**
$\mathbf{H_1}$	$\rho > 0$	$\rho < 0$	$\rho \neq 0$

Your GDC can find the p-value, which depends on the value of r (the sample correlation coefficient) and the number of pairs in the sample. The calculation of the p-value relies on one distributional assumption.

KEY POINT 9.12

To calculate the p-value for a hypothesis test for correlation, it must be assumed that both variables are normally distributed.

WORKED EXAMPLE 9.20

The table shows the distance from the nearest train station and the average house price for seven villages.

Distance (km)	0.8	1.2	2.5	3.7	4.1	5.5	7.4
Average house price (£1000)	240	185	220	196	187	156	162

a Is there evidence, at the 5% significance level, of a negative correlation between the distance and the average house price?

b State any distributional assumptions you have made.

State the hypotheses. You are testing for negative correlation

a H_0: $\rho = 0$, H_1: $\rho < 0$

Enter the two lists into the GDC and select the correct alternative hypothesis

From the GDC:
$p = 0.0174 < 0.05$

There is sufficient evidence to reject H_0.
There is significant evidence of a negative correlation between the distance from the train station and the average house price.

The normal distribution assumption applies to both variables

b The populations of distances and house prices are normally distributed.

Exercise 9F

For questions 1 to 4, you are given hypotheses about the proportion in a binomial distribution $X \sim B(n, p)$ and the observed value of X. Use the technique from Worked Example 9.16 to find the p-value and state the conclusion of the test.

1 **a** H_0: $p = \frac{1}{6}$, H_1: $p < \frac{1}{6}$, $n = 45$, SL = 5%, $x = 4$

b H_0: $p = \frac{1}{3}$, H_1: $p < \frac{1}{3}$, $n = 60$, SL = 5%, $x = 16$

2 **a** H_0: $p = 0.8$, H_1: $p < 0.8$, $n = 20$, SL = 10%, $x = 12$

b H_0: $p = 0.7$, H_1: $p < 0.7$, $n = 50$, SL = 10%, $x = 29$

3 **a** H_0: $p = \frac{3}{4}$, H_1: $p > \frac{3}{4}$, $n = 80$, SL = 2%, $x = 68$

b H_0: $p = \frac{5}{6}$, H_1: $p > \frac{5}{6}$, $n = 60$, SL = 2%, $x = 57$

4 **a** H_0: $p = 0.15$, H_1: $p > 0.15$, $n = 100$, $SL = 5\%$, $x = 22$

b H_0: $p = 0.2$, H_1: $p > 0.2$, $n = 75$, $SL = 5\%$, $x = 21$

For questions 5 to 8, use the technique demonstrated in Worked Example 9.17 to find the critical region for the hypothesis test.

5 **a** $X \sim B(50, p)$, H_0: $p = 0.6$, H_1: $p < 0.6$, SL = 5%

b $X \sim B(40, p)$, H_0: $p = 0.7$, H_1: $p < 0.7$, SL = 5%

6 **a** $X \sim B(80, p)$, H_0: $p = 0.2$, H_1: $p < 0.2$, SL = 10%

b $X \sim B(100, p)$, H_0: $p = 0.15$, H_1: $p < 0.15$, SL = 10%

7 **a** $X \sim B(50,\ p)$, H_0: $p = 0.6$, H_1: $p > 0.6$, SL = 5%

b $X \sim B(40,\ p)$, H_0: $p = 0.7$, H_1: $p > 0.7$, SL = 5%

8 **a** $X \sim B(80,\ p)$, H_0: $p = 0.2$, H_1: $p > 0.2$, SL = 10%

b $X \sim B(100,\ p)$, H_0: $p = 0.15$, H_1: $p > 0.15$, SL = 10%

For questions 9 to 12, you are given hypotheses about the mean of a Poisson distribution, $X \sim \text{Po}(\lambda)$, and the observed value of X. Use the technique from Worked Example 9.18 to find the p-value and state the conclusion of the test.

9 **a** H_0: $\lambda = 12$, H_1: $\lambda < 12$, SL = 5%, $x = 8$

b H_0: $\lambda = 18$, H_1: $\lambda < 18$, SL = 5%, $x = 15$

10 **a** H_0: $\lambda = 9.3$, H_1: $\lambda < 9.3$, SL = 10%, $x = 5$

b H_0: $\lambda = 6.5$, H_1: $\lambda < 6.5$, SL = 10%, $x = 2$

11 **a** H_0: $\lambda = 12$, H_1: $\lambda > 12$, SL = 5%, $x = 16$

b H_0: $\lambda = 18$, H_1: $\lambda < 18$, SL = 5%, $x = 22$

12 **a** H_0: $\lambda = 0.55$, H_1: $\lambda > 0.55$, SL = 2%, $x = 3$

b H_0: $\lambda = 0.82$, H_1: $\lambda > 0.82$, SL = 2%, $x = 4$

For questions 13 to 16, use the technique demonstrated in Worked Example 9.19 to find the critical region for the hypothesis test.

13 **a** H_0: $\lambda = 12$, H_1: $\lambda < 12$, SL = 5%

b H_0: $\lambda = 20$, H_1: $\lambda < 20$, SL = 5%,

14 **a** H_0: $\lambda = 8.2$, H_1: $\lambda < 8.2$, SL = 10%

b H_0: $\lambda = 9.5$, H_1: $\lambda < 9.5$, SL = 10%

15 **a** H_0: $\lambda = 12$, H_1: $\lambda > 12$, SL = 5%

b H_0: $\lambda = 20$, H_1: $\lambda > 20$, SL = 5%

16 **a** H_0: $\lambda = 0.45$ H_1: $\lambda > 0.45$, SL = 2%

b H_0: $\lambda = 0.63$, H_1: $\lambda > 0.63$, SL = 2%

For questions 17 to 20, use the technique demonstrated in Worked Example 9.20 to test the hypotheses at the given significance level.

17 **a** H_0: $\rho = 0$, H_1: $\rho > 0$, SL = 5%

x	2.1	4.5	3.8	5.7	3.1	6.3	1.8
y	12.7	14.1	11.3	21.3	16.8	20.5	12.1

b H_0: $\rho = 0$, H_1: $\rho > 0$, SL = 5%

x	13	21	18	32	28	17	31
y	8	11	6	18	17	21	22

18 **a** H_0: $\rho = 0$, H_1: $\rho < 0$, SL = 10%

x	2.1	4.5	3.8	5.7	3.1	6.3	1.8
y	21	12	23	18	31	14	24

b H_0: $\rho = 0$, H_1: $\rho < 0$, SL = 10%

x	13	21	18	32	28	17	31
y	4.7	1.2	0.8	0.6	1.1	3.2	0.5

19 **a** $H_0: \rho = 0$, $H_1: \rho \neq 0$, SL = 10%

x	27	32	41	53	66	78	82	87
y	12	14	18	18	22	14	22	31

b $H_0: \rho = 0$, $H_1: \rho \neq 0$, SL = 10%

x	4.5	5.3	5.8	6.1	6.6	6.6	7.1	8.3
y	9.7	9.6	8.5	9.1	8.7	7.3	7.8	6.1

20 **a** $H_0: \rho = 0$, $H_1: \rho \neq 0$, SL = 2%

x	1.3	1.5	1.7	1.7	1.8	2.0	2.5	2.6
y	16	11	12	14	13	9	8	5

b $H_0: \rho = 0$, $H_1: \rho \neq 0$, SL = 2%

x	3	6	5	1	1	4	3	7
y	12	32	35	18	21	47	33	54

21 An established treatment for a particular disease is known to be effective in 78% of the cases. A doctor devises a new treatment which she believes is more effective. She uses the treatment on a random sample of 60 patients and finds that the new treatment is effective in 52 cases. Does this data support the doctor's belief at the 2% significance level?

22 The proportion of students who have summer jobs is known to be 48% nationally. Jay suspects that the proportion in his school might be lower. Out of a random sample of 30 students he finds that 10 of them had a summer job. Is there evidence, at the 10% significance level, that Jay's suspicion is justified?

23 Data collected over a long period of time suggests that, at a certain location, it rains on average 82 days per year. Last year, there were 96 rainy days. Does this provide evidence, at the 5% significance level, that it now rains more often on average?

24 Avi is trying to improve his typing. At the moment, the number of typing errors he makes per page can be modelled by a Poisson distribution with mean 4.7.

a State the distribution of the number of errors Avi makes per ten pages.

After attending a course, Avi checks a random sample of ten pages and finds that he made 35 errors.

b Is there evidence, at the 5% significance level, that his average number of errors per page has decreased?

25 A machine produces cables which have an average of 12 flaws per 100 m of cable.

a Write down the distribution of flaws per 20 m of cable.

After the machine is moved, the operator needs to check whether the rate of flaws has increased. He takes a random 20 m length of cable and finds that it has 5 flaws.

b Test, using a 5% significance level, whether the average rate of flaws has increased.

26 A group of five students took a maths test and a chemistry test. Their marks on the two tests are shown in the table.

Maths test mark	63	81	57	72	93
Chemistry test mark	40	57	46	51	60

a Find the Pearson's product-moment correlation coefficient between the two sets of marks.

Rehana wants to test whether there is evidence of positive correlation between the two sets of marks.

b State suitable null and alternative hypotheses.

c Find the p-value.

d Hence state the conclusion of the test at the 5% significance level.

27 A small company records the amount spent on advertising and the profit they made the following month. The results are summarized in the table.

Amount ($\$x$)	120	90	65	150	80	95
Profit ($\$y$)	1600	1300	450	1650	1480	1150

Is there evidence, at the 10% significance level, of a positive correlation between the amount spent on advertising and the profit? State your hypothesis, the p-value and the conclusion clearly.

28 The table shows the age (in years) and value (in thousands of $) of seven cars.

Age	3	5	12	7	9	7	4
Value	3.2	1.1	0.6	3.1	1.8	2.4	3.4

Lewis believes that there is a negative correlation between the age and the value of a car.

a State suitable hypotheses for this test.

b Conduct the test using a 5% significance level.

29 At the last election, party Z won 38% of the vote. An activist wants to find out whether the support for the party has increased. They decide to conduct a poll of 200 people and then conduct a hypothesis test at the 5% significance level.

a State suitable hypotheses.

b Find the critical region for the test.

c In the poll, 86 out of the 200 respondents say they support party Z. State the conclusion of the test.

30 A class of eight students took a maths test and a chemistry test. The table shows the marks on the two tests.

Maths mark (x)	72	47	82	65	71	83	81	57
Chemistry mark (y)	51	60	46	37	52	48	57	41

Test, using a 5% significance level, whether there is any correlation between the two sets of marks.

31 Theo investigates whether spending more time practising his spellings leads to better marks in the weekly spelling test. Each week he records the length of time spent practising and his spelling test mark (out of 20).

Time (t minutes)	17	5	10	7	25	14	20
Mark (m/20)	20	8	6	12	19	16	18

a Find the Pearson's product-moment correlation coefficient.

b Test, using a 2% significance level, whether there is evidence of correlation between the time spent and the mark.

c Is there evidence, at a 2% significance level of *positive* correlation between the time spent and the mark?

32 A bus company claims that, on average, at least one bus arrives at a certain stop every ten minutes. Sanjana says that, in her experience, the average rate is less than this. She decides to use a hypothesis test with a 10% level of significance to find out whether evidence supports her suspicion.

a Write down suitable hypotheses for this test. Define any parameters you use.

b Sanjana decides to conduct her test by counting the number of buses during a one-hour interval. Find the critical region for this test.

c Rahul says that it would be better to use a longer time interval, and suggests five hours. Find the critical region for Rahul's test.

33 Gyan is investigating whether floods have become more common in her district. She knows that the long-term average is 6.3 floods per year.

a State suitable hypotheses Gyan should use.

Gyan plans to look at the records of the last five years and note the number of floods.

b Find the critical region for the test at the 5% significance level.

34 A town council proposes to close a post office because they claim that fewer than 10% of residents use it. A campaigner says that this claim should be tested by asking a random sample of n residents whether they use the post office. The hypotheses for the test are

H_0: $p = 0.1$, H_1: $p < 0.1$

where p is the percentage of residents who use the post office, and the significance level for the test is 5%.

The post office will be closed if there is sufficient evidence to reject the null hypothesis.

- a A sample of 100 residents is used. What *percentage* would need to say that they use the post office in order to avoid the closure?
- b What percentage would be needed if a sample of 200 residents was used?

35 A delivery company models the number of complaints received per day by a Poisson distribution with mean 1.7. Following a change in procedures, they want to test whether the average rate of complaints has decreased. A manager suggests selecting a random day and using the number of complaints received on this day to perform a test at the 5% significance level.

- a State suitable hypotheses for this test.
- b Show that there is no critical region for this test. What does this mean in terms of rejecting the null hypothesis?
- c What is the smallest integer n such that the test using an n% significance level has a critical region?
- d An intern suggests that they should look at the number of complaints during a random sample of 10 days instead. Find the critical region for the new test using the 5% significance level.
- e What is the smallest number of days required to ensure that the test at the 1% significance level has a critical region?

36 Following a large storm, a climate scientist says that this is a 1-in-100-years event. Greta interprets this as meaning that, in any given year, there is a 1% chance of such a storm happening. She claims that, given that two such storms have happened in the last ten years, there is evidence that such storms have become more frequent than 1 in 100 years.

- a State suitable hypotheses and test Greta's claim at the 5% significance level.

David says that '1 in 100 years' should be interpreted as an average rate of 0.01 per year, and that a Poisson distribution should be used to model the number of storms in a 10-year period.

- b Using David's interpretation and the fact that two storms were observed in 10 years, test the hypothesis that the average rate of storms has increased, using a 5% significance level.
- c Find a significance level at which the two tests would give a different conclusion.

9G Type I and Type II errors

In conducting any hypothesis test we come to one of two conclusions:

- There is sufficient evidence to reject H_0 at the specified significance level.
- There is insufficient evidence to reject H_0 at the specified significance level.

However, it is possible that the conclusion we reach is wrong, and this leads to two possible types of error.

KEY POINT 9.13

- **Type I error**: reject H_0 when it was true
- **Type II error**: fail to reject H_0 when it was false

We cannot eliminate the possibility of these errors, but we can find the probability of making them.

Tip

In other books you might see the terms 'nominal significance level' (the stated significance level for a test) and 'actual significance level' (the actual probability of making a Type I error) used in the discrete case.

We will make a Type I error when H_0 is true and the test statistic falls within the critical region (which will cause us to incorrectly reject H_0). However, we already know that, for continuous distributions, the probability of being in the critical region is the significance level of the test, so this immediately gives us the probability of making a Type I error.

It is a little more complicated for discrete distributions where the probability of being in the critical region is as large as possible while never exceeding the significance level. So, in this case, the probability of a Type I error will be less than or equal to the stated significance level.

KEY POINT 9.14

$\mathrm{P}(\text{Type I error}) = \mathrm{P}(\text{being in critical region} \mid \mathrm{H}_0 \text{ true})$

- For a continuous distribution, $\mathrm{P}(\text{Type I error}) = \text{significance level of test}$.
- For a discrete distribution, $\mathrm{P}(\text{Type I error}) \leqslant \text{significance level of test}$.

WORKED EXAMPLE 9.21

$X \sim \mathrm{N}(\mu, 4)$ and the following hypotheses are tested at the 10% level with a sample size of $n = 15$:

$\mathrm{H}_0: \mu = 6$

$\mathrm{H}_1: \mu \neq 6$

Find the probability of making a Type I error.

X follows a continuous distribution and so the probability of a Type I error is exactly the significance level of the test

$\mathrm{P}(\text{Type I error}) = 0.1$

WORKED EXAMPLE 9.22

$X \sim \mathrm{B}(8, p)$ and the following hypotheses are tested:

$\mathrm{H}_0: p = \frac{1}{3}$

$\mathrm{H}_1: p > \frac{1}{3}$

The critical region is $X \geqslant 6$.

Find the probability of making a Type I error.

The probability of a Type I error is the probability of being in the critical region given that H_0 is true

Assuming H_0 is true

$X \sim \mathrm{B}\left(8, \frac{1}{3}\right)$

Use the GDC to find the required probability

$$\begin{aligned}\mathrm{P}(\text{Type I error}) &= \mathrm{P}(X \geqslant 6)\\ &= 1 - \mathrm{P}(X \leqslant 5)\\ &= 1 - 0.980\\ &= 0.020\end{aligned}$$

Since a Type II error occurs when we fail to reject H_0 when it was false, to find the probability of this happening we need to know what the true distribution actually was. We first find the critical region as normal, and then find the probability of being outside the critical region, but now using the true distribution.

KEY POINT 9.15

$\mathrm{P}(\text{Type II error}) = \mathrm{P}(\text{Not being in critical region} \mid \text{true distribution})$

WORKED EXAMPLE 9.23

Given the situation from Worked Example 9.21 and the fact that the true mean is $\mu = 7.5$, find the probability of making a Type II error.

We first need to find the region where we fail to reject H_0 assuming it is true

If H_0 is true then,

$$\bar{X} \sim \mathrm{N}\left(6, \frac{4}{15}\right)$$

The test is two-tailed at 10% so each tail is a 5% region

$$\mathrm{P}(\bar{X} > a) = 0.95$$
$$a = 6.85$$

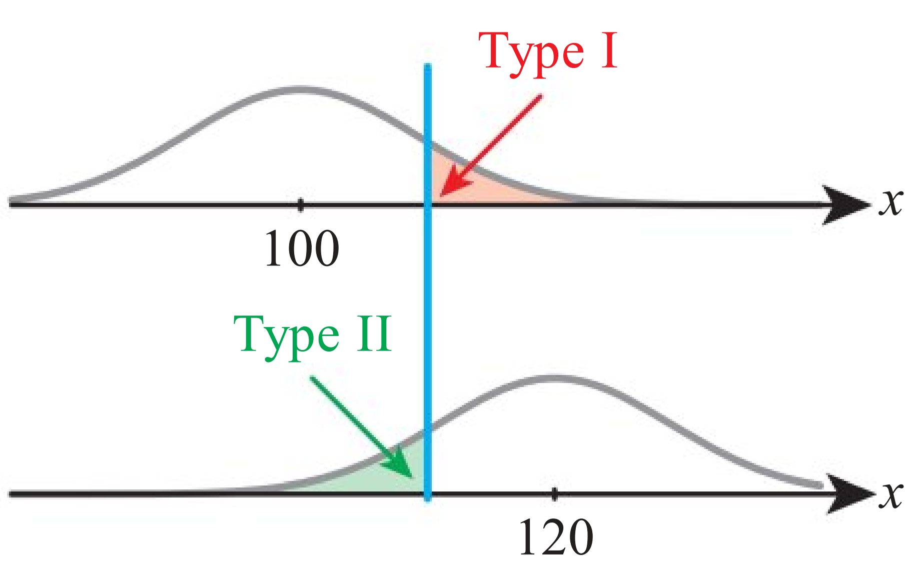
By symmetry about the mean of 6, the lower boundary is $6 - 0.85 = 5.15$ (The critical region is $\bar{X} < 5.15$ or $\bar{X} > 6.85$)

So, fail to reject H_0 when $5.15 < \bar{X} < 6.85$

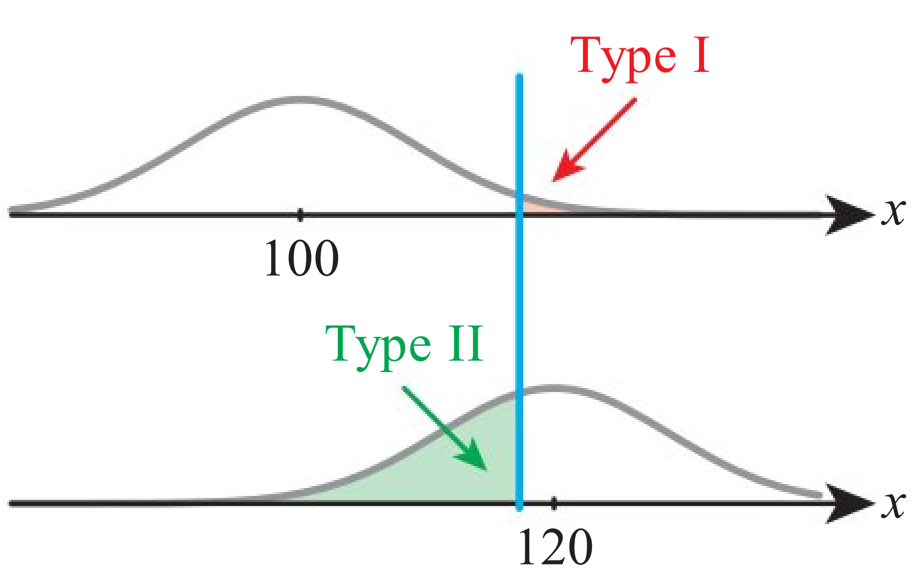
The probability of a Type II error is the probability of not being in the critical region given the true distribution

Given, in fact, that $\bar{X} \sim \mathrm{N}\left(7.5, \frac{4}{15}\right)$,

$$\mathrm{P}(\text{Type II error}) = \mathrm{P}(5.15 < \bar{X} < 6.85) = 0.104$$



Large significance level means small probability of Type II error.



Small significance level means large probability of Type II error.

We can reduce the probability of making a Type I error by reducing the significance level of the test, but this will have the effect of increasing the probability of making a Type II error and vice versa.

The diagrams on the left represent a one-tail hypothesis test with a null hypothesis $\mu = 100$. The first pair has a larger significance level than the second pair. In each pair, the top diagram shows the distribution assuming the null hypothesis is true ($\mu = 100$). The bottom diagram shows the distribution assuming one possible alternative hypothesis ($\mu = 120$ has been picked to illustrate this). Remember that before doing the test, we do not know which of these two curves is correct. The blue vertical line shows the critical value. The red region in the top diagram shows the probability of the null hypothesis being rejected even though it is true – a Type I error. The green region in the bottom diagram shows the null hypothesis not being rejected even though it is false – a Type II error. You can see that reducing the Type I error leads to a larger Type II error.

The only way of reducing the probability of a Type II error in a test on the sample mean without increasing the significance level of the test, is to increase the sample size.

CONCEPT – RELATIONSHIPS

Given the relationship between Type I and Type II errors, can you think of situations where it is more important to prioritise reducing the probability of one over the other in hypothesis tests?

Exercise 9G

For questions 1 and 2, use the techniques of Worked Example 9.21 to find the probability of making a Type I error in each situation. The sample size is n.

1 **a** $X \sim N(\mu, 7^2)$, $n = 10$, 5% significance level:

$H_0: \mu = 61$

$H_1: \mu < 61$

b $X \sim N(\mu, 1.2^2)$, $n = 6$, 10% significance level:

$H_0: \mu = 7.1$

$H_1: \mu > 7.1$

2 **a** $X \sim N(\mu, 9^2)$, $n = 20$, 10% significance level:

$H_0: \mu = 80$

$H_1: \mu \neq 80$

b $X \sim N(\mu, 5^2)$, $n = 8$, 1% significance level:

$H_0: \mu = 12.5$

$H_1: \mu \neq 12.5$

For questions 3 to 8, use the techniques of Worked Example 9.22 to find the probability of making a Type I error in each situation.

3 **a** $X \sim N(\mu, 7^2)$, $n = 10$:

$H_0: \mu = 61$

$H_1: \mu < 61$

Critical region: $\bar{X} < 56$

b $X \sim N(\mu, 1.2^2)$, $n = 6$:

$H_0: \mu = 7.1$

$H_1: \mu > 7.1$

Critical region: $\bar{X} > 8$

4 **a** $X \sim N(\mu, 9^2)$, $n = 20$:

$H_0: \mu = 80$

$H_1: \mu \neq 80$

Critical region: $\bar{X} < 77$ or $\bar{X} > 83$

b $X \sim N(\mu, 5^2)$, $n = 8$:

$H_0: \mu = 12.5$

$H_1: \mu \neq 12.5$

Critical region: $\bar{X} < 9$ or $\bar{X} > 16$

5 **a** $X \sim B(15, p)$:

$H_0: p = 0.8$

$H_1: p < 3.2$

Critical region: $X < 9$

b $X \sim B(15, p)$:

$H_0: p = 0.5$

$H_1: p < 0.5$

Critical region: $X \leqslant 4$

6 **a** $X \sim B(10, p)$:

$H_0: p = 0.4$

$H_1: p > 0.4$

Critical region: $X \geqslant 7$

b $X \sim B(10, p)$:

$H_0: p = 0.25$

$H_1: p > 0.25$

Critical region: $X > 4$

7 **a** $X \sim Po(\lambda)$:

$H_0: \lambda = 4$

$H_1: \lambda < 4$

Critical region: $X \leqslant 2$

b $X \sim Po(\lambda)$:

$H_0: \lambda = 5.5$

$H_1: \lambda < 5.5$

Critical region: $X < 4$

8 **a** $X \sim Po(\lambda)$:

$H_0: \lambda = 3.8$

$H_1: \lambda > 3.8$

Critical region: $X > 6$

b $X \sim Po(\lambda)$:

$H_0: \lambda = 2$

$H_1: \lambda > 2$

Critical region: $X \geqslant 5$

For questions 9 to 14, use the technique of Worked Example 9.23 to find the probability of making a Type II error in each situation from questions 3 to 8.

9 **a** $X \sim \mathrm{N}(\mu, 7^2)$, $n = 10$:
H_0: $\mu = 61$
H_1: $\mu < 61$
Critical region: $\bar{X} < 56$
True $\mu = 58.1$

b $X \sim \mathrm{N}(\mu, 1.2^2)$, $n = 6$:
H_0: $\mu = 7.1$
H_1: $\mu > 7.1$
Critical region: $\bar{X} > 8$
True $\mu = 7.9$

10 **a** $X \sim \mathrm{N}(\mu, 9^2)$, $n = 20$:
H_0: $\mu = 80$
H_1: $\mu \neq 80$
Critical region: $\bar{X} < 77$ or $\bar{X} > 83$
True $\mu = 77.2$

b $X \sim \mathrm{N}(\mu, 5^2)$, $n = 8$:
H_0: $\mu = 12.5$
H_1: $\mu \neq 12.5$
Critical region: $\bar{X} < 9$ or $\bar{X} > 16$
True $\mu = 15.6$

11 **a** $X \sim \mathrm{B}(15, p)$:
H_0: $p = 0.8$
H_1: $p < 0.8$
Critical region: $X < 9$
True $p = 0.5$

b $X \sim \mathrm{B}(15, p)$:
H_0: $p = 0.5$
H_1: $p < 0.5$
Critical region: $X \leqslant 4$
True $p = 0.3$

12 **a** $X \sim \mathrm{B}(10, p)$:
H_0: $p = 0.4$
H_1: $p > 0.4$
Critical region: $X \geqslant 7$
True $p = 0.6$

b $X \sim \mathrm{B}(10, p)$:
H_0: $p = 0.25$
H_1: $p > 0.25$
Critical region: $X > 4$
True $p = 0.4$

13 **a** $X \sim \mathrm{Po}(\lambda)$:
H_0: $\lambda = 4$
H_1: $\lambda < 4$
Critical region: $X \leqslant 2$
True $\lambda = 3.8$

b $X \sim \mathrm{Po}(\lambda)$:
H_0: $\lambda = 5.5$
H_1: $\lambda < 5.5$
Critical region: $X < 4$
True $\lambda = 5$

14 **a** $X \sim \mathrm{Po}(\lambda)$:
H_0: $\lambda = 3.8$
H_1: $\lambda = 3.8$
Critical region: $X > 6$
True $\lambda = 7$

b $X \sim \mathrm{Po}(\lambda)$:
H_0: $\lambda = 2$
H_1: $\lambda > 2$
Critical region: $X \geqslant 5$
True $\lambda = 6$

15 For a court case, the defendant is presumed innocent until proven guilty. In this context, state

a the null and alternative hypotheses

b what a Type I error would be

c what a Type II error would be.

16 For a casino, customers assume by default that a game involves fair dice.

a If, in fact, the casino owner is disreputable and uses biased dice, explain in context whether customers will make a Type I or Type II error.

A customer becomes suspicious and decides to roll a dice 15 times. She will conclude that the dice is biased if she does not throw any 6s.

b State the null and alternative hypotheses for this test.

c Find the probability that she makes a Type I error.

17 A particular brand of ice cream is sold in 2 litre containers. The manufacturer knows that the volume of ice cream is normally distributed with standard deviation 0.1 litres.

They test the following hypotheses by taking a sample of 10 containers:

H_0: $\mu = 2$

H_1: $\mu \neq 2$

They decide to conduct the test with a 5% significance level.

a What is the probability that they conclude that the mean is different from 2 litres when in fact it is not?

They now decide to re-do the test by rejecting the null hypothesis if the sample mean is less than 1.95 litres or more than 2.05 litres.

b Find the probability, in this case, that they conclude that the mean is different from 2 litres when in fact it is not.

18 The number of people attending a gym over the course of an hour follows a Poisson process with mean 20. Having renovated the gym and added new equipment, the manager wants to test whether the number of people attending has increased.

a State suitable null and alternative hypotheses.

The manager decides to reject the null hypothesis if the number attending in a randomly chosen hour is greater than 28.

b Find the probability that he concludes that the number has increased when in fact it has not.

19 The mean length of adult field mice is known to follow a normal distribution with mean 10.5 cm and standard deviation 1.8 cm.

A conservationist believes that field mice in a particular habitat are not growing as well as they should and so carries out a hypothesis test at the 5% level with a sample of six adult field mice for the hypotheses:

H_0 : $\mu = 10.5$

H_1 : $\mu < 10.5$

a Find the probability of a Type I error in this test.

In reality, $\mu = 9.8$ cm.

b Find the probability of a Type II error in this test.

c What could be done to decrease the probability of a Type II error without changing the probability of a Type I error?

20 A distributor of wildflower seeds states that they have an 80% chance of germinating. Daniel suspects that the proportion is not as high as stated and so plants a sample of 20 seeds. He will decide that the distributor's claim is too high if he sees fewer than 14 germinate.

a Describe, in this context, what is meant by

i a Type I error

ii a Type II error.

b Find the probability that he makes a Type I error.

c Given that the true proportion that germinates is 75%, find the probability that he makes a Type II error.

d State two changes Daniel could make to reduce the probability of a Type II error.

21 The number of beta particles emitted in one second by an isotope of a radioactive element is known to follow a Poisson distribution. Theory suggests that $\lambda = 2$ but a physicist believes that this might be an underestimate.

a State the null and alternative hypotheses.

The physicist decides that she will reject the null hypothesis if she sees at least 15 beta particles in a five second period.

b Find the significance level of this test.

In reality $\lambda = 3$.

c Find the probability of a Type II error.

22 A factory produces widgets that should measure 6 cm in length, but it is known that there is some variability and that they have a standard deviation of 0.5 cm.

The factory supervisor suggests taking regular samples of 40 widgets in order to carry out hypothesis tests to check that the mean length is 6 cm. The manager agrees but says that the sample size should be 100.

Explain the effect of a larger sample size on the chance of

a a Type I error

b a Type II error.

23 An archer knows from experience that her probability of hitting the bullseye is $\frac{2}{3}$. She has recently bought a new bow and wants to test the theory that this has improved her accuracy. To do this she shoots 12 arrows at the target.

a State the null and alternative hypotheses.

She wants the probability of making a Type I error to be less than 10%.

b Given that the probability of her hitting the bullseye is actually $\frac{5}{6}$, find the smallest possible probability that she makes a Type II error.

24 The number of calls to a company's customer helpline follows a Poisson distribution with mean 18.4 calls per hour.

The company launches a new page on their website to try to answer some customer queries in the hope of reducing the number of calls.

The company then conducts a hypothesis test by recording the number of calls in a randomly chosen hour.

a State the null and alternative hypotheses.

The mean number of calls per hour after the new webpage has been launched is actually found to be 16.2.

b The company want the probability of making a Type II error in this test to be less than 0.7. Find the smallest possible probability of making a Type I error.

25 Paul has 16 ties on the top shelf of his wardrobe, r of which are red and the rest blue.

He knows that he either has 6 or 11 red ties but is unsure which.

The top shelf is too high for him to see so he decides to reach up and take a random sample of three ties to test the hypotheses:

$H_0: r = 6$

$H_1: r = 11$

He decides to reject H_0 if the number of red ties in his sample is greater than one.

Find the probability that he makes

a a Type I error

b a Type II error.

Checklist

- You should understand the steps in conducting a statistical investigation: choose a research question, define the population and variables, design data collection methods, choose relevant and appropriate data, choose appropriate statistical processes, evaluate the validity and reliability of conclusions.
- You should understand that:
  - A process is **valid** if it is measuring what you really want to measure.
    - Aspects of validity include content validity (whether the test is relevant) and criterion validity (whether it is consistent with other ways of measuring the same quantity).
  - The conclusions of a test are **reliable** if similar conclusions would be reached on each occasion the test is conducted in similar circumstances.
    - Two possible ways to check reliability are test-retest (repeating the test on a different sample) and parallel forms (measuring the same quantity in a slightly different way).

- You should know that an unbiased estimate of the population mean is $\bar{x}$, and an unbiased estimate of the population variance is $s_{n-1}^2 = \frac{n}{n-1}s_n^2$ where $\bar{x}$ is the sample mean and s_n^2 is the sample variance.
- When performing a χ^2 test you should be able to:
  - combine groups to ensure that all expected frequencies are greater than five
  - find the number of degrees of freedom:
    ν = number of groups − 1 − number of estimated parameters
- You should be able to use technology to find non-linear regression models (quadratic, cubic, exponential, power and sine).
- You should be able to calculate the sum of square residuals, SS_{res}:
  - $SS_{res} = \sum (y_i - \hat{y}_i)^2$, where $\hat{y}_i$ are the values the model predicts.
  - The smaller the value of SS_{res} the better the model fits the data.
- You should be able to find the coefficient of determination, R^2, $0 \leqslant R^2 \leqslant 1$, where a value of 1 indicates that the model perfectly predicts the data values.
- You should understand that a confidence interval is an interval which has a prescribed probability of containing the true population mean.
- You should be able to choose an appropriate type of confidence interval for the population mean:
  - use the z-interval when the population variance is known
  - use the t-interval when the variance has been estimated from the sample.
- You should be able to conduct the following hypothesis tests:
  - test for the population mean of a normal distribution with unknown variance (t-test)
  - test for the population mean of a normal distribution with known variance (z-test)
  - test for the difference between two population means (choosing between z- and t-test)
  - test for the difference in population mean for paired samples
  - test for the population proportion using the binomial distribution
  - test for the population mean/average rate using the Poisson distribution
  - test for the population Pearson's product-moment correlation coefficient.
- You should be able to find a critical region for z-tests, binomial and Poisson tests. This is the set of observed values which lead you to reject the null hypothesis.
- You should know about Type I and Type I errors in hypothesis tests:
  - Type I error: reject H_0 when it was true
  - Type II error: fail to reject H_0 when it was false.
- You should be able to find the probability of making Type I and Type II errors:
  - $P(\text{Type I error}) = P(\text{being in critical region} \mid H_0 \text{ true})$
    – for a continuous distribution, $P(\text{Type I error}) =$ significance level of test
    – for a discrete distribution, $P(\text{Type I error}) \leqslant$ significance level of test.
  - $P(\text{Type II error}) = P(\text{Not being in critical region} \mid \text{true distribution})$

Mixed Practice

1 The heights of five-year-old children are modelled by a normal distribution. The heights of a random sample of 20 children have mean 123.8 cm and standard deviation 4.9 cm.

a Find an unbiased estimate for the population variance of heights.

b Find a 95% confidence interval for the population mean height.

2 Darya is campaigning for a change to the school lunch menu. As a part of her argument, she wants to state whether favourite food is dependent on age. She surveys students from different year groups and obtains the following results.

	Year 7	Year 8	Year 9	Year 10	Year 11
Pizza	12	14	9	12	8
Hot dog	4	6	9	10	9
Lasagne	6	6	8	14	15
Burger	10	13	10	9	7

a In order to carry out a chi-squared test for independence, two of the year groups need to be combined. State, with a reason, which groups should be combined.

b Test, using a 5% significance level, whether favourite food depends on age.

c Darya says that the test shows that favourite food is independent of age. Is this a valid conclusion? Explain your answer.

3 Peter and Qingqing have collected the following data.

x	1	2	3	4	4	5	6	6	7	9
y	0.5	0.9	1.1	1.8	2.2	2.5	3.9	3.6	4.9	9.5

Peter suggests a model of the form $y = ax + b$ and Qingqing suggests a model of the form $y = ae^{bx}$.

a On the basis of the coefficient of determination, decide whether Peter's or Qingqing's model is a better fit for the data.

b Use your chosen model to predict the value of y when $x = 8$.

4 All students in a large school were given a typing test and it was found that the times taken to type one page of text are normally distributed with mean 10.3 minutes and standard deviation 3.7 minutes. The students are given a month-long typing course and then a random sample of 20 students was asked to take the typing test again. The mean time was 9.2 minutes.

a Test at a 10% significance level whether there is evidence that the time the students take to type a page of text has decreased.

b What assumption did you make about the standard deviation of the times after the test? Give one reason why this assumption may not be justified.

c State one way you could check that your results are reliable.

5 A sample of 10 eggs are weighed and the results in grams are

62 57 84 92 77 68 59 80 81 72

Assuming that these masses are a random sample from a normal population, calculate

a unbiased estimates of the mean and variance of this population

b a 90% confidence interval for the mean mass of an egg.

6 The blood oxygen levels (measured in percent) of an individual are known to be normally distributed with a standard deviation of 3%. Based upon six readings, Niamh finds that her blood oxygen levels are on average 88.2%. Find a 95% confidence interval for Niamh's true blood oxygen level.

7 The masses of chickens from two different farms are known to follow normal distributions, both with variance $0.36\,\text{kg}^2$. A sample of 12 chickens from the first farm had a mean mass of 2.4 kg and the sample of 15 chickens from the second farm had a mean mass of 2.1 kg. A vet wants to test whether the mean masses of chickens from the two farms are different.

a State suitable hypotheses for this test.
b Find the p-value for the test.
c Hence state the conclusion of the test at the 10% significance level.

8 A meteorologist is investigating the link between temperature and cloud cover at a particular location during the autumn months. For a random sample of eight days, he records the average daily temperature (T°C) and the average cloud cover (c%).

T (°C)	12.5	8.3	4.7	9.8	11.3	9.2	7.5	11.5
c (%)	63	72	78	51	37	42	75	26

The meteorologist wants to conduct a hypothesis test to see whether there is evidence of a negative correlation.

a State suitable hypotheses for this test.
b What assumption about the distribution of the temperature and cloud cover are needed for the test to be valid?
c Use the data in the table to find the p-value.
d State the conclusion of the test at the 5% significance level.
e The meteorologist would like to extend his study to cover data from the whole year rather than just the autumn months. Suggest why a correlation test might not give a significant result.

9 A taxi company receives 17 bookings an hour on average.

a Find the probability that the company receives at least 60 bookings in three hours.

Following an advertising campaign, the owner wants to test whether the average number of bookings has increased.

b State suitable hypotheses for this test.
c In a randomly selected three-hour period there are 60 bookings. Test, using a 5% significance level, whether the average number of bookings has increased.

10 A teacher is investigating whether the final grades are dependent on time spent revising. He gives his students a questionnaire, asking for their name and how many hours they spent revising in the week before the exam.

a Suggest one possible problem with this questionnaire.

The teacher also records the final grade for each student and summarises all the data in a table.

		Grade		
		5	**6**	**7**
Revision time (hours)	**10–12**	12	16	8
	13–15	12	20	14
	16–20	14	18	18
	21–25	3	12	24

b Explain why some rows need to be combined.
c Why should you not combine the '10–12' and '21–25' rows?
d Conduct a chi-squared test to find whether, at the 10% significance level, there is evidence that the revision time and the grades are dependent.

11 Naveen tosses a coin 30 times and obtains 19 heads. Does this provide sufficient evidence, at the 5% significance level, that the probability of getting heads is less than 0.5? State your hypotheses and show your method.

12 The table shows the marks on an English test and a History test for a random sample of ten students.

	1	**2**	**3**	**4**	**5**	**6**	**7**	**8**	**9**	**10**
English mark	62	81	47	85	39	66	72	98	62	58
History mark	53	47	45	56	28	43	47	48	37	48

It can be assumed that both sets of marks come from a normal distribution. Is there evidence, at the 5% significance level, of any linear correlation between the two sets of marks?

13 A manufacturer of stopwatches employs a large number of people to time the winner of a 100 metre sprint. It is believed that if the true time of the winner is μ seconds, the times recorded are normally distributed with mean μ seconds and standard deviation 0.03 seconds.
The times, in seconds, recorded by six randomly chosen people are 9.765, 9.811, 9.783, 9.797, 9.804, 9.798.

a Calculate a 99% confidence interval for μ. Give your answer correct to three decimal places.
b Interpret the result found in **a**.

Mathematics HL May 2015 Paper 3 Statistics and probability Q3 **a** and **b**

14 The following table gives the average yield of olives per tree, in kg, and the rainfall, in cm, for nine separate regions of Greece. You may assume that these data are a random sample from a bivariate normal distribution, with correlation coefficient ρ.

Rainfall (x)	11	10	15	13	7	18	22	20	28
Yield (y)	56	53	67	61	54	78	86	88	78

A scientist wishes to use these data to determine whether there is a positive correlation between rainfall and yield.

a State suitable hypotheses.
b Determine the product moment correlation coefficient for these data.
c Determine the associated p-value and comment on this value in the context of the question.
d Find the equation of the regression line of y on x.
e Hence, estimate the yield per tree in a tenth region where the rainfall was 19 cm.

Mathematics HL May 2014 Paper 3 Statistics and probability Q2 (excluding part **f**)

15 Sajid is examining staffing levels in his small business. He has collected the following data on weekly profit, P (in thousands of \$), and number of staff employed, n.

n	1	2	3	4	5	6
P	3.2	4.6	5.1	5.4	5.2	4.9

He decides to perform a hypothesis test at the 5% significance using Pearson's product-moment correlation coefficient to assess a theory that a greater number of staff is associated with greater profit.

a State the hypotheses and the p-value for the test.
b Sajid says that this shows that there is no relationship between the number of staff employed and the weekly profit. Explain why this is not a valid conclusion.

Rishi suggests that a quadratic model might be more appropriate.

c Find a model of the form $P = an^2 + bn + c$.
d Use the coefficient of determination to comment on the suitability of a quadratic model.

16 Based on the data below, Sven has proposed two potential models to predict the temperature in a 24-hour period at a certain time of year:

Model A: $T = 5\sin(0.26t) + 14$

Model B: $T = 6\sin(0.27t) + 13$

where in each case t is the time after 09:00.

Time	Temperature, T°C
09:00	13.4
13:00	18.7
17:00	17.1
21:00	14.3
01:00	7.9
05:00	8.9

The sum of square residuals for the first four data points (09:00, 13:00, 17:00, 21:00) for each model are as follows:

Model A: $SS_{res} = 2.15$

Model B: $SS_{res} = 4.69$

a Calculate the sum of square residuals for each model once all 6 data points are included.
b Hence suggest which model he should choose and why.

17 A sample of 50 people in a town have an average wage of £24 506 and an unbiased estimate of the population variance is 144 million.
a Find a 95% confidence interval for the mean wage in the town.
b Is there significant evidence (at 5% significance) that the mean wage in this town is different from £25 000?
c What assumption have you made about the wages?

18 Martin and Oli are trying to estimate the mean time it takes their classmates to complete their homework.
a Martin asks five people and gets the following data (in minutes):
26, 39, 42, 47, 58
Find a 90% confidence interval for the mean time.
b Oli asks eight people. This sample has a mean of 28.4 minutes and a standard deviation of 4.2 minutes. Find a 90% confidence interval for mean time for Oli's data.
c What does this suggest about the reliability of the estimate for the mean time taken to complete homework? Explain your answer.
d Suggest one way to improve the reliability of the estimate.

19 A farmer knows from experience that the heights of apple trees are normally distributed with mean 2.7 m and standard deviation 0.7 m. He buys a new orchard and wants to test whether the average height of apple trees is different. He assumes that the heights are still normally distributed with standard deviation 0.7 m.
a State the hypotheses he should use for his test.

The farmer measures the heights of 45 trees and finds their average.
b Find the critical region for the test at the 10% level of significance.
c If the average height of the 45 trees is 2.3 m, state the conclusion of the hypothesis test.
d Is the conclusion still valid if it is not known that the heights are normally distributed? Give a reason for your answer.

20 Qing is a student guide at a local museum and wants to model the number of visitors per hour. She obtains the following data.

Number of visitors	$\leqslant 3$	4	5	6	7	8	9	10	$\geqslant 11$
Frequency	0	6	9	17	22	18	11	7	0

a Estimate the mean number of visitors per hour.

Qing thinks that the number of visitors per hour can be modelled by a Poisson distribution.

b Write down suitable hypotheses.
c Find the expected frequencies.
d Write down the number of degrees of freedom for the test.
e Conduct the test at the 1% significance level. Interpret your conclusion in context.

21 A catering manager at school knows from past experience that around 72% of students eat school lunch. He hopes that introducing a new menu will encourage more students to have school lunches. Following the publication of the new menu, he plans to take a sample of 30 students and find out how many intend to eat school lunch. He will then conduct a hypothesis test, using a 5% significance level, to test whether the proportion has increased.

a State suitable hypotheses for the test.
b Find the critical region.
c What is the probability of a Type I error for this test?

22 The mass of four burgers, in grams, before and after being cooked for one minute is measured.

Burger	A	B	C	D
Before cooking	148	167	160	142
After cooking	124	135	134	119

a Find the decrease in mass for each burger.
b Find the 90% confidence interval for the mean mass loss.
c What distributional assumption does your calculation require?

23 The number of accidents on a particular stretch of road is modelled by a Poisson distribution with an average rate of 1.4 per month. Following the introduction of new speed limits the town council want to test whether the accident rate has decreased. They decide to look at the number of accidents over a four-month period and use a 10% significance level.

a Find the critical region for this test.
b Find the probability of a Type I error.
c Given that the rate has actually decreased to 1.2. per month, find the probability of a Type II error.

24 A meteorologist wants to test whether maximum daily temperatures during the year can be modelled by a normal distribution. She uses the following data.

Temperature (T°C)	$T < 3$	$3 \leqslant T < 7$	$7 \leqslant T < 10$	$10 \leqslant T < 15$	$15 \leqslant T < 18$	$18 \leqslant T < 20$	$20 \leqslant T < 25$	$25 \leqslant T < 30$	$T > 30$
Number of days	0	4	12	21	26	18	7	3	0

a State suitable hypotheses for a χ^2 test.
b Estimate the population mean and standard deviation.
c Find the expected frequencies, correct to two decimal places.
d Explain why some groups need to be combined, and state the number of degrees of freedom (after combining the groups).
e Conduct a test at the 5% significance level.

25 A teacher claims that 60% of students at a college are satisfied with the new uniform. The student president believes that the proportion is lower than this. She suggests asking a random sample of 50 students for their opinion.

a The teacher offers to ask the students in person. Explain why this may not produce valid results.

They decide to distribute an anonymous questionnaire with the question 'Are you satisfied with the new uniform?' The number of students who answer 'Yes' is denoted by X.

b Write suitable hypotheses to test whether the proportion of students who are satisfied with the uniform is lower than 60%.

c Find the critical region for the test using a 5% significance level.

d Given that the real proportion is 43%, find the probability of a Type II error.

e Given instead that the real proportion is 63%, find the probability of a Type II error.

26 A train company claims that the probability of its trains arriving late on a particular service is 0.1. A traveller who regularly uses this service believes that the probability is greater than 0.1. He randomly selects 16 trains and observes that 4 are late.

a Carry out a hypothesis test, at the 5% significance level, to investigate the traveller's belief.

b Find the probability of a Type I error.

The true probability of trains being late is 0.18.

c Find the probability of a Type II error.

d Find the probability of rejecting the null hypothesis when it was not true.

27 The hens on a farm lay either white or brown eggs. The eggs are put into boxes of six. The farmer claims that the number of brown eggs in a box can be modelled by the binomial distribution, B(6, p). By inspecting the contents of 150 boxes of eggs, she obtains the following data.

Number of brown eggs	0	1	2	3	4	5	6
Number of boxes	7	32	35	50	22	4	0

a Show that this data leads to an estimated value of $p = 0.4$.

b Stating null and alternative hypotheses, carry out an appropriate test at the 5% level to decide whether the farmer's claim can be justified.

Mathematics HL May 2008 Paper 3 Statistics and probability Q1

28 The mean mass of a certain breed of bird is believed to be 2.5 kg. In order to test this belief, it is planned to determine the masses $x_1, x_2, x_3, \ldots, x_{16}$ (in kg) of sixteen of these birds and then to calculate the sample mean $\bar{x}$. You may assume that these masses are a random sample from a normal distribution with standard deviation 0.1 kg.

a State suitable hypotheses for a two-tailed test.

b Find the critical region for $\bar{x}$ having a significance level of 5%.

c Given that the mean mass of birds of this breed is actually 2.6 kg, find the probability of making a Type II error.

Mathematics HL November 2009 Paper 3 Statistics and probability Q1

29 **a** The random variable X represents the height of a wave on a particular surf beach. It is known that X is normally distributed with unknown mean μ (metres) and known variance $\sigma = 0.25$ (metres^2). Sally wishes to test the claim made in a surf guide that $\mu = 3$ against the alternative that $\mu < 3$. She measures the heights of 36 waves and calculates their sample mean $\bar{x}$. She uses this value to test the claim at the 5% level.

i Find a simple inequality, of the form $\bar{x} < A$, where A is a number to be determined to four significant figures, so that Sally will reject the null hypothesis that $\mu = 3$, if and only if this inequality is satisfied.

ii Define a Type I error.

iii Define a Type II error.

iv Write down the probability that Sally makes a Type I error.

v The true value of μ is 2.75. Calculate the probability that Sally makes a Type II error.

b The random variable Y represents the height of a wave on another surf beach. It is known that Y is normally distributed with unknown mean μ (metres) and unknown variance σ^2 (metres^2). David wishes to test the claim made in a surf guide that $\mu = 3$ against the alternative that $\mu < 3$. He is also going to perform this test at the 5% level. He measures the heights of 36 waves and finds that the sample mean, $\bar{y} = 2.860$ and the unbiased estimate of the population variance, $s_{n-1}^2 = 0.25$.

i State the name of the test that David should perform.

ii State the conclusion of David's test, justifying your answer by giving the p-value.

iii Using David's results, calculate the 90% confidence interval for μ, giving your answers to four significant figures.

Mathematics HL November 2012 Paper 3 Statistics and probability Q3

30 Marc and Nathan are researching factors that might influence blood pressure.

The blood pressure of 10 men is recorded. Marc is interested in the impact of age on blood pressure while Nathan is interested in the effect of mass, so these data are also collected.

Age/years (y)	**Mass/kg (m)**	**Blood pressure/ mm Hg (p)**
21	72	118
27	85	125
32	79	122
35	75	122
40	91	134
46	88	130
51	97	145
55	80	131
63	86	137
69	70	125

Both Marc and Nathan propose cubic models.

a Find Marc's model where p is a function of y.

b Find Nathan's model where p is a function of m.

c Explain which of the two models accounts for the greater proportion of variability in blood pressure.

d Suggest how the models could be improved.

31 Irina wants to model the total cost to her business, $\$T$, as function of output quantity, Q.

She proposes a quadratic model to fit the data she has gathered.

Quantity, (Q)	Total cost, (T)
4	390
8	770
12	830
16	900
20	1400
24	1900
28	2800

a Find a regression model of the form $T = aQ^2 + bQ + c$.
b Find the coefficient of determination for this model.
c Use the model to find the total cost of producing
  i 10 items
  ii 40 items.
d Comment on the reliability of the predictions in part **c**.

Julian decides to model the total cost for his business in the same way.

He gathers the following data.

Quantity, (Q)	Total cost, (T)
5	310
10	620
15	750
20	810
25	1200

e Find a regression model of the form $T = aQ^2 + bQ + c$ to fit these data.

Julian suggests using the sum of square residuals to compare the fit of his model with the fit of Irina's model.
f Explain why this is not a good way of comparing the fit of the two models.
g Instead, use the coefficient of determination to decide which model is a better fit.

By comparing the value of R^2 for her original quadratic model to that of a cubic model, Irina claims that a cubic model provides a better fit for her data.
h Comment on Irina's claim.

32 The viewing figures of a long-running television series are X million. In the past, it was known to follow a normal distribution with a mean of 0.3 million. A producer wants to know if a new presenter has changed the viewing figures across 12 episodes. He conducts a hypothesis test assuming that the viewing figures are still normally distributed with mean 0.3 million. The acceptance region (the values which do not lead to rejecting H_0) is found to be $6.258 < \overline{X} < 6.542$.
a Deduce the null and alternative hypotheses.
b Find the significance level of this hypothesis test, giving your answer as a percentage to the nearest whole number.

33 An athletic coach wants to find out whether an improved diet leads to better results. He proposes:

Test A: Randomly select two groups of five athletes. One group is given a healthy diet for two weeks. Then compare their 400 m running times.

The assistant coach suggests instead:

Test B: Randomly select a group of ten athletes and time their 400 m run. Then give them a healthy diet for two weeks, time them again and look at the change in their times.

a Which test is more valid? Explain your answer.

The running times are known to be normally distributed. The coach decides to carry out Test B, using a 10% significance level. The times (in seconds) before and after the diet are recorded in the table.

Athlete	**1**	**2**	**3**	**4**	**5**	**6**	**7**	**8**	**9**
Before	53.7	49.4	52.8	43.9	54.5	53.5	53.1	52.9	56.3
After	53.1	48.3	52.7	44.1	54.2	53.8	52.7	53.1	55.9

b Let D be the difference between the times (after – before). Find the standard deviation of D.

c Is there evidence that an improved diet leads to better results? State your hypotheses and conclusion clearly.

34 Long-term observations suggest that the average number of cars passing through a remote village is 23 per day, and the number of lorries is 8 per day. The vehicles are regarded as independent.

Aline wants to test whether the number of vehicles has increased recently by counting the total number of vehicles during a random sample of seven days.

a Aline wants her test to have a significance level of 5%. Find the critical region for this test.

b Reto suggests testing for the increase in the number of cars and the number of lorries separately, but using a 2.5% significance level for each. He decides to conclude that the number of vehicles has increased if either of the numbers fall in the critical region. Who has the larger probability of making a Type I error?

35 The number of machine breakdowns occurring in a day in a certain factory may be assumed to follow a Poisson distribution with mean μ. The value of μ is known, from past experience, to be 1.2. In an attempt to reduce the value of μ, all the machines are fitted with new control units. To investigate whether or not this reduces the value of μ, the total number of breakdowns, x, occurring during a 30-day period following the installation of these new units is recorded.

a State suitable hypotheses for this investigation.

b It is decided to define the critical region by $x \leqslant 25$.

i Calculate the significance level.

ii Assuming that the value of μ was actually reduced to 0.75, determine the probability of a Type II error

Mathematics HL May 2013 Paper 3 Statistics and probability Q3

36 The random variable X has a Poisson distribution with mean μ. The value of μ is known to be either 1 or 2 so the following hypotheses are set up.

$H_0: \mu = 1; H_1: \mu = 2$

A random sample $x_1, x_2, ..., x_{10}$ of 10 observations is taken from the distribution of X and the following critical region is defined.

$$\sum_{i=1}^{10} x_i \geqslant 15$$

Determine the probability of

a a Type I error

b a Type II error.

Mathematics HL May 2010 Paper 3 Statistics and probability Q2

37 Two species of plant, A and B, are identical in appearance though it is known that the mean length of leaves from a plant of species A is 5.2 cm, whereas the mean length of leaves from a plant of species B is 4.6 cm. Both lengths can be modelled by normal distributions with standard deviation 1.2 cm.

In order to test whether a particular plant is from species A or species B, 16 leaves are collected at random from the plant. The length, x, of each leaf is measured and the mean length evaluated. A one-tailed test of the sample mean, $\bar{X}$, is then performed at the 5% level, with the hypotheses

$H_0: \mu = 5.2$ and $H_1: \mu < 5\ 2$.

a Find the critical region for this test.

b Find the probability of a Type II error if the leaves are in fact from a plant of species B.

It is now known that in the area in which the plant was found 90% of all the plants are of species A and 10% are of species B.

c Find the probability that $\bar{X}$ will fall within the critical region of the test.

d If, having done the test, the sample mean is found to lie within the critical region, find the probability that the leaves came from a plant of species A.

Mathematics HL November 2014 Paper 3 Statistics and probability Q5

10 Differentiation

ESSENTIAL UNDERSTANDINGS

- Calculus describes rates of change between two variables.
- Calculus helps us to understand the behaviour of functions and allows us to interpret features of their graphs.

In this chapter you will learn…

- how to differentiate functions such as $\sqrt{x}$, $\sin x$ and $\ln x$
- how to use the chain rule to differentiate composite functions
- how to differentiate products and quotients
- how to interpret the second derivative of a function
- how to classify local minimum and maximum points on a graph
- how to find related rates of change.

CONCEPTS

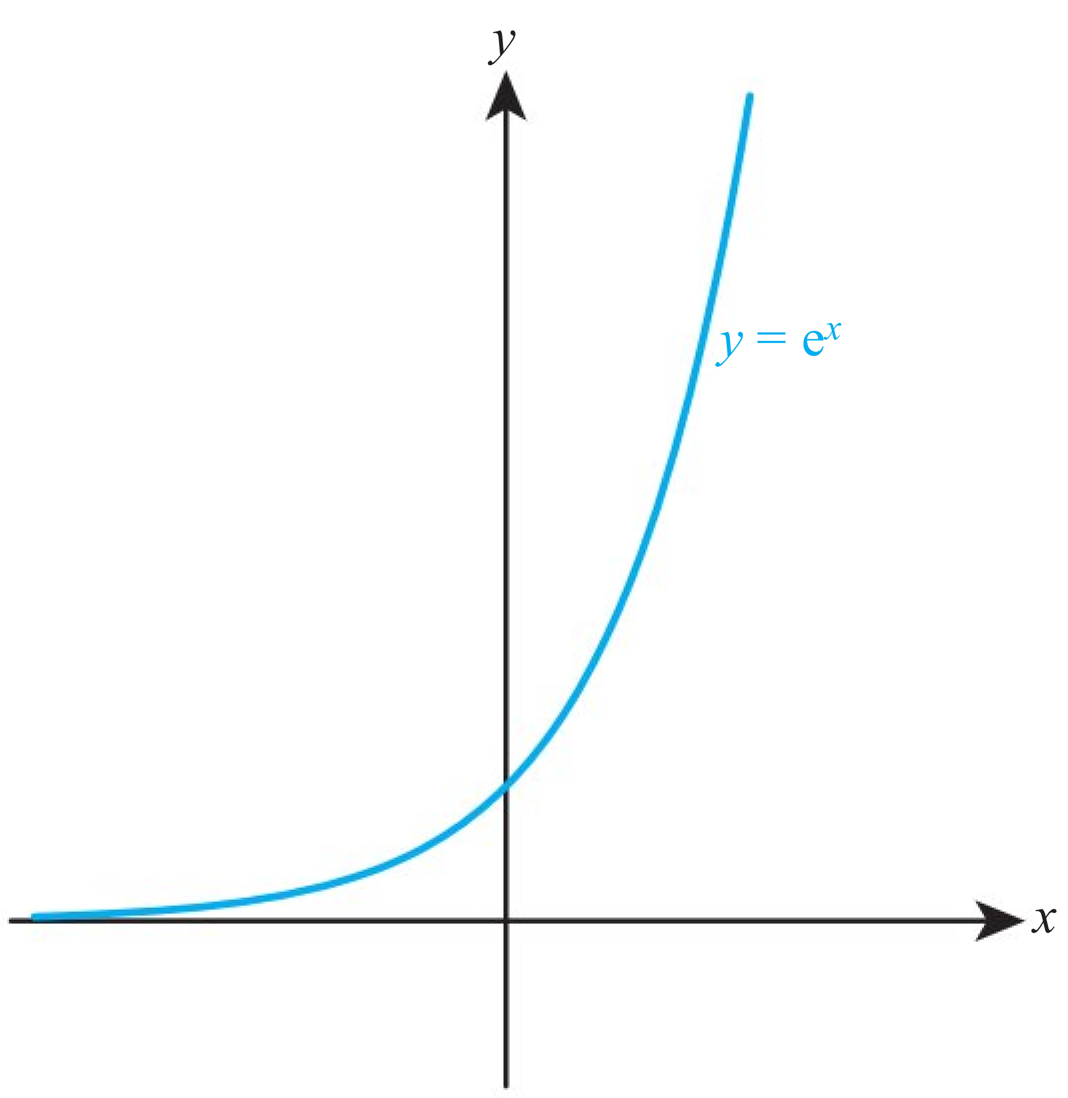
The following concepts will be addressed in this chapter:

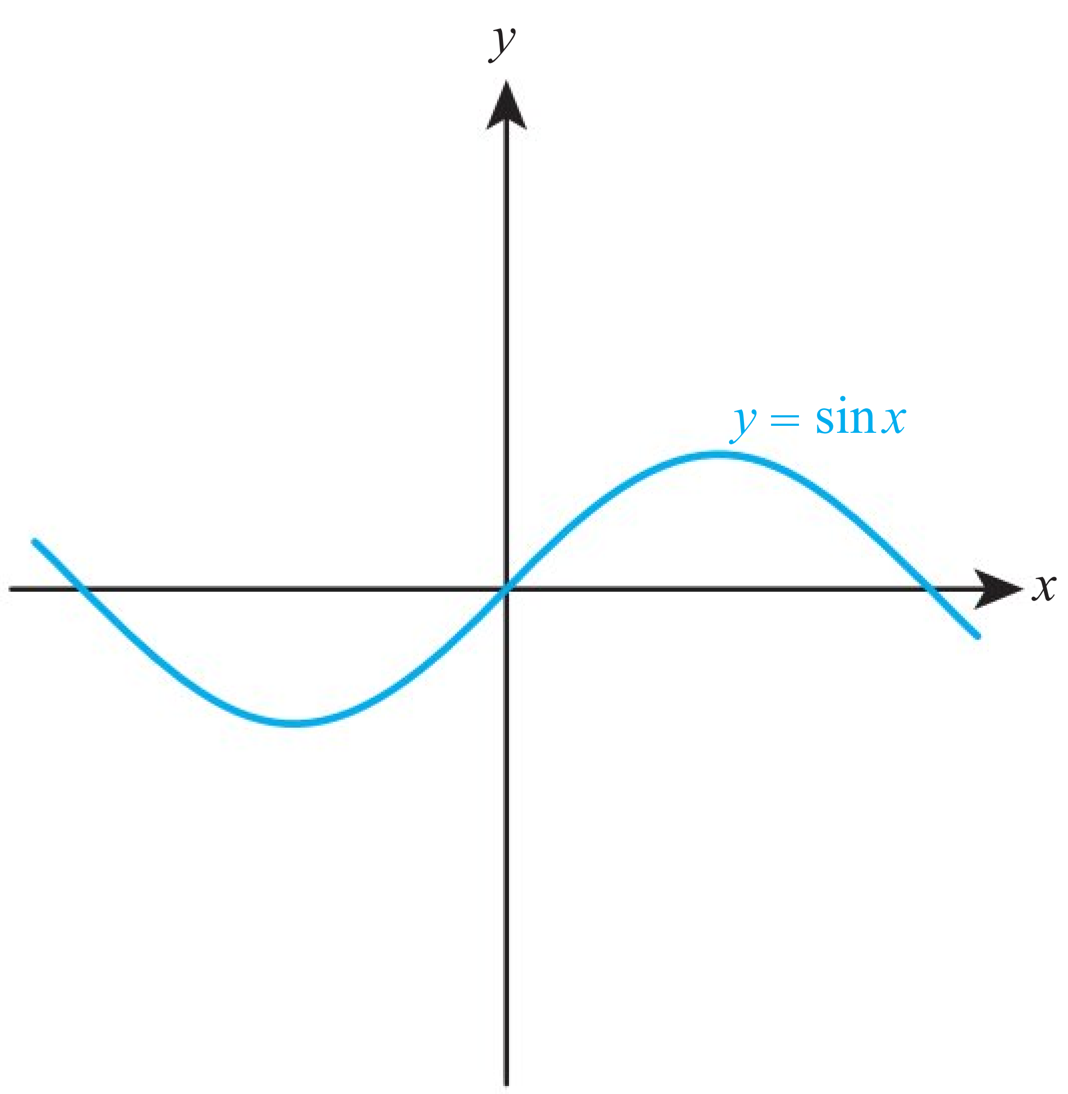
- The derivative may be **represented** physically as a rate of change and geometrically as the gradient or slope function.
- Mathematical **modelling** can provide effective solutions to real-life problems in optimization by maximizing or minimizing a quantity.

LEARNER PROFILE – Caring

When giving to charity, are you influenced more by mathematical measures of the impact your donation will make or by an emotional connection to the cause? Can all outcomes be measured?

■ **Figure 10.1** What real-world scenarios might these graphs be used to model?





PRIOR KNOWLEDGE

Before starting this chapter, you should already be able to complete the following:

1. If $f(x) = x^2 + 2$ find $f''(x)$.
2. Find the equation of the tangent to the curve $y = x^3$ at $x = 1$.
3. Solve $e^x = 5$.
4. Find the coordinates of any stationary points on the curve $y = x^3 - 12x$.

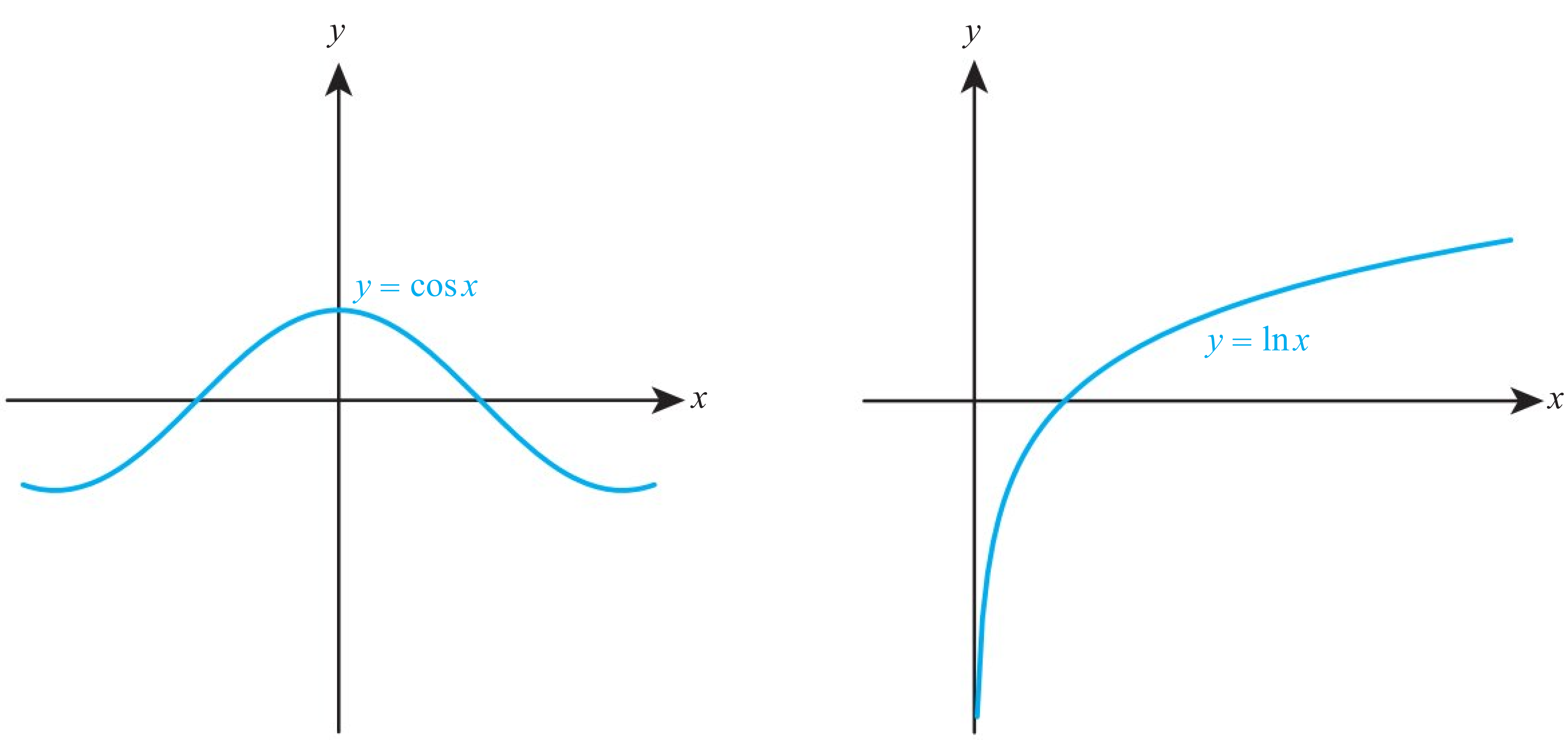
You already know how to differentiate some simple functions, but in this chapter you will significantly extend the number of functions you can differentiate. You will also meet further applications of differentiation used to describe graphs and see how this can be translated into many real-world situations where optimal solutions are required.

Starter Activity

The curves in Figure 10.1 show the graphs of e^x, $\sin x$, $\cos x$ and $\ln x$. Suggest possible real-world situations modelled by each graph. Sketch the derivative of each graph. Hence suggest possible functions that could be the derivative of each graph. What would be the interpretation of the derivatives in each of the real-world situations you have suggested?

Now look at this problem:

The gradient of the curve $y = f(x)$ when $x = 2$ is 10. The function $g(x)$ is defined as $f(2x)$. What is the gradient of the graph of $y = g(x)$ when $x = 1$?



10A Extending differentiation

So far, you have only seen how to differentiate functions of the form $f(x) = ax^n + bx^m + \ldots$, where $n, m \in \mathbb{Z}$. It would be useful, however, to be able to differentiate the many other types of function you have met, too.

You saw in Section 9C of the Mathematics: applications and interpretation SL book that the derivative of $f(x) = x^n$, where $n \in \mathbb{Z}$, is $f'(x) = nx^{n-1}$.

Derivative of x^n, $n \in \mathbb{Q}$

The same result that you already know for differentiating integer powers of x also applies when n is any rational number.

KEY POINT 10.1

If $f(x) = x^n$, where $n \in \mathbb{Q}$, then $f'(x) = nx^{n-1}$.

WORKED EXAMPLE 10.1

Find $f'(x)$ for $f(x) = \sqrt[3]{x}$.

Write $f(x)$ in the form x^n

$$f(x) = \sqrt[3]{x}$$
$$= x^{\frac{1}{3}}$$

Use $f'(x) = nx^{n-1}$

$$f'(x) = \frac{1}{3}x^{\frac{1}{3}-1}$$
$$= \frac{1}{3}x^{-\frac{2}{3}}$$

As before, this result applies to multiples and sums of terms, and you might need to use the laws of exponents to get the expression in the correct form first.

WORKED EXAMPLE 10.2

Find $\frac{dy}{dx}$ for $y = \frac{3x^2 - 4}{\sqrt{x}}$.

Use the laws of exponents to get the expression into the form $ax^n + bx^m$

$$y = \frac{3x^2 - 4}{\sqrt{x}}$$
$$= \frac{3x^2}{x^{\frac{1}{2}}} - \frac{4}{x^{\frac{1}{2}}}$$
$$= 3x^{\frac{3}{2}} - 4x^{-\frac{1}{2}}$$

Then differentiate term by term

$$\frac{dy}{dx} = 3 \times \frac{3}{2}x^{\frac{3}{2}-1} - 4\left(-\frac{1}{2}\right)x^{-\frac{1}{2}-1}$$
$$= \frac{9}{2}x^{\frac{1}{2}} + 2x^{-\frac{3}{2}}$$

Derivatives of sin x, cos x and tan x

A different class of function that is very familiar, but that you have not yet learnt how to differentiate, is trigonometric functions. It turns out that the results given are only true if the angle x is measured in radians.

You investigated these derivatives in the Starter Activity.

KEY POINT 10.2

If $f(x) = \sin x$, then $f'(x) = \cos x$.

KEY POINT 10.3

If $f(x) = \cos x$, then $f'(x) = -\sin x$.

TOOLKIT: Problem Solving

Try sketching the function $y = \sin x$ and sketching its derivative. How is the sketch different if x is in degrees to if it is in radians?

In Section 10C, you will see how to prove the derivative of tan x.

KEY POINT 10.4

If $f(x) = \tan x$, then $f'(x) = \frac{1}{\cos^2 x}$.

Again, these results can be applied to multiples and sums of terms.

WORKED EXAMPLE 10.3

Differentiate $f(x) = 2\sin x - 5\cos x$.

Differentiate term by term

$$f'(x) = 2\cos x - 5(-\sin x)$$
$$= 2\cos x + 5\sin x$$

Derivatives of e^x and ln x

Two other commonly occurring functions you have met are the exponential and natural logarithm functions.

The exponential function has the unique property that its derivative is itself.

KEY POINT 10.5

If $f(x) = e^x$, then $f'(x) = e^x$.

TOOLKIT: Problem Solving

Use technology to sketch the graph of $y = a^x$ and its gradient function for various values of a. What do you notice?

TOK Links

Does this key point reflect a property of e, or is it the definition of e? Who decides the starting points (called axioms) of mathematics? What criteria do you use to determine if one fact is more fundamental than another?

The derivative of the natural logarithm function follows from the fact that it is the inverse of the exponential function.

KEY POINT 10.6

If $f(x) = \ln x$, then $f'(x) = \frac{1}{x}$.

Proof 10.1

Given that, for $y = e^x$, $y' = e^x$, prove that, for $y = \ln x$, $y' = \frac{1}{x}$.

Pick any point on the curve $y = e^x$

Let the point P lie on $y = e^x$ and have x-coordinate p.
So, P is the point (p, e^p).

Reflecting a point in the line $y = x$ swaps the x- and y-coordinates

Let the point Q be the reflection of P in the line $y = x$.
So, Q is the point (e^p, p).

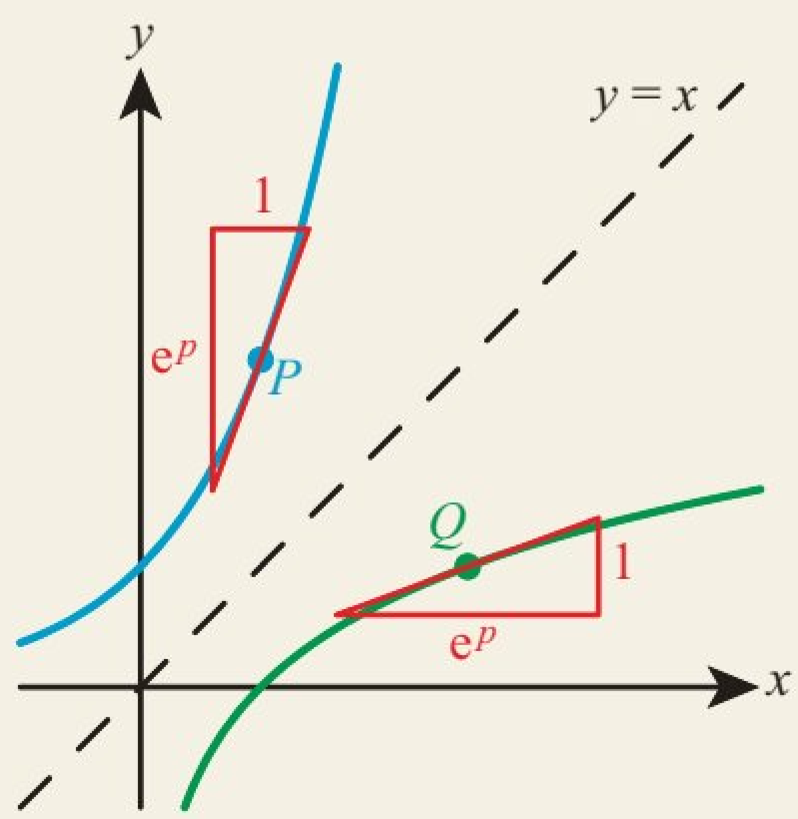
The graphs of a function and its inverse are reflections in the line $y = x$

But since $y = e^x$ and $y = \ln x$ are inverse functions, Q lies on $y = \ln x$.

The gradient of $y = e^x$ at P is e^p.

The gradient of the tangent to $y = e^x$ at P is e^p, so reflecting this tangent in the line $y = x$ gives the gradient of the tangent to $y = \ln x$ at Q

So, the gradient of $y = \ln x$ at Q is $\frac{1}{e^p}$.



But $\frac{1}{e^p}$ is the x-coordinate of Q so the gradient of $y = \ln x$ is $\frac{1}{x}$.

You can use the laws of logarithms to get the expression into the correct form for differentiating.

WORKED EXAMPLE 10.4

Given $y = \ln x^3$, find y'.

Use $\ln x^m = m\ln x$	$y = \ln x^3$ $= 3\ln x$
Now differentiate	$y' = 3\left(\frac{1}{x}\right)$ $= \frac{3}{x}$

In Section 10B, you will see how you can differentiate an expression such as $\ln x^3$ without needing to use the laws of logarithms first.

You can use these results to find the equations of tangents and normals.

WORKED EXAMPLE 10.5

Find the equation of the tangent to the curve $y = \cos x + 2e^x$ at the point $(0, 3)$.

Differentiate term by term	$y' = -\sin x + 2e^x$
Find the value of the gradient (y') at $x = 0$	When $x = 0$, $y' = -\sin 0 + 2e^0$ $= 0 + 2$ $= 2$
Substitute into the equation of a straight line: $y - y_1 = m(x - x_1)$	So, the equation of the tangent is $y - 3 = 2(x - 0)$ $y = 2x + 3$

You saw how to find the equations of tangents and normals in Section 9D of the Mathematics: applications and interpretation SL book.

Remember that, in Worked Example 10.5, you could have found the gradient at $x = 0$ using your calculator. However, it is still useful to practice these manual techniques in case questions use parameters rather than specific values.

Exercise 10A

For questions 1 to 6, use the method demonstrated in Worked Example 10.1 to find $f'(x)$ for the following functions.

1 a $f(x) = x^{\frac{2}{3}}$
b $f(x) = x^{\frac{3}{4}}$

2 a $f(x) = x^{-\frac{1}{2}}$
b $f(x) = x^{-\frac{4}{3}}$

3 a $f(x) = 6x^{\frac{3}{2}}$
b $f(x) = 8x^{\frac{1}{4}}$

4 a $f(x) = 9x^{-\frac{2}{3}}$
b $f(x) = 10x^{-\frac{2}{5}}$

5 a $f(x) = 2\sqrt{x}$
b $f(x) = 6\sqrt[4]{x}$

6 a $f(x) = \frac{-12}{\sqrt[3]{x}}$
b $f(x) = \frac{-5}{\sqrt[5]{x}}$

For questions 7 to 9, use the method demonstrated in Worked Example 10.2 to differentiate the following expressions.

7 **a** $\dfrac{9x+1}{x^{\frac{4}{3}}}$ **b** $\dfrac{2x-1}{x^{\frac{3}{2}}}$

8 **a** $\dfrac{12x^2-x}{5x^{\frac{3}{4}}}$ **b** $\dfrac{6x^3+x}{7x^{\frac{2}{3}}}$

9 **a** $\dfrac{2-3x}{\sqrt[3]{x}}$ **b** $\dfrac{5+8x}{\sqrt[4]{x}}$

For questions 10 to 12, use the method demonstrated in Worked Example 10.3 to differentiate the following functions.

10 **a** $f(x)=3\sin x$ **b** $f(x)=4\cos x$

11 **a** $f(x)=\frac{1}{2}\cos x-5\sin x$ **b** $f(x)=\frac{3}{4}\sin x-2\cos x$

12 **a** $f(x)=2\tan x+1$ **b** $f(x)=x-\tan x$

For questions 13 and 14, use the method demonstrated in Worked Example 10.4 to find $\frac{dy}{dx}$ for the following graphs.

13 **a** $y=3\ln x$ **b** $y=-4\ln x$

14 **a** $y=\ln x^2$ **b** $y=\ln x^{-1}$

For questions 15 and 16, use the technique of differentiating e^x demonstrated in Worked Example 10.5 to find y' for the following graphs.

15 **a** $y=5e^x$ **b** $y=-6e^x$

16 **a** $y=-\dfrac{e^x}{2}$ **b** $y=\dfrac{3e^x}{4}$

17 Find the gradient of the graph of $y=\sin x$ at $x=\frac{\pi}{3}$.

18 The height (h) of a falcon above the ground, in metres, during a swooping attack t seconds after it spots some prey is modelled by $h=\dfrac{(10t+20)}{\sqrt{t}}$.

What is the rate of change of height when $t=1$? What is the significance of the sign of your answer?

19 Find the point on the curve $y=2+5x-e^x$ which has gradient 4.

20 Find the exact value of the gradient of the graph $y=e^x+3x+4$ when $x=\ln 2$.

21 Find the equation of the tangent to $y=3\cos x$, $0<x<\frac{\pi}{2}$, that is parallel to $2y+3x+3=0$.

22 Find the equation of the tangent to $y=e^x-2x$ that is perpendicular to $y=-x+4$.

23 Find the point on the curve $y=\sqrt{x}$ that has a tangent of $y=\frac{x}{6}+\frac{3}{2}$.

24 **a** Find the equation of the normal to $y=2x-e^x$ at $x=\ln\frac{5}{2}$.

b Show that this normal does not meet the original curve at any other point.

25 Find the equation of the tangent to $y=\tan x$ at the point where $x=k$.

26 The number of bacteria in millions, N, after t hours of being introduced to an agar dish is modelled by $N=(1+\sqrt{t})^2$.

a What is the initial population introduced?

b How long does it take for the population to reach 16 million?

c Find the rate of increase of bacteria when $t=4$.

d State one limitation of this model.

27 Find the interval in which $\ln x-2x$ is a decreasing function.

28 The tangent to the curve $y=\sqrt{x}$ at $x=k$ is perpendicular to the tangent to the curve $y=\dfrac{1}{\sqrt{x}}$, also at $x=k$. Find the value of k.

29 **a** The curve $y=\cos x$ for $-\pi<x<\pi$ meets the x-axis at two points, labelled P and Q. The tangents at P and Q meet at R. Find the area of the triangle PQR.

b The curve $y=\sin x$ for $0\leqslant x<2\pi$ meets the x-axis at two points, labelled A and B. The tangents at A and B meet at C. Find the area of the triangle ABC.

30 Find the coordinates of the point on the curve $y=\sqrt{x}$ that has a tangent passing through the point $(0,1)$.

31 The line $2\sqrt{3}y=3x-\dfrac{13\pi}{2}$ is a tangent to the curve $y=\sin(x)-\dfrac{1}{2}$ at the point P. Find the coordinates of P.

32 Find the x-coordinate of all points on the graph of $y=x^2-2\ln x$ where the tangent is parallel to $y+3x=9$.

10B The chain rule

In Worked Example 10.4, we used one of the laws of logarithms to enable us to differentiate $\ln x^3$. However, there is no equivalent method to enable us to differentiate, say, $\sin x^3$ or e^{x^3}.

Both of these are examples of composite functions, $f(g(x))$, where $g(x) = x^3$, so we need a method for differentiating composite functions. This is provided by the **chain rule**.

KEY POINT 10.7

The chain rule:

If $y = f(u)$, where $u = g(x)$, then $\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx}$.

WORKED EXAMPLE 10.6

Differentiate $y = (x^2 - 3x)^6$.

This is a composite function .... $y = u^6$, where $u = x^2 - 3x$

Use $\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx}$, with $\frac{dy}{du} = 6u^5$ and $\frac{du}{dx} = 2x - 3$ .... $\frac{dy}{dx} = 6u^5 \times (2x - 3)$

Replace u with $x^2 - 3x$ .... $= 6(x^2 - 3x)^5(2x - 3)$

WORKED EXAMPLE 10.7

Differentiate $y = \sin 4x$.

This is a composite function .... $y = \sin u$, where $u = 4x$

Use $\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx}$, with $\frac{dy}{du} = \cos u$ and $\frac{du}{dx} = 4$ .... $\frac{dy}{dx} = \cos u \times 4$

Replace u with $4x$ .... $= 4\cos 4x$

WORKED EXAMPLE 10.8

Differentiate $y = e^{3x^2}$.

This is a composite function .... $y = e^u$, where $u = 3x^2$

Use $\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx}$, with $\frac{dy}{du} = e^u$ and $\frac{du}{dx} = 6x$ .... $\frac{dy}{dx} = e^u \times 6x$

Replace u with $3x^2$ .... $= 6xe^{3x^2}$

If you have a composite of three functions, you'll need an extra derivative in the chain rule.

WORKED EXAMPLE 10.9

Differentiate $y = (\ln 2x)^5$.

This is a composite of three functions

$y = u^5$, where $u = \ln v$ and $v = 2x$

Use the chain rule with three derivatives: $\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dv} \times \frac{dv}{dx}$, where $\frac{dy}{du} = 5u^4$, $\frac{du}{dv} = \frac{1}{v}$ and $\frac{dv}{dx} = 2$

$$\frac{dy}{dx} = 5u^4 \times \frac{1}{v} \times 2$$

Replace u with $\ln v$ and v with $2x$

$$= 5(\ln 2x)^4 \times \frac{1}{2x} \times 2$$

$$= \frac{5}{x}(\ln 2x)^4$$

Exercise 10B

For questions 1 to 3, use the method demonstrated in Worked Example 10.6 to find $\frac{dy}{dx}$ for the following graphs.

1 a $y = (3x+2)^4$
b $y = (2x-7)^5$

2 a $y = (x^2+3)^{\frac{3}{4}}$
b $y = (4-x^2)^{\frac{2}{3}}$

3 a $y = \sqrt{2x^3+x}$
b $y = \sqrt[3]{5x-x^3}$

For questions 4 to 7, use the method demonstrated in Worked Example 10.7 to find $f'(x)$ for the following functions.

4 a $f(x) = \sin 2x$
b $f(x) = \sin \frac{1}{3}x$

5 a $f(x) = \cos \pi x$
b $f(x) = \cos 5x$

6 a $f(x) = \sin(3x+1)$
b $f(x) = \sin(1-4x)$

7 a $f(x) = \cos(2-3x)$
b $f(x) = \cos\left(\frac{1}{2}x+4\right)$

For questions 8 to 11, use the method demonstrated in Worked Example 10.8 to differentiate the following expressions.

8 a e^{3x}
b $e^{\frac{x}{2}}$

9 a e^{-x^3}
b e^{4x^2}

10 a $\ln 4x$
b $\ln \frac{x}{3}$

11 a $\ln(x^2+1)$
b $\ln(3-4x^2)$

For questions 12 to 14, use the method demonstrated in Worked Example 10.9 to find the derivatives of the following graphs.

12 a $y = \sin^2 3x$
b $y = \cos^2 4x$

13 a $y = e^{\cos 2x}$
b $y = e^{\sin 5x}$

14 a $y = (\ln 3x)^{\frac{1}{2}}$
b $y = (\ln 2x)^{\frac{1}{3}}$

15 If $f(x) = \frac{1}{e^x}$ find $f'(x)$.

16 Find the gradient of the curve $y = (e^x)^2 + 5x$ when $x = \ln 3$.

17 Find the equation of the tangent to the curve $y = \sqrt{x^2+9}$ at $x = 4$.

18 Find the coordinates of the point on the graph of $y = \ln(2x-5)$ at which the tangent is parallel to $y = 2x$.

19 Find the tangent to $y = \cos\left(\frac{1}{x}\right)$ at $x = \frac{2}{\pi}$.

20 Find the tangent to the curve $y = \ln(x-2)$ that is perpendicular to $y = -3x+2$.

21 **a** Find the derivative of $\sin^2 x$.

b Hence find all points on the curve of $y = \sin^2 x$, $0 < x < 2\pi$, where the gradient of the tangent is zero.

22 **a** Show that $(\cos x + \sin x)^2 = 1 + \sin 2x$.

b Use differentiation to deduce that $\cos 2x = \cos^2 x - \sin^2 x$.

23 Given that $\sin 3x = 3\sin x - 4\sin^3 x$, prove that $\cos 3x = 4\cos^3 x - 3\cos x$.

24 The tangent to the curve $y = e^{kx}$ at $x = \frac{1}{k}$ is called L. Prove that, for all $k > 0$, the y-intercept of L is independent of k.

25 Find the x-coordinate of all points of the curve $y = \ln(x^2 - 8)$ where the tangent is parallel to $y = 6x - 5$.

26 **a** Show that if $y = e^{2x}$, then $\frac{dy}{dx} = 2y$.

b Is it true that if $\frac{dy}{dx} = 2y$, then $y = e^{2x}$? Justify your answer.

10C The product and quotient rules

You know that you cannot differentiate a product of functions term by term as you can with a sum. Sometimes you can get round this: for example you can expand the product $x^2(2x+1)$ to give $2x^3 + x^2$ and then differentiate.

However, clearly you cannot do this with a product such as $x^2 \ln x$, so we need a method for differentiating products of functions. This is provided by the **product rule**.

KEY POINT 10.8

If $y = u(x)v(x)$, then $\frac{dy}{dx} = u\frac{dv}{dx} + v\frac{du}{dx}$.

WORKED EXAMPLE 10.10

Differentiate $y = x^2 \ln x$.

Define the functions $u(x)$ and $v(x)$

Let $u = x^2$ and $v = \ln x$

Find their derivatives

Then $\frac{du}{dx} = 2x$ and $\frac{dv}{dx} = \frac{1}{x}$

Substitute into $\frac{dy}{dx} = u\frac{dv}{dx} + v\frac{du}{dx}$

$$\frac{dy}{dx} = x^2\left(\frac{1}{x}\right) + (\ln x)(2x)$$
$$= x + 2x\ln x$$

You may need to use the product rule and the chain rule together.

WORKED EXAMPLE 10.11

Differentiate $f(x) = e^x (x^2 + 3)^4$.

Define the functions $u(x)$ and $v(x)$ for use in the product rule

Let $u = e^x$ and $v = (x^2 + 3)^4$

Then $u' = e^x$

For v' you need the chain rule. Remember to multiply by the derivative of $x^2 + 3$

and $v' = 4(x^2 + 3)^3 \times 2x = 8x(x^2 + 3)^3$

Substitute into $f'(x) = uv' + vu'$

$$f'(x) = e^x \times 8x(x^2 + 3)^3 + (x^2 + 3)^4 \times e^x$$
$$= 8xe^x (x^2 + 3)^3 + e^x (x^2 + 3)^4$$

Be the Examiner 10.1

Differentiate $f(x) = \sin(x^2 - 5x)$.

Which is the correct solution? Identify the errors in the incorrect solutions.

Solution 1	Solution 2	Solution 3
Let $u = \sin$ and $v = x^2 - 5x$ Then $u' = \cos$ and $v' = 2x - 5$ $f'(x) = \sin(2x - 5) + \cos(x^2 - 5x)$	$f'(x) = (2x - 5)\cos(x^2 - 5x)$	$f'(x) = \cos(2x - 5)$

As with products, you know that you cannot differentiate the top and bottom of a quotient separately. You could rewrite a quotient as a product and use the product rule to differentiate: for example, $\frac{x^2 + 6x - 2}{(x + 3)^2} = (x^2 + 6x - 2)(x + 3)^{-2}$. However, it is often simpler to use a method that allows you to differentiate quotients directly. This is called the **quotient rule**.

KEY POINT 10.9

If $y = \frac{u(x)}{v(x)}$, then $\frac{dy}{dx} = \frac{v\frac{du}{dx} - u\frac{dv}{dx}}{v^2}$.

WORKED EXAMPLE 10.12

Differentiate $y = \frac{x^2+6x-1}{(x+3)^2}$, giving your answer in the form $\frac{a}{(x+3)^b}$, where $a, b \in \mathbb{Z}$.

Define the functions $u(x)$ and $v(x)$

Let $u = x^2 + 6x - 1$ and $v = (x+3)^2$

Note that for $\frac{\mathrm{d}v}{\mathrm{d}x}$ you need the chain rule, but the derivative of $x+3$ is just 1

Then $\frac{\mathrm{d}u}{\mathrm{d}x} = 2x + 6$ and $\frac{\mathrm{d}v}{\mathrm{d}x} = 2(x+3)$

Substitute into $\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{v\frac{\mathrm{d}u}{\mathrm{d}x} - u\frac{\mathrm{d}v}{\mathrm{d}x}}{v^2}$

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{(x^2+6x-1)2(x+3) - (x+3)^2(2x+6)}{\left((x+3)^2\right)^2}$$

Cancel a factor of $(x+3)$

$$= \frac{2(x^2+6x-1) - (x+3)(2x+6)}{(x+3)^3}$$

Expand the numerator and simplify

$$= \frac{2x^2+12x-2-2x^2-12x-18}{(x+3)^3}$$

$$= -\frac{20}{(x+3)^3}$$

WORKED EXAMPLE 10.13

Show that if $\mathrm{f}(x) = \tan x$, then $\mathrm{f}'(x) = \frac{1}{\cos^2 x}$.

Use $\tan x = \frac{\sin x}{\cos x}$ to write $\tan x$ in terms of functions you can differentiate

$$\mathrm{f}(x) = \tan x = \frac{\sin x}{\cos x}$$

Now use the quotient rule

Let $u = \sin x$ and $v = \cos x$

Then $u' = \cos x$ and $v' = -\sin x$

Substitute into $\mathrm{f}'(x) = \frac{vu' - uv'}{v^2}$

$$\mathrm{f}'(x) = \frac{\cos x \cos x - \sin x(-\sin x)}{\cos^2 x}$$

$$= \frac{\cos^2 x + \sin^2 x}{\cos^2 x}$$

Use $\cos^2 x + \sin^2 x \equiv 1$

$$= \frac{1}{\cos^2 x}$$

Be the Examiner 10.2

Differentiate $f(x) = \frac{5}{e^{2x}}$.

Which is the correct solution? Identify the errors in the incorrect solutions.

Solution 1	Solution 2	Solution 3
Let $u = 5$ and $v = e^{2x}$ Then $u' = 0$ and $v' = 2e^{2x}$ $f'(x) = \frac{e^{2x} \times 0 - 5 \times 2e^{2x}}{(e^{2x})^2}$ $= -\frac{10e^{2x}}{(e^{2x})^2} = -\frac{10}{e^{2x}}$	$f(x) = \frac{5}{e^{2x}} = 5e^{-2x}$ Let $u = 5$ and $v = e^{-2x}$ Then $u' = 0$ and $v' = -2e^{-2x}$ $f'(x) = 5(-2e^{-2x}) + e^{-2x} \times 0$ $= -10e^{-2x}$	$f(x) = \frac{5}{e^{2x}} = 5e^{-2x}$ $f'(x) = -10e^{-2x}$

Exercise 10C

For questions 1 to 4, use the method demonstrated in Worked Example 10.10 to find $\frac{dy}{dx}$ for the following graphs.

1 a $y = x \sin x$
 b $y = x^{\frac{1}{2}} \sin x$

2 a $y = x^2 \cos x$
 b $y = x^{-1} \cos x$

3 a $y = x^{-\frac{1}{2}} e^x$
 b $y = x^3 e^x$

4 a $y = x^{\frac{2}{3}} \ln x$
 b $y = x^4 \ln x$

For questions 5 to 9, use the method demonstrated in Worked Example 10.11 to find $f'(x)$ for the following functions.

5 a $f(x) = x(2x+1)^{\frac{1}{2}}$
 b $f(x) = x^2(3x-4)^{\frac{3}{2}}$

6 a $f(x) = x^2 \sin 3x$
 b $f(x) = x^{\frac{3}{4}} \sin 2x$

7 a $f(x) = x\cos(4x-1)$
 b $f(x) = x^{-2} \cos 5x$

8 a $f(x) = x^{\frac{1}{2}} e^{4x+5}$
 b $f(x) = x^{-2} e^{1-x}$

9 a $f(x) = x^{-1} \ln(2x-3)$
 b $f(x) = x^3 \ln(5-x)$

For questions 10 to 12, use the method demonstrated in Worked Example 10.12 to find the derivative of the following graphs.

10 a $y = \frac{x^2}{x+2}$
 b $y = \frac{3x}{x-4}$

11 a $y = \frac{x-2}{\sqrt{x+1}}$
 b $y = \frac{4-x}{(x-3)^2}$

12 a $y = \frac{e^x}{(3x+1)^2}$
 b $y = \frac{e^x}{(2x-1)^3}$

13 Find the equation of the tangent to the curve $y = \frac{\sin x}{x}$ at $x = \frac{\pi}{2}$.

14 Find the equation of the normal to the curve $y = x\cos 2x$ at $x = \frac{\pi}{4}$.

15 If $f(x) = xe^{2x}$, find $f'(3)$.

16 Differentiate $xe^x \ln x$.

17 Differentiate $e^{x \sin x}$.

18 Find the point on the curve $y = e^{3x} - 11x$ where the tangent has gradient 13.

19 Find the coordinates of the points on the curve $y = xe^{-x}$ where the gradient is zero.

20 Find the coordinates of the points on the curve $y = x \ln x$ where the gradient is 2.

21 Find and simplify an expression for the derivative of $f(x) = \frac{x}{\sqrt{k + x^2}}$, where k is a positive constant.

22 If $f(x) = x\sqrt{2+x}$, show that $f'(x) = \frac{ax+b}{2\sqrt{2+x}}$, where a and b are integers to be found.

23 Show that $f(x) = \frac{x^2}{1+x}$ is an increasing function if $x > a$ or if $x < b$, where a and b are constants to be determined.

24 The population of rabbits on an island, P thousands, n years after they were introduced is modelled by

$$P = \frac{4}{1+3e^{-n}}$$

a Find an expression for the population growth rate, $\frac{dP}{dn}$, and explain why this predicts that the population is always growing.

b Find

i the initial population

ii the initial population growth rate.

c What is the value of P as n gets very large?

d Hence sketch the population as a function of time.

25 **a** Show that if $x > 0$, then $x^x = e^{x \ln x}$.

b Hence find the equation of the tangent to the curve $y = x^x$ at $x = 1$.

10D Related rates of change

There are many situations where we are given one rate and want to link it to another rate. For example, we might know the rate at which a barrel is leaking, and want to know the rate at which the height of water is decreasing.

One of the most common tricks to do this is to use the chain rule to link the two rates. If we know $\frac{dx}{dt}$ and want to know $\frac{dy}{dt}$ we use

$$\frac{dy}{dt} = \frac{dy}{dx} \times \frac{dx}{dt}$$

So, the problem becomes finding $\frac{dy}{dx}$. This can be done using some other link between y and x, such as a given equation.

WORKED EXAMPLE 10.14

If $y = x^2$, find $\frac{dy}{dt}$ when $x = 4$ and $\frac{dx}{dt} = 12$.

Use the chain rule to relate $\frac{dy}{dt}$ and $\frac{dx}{dt}$: $\frac{dy}{dt} = \frac{dy}{dx} \times \frac{dx}{dt}$

Use the fact that $y = x^2$ to find $\frac{dy}{dx}$: $= 2x \times \frac{dx}{dt}$

Substitute in values for x and $\frac{dx}{dt}$: $= 8 \times 12 = 96$

Frequently, the link between the variables comes from a geometric context.

WORKED EXAMPLE 10.15

A rectangle has width x and height y. When $x = 2$ cm and $y = 4$ cm, these values are changing according to $\frac{dx}{dt} = 3\text{ cm s}^{-1}$ and $\frac{dy}{dt} = -1\text{ cm s}^{-1}$.
What is the rate of change of the area at this time?

Define a variable for the area — Let the area be $A\text{ cm}^2$.

$A = xy$

Differentiate both sides with respect to t — $\frac{dA}{dt} = \frac{d}{dt}(xy)$

This requires the product rule — $= x\frac{dy}{dt} + y\frac{dx}{dt}$

Substitute in the given values — $= 2 \times -1 + 4 \times 3$

$= 10$

So, the area is increasing at a rate of 10 cm s^{-1}.

CONCEPTS – QUANTITY AND CHANGE

Worked Example 10.15 illustrates that the rate of **change** can depend on its current size. For example, if the radius of a circle increases at a constant rate, then the area will increase faster as the circle gets larger; when a balloon is inflated at a constant rate (so that the rate of increase of volume is constant) the rate of change of the radius will decrease with the size of the balloon. The chain rule is an important tool for **quantifying** the relationship between the different rates.

Exercise 10D

For questions 1 to 4, use the method demonstrated in Worked Example 10.14 to find the required rate of change.

1 a If $y = x^3$, find $\frac{dy}{dt}$ when $x = 1$ and $\frac{dx}{dt} = -1$.

b If $y = x^2 + x$, find $\frac{dy}{dt}$ when $x = 0$ and $\frac{dx}{dt} = 4$.

2 a If $A = e^{2z}$, find the rate of increase of A when $z = 0$ and $\frac{dz}{dt} = 6$.

b If $p = e^q$, find the rate of increase of p when $q = 1$ and $\frac{dq}{dt} = 7$.

3 a If $a = \frac{1}{b}$, find the rate of increase of a when $b = 2$ and b is increasing at a rate of 3 per second.

b If $a = \frac{1}{b^2}$, find the rate of increase of a when $b = 1$ and b is increasing at a rate of 2 per second.

4 a If $y = \ln x$, find the rate of increase of y when $x = 2$ and x is decreasing at a rate of 4 per hour.

b If $y = \ln(x+1)$, find the rate of increase of y when $x = 1$ and x is decreasing at a rate of 3 per hour.

5 Given that $A = x^2 + y^2$, find $\frac{dA}{dt}$ when $x = 3,\ y = 4,\ \frac{dx}{dt} = 1,\ \frac{dy}{dt} = -1$.

6 Given that $B = x^3 + y^3$, find $\frac{dB}{dt}$ when $x = 1,\ y = 2,\ \frac{dx}{dt} = 1,\ \frac{dy}{dt} = -2$.

7 Given that $C = \frac{x}{y}$, find $\frac{dC}{dt}$ when $x = 3,\ y = 4,\ \frac{dx}{dt} = 3,\ \frac{dy}{dt} = -1$.

8 The sides of a square are increasing at a rate of $2\,\text{cm}\,\text{s}^{-1}$. Find the rate of increase of the area when the area is $25\,\text{cm}^2$.

9 Circular mould is spreading on a leaf. When the radius is 3 mm, the rate of increase is 1.2 mm per day. What is the rate of increase of the area?

10 The volume of a spherical balloon is increasing at a rate of $200\,\text{cm}^3$ per second. Find the rate of increase of the radius when the volume is $100\,\text{cm}^3$.

11 An x by y rectangle is expanding, with $\frac{dx}{dt} = 4\,\text{cm}\,\text{s}^{-1}$ and $\frac{dy}{dt} = -2\,\text{cm}\,\text{s}^{-1}$. When $x = 3$ cm and $y = 4$ cm, find

a the rate of increase of the rectangle's area

b the rate of increase of the length of the diagonal.

12 An inverted cone is being filled with water at a constant rate of $5\,\text{cm}^3\,\text{s}^{-1}$. The surface of the water is always horizontal as it is being filled. The largest diameter of the cone is 10 cm and its height is 30 cm. If the volume of water in the cone is V at time t, and h is the height of the water above the vertex of the cone,

a show that $V = \frac{\pi h^3}{108}$

b find the rate that the height is increasing when $h = 18$ cm.

13 A circular stain of radius r cm and area A cm^2 is increasing in size. At a certain time, the rate of increase of the radius is $1.8\,\text{cm}\,\text{s}^{-1}$ and the rate of increase of the area is $86.5\,\text{cm}^2\,\text{s}^{-1}$. Find the radius of the stain at this point.

14 A sportsman throws a ball. When it is 2 m above the sportsman and 4 m away horizontally, it is moving purely horizontally with a speed of $3\,\text{m}\,\text{s}^{-1}$. Find the rate at which the ball is moving away from the sportsman.

15 The density of a reactive substance is given by its mass divided by its volume. When the density is $5\,\text{g}\,\text{cm}^{-3}$, the mass is decreasing at a rate of $2\,\text{g}\,\text{s}^{-1}$ and the volume is decreasing at a rate of $1\,\text{cm}^3\,\text{s}^{-1}$. Determine, with justification, whether the density is increasing or decreasing.

16 A ladder of length 3 m is sliding down a vertical wall. The foot of the ladder is on horizontal ground. When the point of contact with the wall is 2 m above the horizontal, that point is moving down at a rate of $0.1\,\text{m}\,\text{s}^{-1}$. At what speed is the foot of the ladder moving away from the wall, assuming that the ladder always stays in contact with both the wall and the ground?

10E The second derivative

Finding the second derivative

Differentiating the derivative $f'(x)$ gives the second derivative $f''(x)$, which measures the rate of change of the gradient.

If the notation $\frac{dy}{dx}$ is used for the derivative, then $\frac{d^2y}{dx^2}$ is used for the second derivative.

WORKED EXAMPLE 10.16

If $f(x) = x^3 + 3x$, find $f''(2)$.

First find $f'(x)$ .... $f'(x) = 3x^2 + 3$

Then differentiate again .... $f''(x) = 6x$

Now substitute $x = 2$ .... $f''(2) = 12$

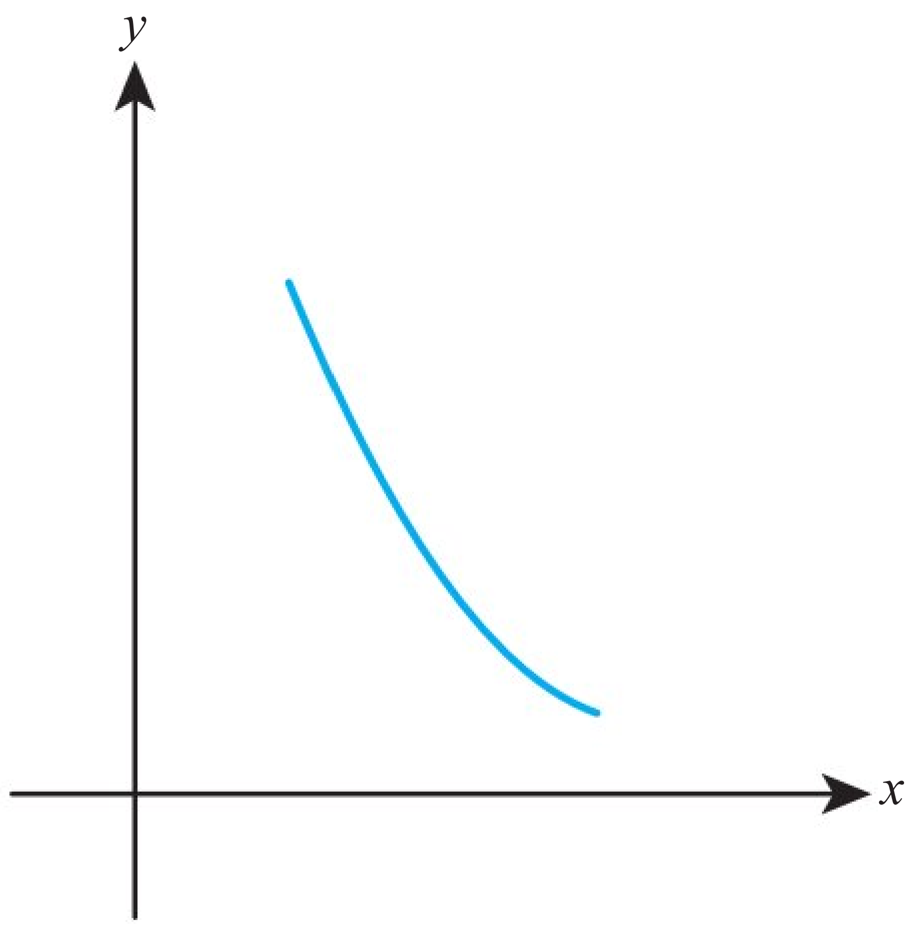
Concave-up and concave-down

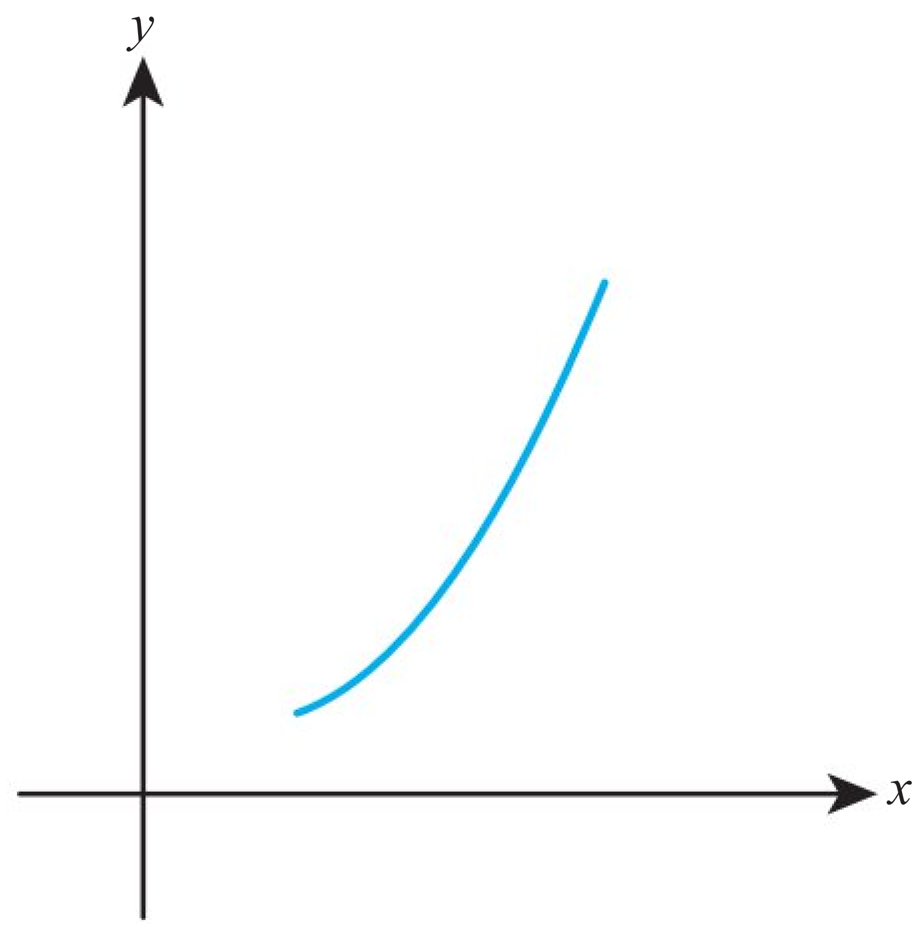
In Section 9B of the Mathematics: applications and interpretation SL book, you saw that a function is increasing where $f'(x) > 0$ and decreasing where $f'(x) < 0$.

You already know that you can classify different parts of the graph of a function as being increasing or decreasing depending on whether the gradient (the derivative) is positive or negative.

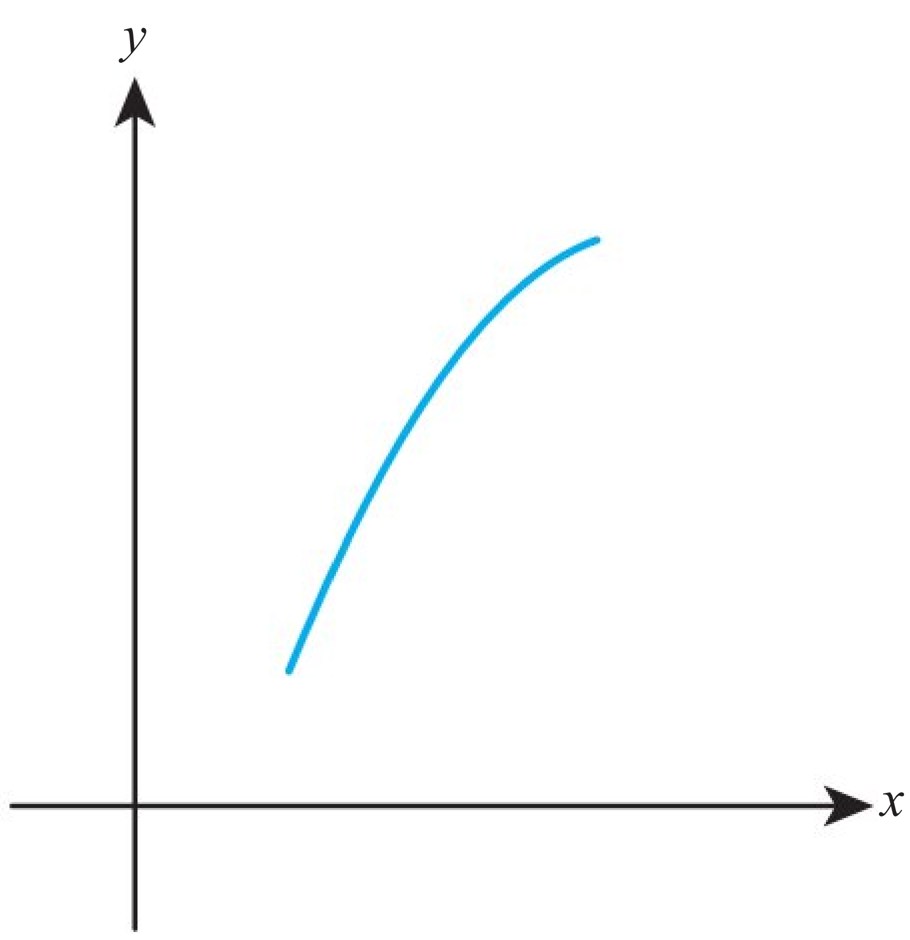
There is another classification based on the gradient of the gradient (the second derivative).

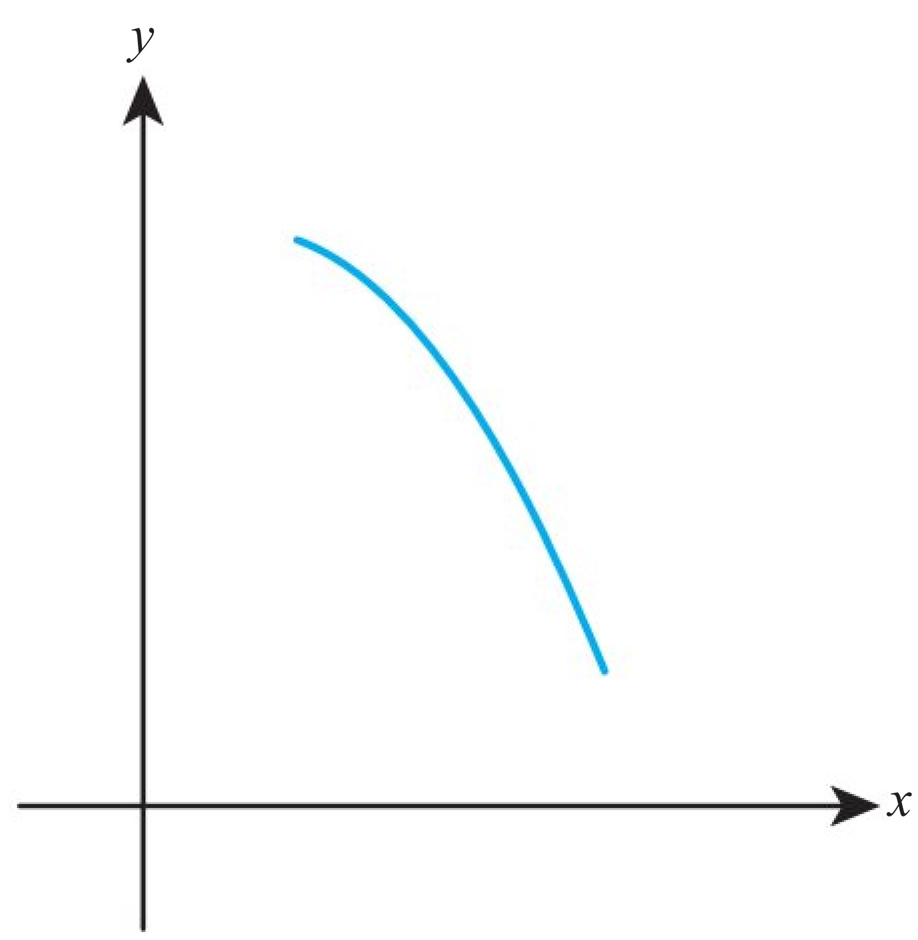
If the second derivative is positive, it means that the gradient is increasing. We say that the curve is **concave-up**. Depending upon whether the gradient is positive or negative it can look like:





If the second derivative is negative, it means that the gradient is decreasing. This is called **concave-down**. Depending upon whether the gradient is positive or negative it can look like:





Tip

A curve does not have to have a turning point to be described as concave-up or concave-down. Any part of the graphs above can be described as concave-up or concave-down. One useful trick is to think where you would be naturally put your compass (or your elbow) when sketching small sections of the curve. If it is above the curve it is concave-up. If it is below the curve it is concave-down.

KEY POINT 10.10

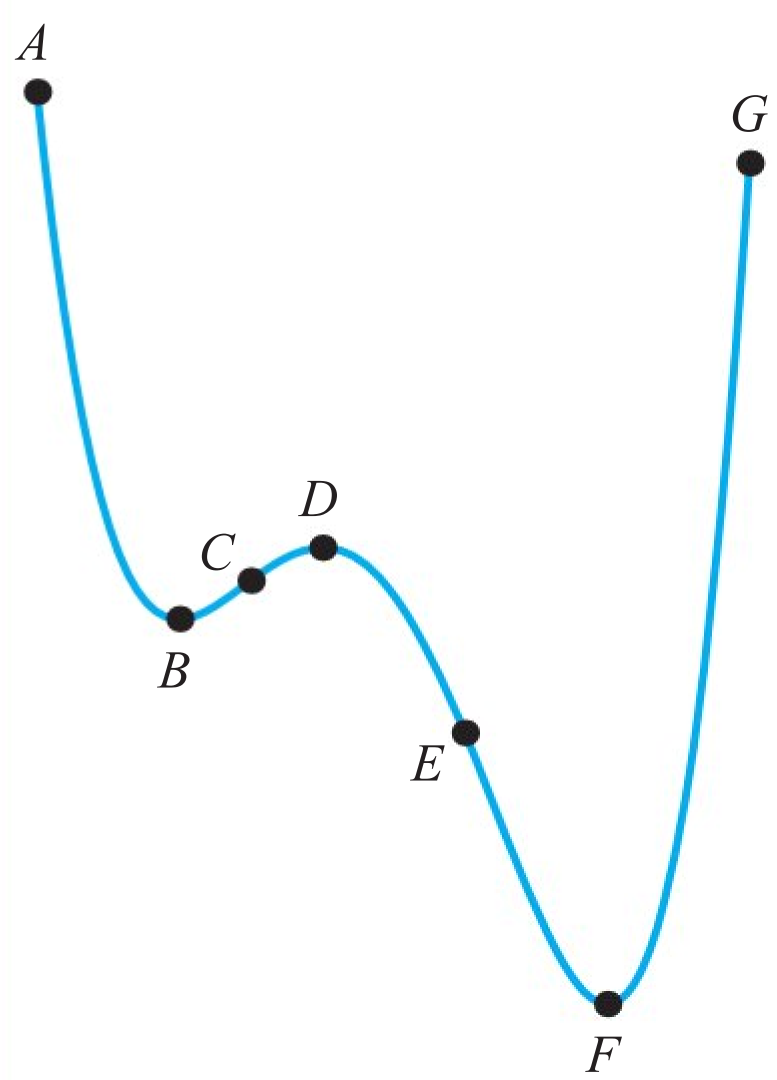
A function $f(x)$ is

- **concave-up** where $f''(x) > 0$
- **concave-down** where $f''(x) < 0$.

WORKED EXAMPLE 10.17

Using one or more of the terms 'increasing', 'decreasing', 'concave-up', 'concave-down', describe these sections of the graph.

a A to B **b** C to E **c** F to G



a Decreasing, concave-up

The curve is increasing C to D but decreasing D to E. It is concave-down from C to E

b Concave-down

c Increasing, concave-up

You are the Researcher

Just like the first derivative is related to the graphical concept of gradient of a graph, the second derivative is related to the graphical concept of curvature. You might like to investigate how to find a formula for the radius of curvature of a graph. This is a core problem in an area of mathematics called differential geometry.

WORKED EXAMPLE 10.18

Find the values of x for which the function $f(x) = x^3 + 3x^2 + 2x + 1$ is concave-down.

Find the second derivative

$$f'(x) = 3x^2 + 6x + 2$$
$$f''(x) = 6x + 6$$

For concave-down $f''(x) < 0$

$$f''(x) < 0$$
$$6x + 6 < 0$$

Solve the inequality

$$x < -1$$

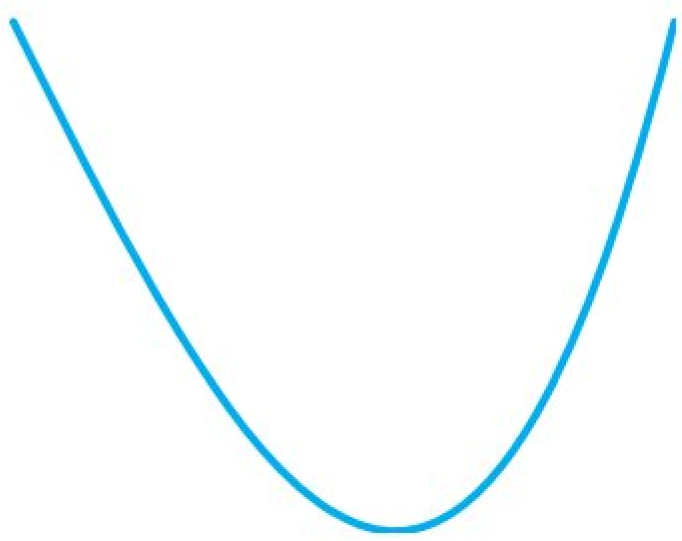
Tip

You can check your answer by graphing the function on your GDC.

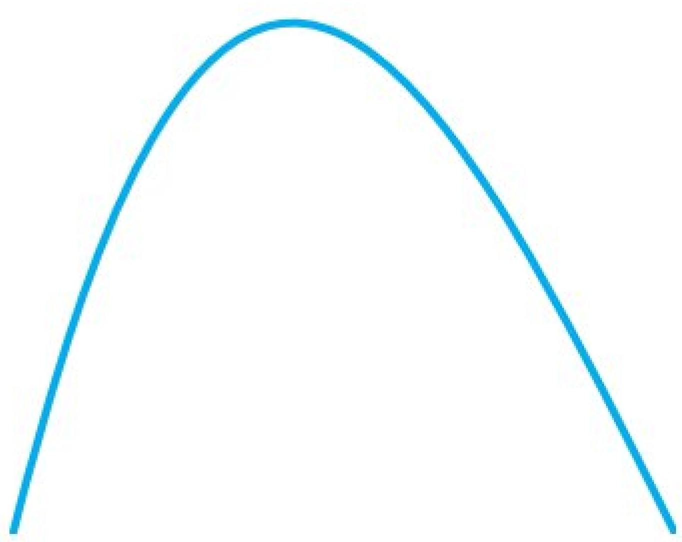
Classifying local maximum and minimum points

You have already found that at a maximum or minimum point the derivative is zero. This fact can be used to find local maxima and minima. However, you can now use the concavity of the graph to decide whether the stationary point is a maximum or a minimum.

If the graph is concave-up at the stationary point, then it must be a minimum:



If the graph is concave-down at the stationary point, then it must be a maximum:



We can rephrase this in terms of derivatives.

KEY POINT 10.11

Given $f'(a) = 0$, if

- $f''(a) < 0$, then there is a local maximum at $x = a$
- $f''(a) > 0$, then there is a local minimum at $x = a$.

WORKED EXAMPLE 10.19

Find and classify the stationary points on the curve $y = x^3 - 3x^2$.

Differentiate the curve to find the gradient

$$\frac{dy}{dx} = 3x^2 - 6x$$

At stationary points, the gradient is zero

$$3x^2 - 6x = 0$$

This equation is best solved by factorising

$$3x(x-2) = 0$$

$x = 0$ or $x = 2$

We need to find the y-coordinates corresponding to the x-coordinates

When $x = 0$, $y = 0$

When $x = 2$, $y = -4$

We find the second derivative to classify the stationary points

$$\frac{d^2y}{dx^2} = 6x - 6$$

Substitute in the x value of each stationary point to determine whether the second derivative is positive or negative

When $x = 0$, $\frac{d^2y}{dx^2} = -6 < 0$

So, $(0,0)$ is a local maximum.

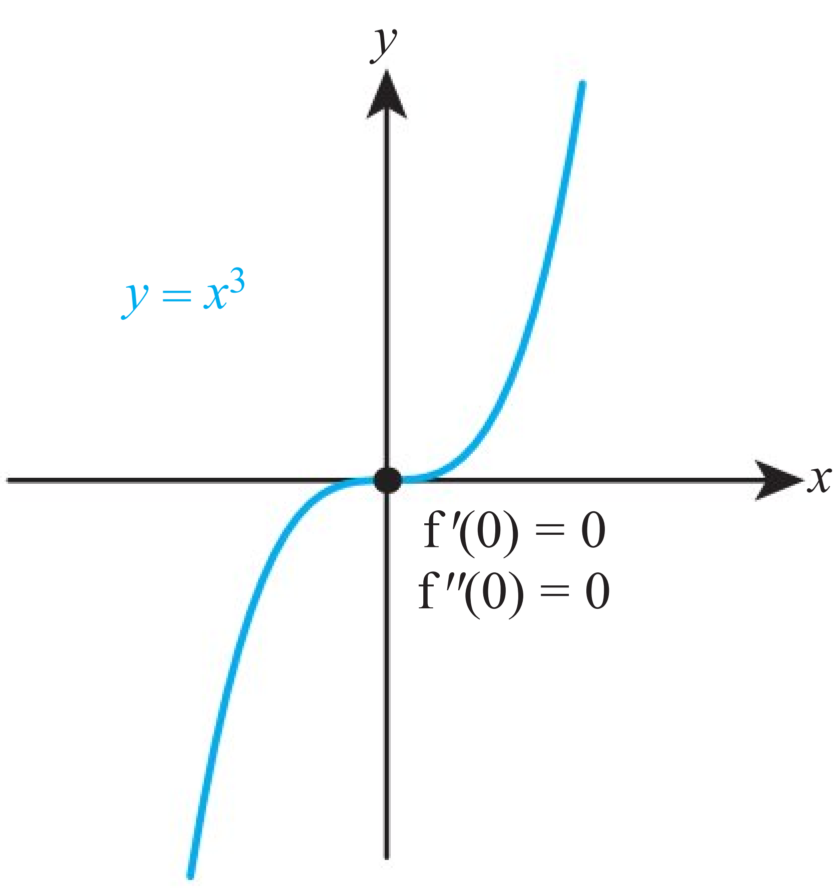
When $x = 2$, $\frac{d^2y}{dx^2} = 6 > 0$

So, $(2,-4)$ is a local minimum.

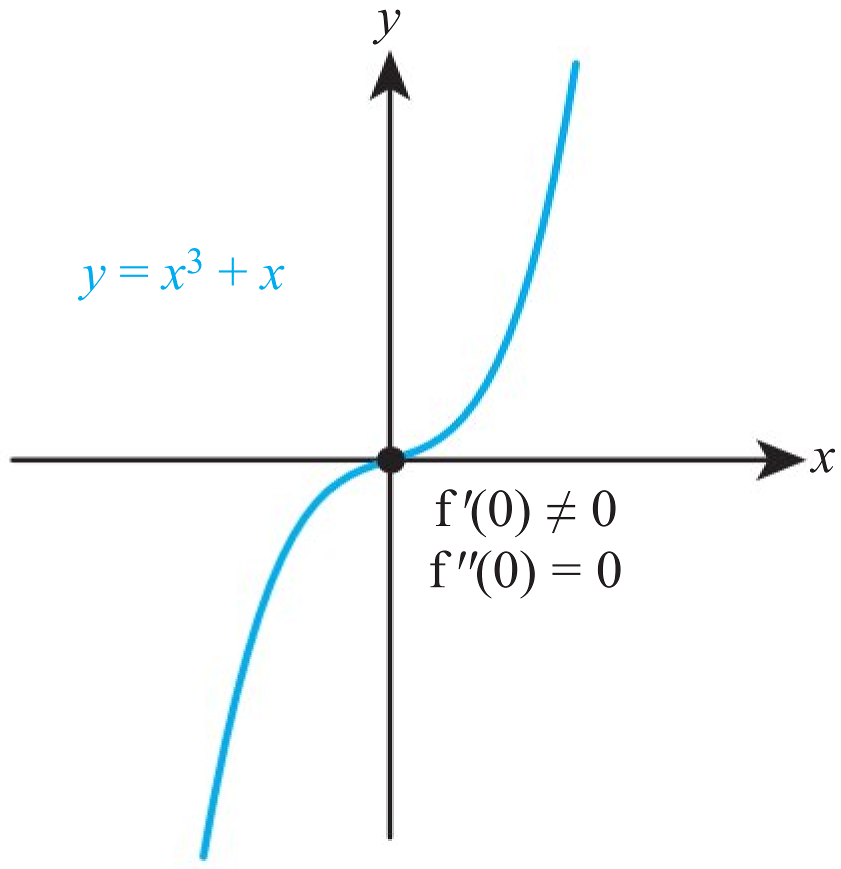
Tip

You can find stationary points using your calculator, but it is useful to practise this method in case a question uses parameters.

In Key Point 10.11, it turns out that if $f''(a) = 0$, the situation is more complicated. It might be a minimum (for example, $y = x^4$ at $x = 0$) or a maximum, or it might be a **point of inflection**. This is a point where the concavity changes, for example $y = x^3$ at $x = 0$.



Points of inflection do not have to have zero gradient, they occur whenever the concavity changes:



This means that at any point of inflection, $\frac{d^2y}{dx^2} = 0$.

Another equivalent interpretation of points of inflection is where the gradient reaches a local maximum or minimum.

Exercise 10E

For questions 1 to 4, use the method demonstrated in Worked Example 10.16 to evaluate the second derivative of the following functions when $x = 2$.

1 **a** $f(x) = 5x^2 + 2x + 1$
 b $f(x) = 6x^2 + 3x + 5$

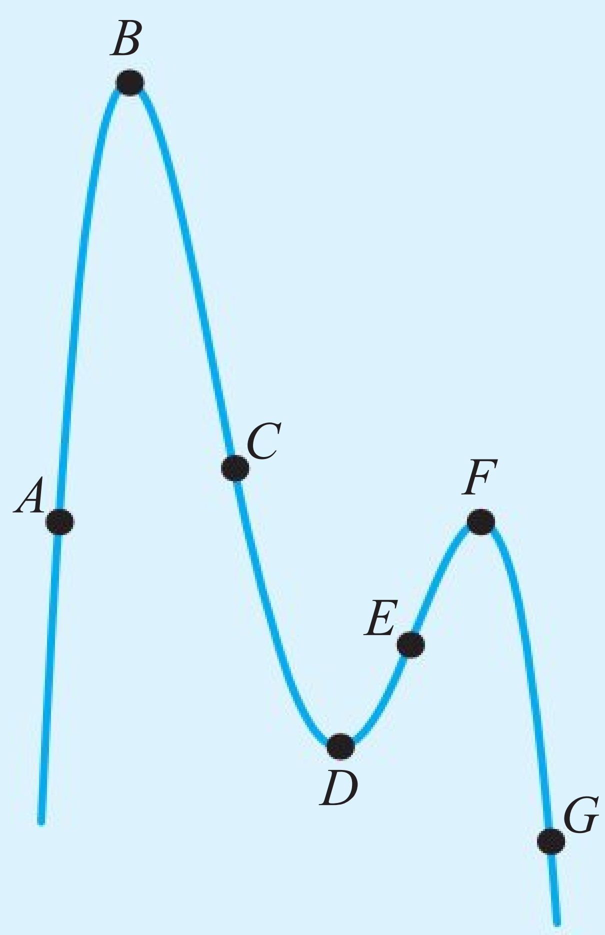
2 **a** $f(x) = 5x^3 + x^2 + 3x + 7$
 b $f(x) = x^3 - 2x^2 + 4x + 11$

3 **a** $y = e^{-2x}$
 b $y = 2e^{3x}$

4 **a** $y = \ln x$
 b $y = -4\ln(x) + 2$

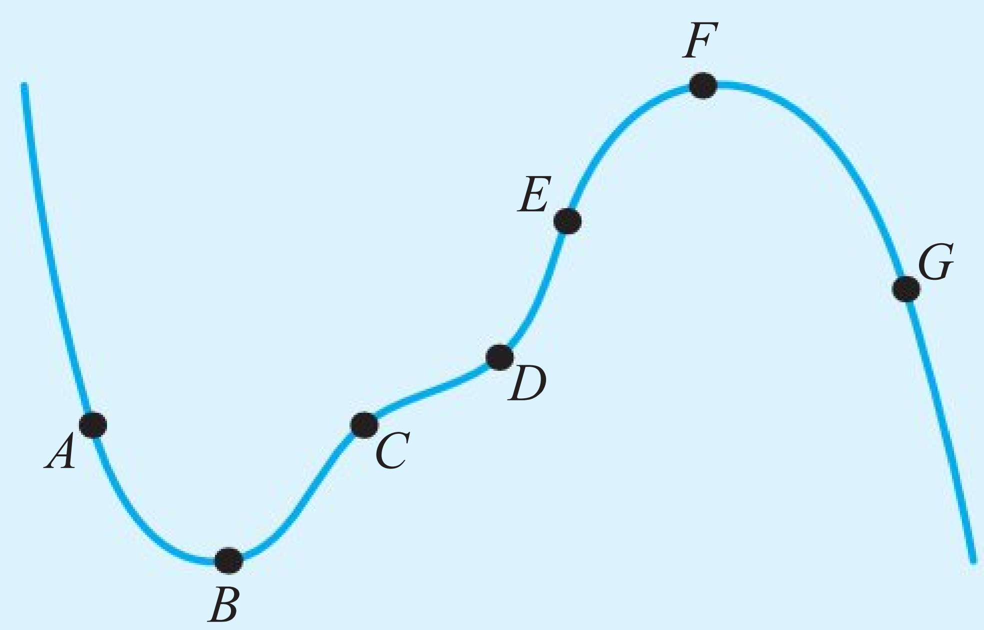
5 Use the method demonstrated in Worked Example 10.17 to give one or more of the terms 'increasing', 'decreasing', 'concave-up', 'concave-down' to describe the following sections of each graph.

a



i A to B
ii C to E
iii E to G

b



i A to B
ii C to F
iii F to G

For questions 6 and 7, use the method demonstrated in Worked Example 10.18 to find the values of x for which the function has the given description.

6 a $f(x) = x^3 - 4x^2 + 5x - 2$ is concave-down
b $f(x) = x^3 + 9x^2 - 4x + 1$ is concave-up

7 a $f(x) = 18x^2 + 2x - 1$ is concave-up
b $f(x) = 2x - 7 - x^2$ is concave-up

For questions 8 to 12, use the method demonstrated in Worked Example 10.19 to find and classify the stationary points on the given graph.

8 a $y = 2x^3 - 9x^2 + 12x + 6$
b $y = x^3 - 12x + 4$

9 a $y = x^4 - 18x^2 - 3$
b $y = 2x^4 - x + 1$

10 a $y = 2x^{\frac{1}{2}} - 3x + 1$
b $y = x^{-\frac{1}{2}} + 4x - 2$

11 a $y = e^x - 5x$
b $y = \dfrac{x}{2} - e^x$

12 a $y = 3\ln x - 2x$
b $y = x^{\frac{1}{2}} - \ln x$

13 If $f(x) = x^3 + kx^2 + 3x + 1$ and $f''(1) = 10$, find the value of k.

14 If $f(x) = x^3 + ax^2 + bx + 1$, with $f''(-1) = -4$ and $f'(1) = 4$, find a and b.

15 If $y = xe^x$, find $\dfrac{d^2y}{dx^2}$.

16 Sketch a graph where, for all x:
$f(x) < 0$, $f'(x) > 0$, $f''(x) < 0$

17 Sketch a graph where, for all positive x, $f'(x) > 0$ and $f''(x) > 0$ and, for all negative x, $f'(x) > 0$ and $f''(x) < 0$.

18 a The graph of $y = x^3 - kx^2 + 4x + 7$ is concave-up for all $x > 1$ and this is the only section which is concave-up. Find the value of k.
b What can be said about the graph at $x = 1$?

19 If $f(x) = x^n$ and $f''(1) = 12$, find the possible values of n.

20 The curve $y = x^2 + bx + c$ has a minimum value at (1, 2). Find the values of b and c.

21 The curve $y = x^4 + bx + c$ has a minimum value at (1, –2). Find the values of b and c.

22 Find and classify the stationary point on the curve $y = x - 2\sqrt{x}$.

23 Show that the graph of $y = x^2 + bx + c$ is always concave-up.

24 If $y = \sin 3x + 2\cos 3x$, show that $\dfrac{d^2y}{dx^2} = -9y$.

25 If $y = \ln(a + x)$, find $\dfrac{d^2y}{dx^2}$. Hence show that the graph of $y = \ln(a + x)$ is always concave-down.

26 a Find the point on the graph $y = x^2 \ln x$ where the second derivative equals 1.
b Given that the graph has a point of inflection, find its x-coordinate.

27 a Find the interval in which the gradient of the graph $y = x^3 - 5x^2 + 4x - 2$ is decreasing.
b Hence find the point of inflection on this graph.

28 The function f is defined by $f(x) = \dfrac{\ln x}{x}$ for $x > 0$.
a Find $f'(x)$.
b Find the coordinates of the stationary point on the curve $y = f(x)$.
c Show that $f''(x) = \dfrac{2\ln x - 3}{x^3}$.
d Hence classify the stationary point.

29 Find and classify the stationary points on the curve $y = \dfrac{1}{x} + p^2x$ with $p > 0$, giving your answer in terms of p.

30 Find and classify the stationary points on the curve $y = e^{\sin x}$ for $0 < x < 2\pi$.

31 Find and classify the stationary points on the curve $y = \sin x - \frac{x}{2}$, $0 < x < 2\pi$.

32 Find and classify the stationary points on the curve $y = \frac{x^2}{1+x}$.

33 A business produces x thousand of a particular product each month, where $x > 0$. The profit P (in thousands of dollars) is given by $P = 16x^{\frac{3}{2}} - x^3 - 25$.

a Find the number of items the business should produce each month to maximise its profit, showing that this is indeed a maximum.

b Find this maximum monthly profit.

34 A cuboid has a square base of side length x cm. The other side of the cuboid is of length $a - x$ cm. Find the maximum volume of the cuboid in terms of a, justifying that it is a maximum.

35 Find and classify the stationary point on the curve $y = xe^{px}$, $p \neq 0$, giving your answer in terms of p.

36 The volume of water, V litres, in a barrel filled by rain is modelled by $V = t^3 - 3t^2 + 5t$, where t is the time in hours.

Find the time at which the barrel is filling most slowly, and state that minimum rate.

37 The population of fish in a lake, P thousands, is modelled by $P = \frac{t^2}{1+t^2}$, where t is the time in years since the fish were introduced. At what population is the fish population growing most quickly?

38 Find the range of the function $f(x) = e^x - ax$ in terms of a.

39 Find the exact coordinates of the stationary points on the curve $y = e^{2x} - 6e^x + 4x + 8$, classifying them.

40 The height, h metres of a point on a hanging chain at a given x-coordinate is modelled by $h(x) = e^x + e^{-2x}$.

Find the exact value of the minimum height of the chain.

41 The curve $y = x^3 + ax^2 + bx + 7$ is concave-up for all $x > 2$ and this is the only section which is concave-up.

a Find the value of a.

b Find a condition on b for the curve to be strictly increasing.

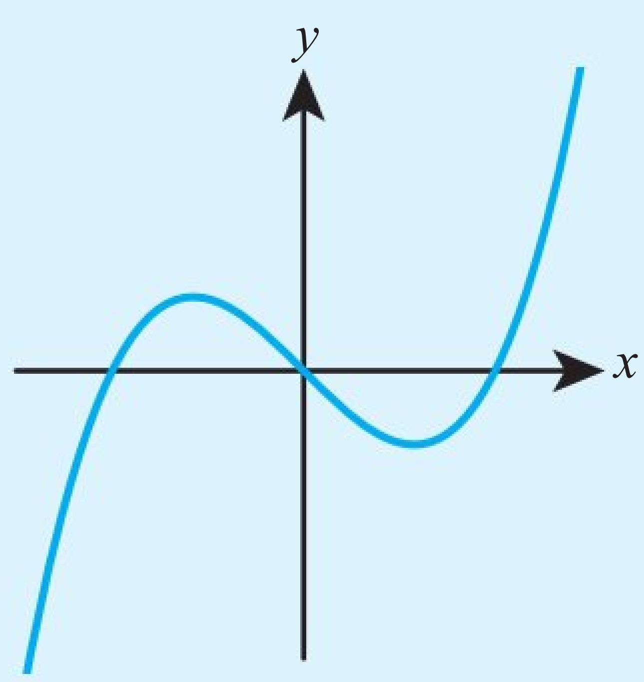
42 The curve $y = ax^3 - bx^2$ (where a and b are positive) is increasing for $x < 0$ and $x > 4$.

a Evaluate $\frac{b}{a}$.

b Prove that the curve is concave-up for $x > 2$.

43 If $y''(x) = 10$, $y'(0) = 10$ and $y(0) = 2$, find $y(x)$.

44 The graph below shows the graph of $y = f'(x)$.



On a copy of the diagram

a mark all points corresponding to a local minimum of $f(x)$ with a P

b mark all points corresponding to a local maximum of $f(x)$ with a Q

c mark all points corresponding to a point of inflection of $f(x)$ with a R.

d Are any of the points of inflection stationary points of inflection? Explain your answer.

Checklist

- You should be able to differentiate functions such as $\sqrt{x}$, $\sin x$ and $\ln x$:
  - If $f(x) = x^n$, where $n \in \mathbb{Q}$, then $f'(x) = nx^{n-1}$.
  - If $f(x) = \sin x$, then $f'(x) = \cos x$.
  - If $f(x) = \cos x$, then $f'(x) = -\sin x$.
  - If $f(x) = e^x$, then $f'(x) = e^x$.
  - If $f(x) = \ln x$, then $f'(x) = \frac{1}{x}$.
- You should know how to use the chain rule to differentiate composite functions:
  - If $y = f(u)$, where $u = g(x)$, then

    $$\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx}$$
- You should know how to differentiate products and quotients:
  - If $y = u(x)v(x)$, then

    $$\frac{dy}{dx} = u\frac{dv}{dx} + v\frac{du}{dx}$$
  - If $y = \frac{u(x)}{v(x)}$, then

    $$\frac{dy}{dx} = \frac{v\frac{du}{dx} - u\frac{dv}{dx}}{v^2}$$
- You should know how to convert related rates of change.
- You should know how to interpret the second derivative of a function. A function $f(x)$ is
  - **concave-up** where $f''(x) > 0$
  - **concave-down** where $f''(x) < 0$.
- You should know how to find local minimum and maximum points on a graph.
  - Given $f'(a) = 0$, if
    - $f''(a) < 0$, then there is a local maximum at $x = a$
    - $f''(a) > 0$, then there is a local minimum at $x = a$.

Mixed Practice

1 The curve $y = x^2 + bx + c$ has a minimum value at $(2, 3)$. Find the values of b and c.

2 Find the derivative of e^{x^2}.

3 Find $\frac{d^2y}{dx^2}$ if $y = \frac{1}{x+2}$.

4 Differentiate $\frac{\sin x}{x}$.

5 Find $f'(x)$ if $f(x) = x\ln x$.

6 Find the gradient of the tangent to the curve $y = \sqrt{x-1}$ at $x = 5$.

7 Find the equation of the tangent to the curve $y = \ln x$ at $x = 1$.

8 Find the equation of the normal to the curve $y = e^{2x}$ at $x = 0$.

9 The side of a cube is increasing at a rate of $0.6\,\text{cm s}^{-1}$. Find the rate of increase of the volume of the cube when the side length is 12 cm.

10 Given that $y = 3\sin 2\pi x$, find the rate of change of y when $x = \frac{7}{12}$.

11 A cuboid has a square base of side a cm and height h cm. The volume of the cuboid is $1000\,\text{cm}^3$.

a Show that the surface area of the cuboid is given by $S = 2a^2 + \frac{4000}{a}$.

b Find the value of a for which $\frac{dS}{da} = 0$.

c Show that this value of a gives the minimum value of the surface area and find this minimum value.

12 Find the equation of the tangent to the curve $y = x\cos x$ at $x = 0$.

13 If $y = x^2e^{2x}$, find $\frac{d^2y}{dx^2}$.

14 The two shorter sides of a right-angled triangle are x cm and $6 - x$ cm.

a Find the maximum area of the triangle.

b Find the minimum perimeter of the triangle.

15 If $f(x) = xe^{kx}$ and $f''(0) = 10$, find $f'(1)$.

16 **a** The graph of $x^3 - kx^2 + 8x + 2$ is concave-up for all $x > 4$ and this is the only section which is concave-up. Find the value of k.

b Hence find the coordinates of the point of inflection on the graph.

17 Differentiate $xe^x \sin x$.

18 If $f(x) = \frac{x}{\sqrt{4+x}}$, show that $f'(x) = \frac{ax+b}{2(4+x)^c}$, where a, b and c are constants to be found.

19 **a** Find the set of values for which $f(x) = xe^{-x}$ is concave-down.

b Hence find the point of inflection on the curve $y = f(x)$.

20 Find the coordinates of the stationary point on the curve $y = e^{-x}\sin x$ for $0 < x < \pi$.

21 Consider $z = xe^y$. When $x = 2$ and $y = 1$, $\frac{dx}{dt} = -1$ and $\frac{dy}{dt} = 0.5$, find $\frac{dz}{dt}$.

22 The volume of water, V million cubic metres, in a lake t hours after a storm is modelled by $V = 2te^{-t} + 5$.

a What is the initial volume of the lake?
b What is the maximum volume of the lake?
c When is the lake emptying fastest?

23 A tranquilizer is injected into a muscle from which it enters the bloodstream. The concentration C in mg l^{-1}, of tranquilizer in the bloodstream can be modelled by the function $C(t) = \frac{2t}{3+t^2}, t \geqslant 0$ where t is the number of minutes after the injection.

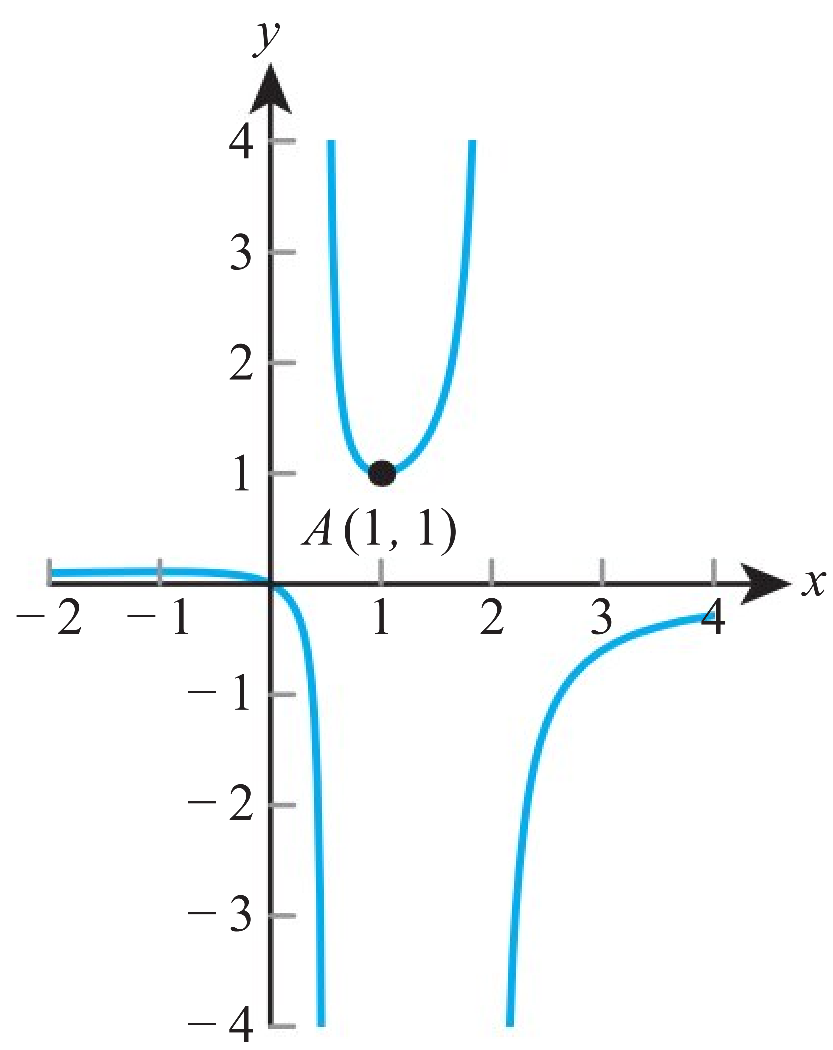
Find the maximum concentration of tranquilizer in the bloodstream.

Mathematics HL November 2014 Paper 1 Q5

24 Two cyclists are at the same road intersection. One cyclist travels north at $20\,\text{km h}^{-1}$; the other travels west at $15\,\text{km h}^{-1}$.
Use calculus to show that the rate at which the distance between the two cyclists changes is independent of time.

Mathematics HL November 2014 Paper 2 Q4

25 Let $f(x) = \frac{x}{-2x^2 + 5x - 2}$ for $-2 \leqslant x \leqslant 4$, $x \neq \frac{1}{2}$, $x \neq 2$. The graph of f is given below.



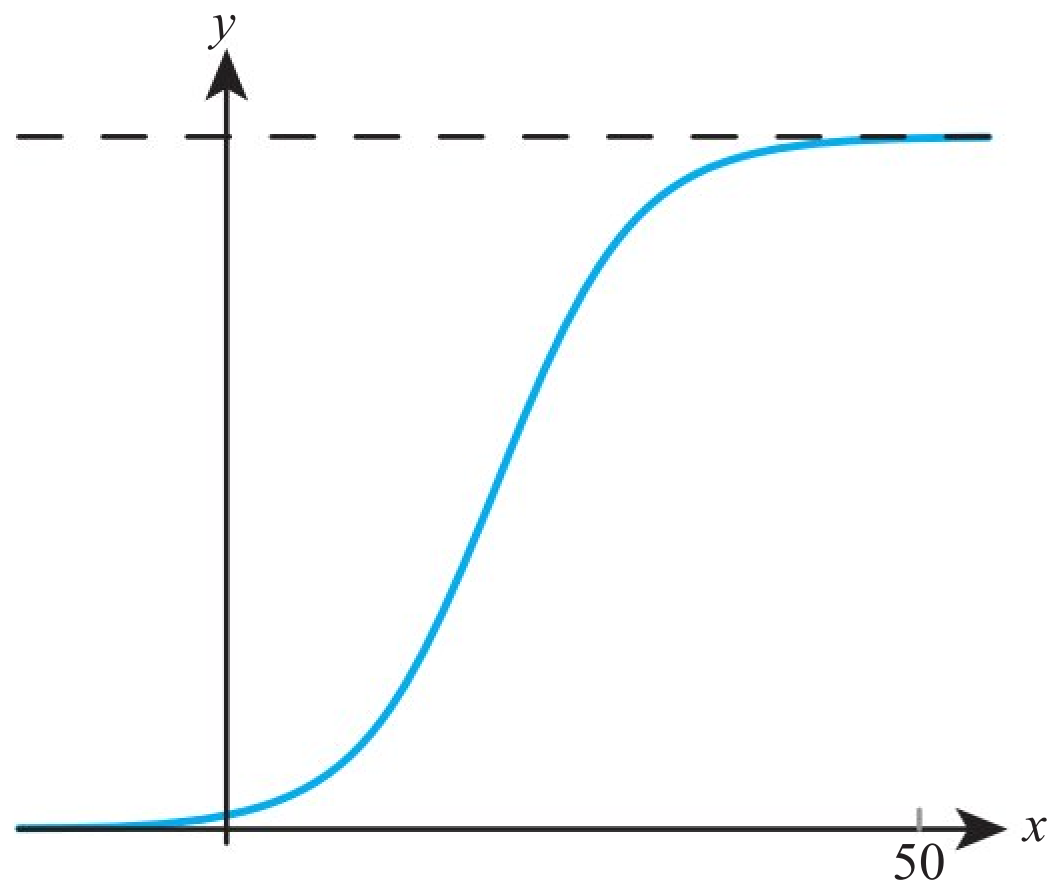
The graph of f has a local minimum at A (1, 1) and a local maximum at B.

a Use the quotient rule to show that $f'(x) = \frac{2x^2 - 2}{(-2x^2 + 5x - 2)^2}$.
b Hence find the coordinates of B.
c Given that the line $y = k$ does not meet the graph of f, find the possible values of k.

Mathematics SL May 2012 Paper 1 TZ2 Q10

26 Prove that the equation of the tangent to the curve $y = \ln x + kx$ at $x = 1$ always passes through the point (0, −1) independent of the value of k.

27 Let $\mathrm{f}(x) = \dfrac{100}{\left(1 + 50\mathrm{e}^{-0.2x}\right)}$. Part of the graph of f is shown below.



a Write down f(0).

b Solve $\mathrm{f}(x) = 95$.

c Find the range of f.

d Show that $\mathrm{f}'(x) = \dfrac{1000\mathrm{e}^{-0.2x}}{\left(1 + 50\mathrm{e}^{-0.2x}\right)^2}$.

e Find the maximum rate of change of f.

Mathematics SL May 2013 Paper 2 TZ1 Q9

28 The function $\mathrm{f}(x) = \dfrac{x - a}{x - b}$ is defined for $x \neq b$.

a Show that if $a > b$, then $\mathrm{f}'(x) > 0$ at all values of x in the domain.

b Does this mean that if $p > q$, then $\mathrm{f}(p) > \mathrm{f}(q)$? Justify your answer.

29 **a** State the largest possible domain for the function $y = \sqrt{4 - x^2}$.

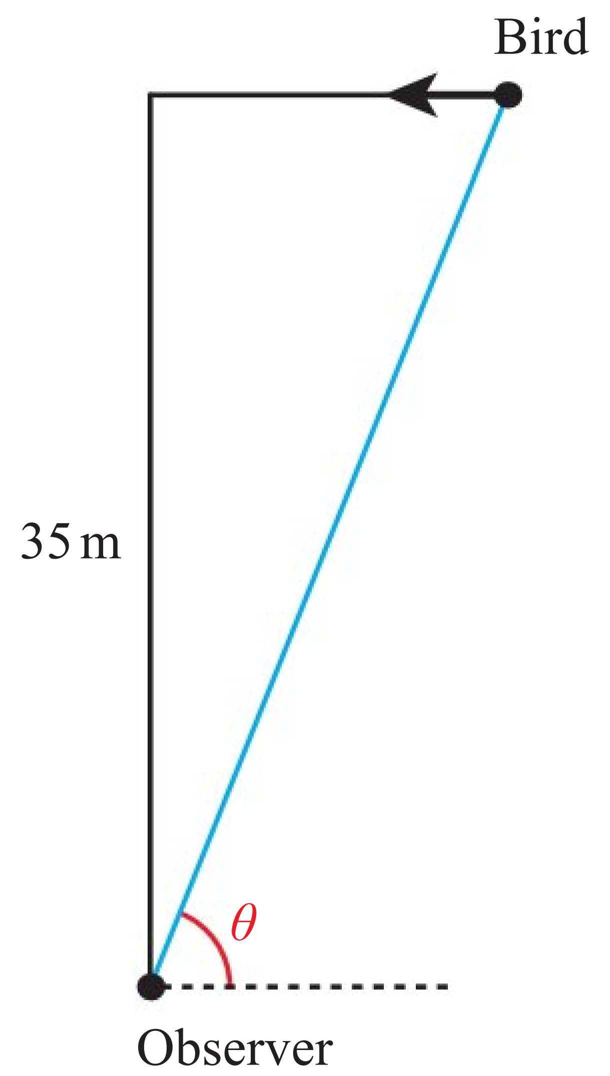
b Prove that, for any point on the graph of $y = \sqrt{4 - x^2}$, the normal passes through the origin.

30 The graph $y = kx^n$ has the property that, at every point, the product of the gradient and the y value equals 1.

a Find the value of n.

b Find the possible values of k.

31 A bird is flying at a constant speed at a constant height of 35 m in a straight line that will take it directly over an observer at ground level. At a given instant, the observer notes that the angle θ is 1.2 radians and is increasing at $\frac{1}{60}$ radians per second. Find the speed, in metres per second, at which the distance between the bird and the observer is decreasing.



11 Integration

ESSENTIAL UNDERSTANDINGS

- Calculus describes rates of change between two variables and the accumulation of limiting areas.
- Understanding these rates of change and accumulations allows us to model, interpret and analyse real-world problems and situations.
- Calculus helps us to understand the behaviour of functions and allows us to interpret features of their graphs.

In this chapter you will learn…

- how to integrate functions such as $\sqrt{x}$, $\sin x$ and e^x
- how to integrate by inspection
- how to find definite integrals
- how to link definite integrals to areas between a curve and the x-axis, and also between two curves
- how to use integration to find volumes of certain solids.

CONCEPTS

The following concepts will be addressed in this chapter:

- The area under a function on a graph has a meaning and has applications in **space** and time.

In Chapter 12, you will learn how to apply integration to one of the most important topics in physics – the study of motion.

Figure 11.1 What is being accumulated in each photograph?

PRIOR KNOWLEDGE

Before starting this chapter, you should already be able to complete the following:

1. Find y if $\dfrac{\mathrm{d}y}{\mathrm{d}x} = x^2 + 2$ and $y = 4$ when $x = 0$.
2. Use technology to evaluate the area between the curve $y = x^4$ and the x-axis between $x = 1$ and $x = 3$.
3. If $y = \mathrm{e}^{2x} + \sin x + 2$, find $\dfrac{\mathrm{d}y}{\mathrm{d}x}$.
4. If $\mathrm{f}(x) = \sqrt{x^2 + 3}$, find $\mathrm{f}'(x)$.

In the Mathematics: applications and interpretation SL book, you saw that the operation which reverses differentiation is called integration, but that surprisingly this has applications in evaluating areas and accumulations. Now that you know how to differentiate more functions, it is natural to extend your knowledge of integration to reverse these new derivatives. We can then apply these new integrals to more complex situations, including finding areas and volumes of various shapes.

Starter Activity

Look at the photos in Figure 11.1 and discuss the following questions. What rate is being shown? What is being accumulated?

Now look at this problem:

If you know that

$$\int_0^a \mathrm{f}(x)\,\mathrm{d}x = A$$

Which of the following can you evaluate?

$$\int_0^a \mathrm{f}(x-1)\,\mathrm{d}x \qquad \int_1^{a+1} \mathrm{f}(x-1)\,\mathrm{d}x$$

$$\int_0^a \mathrm{f}(2x)\,\mathrm{d}x \qquad \int_0^{\frac{a}{2}} \mathrm{f}(2x)\,\mathrm{d}x$$

11A Further indefinite integration

Extending the list of integrals

In Chapter 10, you saw that the derivative of x^n with respect to x is nx^{n-1} for any $n \in \mathbb{Q}$. This means that x^{n+1} differentiates to $(n+1)x^n$ so $\frac{1}{n+1}x^{n+1}$ differentiates to x^n as long as $n \neq -1$ (to avoid dividing by zero). Since integration is the reverse of differentiation, this means the following.

KEY POINT 11.1

$$\int x^n \, dx = \frac{x^{n+1}}{n+1} + c \text{ for } n \neq -1$$

You saw a similar rule in Chapter 10 of the Mathematics: applications and interpretation SL book, but we have now extended it to all rational values of n other than –1. Remember that we always need a '+ c' when integrating since this will disappear when it is differentiated.

WORKED EXAMPLE 11.1

Find $\int 3\sqrt{x} \, dx$.

Rewrite the integral to make it explicitly include x^n

$$\int 3x^{\frac{1}{2}} \, dx$$

The constant factor can be taken out of the integral

$$= 3\int x^{\frac{1}{2}} \, dx$$

Apply Key Point 11.1

$$= \frac{3x^{\left(\frac{1}{2}+1\right)}}{\frac{1}{2}+1} + c$$

Simplify

$$= \frac{3x^{\frac{3}{2}}}{\frac{3}{2}} + c$$

$$= 2x^{\frac{3}{2}} + c$$

You also found, in Chapter 10, that $\sin x$ differentiates to $\cos x$ when x is measured in radians. This can be reversed to turn this statement into an integral.

KEY POINT 11.2

$$\int \cos x \, dx = \sin x + c$$

Since $\cos x$ differentiates to $-\sin x$, it follows that $-\cos x$ differentiates to $\sin x$. Reversing this result gives an integral.

Tip

Because the rules of differentiating and integrating trigonometric functions are so similar they are often confused. Make sure you remember which way round they work.

KEY POINT 11.3

$$\int \sin x \, dx = -\cos x + c$$

Since $\tan x$ differentiates to $\frac{1}{\cos^2 x}$, this also can be reversed.

KEY POINT 11.4

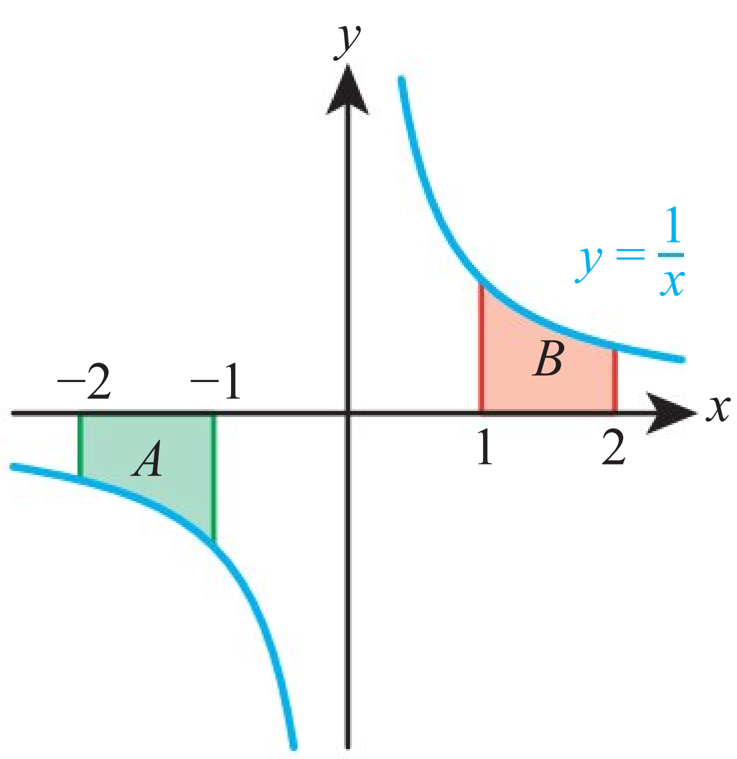
$$\int \frac{1}{\cos^2 x} dx = \tan x + c$$

You also saw, in Chapter 10, that e^x differentiates to e^x. This statement can be reversed.

KEY POINT 11.5

$$\int e^x \, dx = e^x + c$$

You know that $\frac{1}{x}$ differentiates to $\ln x$. This suggests that $\int \frac{1}{x} dx = \ln x + c$. However, there is a small complication that you need to consider. This integral can be linked to the area between the curve and the x-axis and this area exists for negative values of x even though $\ln x$ is not defined in this region:



This fills in the hole in Key Point 11.1 – integrating x^n where $n = -1$.

We can use the symmetry of the graph of $y = \frac{1}{x}$ to see that that area A and area B are equal. This leads to the following interpretation.

KEY POINT 11.6

$$\int \frac{1}{x} dx = \ln|x| + c$$

WORKED EXAMPLE 11.2

If $\frac{dy}{dx} = 3e^x + \cos x + 2$, find a general expression for y.

We reverse differentiation using integration $\quad y = \int 3e^x + \cos x + 2 \; dx$

We can split up the sum into separate integrals and take constant factors out $\quad = 3\int e^x \; dx + \int \cos x \; dx + \int 2 \; dx$

The integrals can be evaluated using the key points above $\quad = 3e^x - \sin x + 2x + c$

Integrating f($ax + b$)

From the chain rule, you know that when differentiating $f(ax + b)$ you get $af'(ax + b)$. In words, this says that you can differentiate the function as normal and then multiply by a. When integrating, we can reverse this logic to say that when integrating $f(ax + b)$ we can integrate as normal and then divide by a.

KEY POINT 11.7

If $\int f(x)\,dx = F(x)$, then $\int f(ax + b)\,dx = \frac{1}{a}F(ax + b)$.

WORKED EXAMPLE 11.3

Find $\int \sin(5x + 2)\,dx$.

We can identify $f(x) = \sin x$. This integrates to $-\cos x$ $\quad \int \sin(5x + 2)\,dx = -\frac{1}{5}\cos(5x + 2) + c$

Tip

In Worked Example 11.3, you might have wondered whether you needed to divide the constant by 5 too. However, $\frac{c}{5}$ is just another constant so it actually does not matter.

Key Point 11.7 is a special case of Key Point 11.8 (slightly rearranged) with $g(x) = ax + b$. You will meet a further generalization, called integration by substitution, in Section 11D.

Integration by inspection

The chain rule says that the derivative of a composite function, $f(g(x))$ is $g'(x)\,f'(g(x))$. This is a very common type of expression to look out for – a composite function multiplied by the derivative of the inner function – because it can easily be integrated.

KEY POINT 11.8

$$\int g'(x)f'(g(x))\;dx = f(g(x)) + c$$

Tip

You can check your answer by differentiating the result.

WORKED EXAMPLE 11.4

Find $\int 2x\, e^{(x^2)}\, dx$.

$f'(x) = e^x$ and $g(x) = x^2$. The composite function is multiplied by $2x$ which is $g'(x)$ so we can apply Key Point 11.8 directly

$$\int 2xe^{(x^2)}\, dx = e^{(x^2)} + c$$

Sometimes the expression multiplying the composite function is not exactly the derivative of the inner function, but is within a constant factor of it. In this situation, we can introduce the required factor in the integral, but to keep the value unchanged we need to divide by the same factor too.

WORKED EXAMPLE 11.5

Find $\int \frac{x}{3x^2 - 4}\, dx$.

We can identify $f'(x) = \frac{1}{x}$ and $g(x) = 3x^2 - 4$; therefore $g'(x) = 6x$. To get this, we need to introduce an extra factor of 6, which must be balanced by $\frac{1}{6}$

$$\int \frac{x}{3x^2 - 4}\, dx = \frac{1}{6}\int \frac{6x}{3x^2 - 4}\, dx$$

The integral is now of the required form to apply Key Point 11.8. We can use the fact that the integral of $\frac{1}{x}$ is $\ln|x| + c$

$$= \frac{1}{6}\ln|3x^2 - 4| + c$$

Tip

Worked Example 11.5 makes use of the very common integral

$$\int \frac{f'(x)}{f(x)}\, dx = \ln|f(x)| + c$$

This integral occurs so often you might like to learn this result.

TOOLKIT: Problem Solving

Consider $\int \frac{1}{2x}\, dx$. Using the method of Worked Example 11.5, we can make the top of the fraction the derivative of the bottom:

$$\int \frac{1}{2x}\, dx = \frac{1}{2}\int \frac{2}{2x}\, dx = \frac{1}{2}\ln|2x| + c$$

Alternatively, we can use algebra to take out a factor of $\frac{1}{2}$ from the integral.

$$\int \frac{1}{2x}\, dx = \frac{1}{2}\int \frac{1}{x}\, dx = \frac{1}{2}\ln|x| + c$$

Which of these two solutions is correct?

Be the Examiner 11.1

Find $\int \sin x \cos x \, dx$.

Which is the correct solution? Identify the errors in the incorrect solutions.

Solution 1	Solution 2	Solution 3
Since $\frac{d}{dx}(\sin x) = \cos x$ $\int (\sin x)^1 \cos x \, dx$ $= \frac{\sin^2 x}{2} + c$	$\int \sin x \cos x \, dx$ $= \frac{1}{2}\int \sin 2x \, dx$ $= -\frac{1}{4}\cos 2x + c$	Since $\frac{d}{dx}(\cos x) = -\sin x$ $-\int -\sin x (\cos x)^1 \, dx$ $= -\frac{\cos^2 x}{2} + c$

Exercise 11A

For questions 1 to 6, use the method demonstrated in Worked Example 11.1 to find the following integrals.

1 a $\int x^{\frac{2}{3}} \, dx$ b $\int x^{\frac{3}{4}} \, dx$

2 a $\int x^{-\frac{1}{2}} \, dx$ b $\int x^{-\frac{4}{3}} \, dx$

3 a $\int 10x^{\frac{3}{2}} \, dx$ b $\int 5x^{\frac{1}{4}} \, dx$

4 a $\int 4x^{-\frac{2}{3}} \, dx$ b $\int 3x^{-\frac{2}{5}} \, dx$

5 a $\int \frac{1}{2}\sqrt[3]{x} \, dx$ b $\int \frac{1}{3}\sqrt[5]{x} \, dx$

6 a $\int \frac{6}{\sqrt[4]{x}} \, dx$ b $\int \frac{7}{\sqrt{x}} \, dx$

For questions 7 to 10, use the method demonstrated in Worked Example 11.2 to find an expression for y.

7 a $\frac{dy}{dx} = 3\cos x$ b $\frac{dy}{dx} = -\cos x$

8 a $\frac{dy}{dx} = -2\sin x$ b $\frac{dy}{dx} = \frac{1}{2}\sin x$

9 a $\frac{dy}{dx} = 5e^x$ b $\frac{dy}{dx} = -\frac{4}{3}e^x$

10 a $\frac{dy}{dx} = \frac{2}{x}$ b $\frac{dy}{dx} = \frac{1}{2x}$

For questions 11 to 16, use the method demonstrated in Worked Example 11.3 to find the following integrals.

11 a $\int \sqrt{2x+1} \, dx$ b $\int \sqrt[3]{1-2x} \, dx$

12 a $\int \frac{1}{\sqrt{3-5x}} \, dx$ b $\int \frac{1}{\sqrt[4]{2x-7}} \, dx$

13 a $\int \cos(2-3x) \, dx$ b $\int \cos(4x+3) \, dx$

14 a $\int \sin\left(\frac{1}{2}x - 5\right) dx$ b $\int \sin(5-2x) \, dx$

15 a $\int e^{5x+2} \, dx$ b $\int e^{1-3x} \, dx$

16 a $\int \frac{1}{4x-5} \, dx$ b $\int \frac{1}{3-2x} \, dx$

For questions 17 to 19, use the method demonstrated in Worked Example 11.4 to find the following integrals.

17 a $\int 2x(x^2+4)^3 \, dx$ b $\int 4x^3(x^4-2)^{\frac{3}{2}} \, dx$

18 a $\int \cos x \sin^3 x \, dx$ b $\int -\sin x \cos^2 x \, dx$

19 a $\int \frac{3x^2+2}{x^3+2x} \, dx$ b $\int \frac{4x^3-5}{x^4-5x} \, dx$

For questions 20 to 22, use the method demonstrated in Worked Example 11.5 to find the following integrals.

20 a $\int x^2\sqrt{x^3+4}\,dx$

b $\int x\sqrt[3]{1-x^2}\,dx$

21 a $\int 4xe^{-x^2}\,dx$

b $\int 9x^2e^{x^3}\,dx$

22 a $\int \frac{x^2}{x^3+5}\,dx$

b $\int \frac{x^3}{x^4-3}\,dx$

23 Given that $\frac{dy}{dx} = 3\sqrt{x}$ and that $y = 12$ when $x = 9$, find an expression for y in terms of x.

24 Function f satisfies $f'(x) = 2\cos x - 3\sin x$ and $f(0) = 5$. Find an expression for $f(x)$.

25 Given that $\frac{dy}{dx} = 2e^x - \frac{5}{x}$, and that $y = 0$ when $x = 1$, find an expression for y in terms of x.

26 Find $\int \frac{4x^2+3}{2x}\,dx$.

27 Given that $\frac{dV}{dt} = t - \frac{1}{2}\sin t$, and that $V = 2$ when $t = 0$, find an expression for V in terms of t.

28 It is given that $x(0) = 5$ and that $\frac{dx}{dt} = 5 - 2e^t$. Find an expression for x in terms of t.

29 Find $\int (5-2x)^6\,dx$.

30 Find $\int (3\sin(2x) - 2\cos(3x))\,dx$.

31 Find $\int 2x(x^2+1)^5\,dx$.

32 Given that $\frac{dy}{dx} = x\sin(3x^2)$, find an expression for y in terms of x.

33 Find $\int \frac{3x}{\sqrt{x^2+2}}\,dx$.

34 Given that $f'(x) = \frac{1}{(2x+3)^2}$, and that the graph of $y = f(x)$ passes through the point $(-1, 1)$, find an expression for $f(x)$.

35 The curve $y = f(x)$ passes through the point $(1, 1)$ and has gradient $f'(x) = \frac{2\ln x}{3x}$. Find an expression for $f(x)$.

36 Given that $f'(x) = \frac{2}{x}$ and that $f(-1) = 5$, find the exact value of $f(-3)$.

37 Find $\int 4\cos^3 x\sin x + kx^2 + 1\,dx$, where k is a constant.

38 Find $\int \frac{e^{2x} - e^{-2x}}{e^x}\,dx$.

39 Find $\int \sqrt{x^4+x^2}\,dx$ for $x > 0$.

40 a Express $\cos 2x$ in terms of $\sin x$.

b Hence find $\int \sin^2 x\,dx$.

41 Find $\int \tan x\,dx$, showing all your working.

42 Find $\int \sin^3 x\,dx$.

43 Find $\int \frac{1}{x\ln x}\,dx$ for $x > 0$.

11B Definite integrals and the area between a curve and the x-axis

In Chapter 10 of the Mathematics: applications and interpretation SL book, you saw that we can represent the area under a curve $y = f(x)$ between $x = a$ and $x = b$ (called the **limits** of the integral) using $\int_a^b f(x)\,dx$ and you evaluated this using technology.

In this section, you will see how this can be evaluated analytically.

It turns out the **definite integral** is found by evaluating the integral at the value of the upper limit and subtracting the value of the integral at the lower limit. This is often expressed using 'square bracket' notation.

KEY POINT 11.9

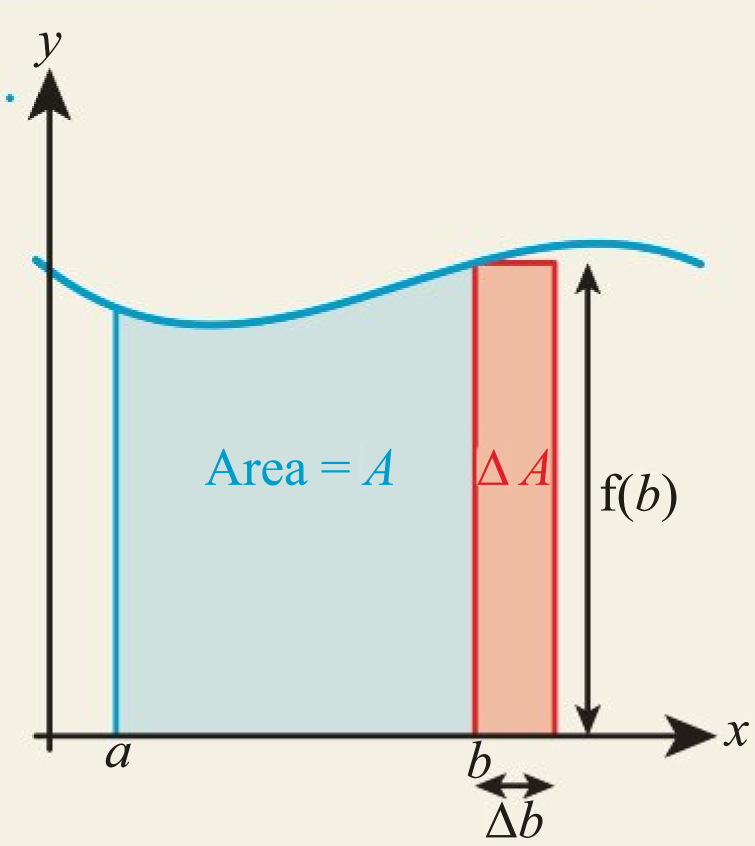
$$\int_a^b f'(x)\,dx = [f(x)]_a^b = f(b) - f(a)$$

It is not at all obvious that the process of reversing differentiation has anything to do with the area under a curve. This is something called the 'fundamental theorem of calculus' and it was formalized by English mathematician and physicist Isaac Newton and German mathematician and philosopher Gottfried Leibniz in the late seventeenth century. Although, today, most historians agree that the two made their discoveries about calculus independently, at the time there was a great controversy over the question of who developed these ideas first, and whether one had stolen them from the other.

Proof 11.1

Prove that the area under the curve $y = f(x)$ is $\int_a^b f(x)\,dx$.

Consider a general representation of $y = f(x)$. The increase in the area when b is changed is approximately a rectangle of height $f(b)$



The change in area, ΔA, is approximately the red rectangle, so $\Delta A = f(b)\,\Delta b$.

So $\frac{\Delta A}{\Delta b} = f(b)$. In the limit as Δb gets very small, this expression becomes $\frac{dA}{db}$.

To find A we need to undo the differentiation – this is integration. We can use the standard notation that the indefinite integral of $f(x)$ is $F(x)$. This is the area from some unspecified starting point up to b (hence it still contains an unknown constant)

Therefore, $A = F(b) + c$

The area between a and b can be found as the difference between the area up to b and the area up to a

So, the area between a and b is

$$(F(b)+c) - (F(a)+c) = F(b) - F(a)$$
$$= \int_a^b f(x)\,dx$$

WORKED EXAMPLE 11.6

Evaluate $\int_1^2 x^3\,dx$.

First, find the indefinite integral. For a definite integral you do not need the $+c$ although it does not matter if you put it in – it will end up cancelling out in the next line

$$\int_1^2 x^3\,dx = \left[\frac{x^4}{4}\right]_1^2$$

Evaluate the expression at the upper limit and subtract the value at the lower limit

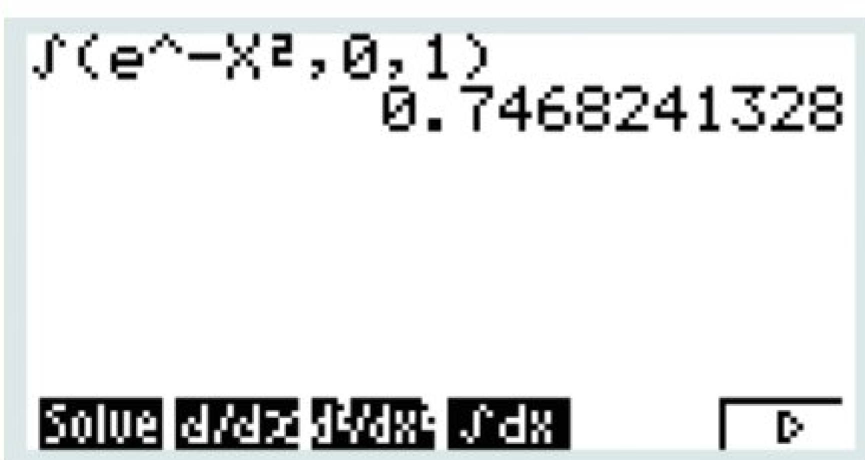
$$= \left(\frac{2^4}{4}\right) - \left(\frac{1^4}{4}\right)$$
$$= 4 - \frac{1}{4}$$
$$= 3.75$$

Not every definite integral can be evaluated analytically. When technology is available, it is often better to use it to evaluate definite integrals.

WORKED EXAMPLE 11.7

Evaluate $\int_0^1 e^{-x^2}\,dx$, giving your answer to three significant figures.

We can do this using a function on the calculator or using the graph



From the GDC: $\int_0^1 e^{-x^2}\,dx \approx 0.747$

TOK Links

e^{-x^2} is an example of a function which you might be told cannot be integrated. This really means that the indefinite integral cannot be written in terms of other standard functions, but who decides what is a 'standard function'? The area under the curve does exist, so we could (and indeed some mathematical communities do) define a new function that is effectively the area under this curve from 0 up to x (if you are interested it is $\frac{\sqrt{\pi}}{2}\mathrm{erf}(x)$). Does just giving the answer a name increase your knowledge?

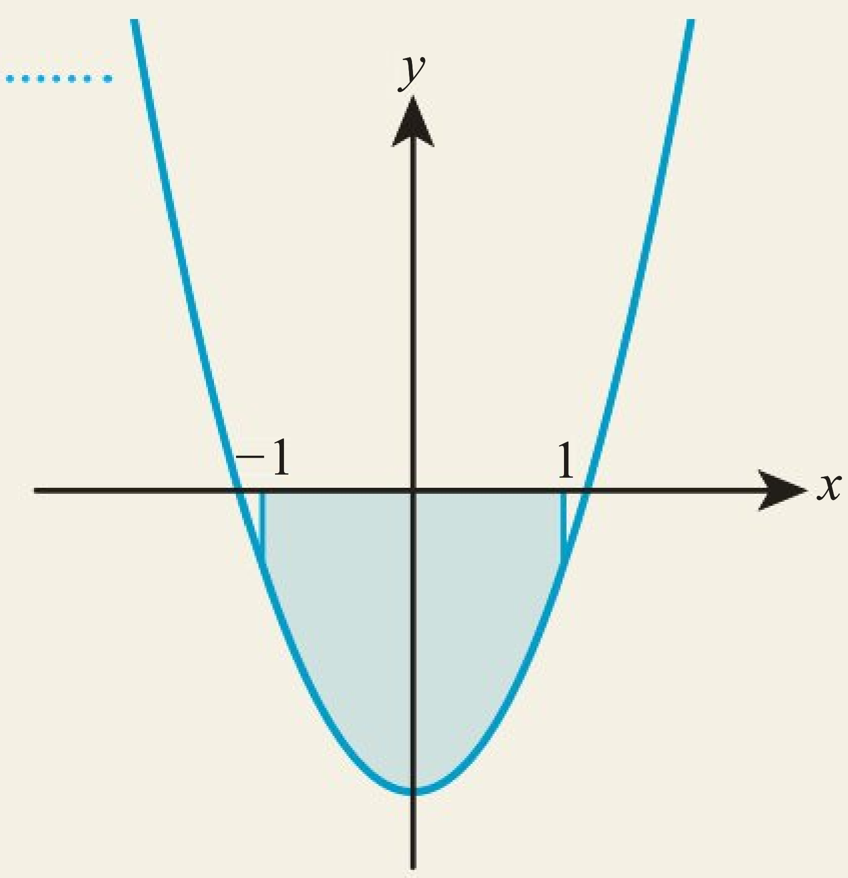
Area between a curve and the x-axis

Definite integrals give the area between a curve and the x-axis when the function between the limits is entirely above the x-axis. If the curve is below the x-axis, then the integral will give a negative value. The modulus of this value is the area.

WORKED EXAMPLE 11.8

Find the area enclosed by $x = -1$, $x = 1$, $y = 3x^2 - 4$ and the x-axis.

It is always useful to sketch the graph to visualise the area



We need to evaluate an appropriate definite integral

$$\int_{-1}^{1} 3x^2 - 4 \, dx = \left[x^3 - 4x\right]_{-1}^{1}$$

$$= (1-4) - (-1-(-4))$$

$$= (-3) - (3)$$

$$= -6$$

The area is the modulus of this answer

$$\text{Area} = |-6| = 6$$

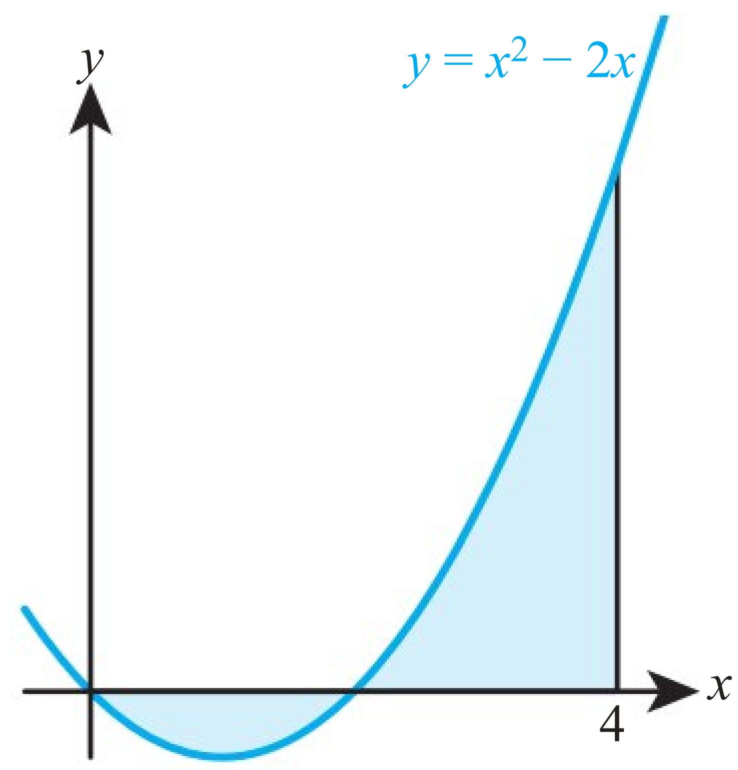
Tip

If you are not told any units for x and y, then you do not give units in the answer.

Sometimes areas are partly above and partly below the x-axis. In that case, you have to find each part separately.

WORKED EXAMPLE 11.9

Find the shaded area on the curve of $y = x^2 - 2x$.



First find where it goes from above the axis to below the axis

It crosses the x-axis when

$$x^2 - 2x = 0$$
$$x(x-2) = 0$$

So, it crosses when $x = 0$ and $x = 2$

Split the integral into two parts. Consider first the region below the x-axis

$$\int_0^2 x^2 - 2x \, dx = \left[\frac{x^3}{3} - x^2\right]_0^2 = -\frac{4}{3} - 0$$

So, the first area is $\left|-\frac{4}{3}\right| = \frac{4}{3}$

Then the second region

$$\int_2^4 x^2 - 2x \, dx = \left[\frac{x^3}{3} - x^2\right]_2^4 = \frac{16}{3} - \left(-\frac{4}{3}\right) = \frac{20}{3}$$

Combine the two areas

Total area is $\frac{4}{3} + \frac{20}{3} = 8$

Tip

A common mistake in problems like that in Worked Example 11.9 is to think that the area is $\int_0^4 x^2 - 2x \, dx$, which is $\frac{16}{3}$. This is the 'net area' – how much more is above the x-axis than below – and this is sometimes very useful, but it is not the answer to the given question.

With a calculator, you can find the total area between $x = a$, $x = b$, the curve $y = f(x)$ and the x-axis using the formula in Key Point 11.10.

KEY POINT 11.10

$$\text{Area} = \int_a^b |f(x)| \, dx$$

Tip

Check that Key Point 11.10 gives the correct answer to Worked Example 11.9.

WORKED EXAMPLE 11.10

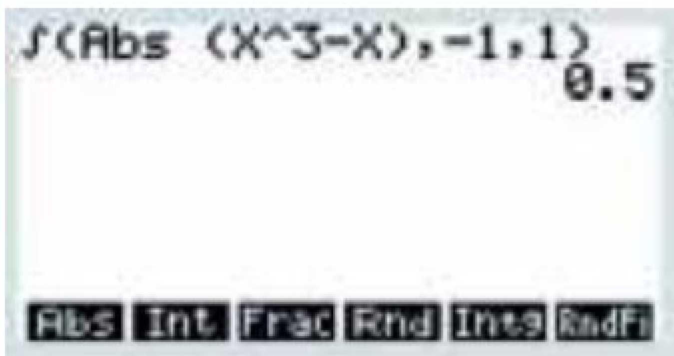
Find the area between the curve $y = x^3 - x$ and the x-axis between $x = -1$ and $x = 1$.

Write the required area in terms of a definite integral from Key Point 21.9

$$\text{Area} = \int_{-1}^{1} |x^3 - x| \, dx$$

Evaluate using technology

$= 0.5$ from the GDC



TOOLKIT: Proof

Justify each of the following rules for definite integrals:

$$\int_a^b f(x)\, dx = -\int_b^a f(x)\, dx$$

$$\int_a^b f(x)\, dx + \int_b^c f(x)\, dx = \int_a^c f(x)\, dx$$

$$\int_a^b f(x)\, dx + \int_a^b g(x)\, dx = \int_a^b f(x) + g(x)\, dx$$

Exercise 11B

For questions 1 to 6, use the method demonstrated in Worked Example 11.6 to find the given definite integrals.

1 **a** $\int_1^4 x^{\frac{1}{2}}\, dx$

b $\int_0^8 x^{\frac{1}{3}}\, dx$

2 **a** $\int_{-2}^{1} x^4\, dx$

b $\int_{-1}^{0} x^5\, dx$

3 **a** $\int_{-3}^{-1} x^2\, dx$

b $\int_{-4}^{-2} x^3\, dx$

4 **a** $\int_{\frac{\pi}{3}}^{\frac{\pi}{2}} \sin x\, dx$

b $\int_0^{\frac{\pi}{2}} \cos x\, dx$

5 **a** $\int_0^{\ln 3} e^x\, dx$

b $\int_{\ln 2}^{\ln 7} e^x\, dx$

6 **a** $\int_1^5 \frac{1}{x}\, dx$

b $\int_2^3 \frac{1}{x}\, dx$

For questions 7 and 8, use the method demonstrated in Worked Example 11.7 to find the given definite integrals.

7 **a** $\int_0^3 \sin(x^2)\,dx$

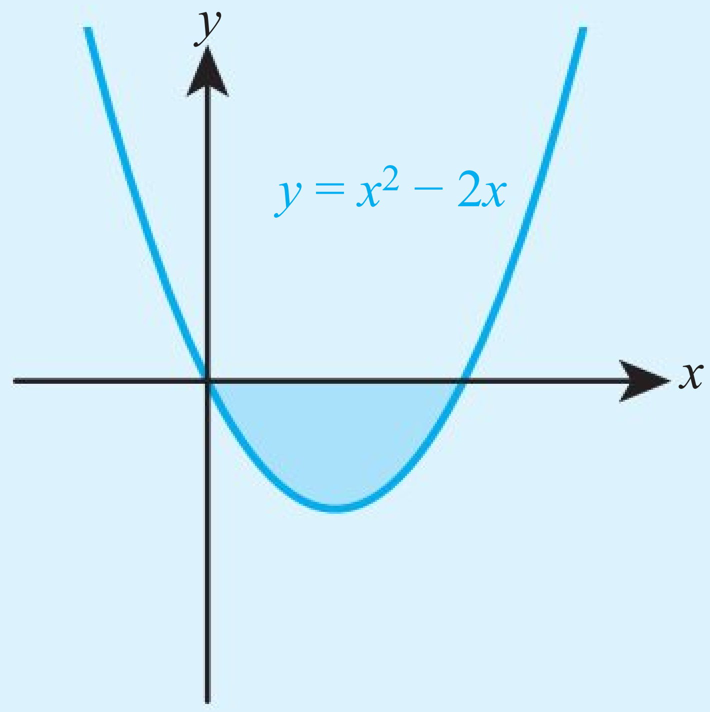
b $\int_1^6 \cos(x^3)\,dx$

8 **a** $\int_2^3 \frac{1}{\ln x}\,dx$

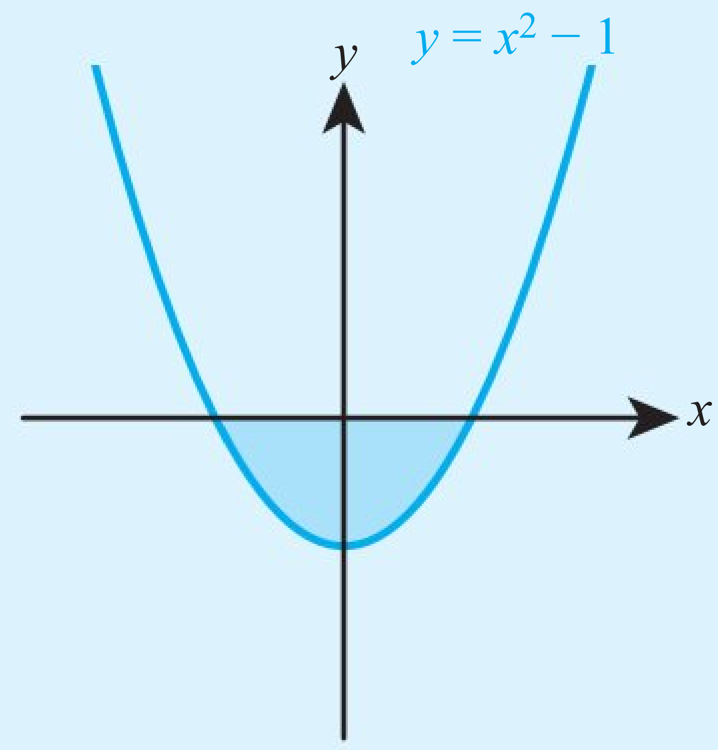
b $\int_4^9 \ln(\ln x)\,dx$

For questions 9 and 10, use the method demonstrated in Worked Examples 11.8 and 11.9 to find the shaded areas.

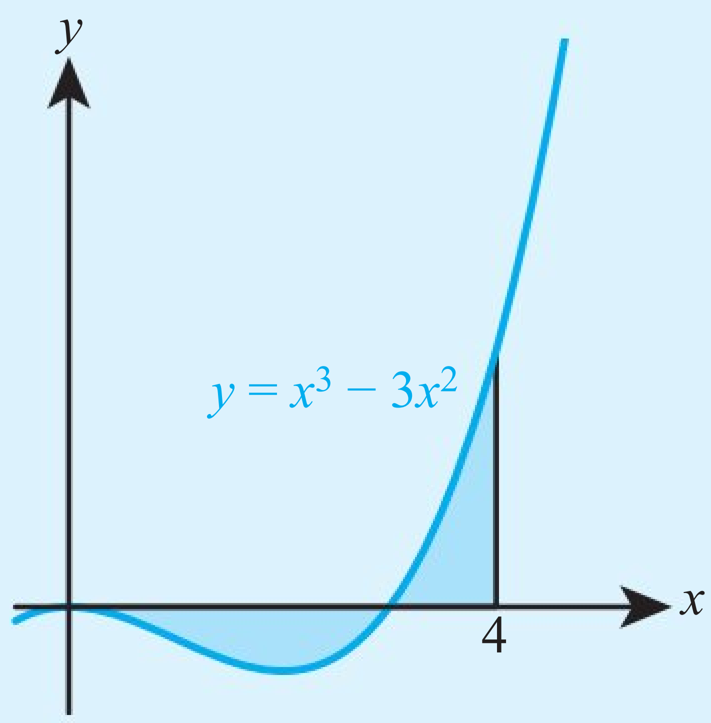
9 **a**



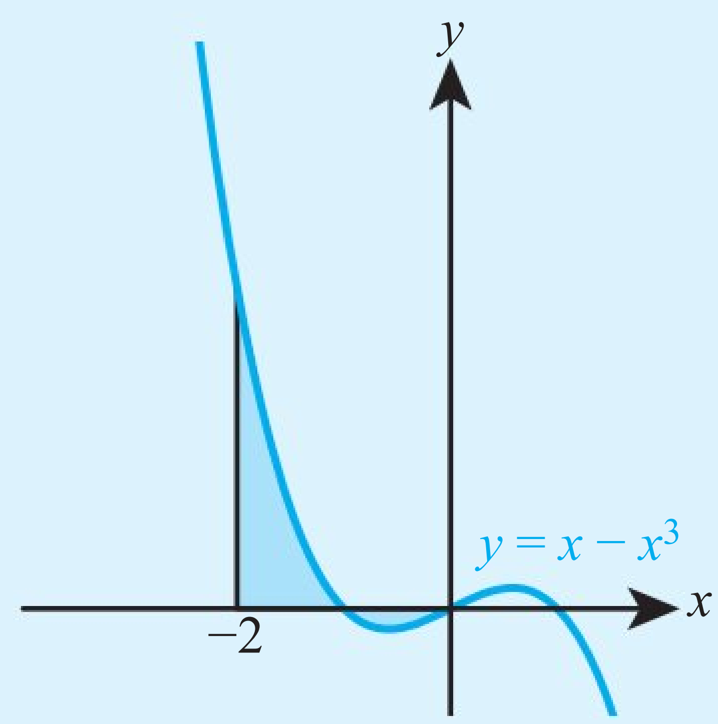
b



10 **a**

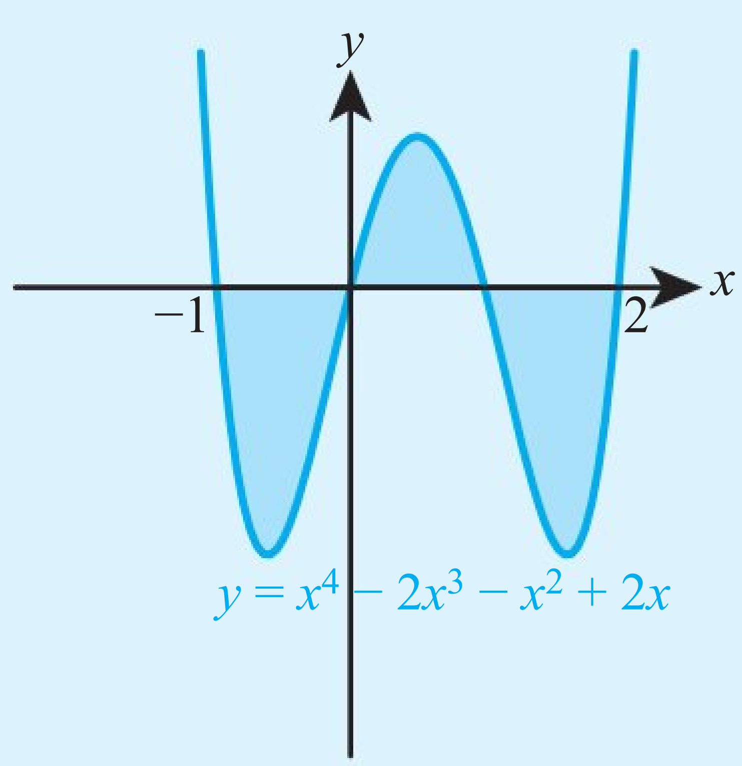


b

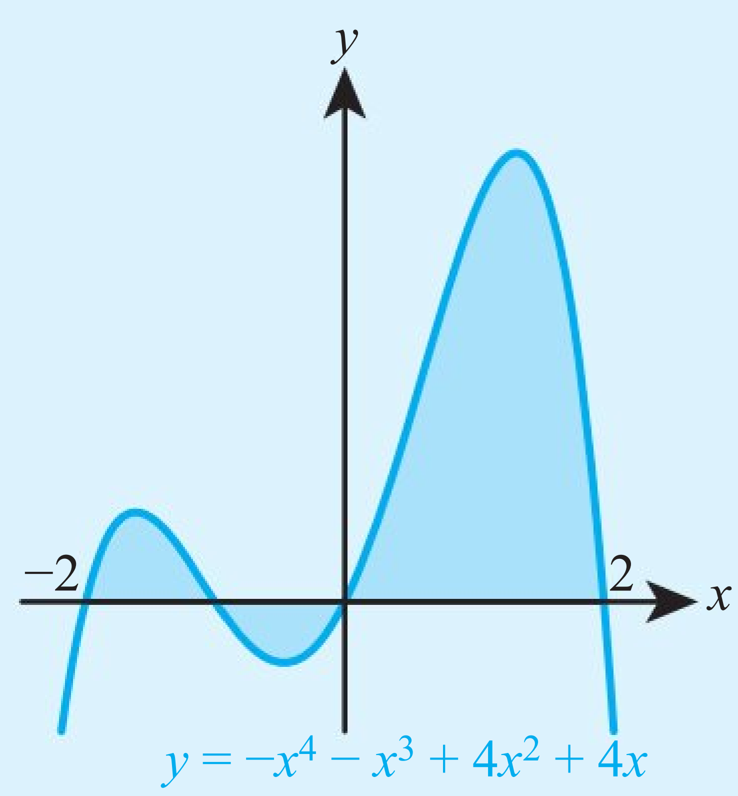


For questions 11 and 12, use the method demonstrated in Worked Example 11.10 to find the shaded areas.

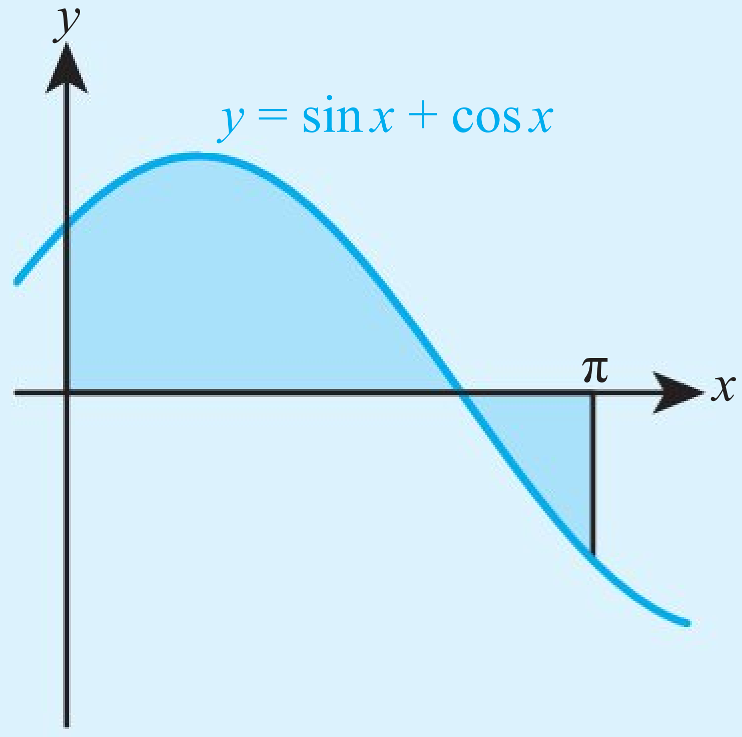
11 **a**



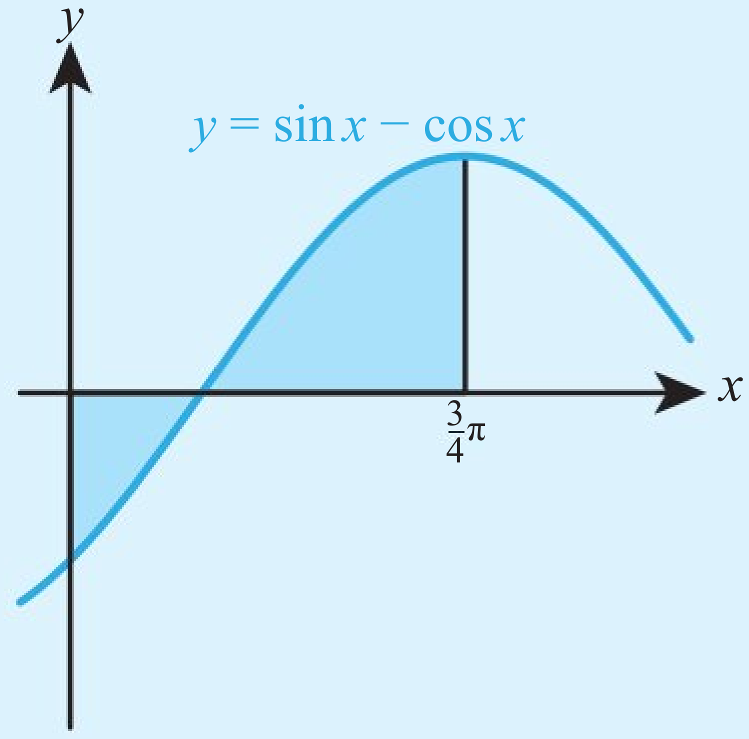
b



12 **a**



b



13 Evaluate $\int_1^5 \sqrt{3x+1}\,\mathrm{d}x$.

14 The curve with equation $y = 3\sin 2x$ crosses the x-axis at $x = 0$ and $x = \frac{\pi}{2}$. Find the exact area enclosed by this part of the curve and the x-axis.

15 Find the area enclosed by the graph of $y = \sqrt{\ln x}$, the x-axis and the lines $x = 2$ and $x = 5$.

16 Find the value of a such that $\int_1^a \frac{3}{\sqrt{x}}\,\mathrm{d}x = 24$.

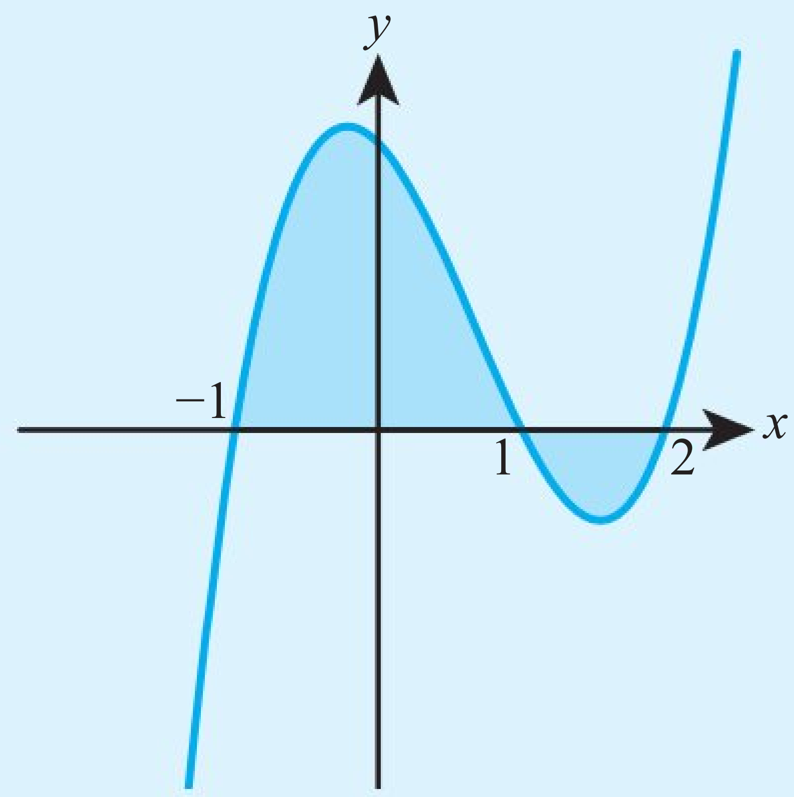
17 **a** Find the coordinates of the points where the curve $y = 9 - x^2$ crosses the x-axis.

b Find the area enclosed by the curve and the x-axis.

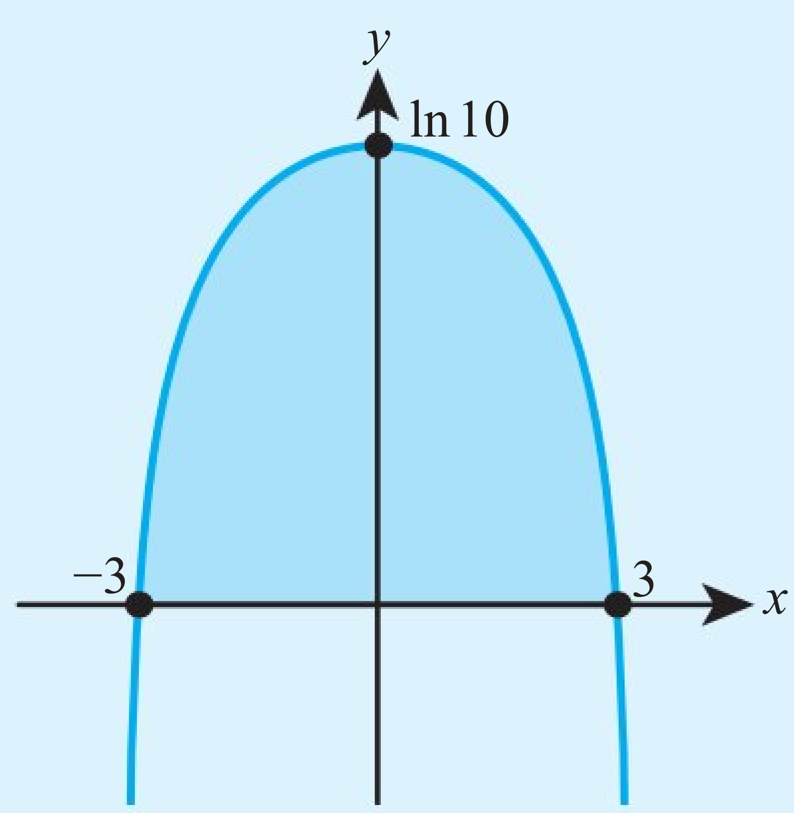
18 The graph of $y = x^3 - 2x^2 - x + 2$ is shown in the diagram.

a Evaluate $\int_{-1}^{1} y\,\mathrm{d}x$ and $\int_{1}^{2} y\,\mathrm{d}x$.

b Hence find the total area of the shaded region.



19 The graph of $y = \ln(10 - x^2)$, shown in the diagram, crosses the x-axis at the points (–3, 0) and (3, 0). Find the shaded area.

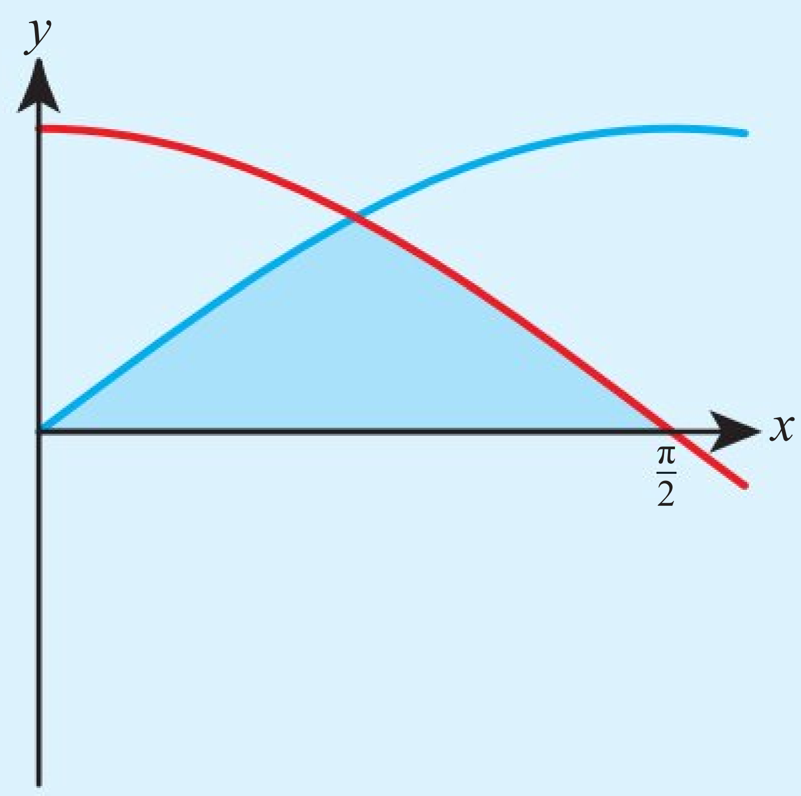


20 Find the exact value of $\int_0^{\ln 3} 2e^{-3x}\,dx$.

21 Find the exact value of $\int_{-9}^{-3} \frac{5}{x}\,dx$.

22 Show that the value of the integral $\int_k^{2k} \frac{1}{x}\,dx$ is independent of k.

23 The diagram shows the graphs of $y = \sin x$ and $y = \cos x$ for $0 \leqslant x \leqslant \frac{\pi}{2}$.



a Write down the coordinates of the point of intersection of the two graphs.

b Find the exact area of the shaded region.

24 **a** On the same axes, sketch the graphs of $y = \cos(x^2)$ and $y = 2 - 2\sin(x^2)$ for $0 \leqslant x \leqslant \frac{\pi}{2}$.

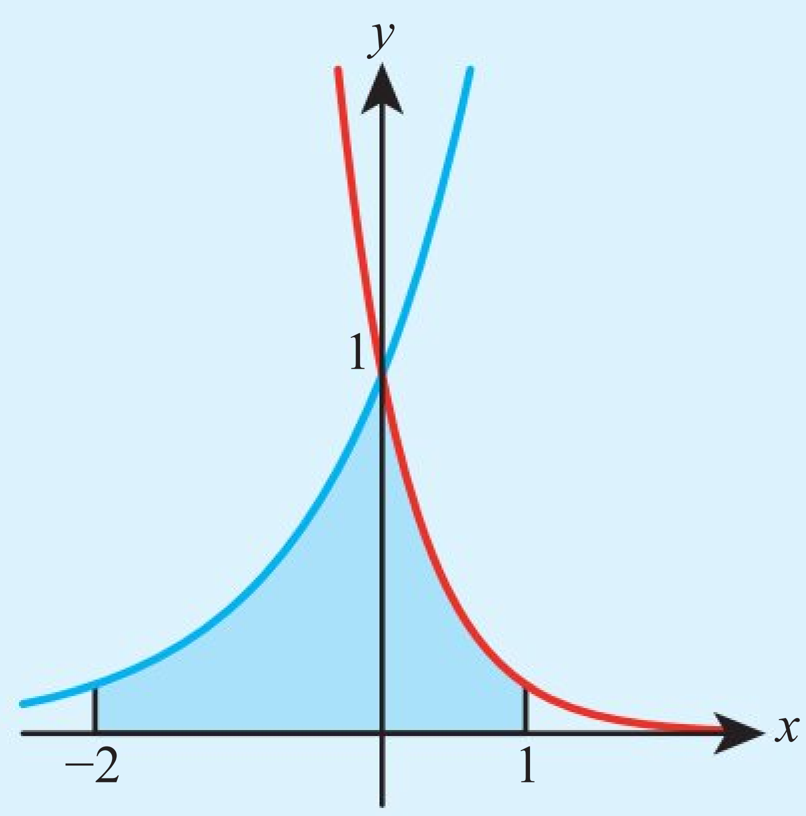
b Find the area enclosed by the two graphs.

25 Show that the area enclosed by the graph of $y = \sin 2x$ and the x-axis for $0 \leqslant x \leqslant \pi$ equals 2.

26 Find the exact value of $\int_0^{\sqrt{3}} x\sqrt{x^2+1}\,dx$.

27 The curves shown in the diagram have equations $y = e^x$ and $y = e^{-2x}$.

Find the area of the shaded region.



28 Find the exact value of $\int_1^4 \frac{1}{2x+3}\,dx$.

29 Evaluate $\int_2^3 \frac{1}{x-5}\,dx$.

30 Solve for a

$$\int_0^a x^2 + x\,dx = 90$$

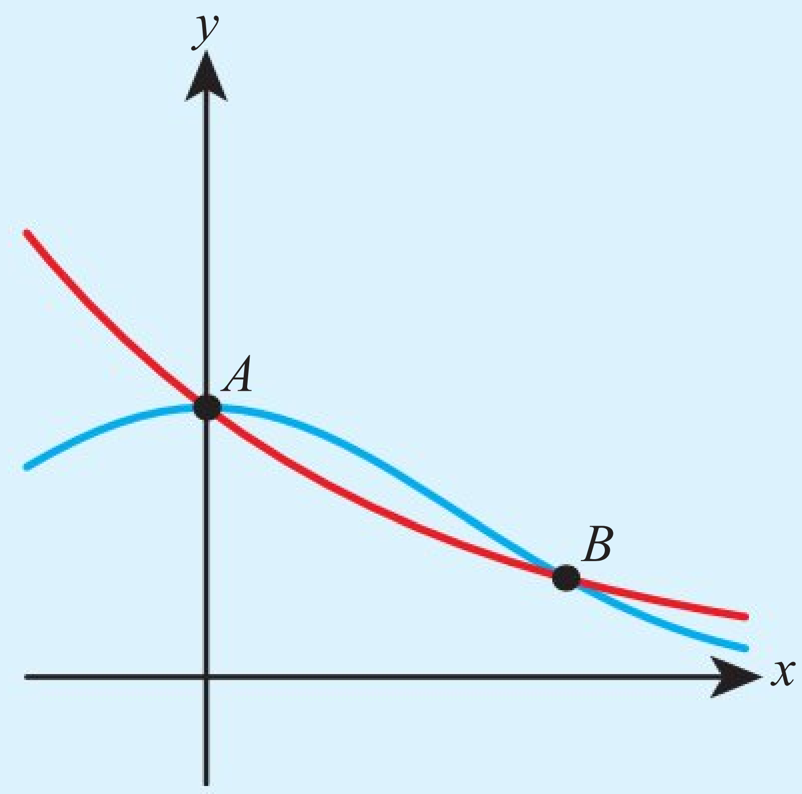
31 Solve for a

$$\int_a^{2a} x + 1\,dx = 8$$

32 The curves with equations $y = e^{-x^2}$ and $y = e^{-x}$ are shown in the diagram.

a Find the coordinates of the points A and B.

b Find the area enclosed by the two curves.



33 Given that $\int_2^5 f(x)\,dx = 10$, evaluate $\int_2^5 (3f(x) - 1)\,dx$.

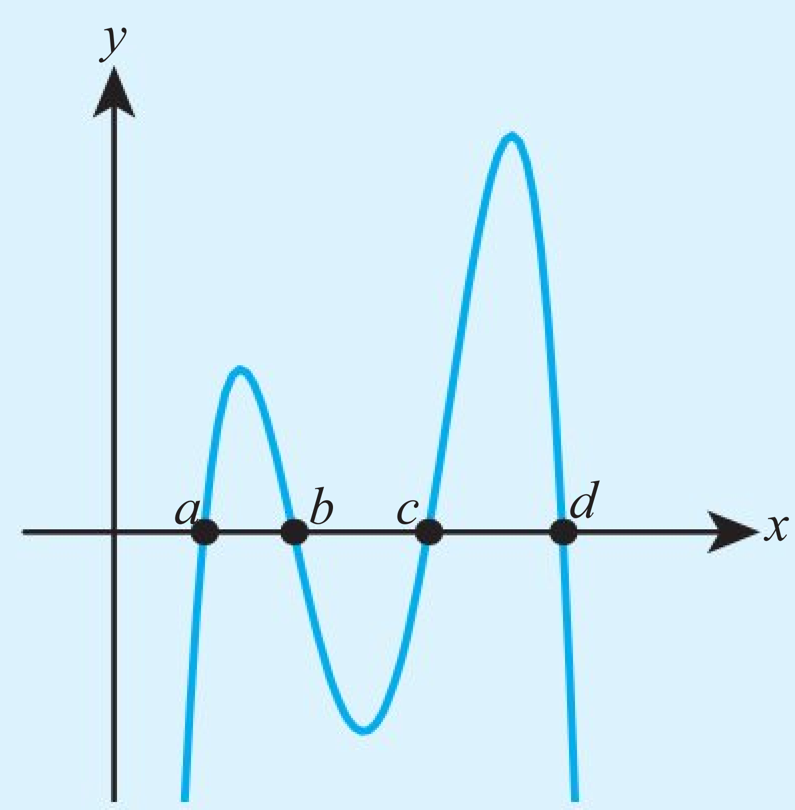
34 Given that $\int_0^6 f(x)\,dx = 7$, evaluate $\int_0^3 5f(2x)\,dx$.

35 **a** Differentiate $x\ln x$.

b Hence evaluate $\int_1^e \ln x\,dx$.

36 The diagram shows the graph of $y = f(x)$.

Given that $\int_b^d f(x)\,dx = 1$, $\int_a^d f(x)\,dx = 5$ and $\int_a^d |f(x)|\,dx = 17$, find $\int_a^c f(x)\,dx$.



You are the Researcher

Use technology to evaluate $\int_0^1 4\sqrt{1-x^2}\,dx$. What do you notice about the answer?

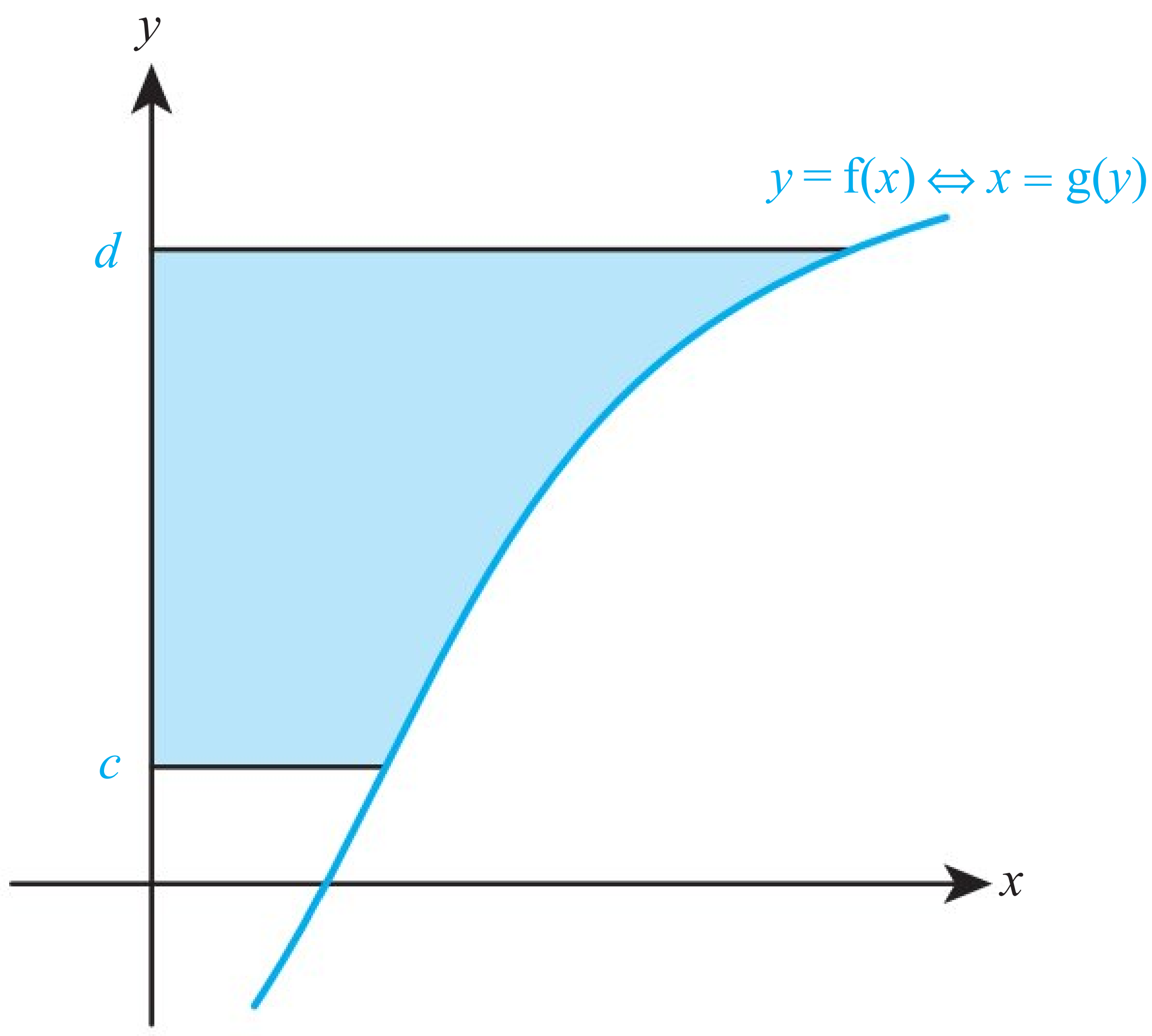
Justify your answer (i) graphically, (ii) using the substitution $x = \sin t$. This is an example of an integral which can be done analytically but is not of the form from Key Point 11.8. Can you explain why the substitution $x = \sin t$ works here? Can you find any other integrals which need surprising substitutions?

11C Further geometric interpretation of integrals

Area of the region enclosed by a curve and the y-axis

You already know that the area between a curve and the x-axis is given by $\int_a^b f(x)\,dx$.

The diagram below shows the area between a curve with equation $y = f(x)$ and the y-axis.



If you imagine swapping the x- and the y-axes, you can find the area by using integration, as before. However, the equation needs to be for x in terms of y, that is, in the form $x = g(y)$, and the limits need to be the y-values.

KEY POINT 11.11

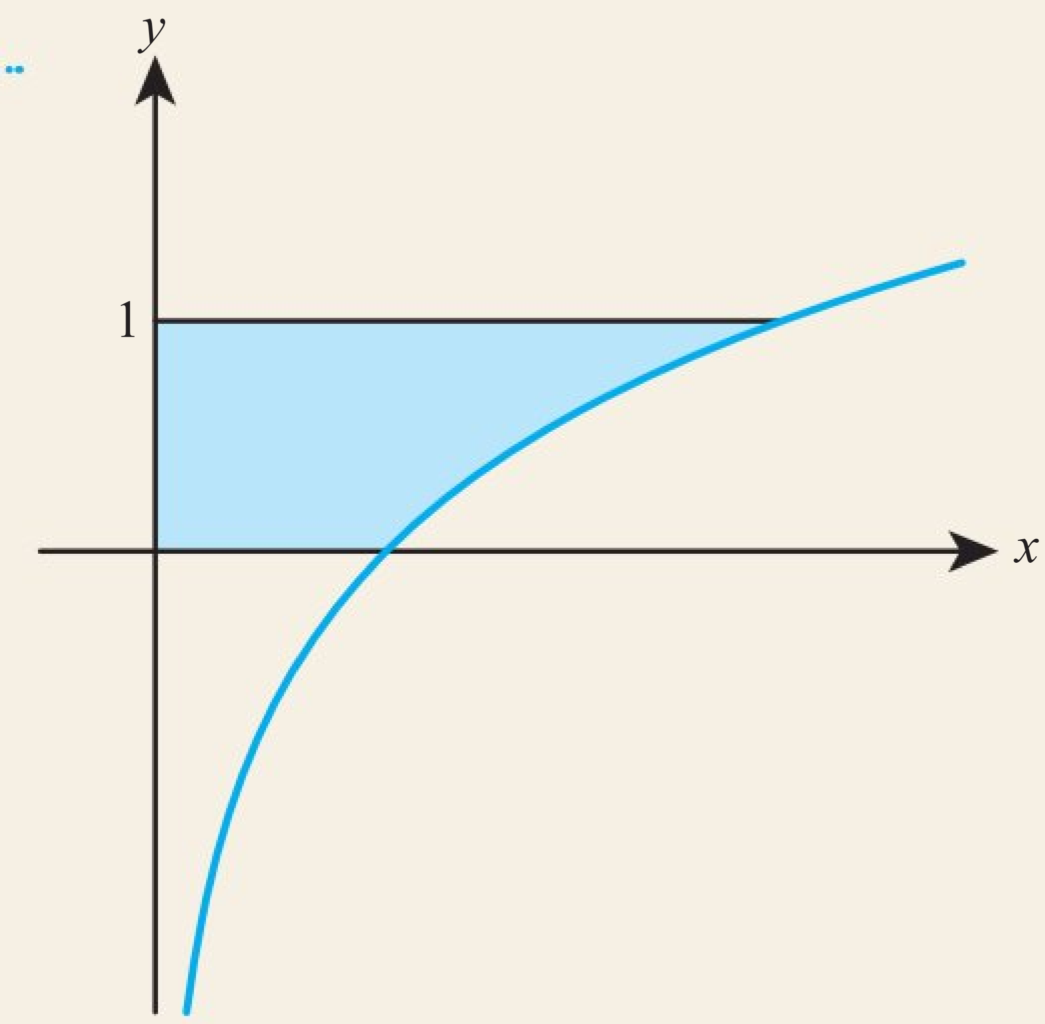
The area bounded by a curve $x = g(y)$, the y-axis and the lines $y = c$ and $y = d$ is given by

$$\int_c^d g(y)\,dy$$

WORKED EXAMPLE 11.11

Find the area enclosed by the curve $y = \ln x$, the y-axis and the lines $y = 0$ and $y = 1$.

A sketch is always helpful to make sure that you are finding the correct area



Rearrange the equation to express x in terms of y

$y = \ln x$ so $x = e^y$

Area $= \int_c^d g(y)\,dy$. The limits for y are given in the question

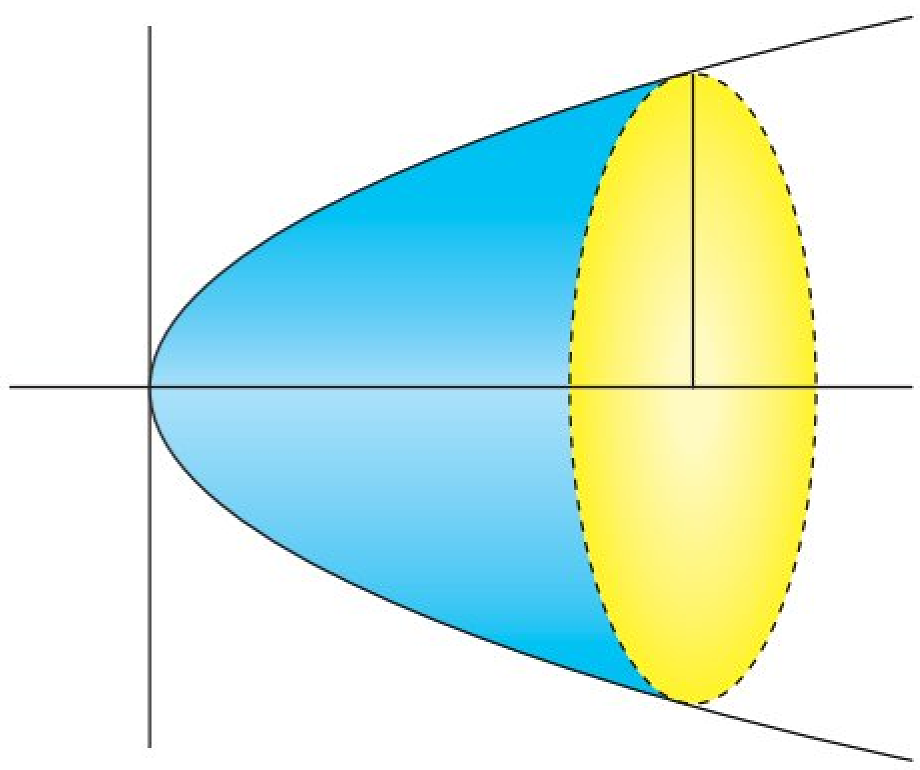
$$\text{Area} = \int_0^1 e^y \, dy$$

Evaluate the integral using your GDC

$= 1.72$ (3 s.f.)

Volumes of revolution about the x-axis or y-axis

When a part of a curve is rotated 360° about the x-axis (or the y-axis), it forms a shape known as a **solid of revolution**. The volume of this solid is a **volume of revolution**.



KEY POINT 11.12

The volume of revolution formed when the part of the curve $y = f(x)$, between $x = a$ and $x = b$, is rotated around the x-axis is given by $V = \int_a^b \pi y^2 \, dx$.

The proof of this result is based on the same idea as calculating the area under a curve: split up the volume into lots of small parts and add them up.

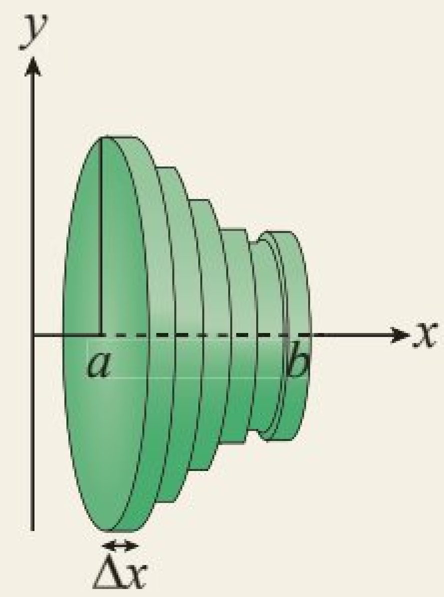
Proof 11.2

Prove that the volume of revolution when $y = f(x)$ (for $a < x < b$) is rotated around the x-axis is given by $\int_a^b \pi y^2 \, dx$.

The volume can be approximated by a sum of small cylinders

The volume can be split up into small cylinders each of length Δx:

It is useful to sketch a diagram to illustrate this



The volume of each cylinder is given by area of cross-section × length. The length of each cylinder is Δx

The volume of each cylinder is $\pi y^2 \Delta x$

The radius is equal to the y-coordinate of a point on the curve

You can now add up the volumes of the cylinders to approximate the volume of revolution

The total volume is approximately

$$V \approx \sum_a^b \pi y^2 \Delta x$$

The approximation becomes more accurate as Δx gets smaller

$$V = \lim_{\Delta x \to 0} \sum_a^b \pi y^2 \Delta x$$

In the limit when $\Delta x \to 0$, the sum becomes an integral

$$= \int_a^b \pi y^2 \, dx$$

WORKED EXAMPLE 11.12

Find the volume of revolution when the curve $y = x^3$, $0 < x < 2$, is rotated around the x-axis. Give your answer in terms of π.

The volume is given by $\int_0^2 \pi y^2 \, dx$

$$V = \int_0^2 \pi (x^3)^2 \, dx$$

$$= \pi \int_0^2 x^6 \, dx$$

Evaluate the integral on your GDC, then multiply by π

$$= \frac{128}{7}\pi$$

You are the Researcher

There are also formulae to find the surface area of a solid formed by rotating a region around an axis. Some particularly interesting examples arise if we allow one end of the region to tend to infinity. For example, rotating the region formed by the lines $y = \frac{1}{x}$, $x = 1$ and the x-axis results in a solid called Gabriel's horn, or Torricelli's trumpet, which has a finite volume but infinite surface area!

When a curve is rotated around the y-axis, you can obtain the formula for the resulting volume of revolution simply by swapping x and y.

KEY POINT 11.13

The volume of revolution formed when the part of the curve $y = f(x)$, between $y = c$ and $y = d$, is rotated around the y-axis is given by $V = \int_c^d \pi x^2 \, dy$.

Notice that, to use this formula, you need to write x in terms of y and find the limits on the y-axis.

WORKED EXAMPLE 11.13

Find the volume of revolution when the curve $y = x^2 - 1$, $1 < x < 5$, is rotated around the y-axis. Give your answer in terms of π.

You need to express x^2 in terms of y: $y = x^2 - 1$, so $x^2 = y + 1$

Find the limits for y: Limits:

when $x = 1$, $y = 0$

when $x = 5$, $y = 24$

The volume is given by $\int_0^{24} \pi x^2 \, dy$:

$$V = \int_0^{24} \pi(y + 1) \, dy$$

Evaluate the integral on your GDC:

$$= 312\pi$$

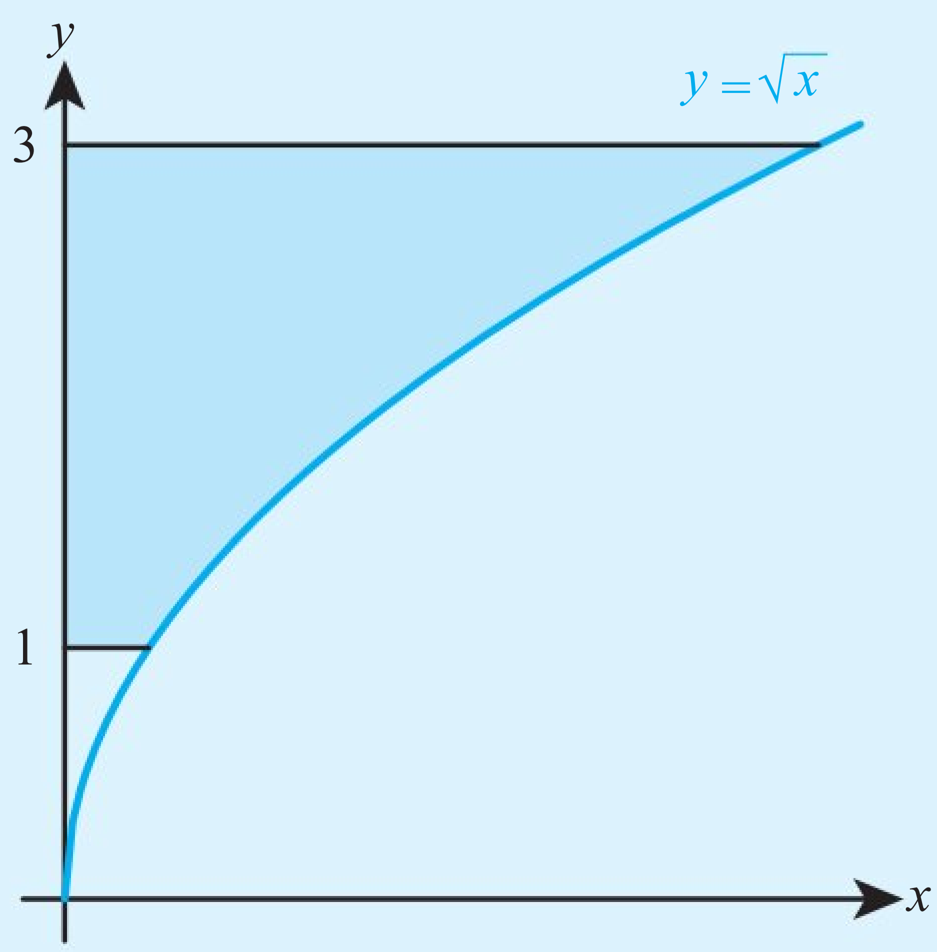
You are the Researcher

An alternative formula for the volume of revolution when $y = f(x)$, $a < x < b$, is rotated around the y-axis is given by $\int_a^b 2\pi xy \, dx$. Can you justify this and find any applications?

Exercise 11C

For questions 1 to 4, use the technique demonstrated in Worked Example 11.11 to find the area between the given curve, the y-axis and the lines $y = c$ and $y = d$.

1 a



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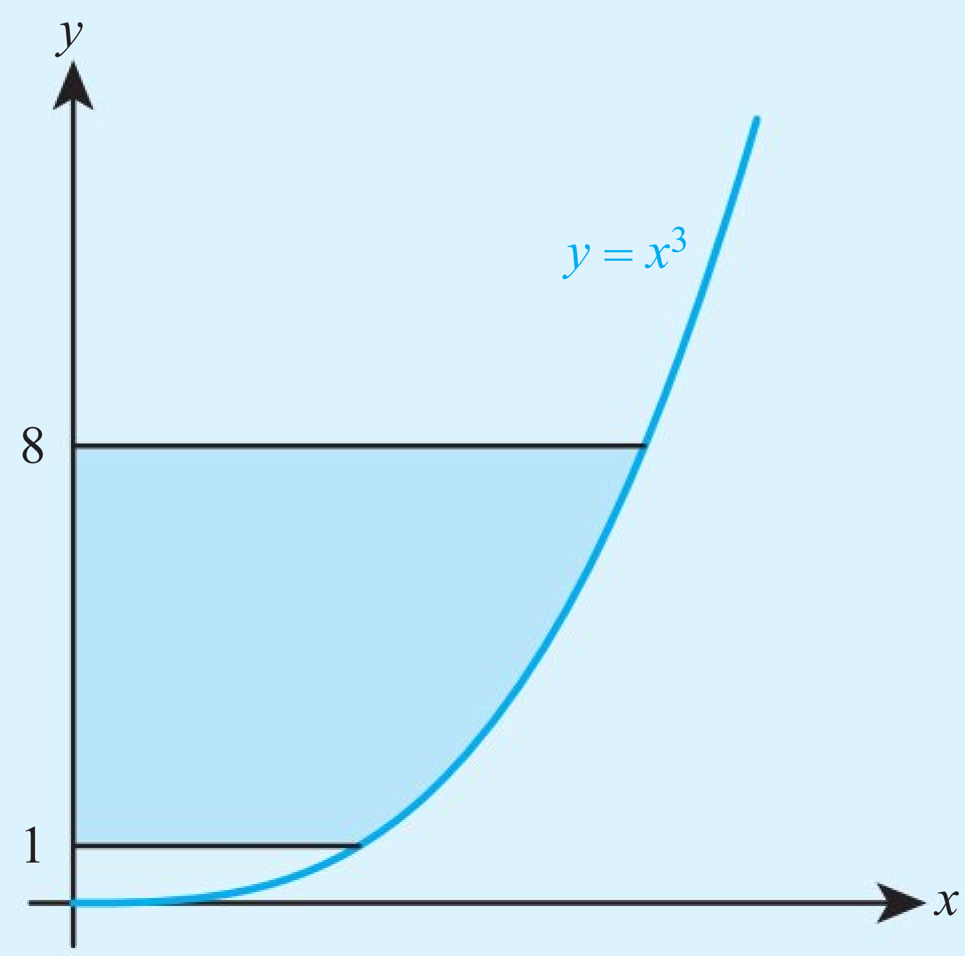
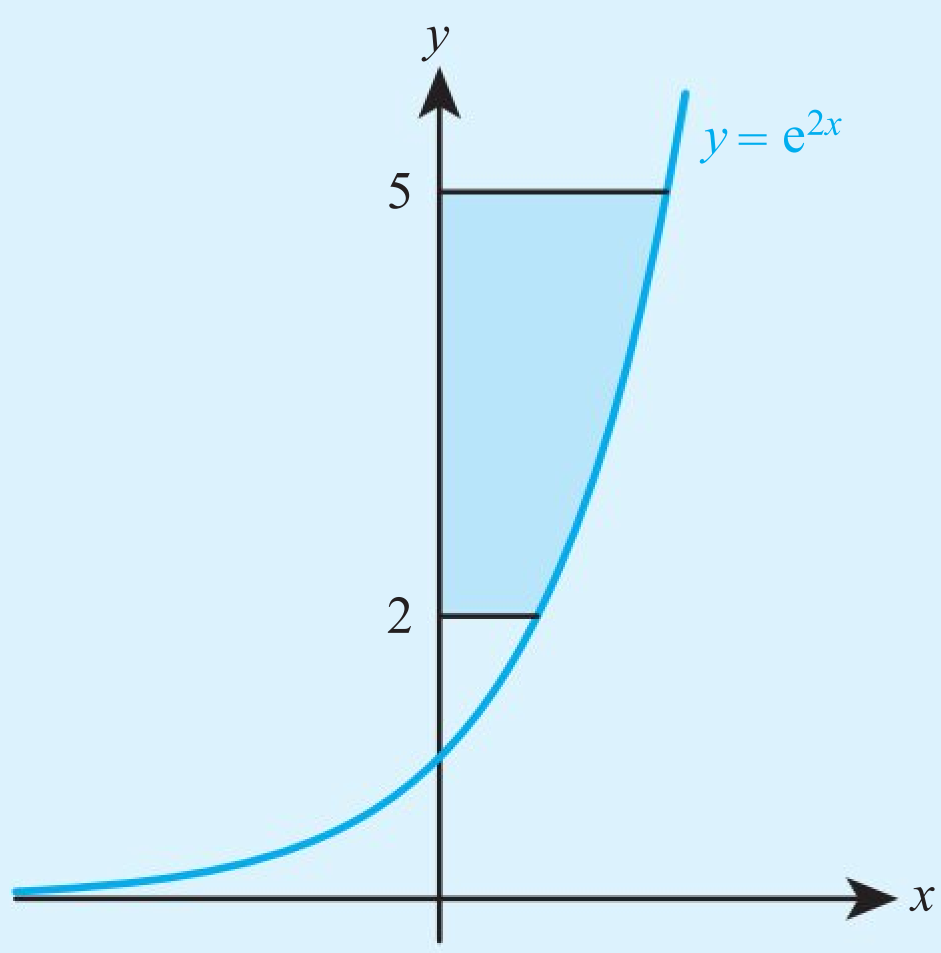


2 a



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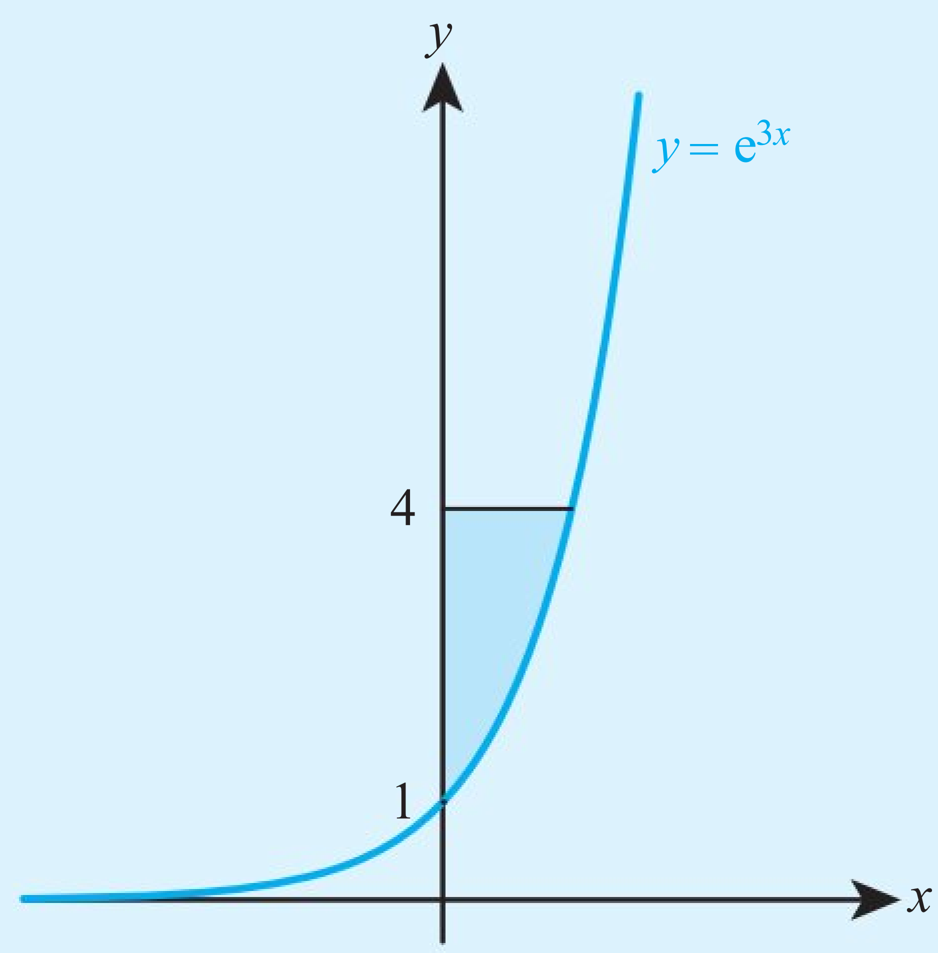
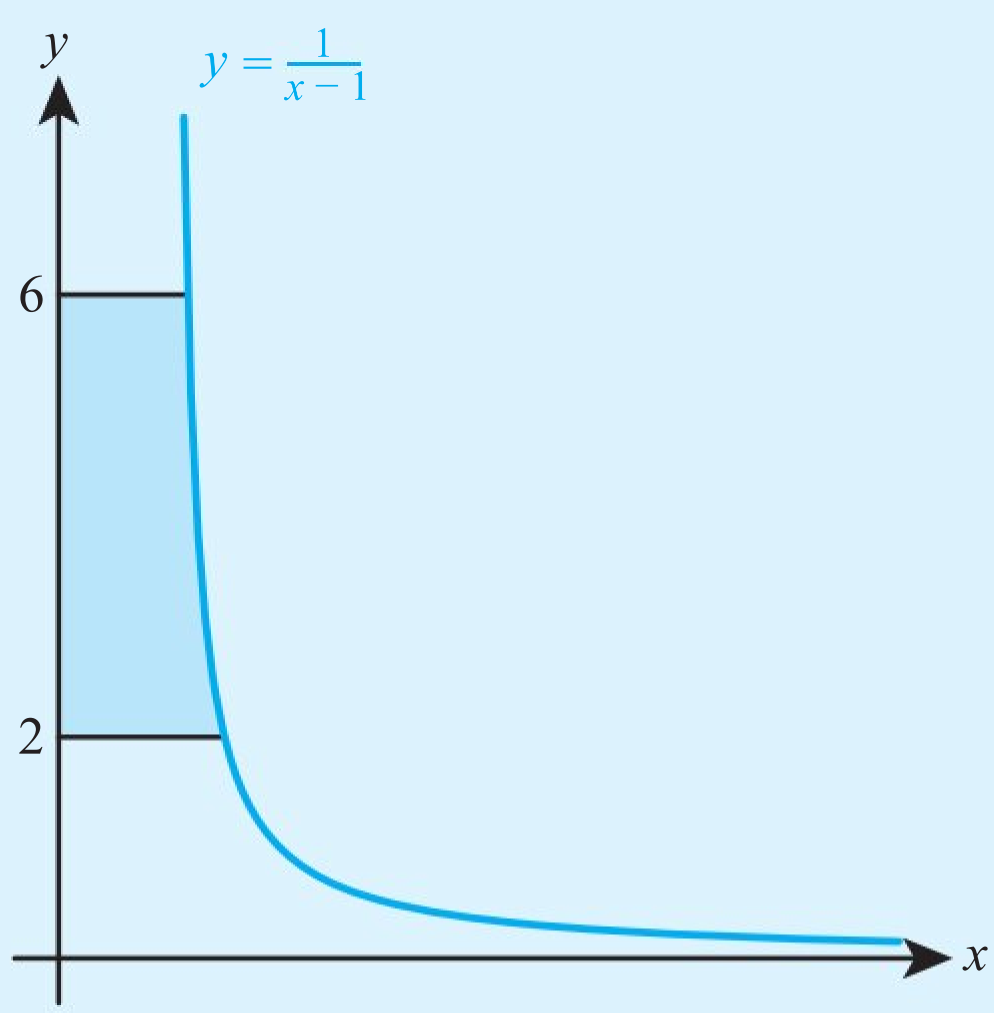


3 a



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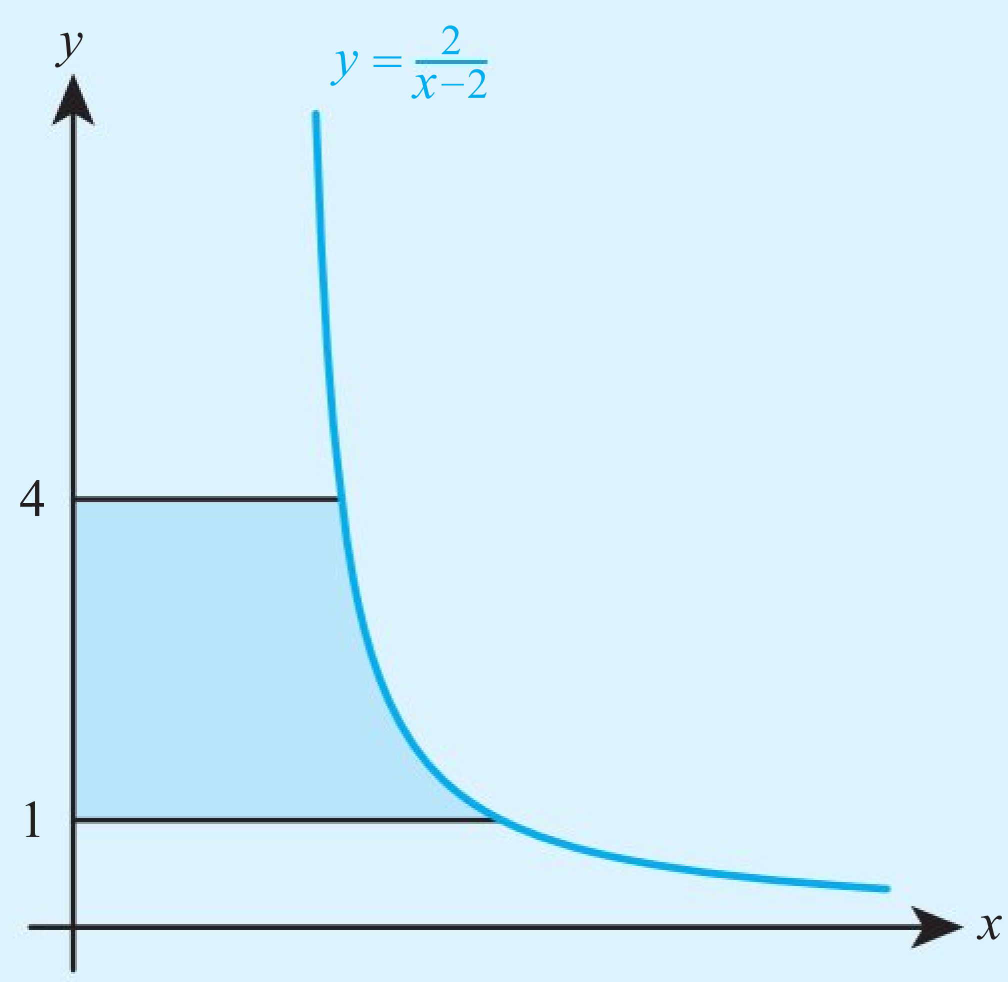
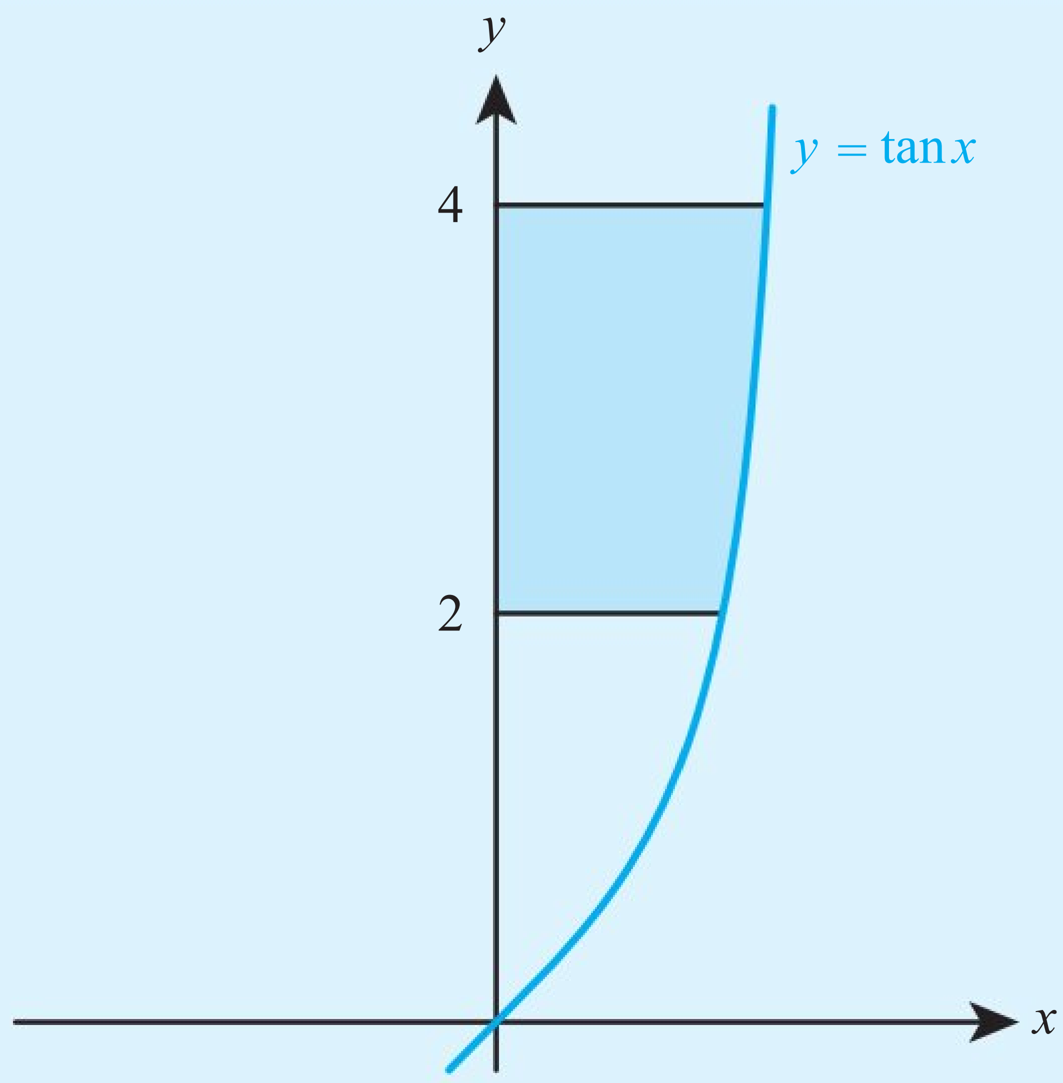


4 **a**



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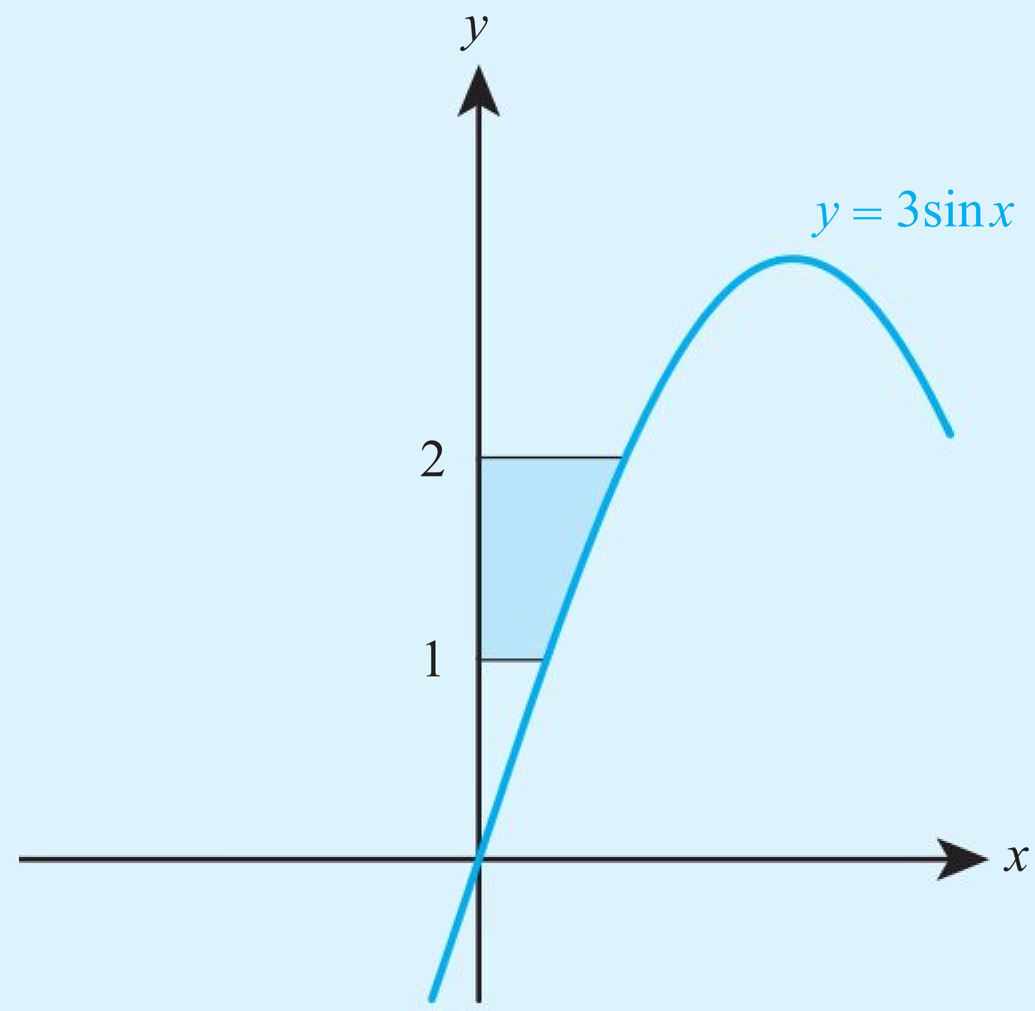


For questions 5 to 8, use the technique demonstrated in Worked Example 11.12 to find the volume of revolution formed when the given part of the curve is rotated 360° about the x-axis. Give your answers to three significant figures.

5 **a** $y = 3x^2$ between $x = 0$ and $x = 3$

b $y = 2x^3$ between $x = 0$ and $x = 2$

6 **a** $y = x^2 + 3$ between $x = 1$ and $x = 2$

b $y = x^2 - 1$ between $x = 2$ and $x = 4$

7 **a** $y = e^{2x}$ between $x = 0$ and $x = \ln 2$

b $y = e^{3x}$ between $x = 0$ and $x = \ln 2$

8 **a** $y = \dfrac{2}{x+1}$ between $x = 1$ and $x = 3$

b $y = \dfrac{3}{x+2}$ between $x = 0$ and $x = 2$

For questions 9 to 12, use the technique demonstrated in Worked Example 11.13 to find the volume of revolution formed when the given part of the curve $y = g(x)$, for $x < a < b$, is rotated 360° about the y-axis.

9 **a** $g(x) = 4x^2 + 1$, $a = 0$, $b = 2$

b $g(x) = \dfrac{x^2 - 1}{3}$, $a = 1$, $b = 4$

10 **a** $g(x) = \ln x + 1$, $a = 1$, $b = 3$

b $g(x) = \ln(2x - 1)$, $a = 1$, $b = 5$

11 **a** $g(x) = \cos x$, $a = 0$, $b = \dfrac{\pi}{2}$

b $g(x) = \tan x$, $a = 0$, $b = \dfrac{\pi}{4}$

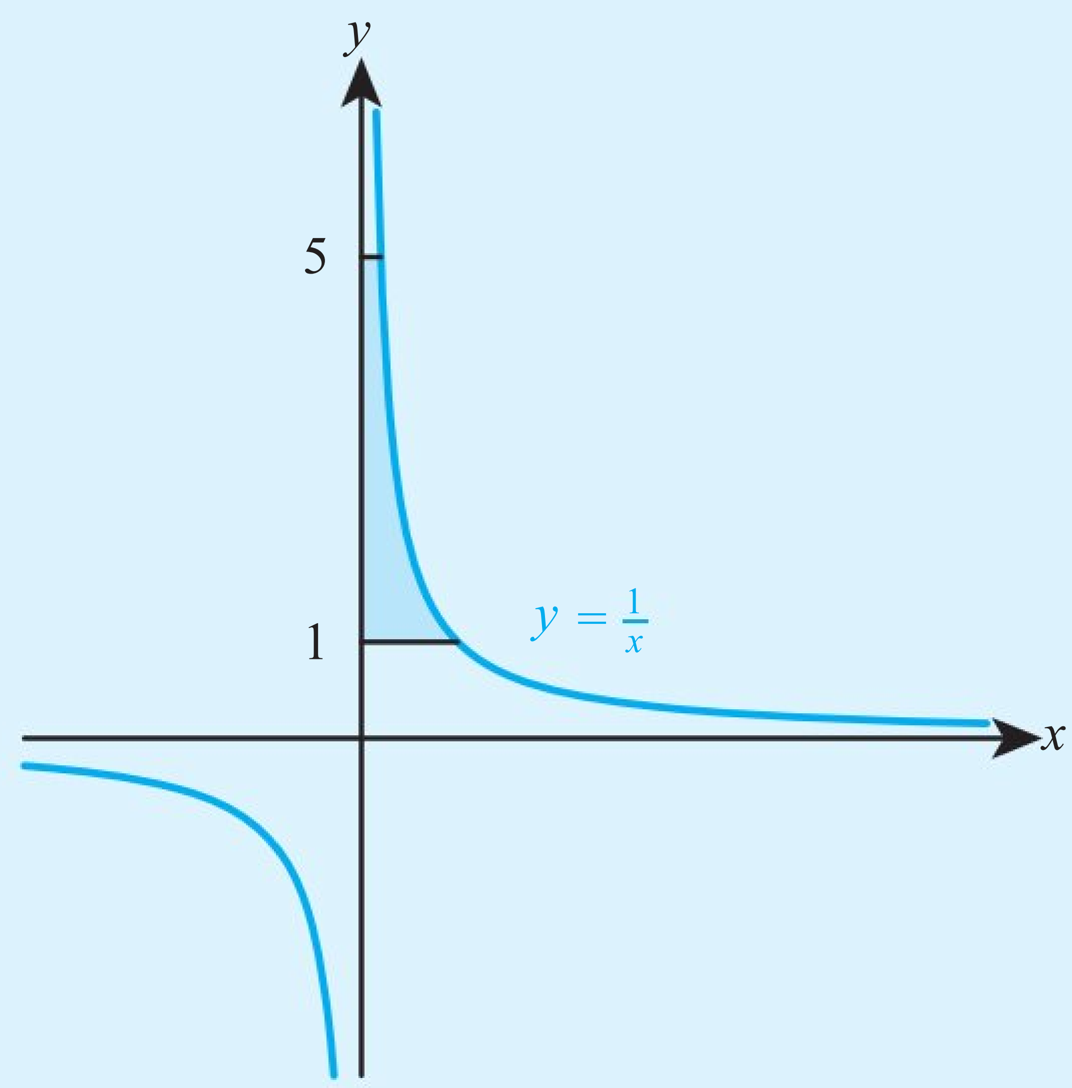
12 **a** $g(x) = \dfrac{1}{x-5}$, $a = 6$, $b = 8$

b $g(x) = \dfrac{1}{x-2}$, $a = 3$, $b = 8$

13 The shaded region in the diagram is bounded by the curve $y = \dfrac{1}{x}$, the y-axis and the lines $y = 1$ and $y = 5$.

a Find the area of the shaded region.

b Find the volume of the solid generated when the shaded region is rotated about the y-axis.



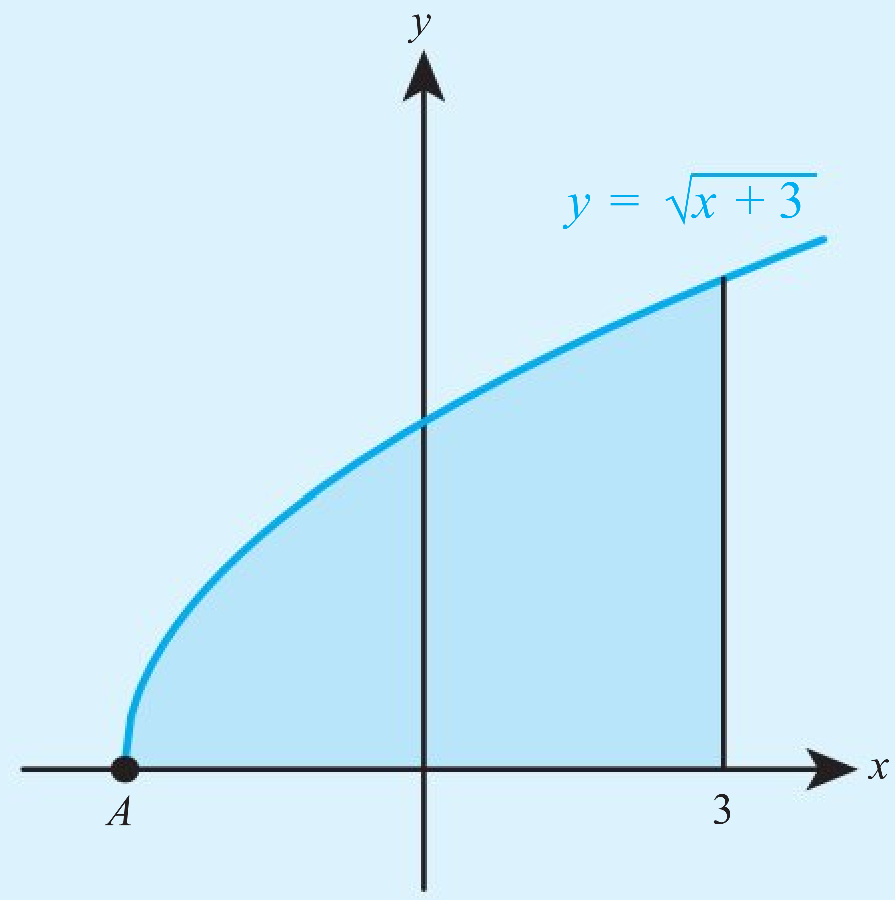
14 The part of the curve with equation $y = \frac{1}{x}$ between $x = 1$ and $x = a$ is rotated 360° about the x-axis. The volume of the resulting solid is $\frac{2\pi}{3}$. Find the value of a.

15 The part of the parabola $y = x^2$ between $x = 0$ and $x = a$ is rotated about the y-axis. The volume of the resulting solid is 8π. Find the value of a.

16 The diagram shows the curve with equation $y = \sqrt{x+3}$.

a Write down the x-coordinates of point A.

b The region bounded by the curve, the x-axis and the line $x = 3$ is rotated completely about the x-axis. Find the volume of the resulting solid.

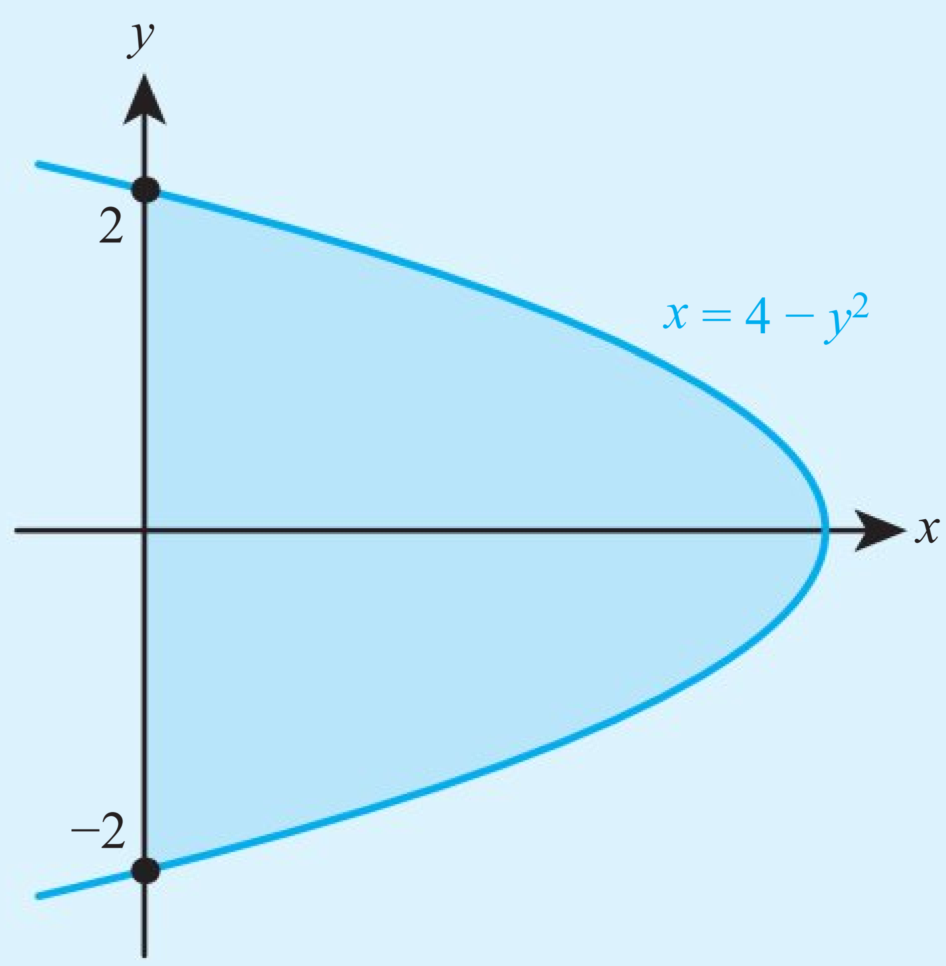


17 The diagram shows the region bounded by the y-axis and the curve with equation $x = 4 - y^2$.

Find

a the area of the region

b the volume of the solid generated when the region is rotated about the y-axis.

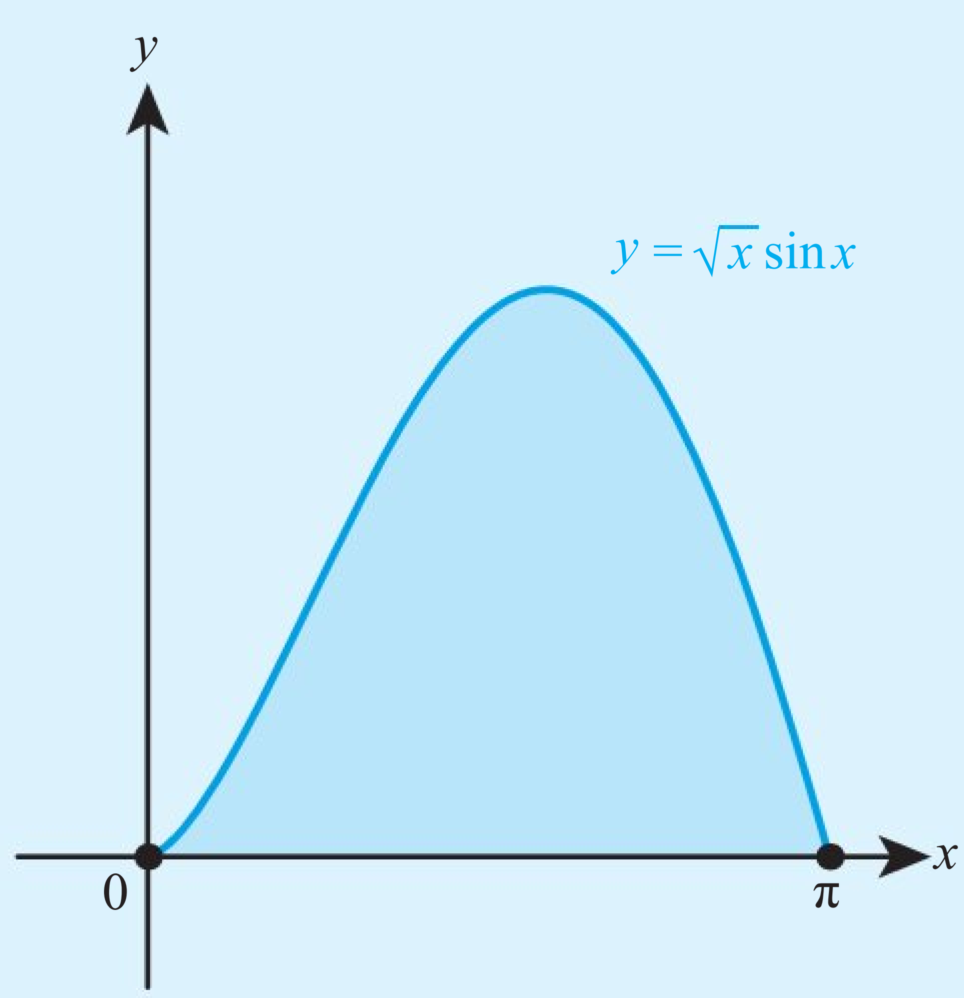


18 The diagram shows the region bounded by the curve $y = \sqrt{x}\sin x$ and the x-axis.

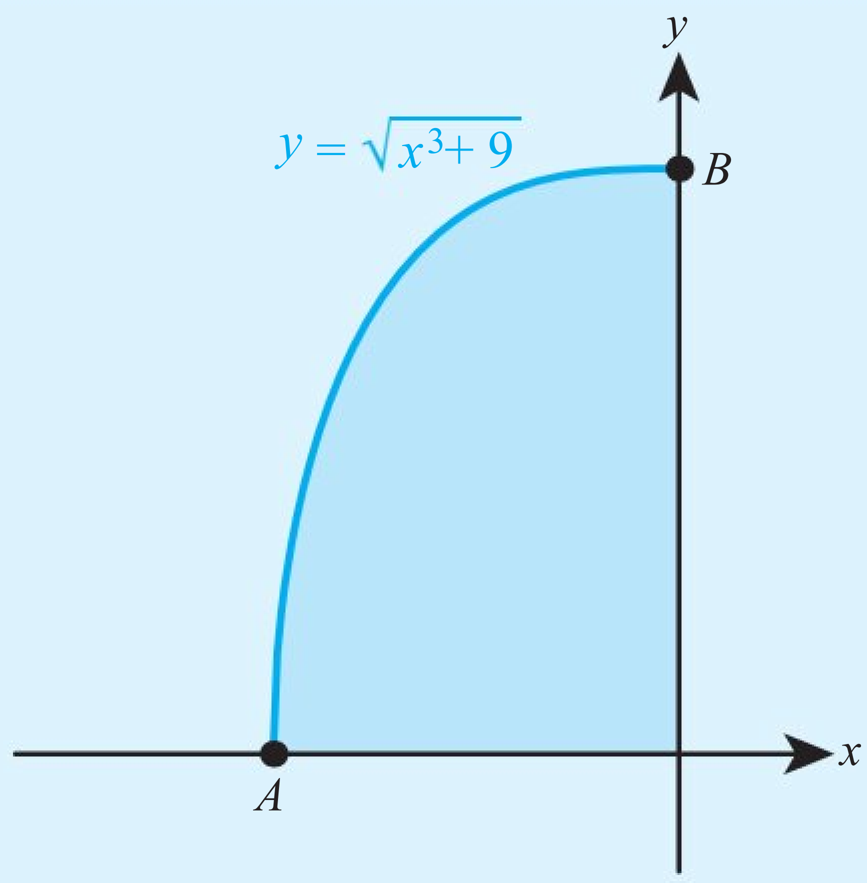
Find

a the area of the region

b the volume of the solid generated when the region is rotated through 2π radians about the x-axis.



19 The curve in the diagram has equation $y = \sqrt{x^3 + 9}$ and intersects the coordinate axes at the points $A\left(-\sqrt[3]{9}, 0\right)$ and $B(0, 3)$. Region R is bounded by the curve, the x-axis and the y-axis.



a Show that $x^2 = \sqrt[3]{(y^2 - 9)^2}$.

b Find the area of R.

c Find the volume of revolution generated when R is rotated fully about the

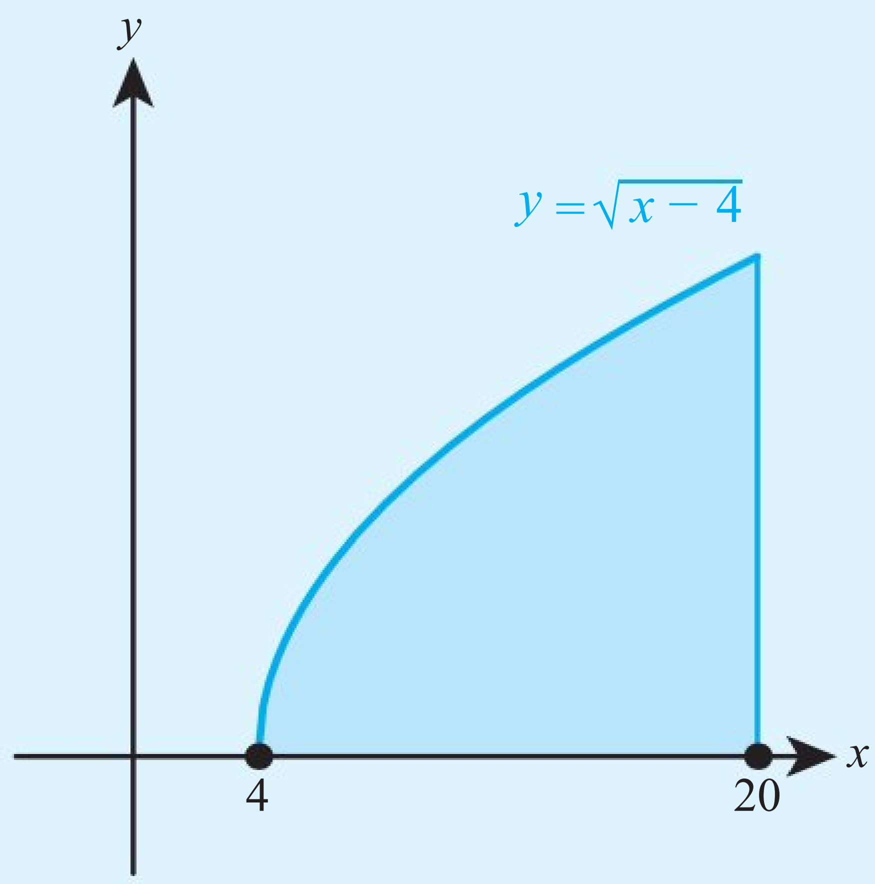
i x-axis

ii y-axis.

20 The part of the curve $y = \frac{1}{x}$, between $x = 1$ and $x = 3$, is rotated about the y-axis. Find the volume of the resulting solid.

21 The diagram shows the part of the curve $y = \sqrt{x - 4}$ between $x = 4$ and $x = 20$.

The shaded region bounded by the curve, the line $x = 20$ and the x-axis is rotated 360° about the y-axis. Find the resulting volume of revolution. Give your answer to the nearest integer.



22 The part of the curve $y = \sin x$ between $x = 0$ and $x = \pi$ is rotated around the x-axis. Find the exact value of the volume generated.

23 **a** Sketch the curve with equation $y = \sqrt{x}$.

b The part of the curve between $x = 0$ and $x = 9$ is rotated about the x-axis. Find the volume of the resulting solid.

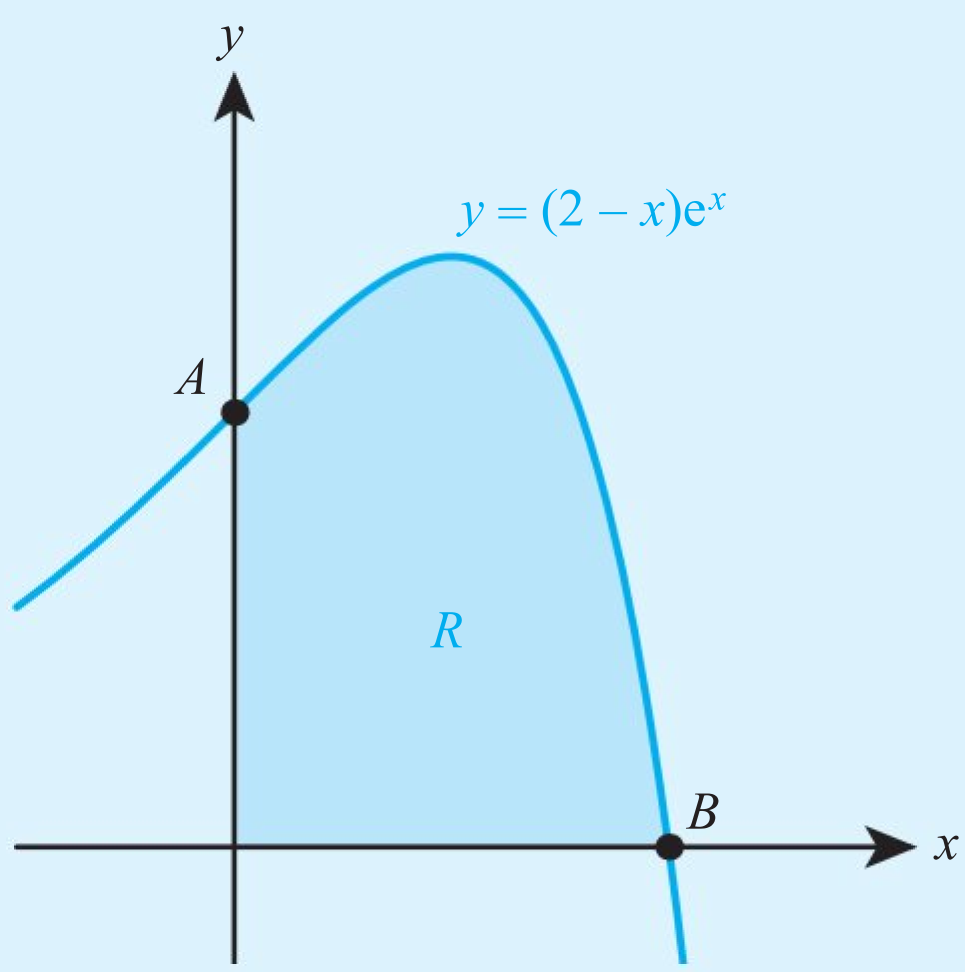
c Find the volume of the solid generated when the same part of the curve is rotated about the y-axis.

24 The diagram shows the graph of $y = (2 - x)e^x$. The region R is bounded by the curve and the coordinate axes.

a Find the coordinates of the points A and B.

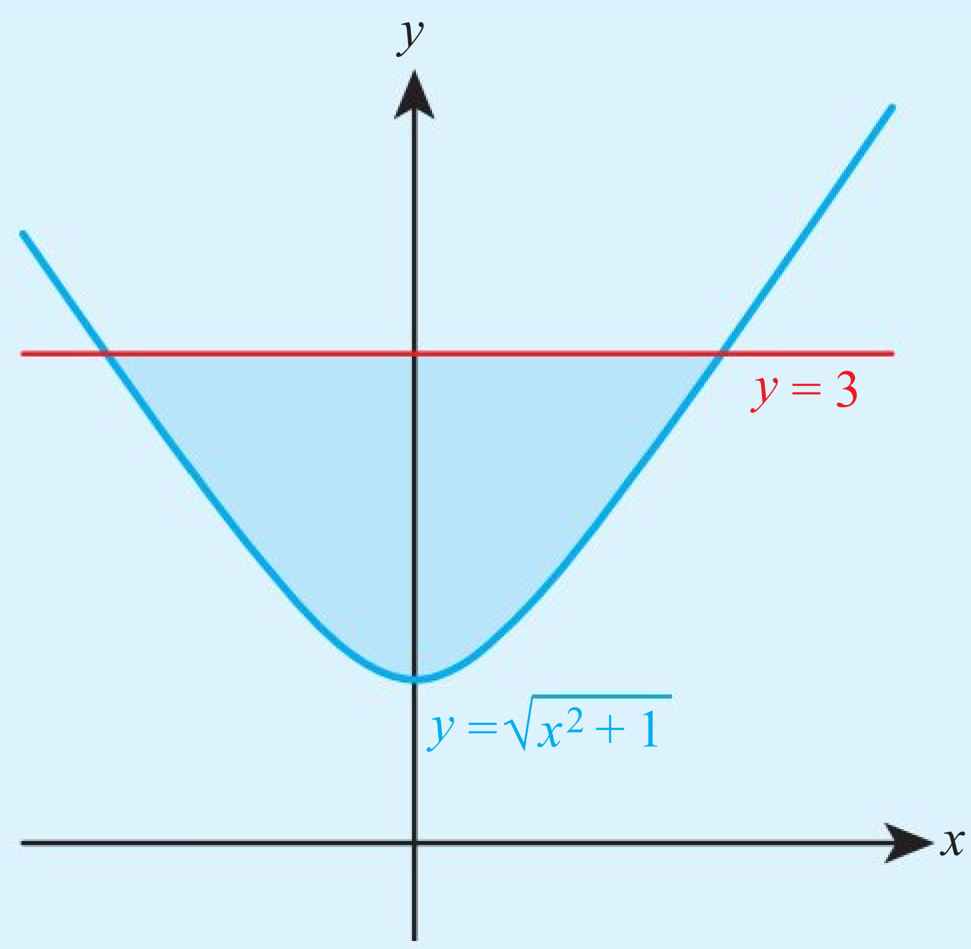
b Find the area of R.

c Find the volume of the solid generated when R is rotated 360° about the x-axis.



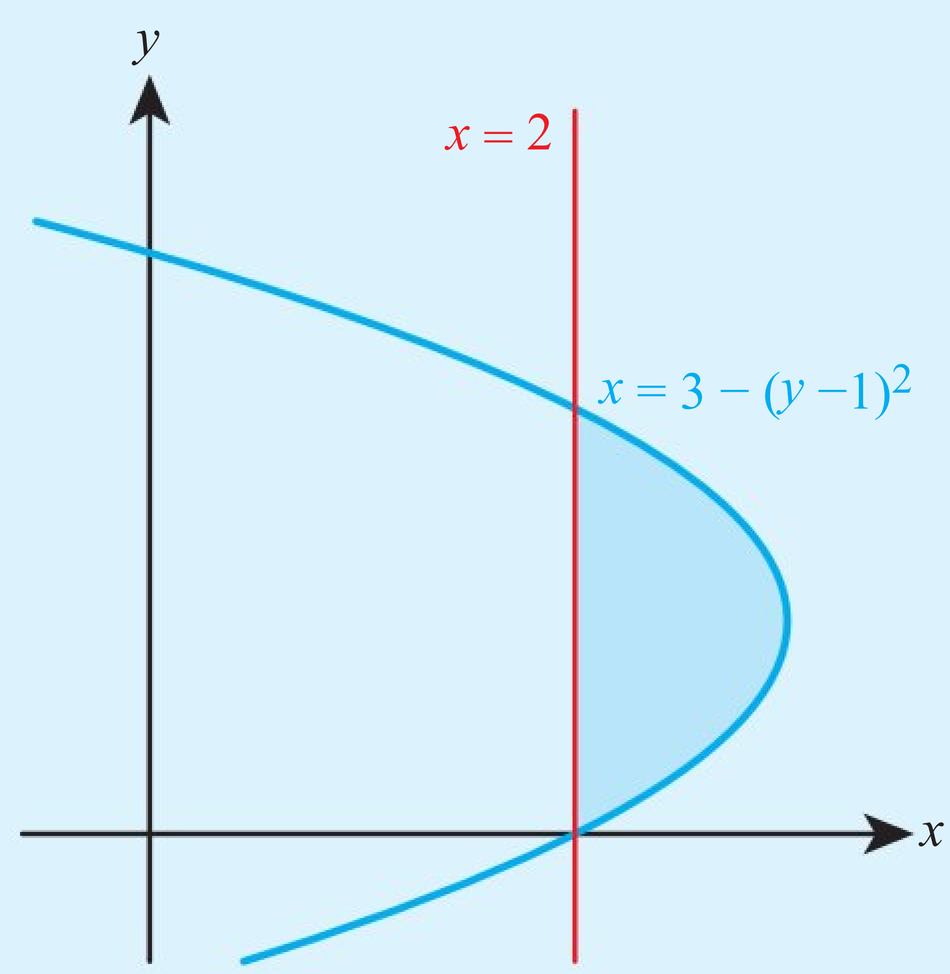
25 The region between the curve $y = \sqrt{x^2 + 1}$ and the line $y = 3$, shown in the diagram, is rotated fully about the y-axis.

Find the volume of the resulting solid, giving your answer as a multiple of π.



26 The diagram shows the region bounded by the curve with equation $x = 3 - (y - 1)^2$ and the line $x = 2$.

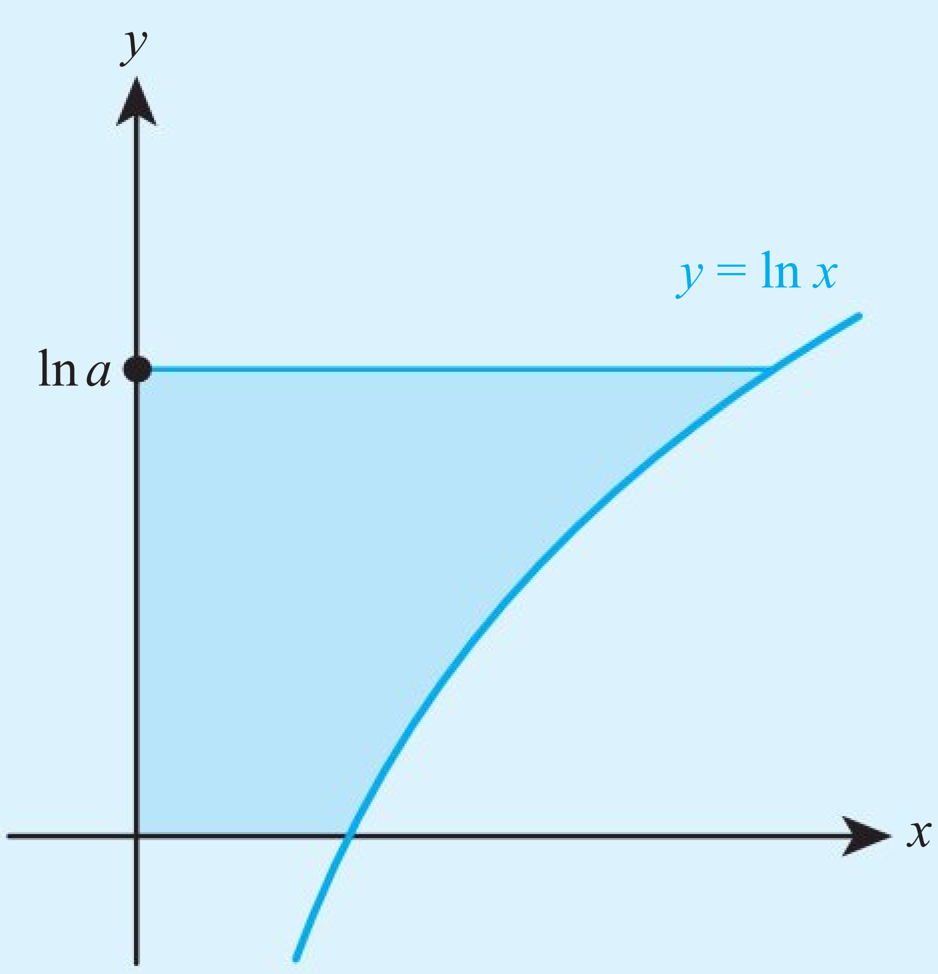
The region is rotated around the y-axis. Find the volume of the resulting solid, giving your answer as a multiple of π.



27 The diagram shows the region bounded by the graph of $y = \ln x$, the coordinates axes and the line $y = \ln a$.

a Find, in terms of a, the area of the shaded region.

b Hence show that $\int_1^a \ln x \, dx = a(\ln a - 1) + 1$.



28 **a** Find the equation of the line passing through the points $(r, 0)$ and $(0, h)$, giving your answer in the form $ax + by = c$.

b By considering the solid formed when a part of this line is rotated about the y-axis, show that the volume of a cone with radius r and height h is given by $\frac{1}{3}\pi r^2 h$.

29 **a** Write down the equation of a circle with centre at the origin and radius r.

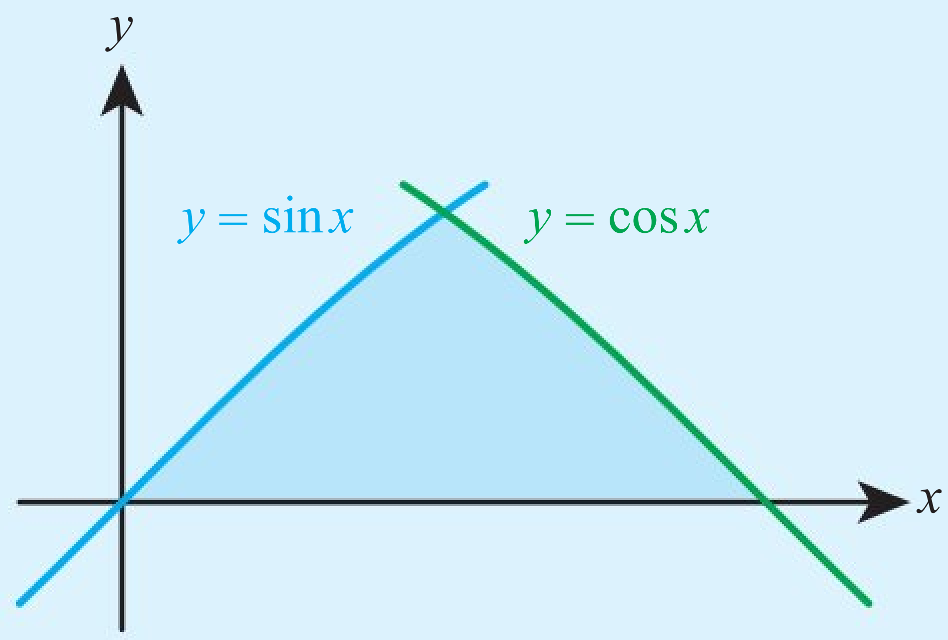
b By considering a suitable solid of revolution, prove that the volume of a sphere with radius r is $\frac{4}{3}\pi r^3$.

30 **a** Find the coordinates of the intersection points of the curves $y = x^2$ and $y = \sqrt{x}$.

b Find the volume of revolution generated when the region between the two curves is rotated about the x-axis.

31 The region shown in the diagram is bounded by the x-axis and the curves $y = \sin x$ and $y = \cos x$.

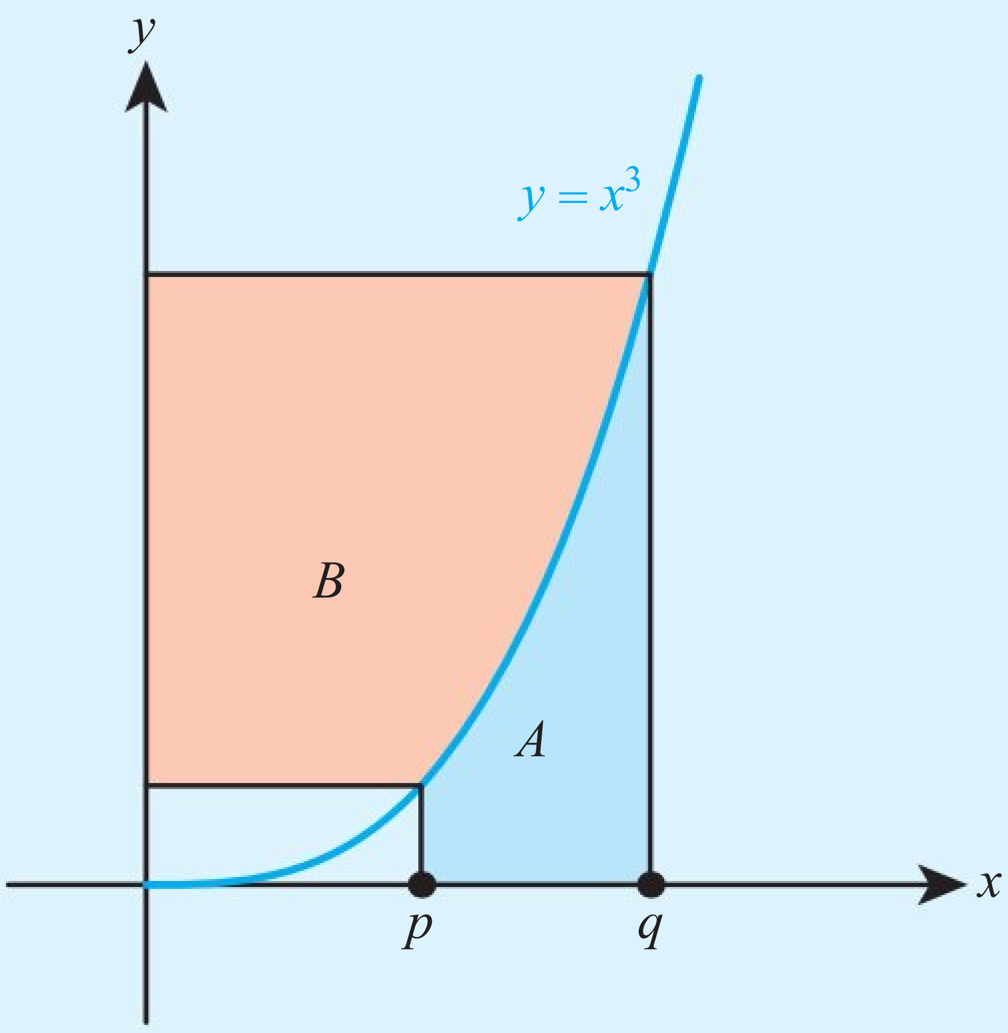
Find the exact volume of the solid generated when the region is rotated around the x-axis.



32 **a** Find the points of intersection of the curve $y = x^2$ and the line $y = 2x$.

b The region bounded by the line and the curve is rotated fully about the y-axis. Find the volume of the resulting solid.

33 The diagram shows the graph of $y = x^3$ and two regions, A and B.



Show that the ratio (area of A) : (area of B) is independent of p and q, and find the value of this ratio.

34 **a** The graph of $y = \ln x$ is translated 2 units to the right. Write down the equation of the resulting curve.

b Hence find the exact volume generated when the region bounded by $x = 1$, $y = 1$ and the curve $y = \ln x$, for $1 \leqslant x \leqslant e$, is rotated around the line $x = -2$.

35 **a** Sketch the graph of $y = \cos x$ for $-\pi \leqslant x \leqslant \pi$.

The curve from part **a** is rotated through 2π radians about the line $y = -1$.

b Show that the volume of the resulting solid is given by

$$\frac{1}{2}\pi \int_{-\pi}^{\pi} (\cos 2x + 4\cos x + 3)\,dx$$

c Find the exact value of the volume.

11D Integration by substitution

Note that this is an optional section. All the questions in this section can be done using Key Point 11.8, but some people find this method easier.

Instead of using Key Point 11.8, you can use a substitution to find integrals. This involves using a substitution $u = \mathrm{f}(x)$ to replace all instances of x and $\mathrm{d}x$ in the integral, turning it into one which is easier to evaluate.

Generally, if trying to integrate $\int k\mathrm{g}'(x)\,\mathrm{f}'(\mathrm{g}(x))\,\mathrm{d}x$, the substitution $u = \mathrm{g}(x)$ will work.

WORKED EXAMPLE 11.14

Evaluate $\int x\sin\left(x^2\right)\mathrm{d}x$ using the substitution $u = x^2$.

Use the given substitution to find $\frac{\mathrm{d}u}{\mathrm{d}x}$ and rearrange it to find $\mathrm{d}x$

If $u = x^2$, then $\frac{\mathrm{d}u}{\mathrm{d}x} = 2x$

So, $\mathrm{d}u = 2x\,\mathrm{d}x$

$$\mathrm{d}x = \frac{\mathrm{d}u}{2x}$$

Replace all instances of x^2 and $\mathrm{d}x$

$$\int x\sin\left(x^2\right)\mathrm{d}x = \int x\sin(u)\frac{\mathrm{d}u}{2x}$$

The factor of x which was causing problems can now be cancelled, leaving something which can be integrated

$$= \int \frac{\sin u}{2}\,\mathrm{d}u$$

Perform the integration and then replace u using the substitution

$$= -\frac{1}{2}\cos u + c$$

$$= -\frac{1}{2}\cos\left(x^2\right) + c$$

TOK Links

Technically, $\frac{\mathrm{d}u}{\mathrm{d}x}$ is not a fraction, so splitting it up as shown in Worked Example 11.14 is frowned upon in formal mathematics (although it will work for all the integrals you will meet in this course). Is analogy a valid way of arguing in mathematics?

Exercise 11D

For questions 1 to 5, use the method demonstrated in Worked Example 11.14 to evaluate the given integral using the given substitution.

1 a $\int e^{2x}\,dx,\ u = 2x$

b $\int \cos(5x)\,dx,\ u = 5x$

2 a $\int \frac{1}{2-5x}\,dx,\ u = 2-5x$

b $\int e^{1+4x}\,dx,\ u = 1+4x$

3 a $\int \frac{e^x}{1+e^x}\,dx,\ u = 1+e^x$

b $\int \frac{x^2}{1+x^3}\,dx,\ u = 1+x^3$

4 a $\int x\sqrt{x^2+2}\,dx,\ u = x^2+2$

b $\int \cos x\sqrt{\sin x}\,dx,\ u = \sin x$

5 a $\int \frac{\sin\sqrt{x}}{\sqrt{x}}\,dx,\ u = \sqrt{x}$

b $\int \frac{\cos(\ln x)}{x}\,dx,\ u = \ln x$

For questions 6 to 10, use a substitution to evaluate the given integral.

6 a $\int \sin(4x)\,dx$

b $\int e^{3x}\,dx$

7 a $\int \frac{1}{3+4x}\,dx$

b $\int \frac{1}{2-x}\,dx$

8 a $\int x^2 e^{(x^3)}\,dx$

b $\int x^4\cos x^5\,dx$

9 a $\int \frac{x}{1+3x^2}\,dx$

b $\int \frac{\cos x}{1+\sin x}\,dx$

10 a $\int \sin x\cos x\,dx$

b $\int \tan x\,dx$

11 Use the substitution $u = x+1$ to integrate

$$\int \sqrt{x^3+x^2}\,dx$$

for $x > 0$.

12 Use the substitution $x = \sin u$ to evaluate

$$\int \frac{1}{\sqrt{1-x^2}}\,dx$$

Checklist

- You should be able to integrate functions such as $\sqrt{x}$, $\sin x$ and e^x:
  - $\int x^n \, dx = \frac{x^{n+1}}{n+1} + c$ for $n \neq -1$
  - $\int \cos x \, dx = \sin x + c$
  - $\int \sin x \, dx = -\cos x + c$
  - $\int e^x \, dx = e^x + c$
  - $\int \frac{1}{x} \, dx = \ln|x| + c$
- You should be able to integrate by inspection or substitution:
  - If $\int f(x) \, dx = F(x)$, then $\int f(ax+b) \, dx = \frac{1}{a} F(ax+b)$.
  - $\int g'(x) f'(g(x)) \, dx = f(g(x)) + c$
- You should be able to find definite integrals:
  - $\int_a^b g'(x) \, dx = [g(x)]_a^b = g(b) - g(a)$
- You should be able to link definite integrals to areas between a curve and the x-axis:
  - Area $= \int_a^b |f(x)| \, dx$
- You should be able to find the area bounded by a curve $x = g(y)$, the y-axis and the lines $y = c$ and $y = d$:
  - $\int_c^d g(y) \, dy$
- You should be able to find volumes of revolution:
  - The volume of revolution formed when the part of the curve $y = f(x)$, between $x = a$ and $x = c$, is rotated around the x-axis is given by $V = \int_a^b \pi y^2 \, dx$.
  - The volume of revolution formed when the part of the curve $y = f(x)$, between $y = c$ and $y = d$, is rotated around the y-axis is given by $V = \int_c^d \pi x^2 \, dy$.

Mixed Practice

1 Evaluate $\int_0^a 3x^2 - 4\,\mathrm{d}x$.

2 Given that $\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{1}{4\sqrt{x}}$ and that $y = 3$ when $x = 4$, find y in terms of x.

3 Find the area between the graph of $y = \cos x - \sin x$, the x-axis and the lines $x = 0$ and $x = \frac{\pi}{6}$.

4 Find $\int \frac{3x-2}{4x^2}\,\mathrm{d}x$.

5 The part of the graph of $y = \ln x$ between $x = 1$ and $x = 2\mathrm{e}$ is rotated $360°$ around the x-axis. Find the volume generated.

6 The part of the graph $y = \cos x$ between $x = 0$ and $x = \frac{\pi}{2}$ is rotated about the x-axis. Find the resulting volume of revolution.

7 Find the area enclosed by the line $y = \sqrt{x}$, the y axis and the lines $y = 1$ and $y = 2$.

8 Let $\mathrm{f}(x) = \int \frac{12}{2x-5}\,\mathrm{d}x$, for $x \geqslant \frac{5}{2}$. The graph of f passes through (4, 0).

Find $\mathrm{f}(x)$.

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9 Find $\int \frac{3x^2-2}{x\sqrt{x}}\,\mathrm{d}x$.

10 A curve has gradient $3\cos x \sin^2 x$ and passes through the point $(\pi, 2)$. Find the equation of the curve.

11 **a** Sketch the graph of $y = \cos x$ for $0 \leqslant x \leqslant \frac{\pi}{2}$.

b The graph intersects the y-axis at the point A and the x-axis at the point B. Write down the coordinates of A and B.

c Find the equation of the straight line which passes through the points A and B.

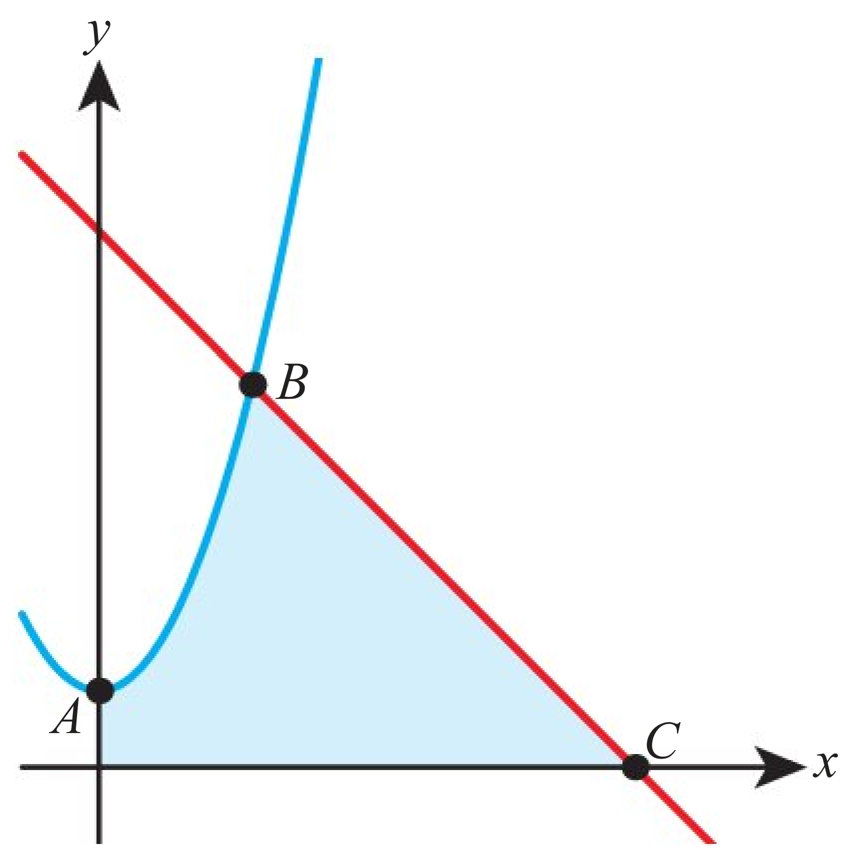
d Calculate the exact area enclosed by the line and the curve between intersections A and B.

12 Function f is defined on the domain $0 \leqslant x \leqslant 2$ by $\mathrm{f}(x) = x^2 - kx$. On the graph of $y = \mathrm{f}(x)$, the area below the x-axis equals the area above the x-axis. Find the value of k.

13 The diagram shows the graphs of $y = x^2 + 1$ and $y = 7 - x$.

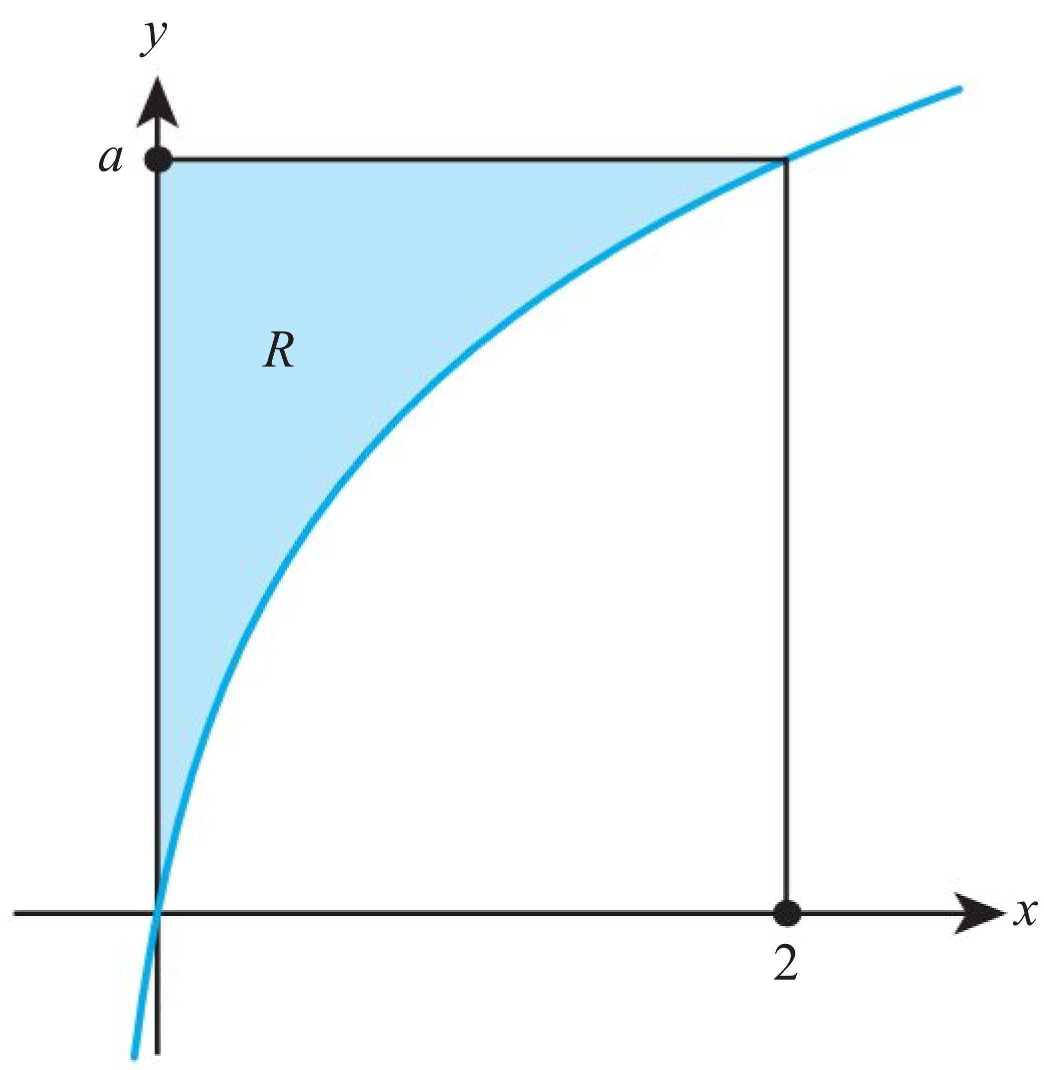
a Find the coordinates of the points A, B and C.

b Find the area of the shaded region.



14 The diagram shows the region R bounded by the curve $y = \ln(5x+1)$, the y-axis and the line $y = a$.

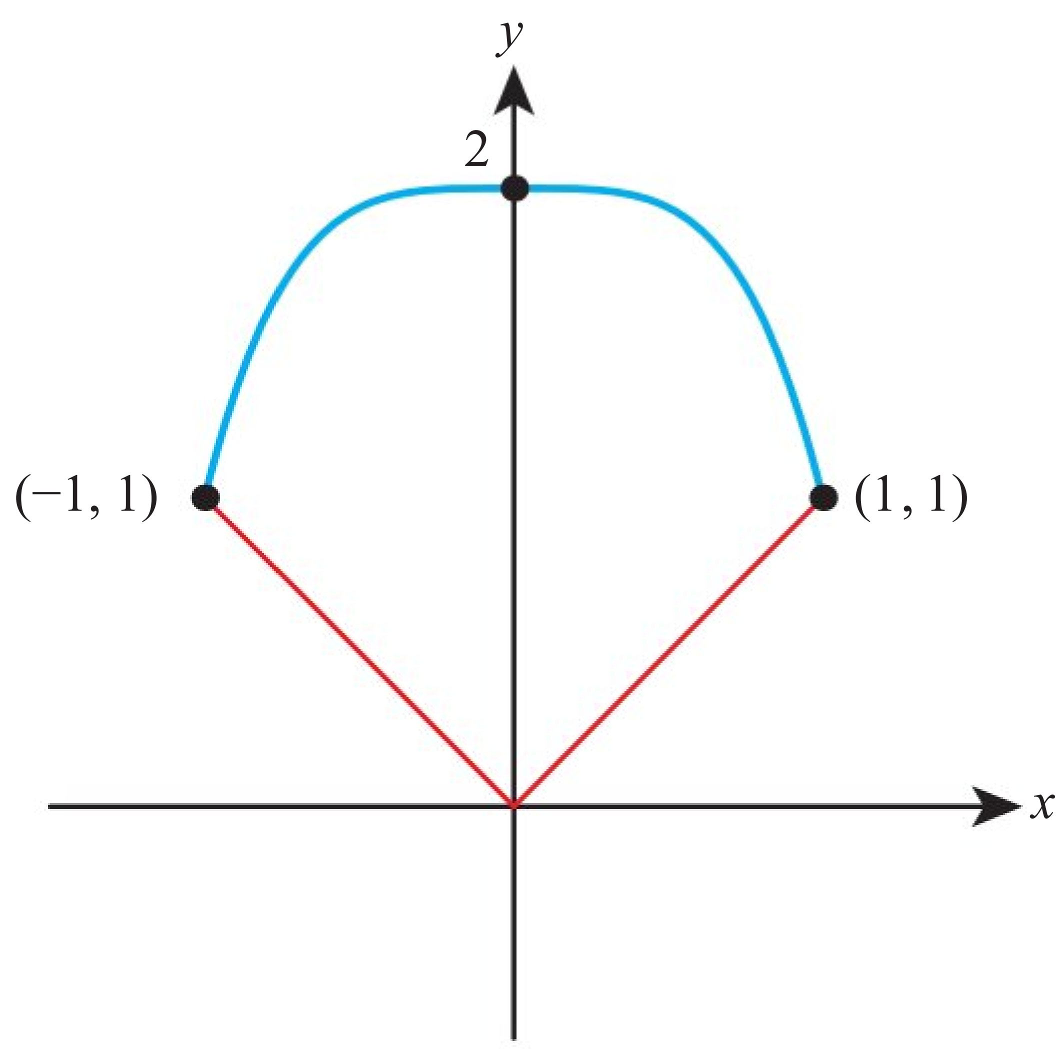
a Write down the exact value of a.
b Find the area of R.
c The region R is rotated fully about the y-axis. Find the volume of the resulting solid.



15 The part of the curve $y = \ln(x^2)$ between $x = 1$ and $x = e^2$ is rotated 360° around the y-axis. Find the exact value of the resulting volume of revolution.

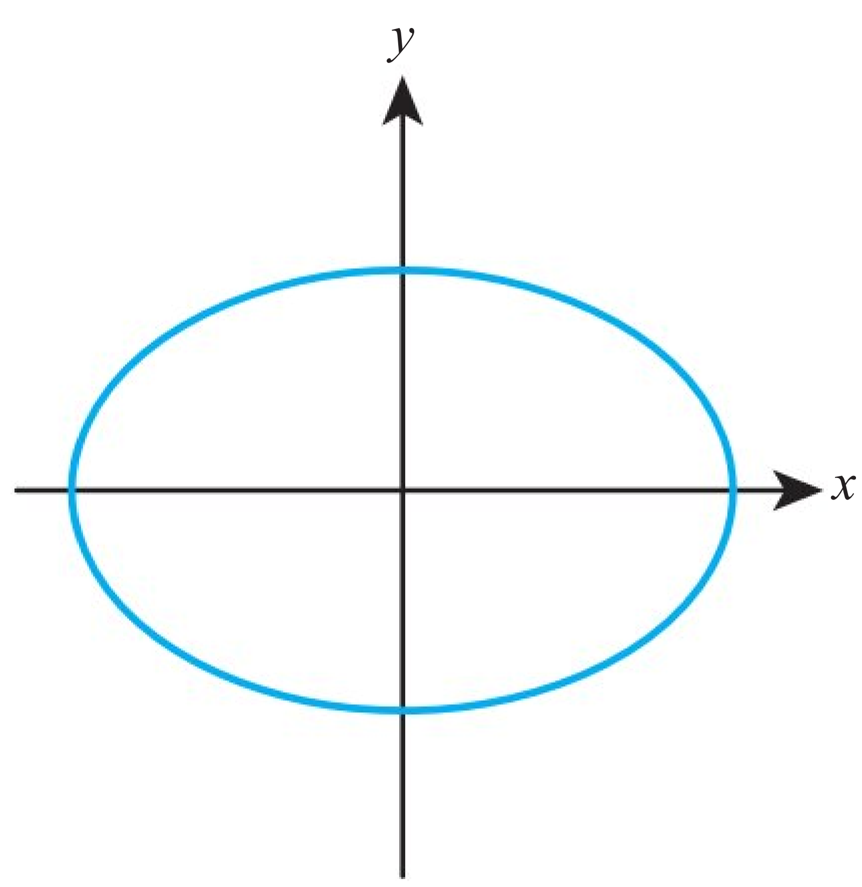
16 The diagram shows the region bounded by the graphs of $y = 2 - x^4$ and $y = |x|$.

The region is rotated about the y-axis to form a solid of revolution. Find the volume of the solid.



17 The diagram shows an ellipse with equation $4x^2 + 9y^2 = 36$.

Show that the volume generated when the ellipse is rotated around the x-axis is not the same as the volume generated when the ellipse is rotated around the y-axis.



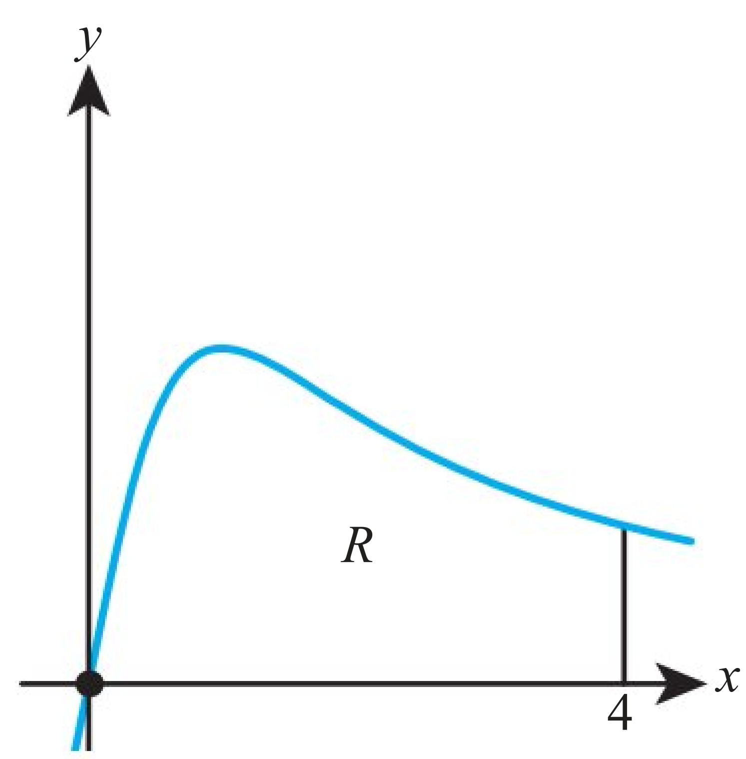
18 **a** Find an expression for the volume when the curve $y = \frac{1}{\cos x}$, for $0 < x < a < \frac{\pi}{2}$, is rotated 2π radians around the x-axis.

b Given that this volume equals π, find the value of a.

19 The following diagram shows the graph of $f(x) = \frac{x}{x^2+1}$, for $0 \leqslant x \leqslant 4$, and the line $x = 4$.

Let R be the region enclosed by the graph of f, the x-axis and the line $x = 4$.

Find the exact area of R.



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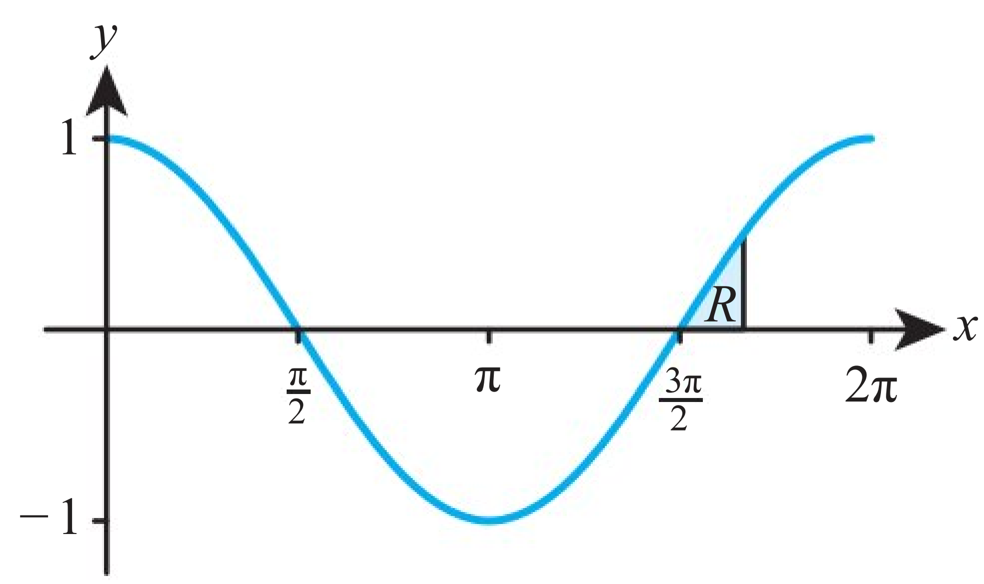
20 Let $f(x) = \cos x$, for $0 \leqslant x \leqslant 2\pi$. The following diagram shows the graph of f.

There are x-intercepts at $x = \frac{\pi}{2}, \frac{3\pi}{2}$.

The shaded region R is enclosed by the graph of f, the line $x = b$, where $b > \frac{3\pi}{2}$, and the x-axis.

The area of R is $\left(1 - \frac{\sqrt{3}}{2}\right)$. Find the value of b.



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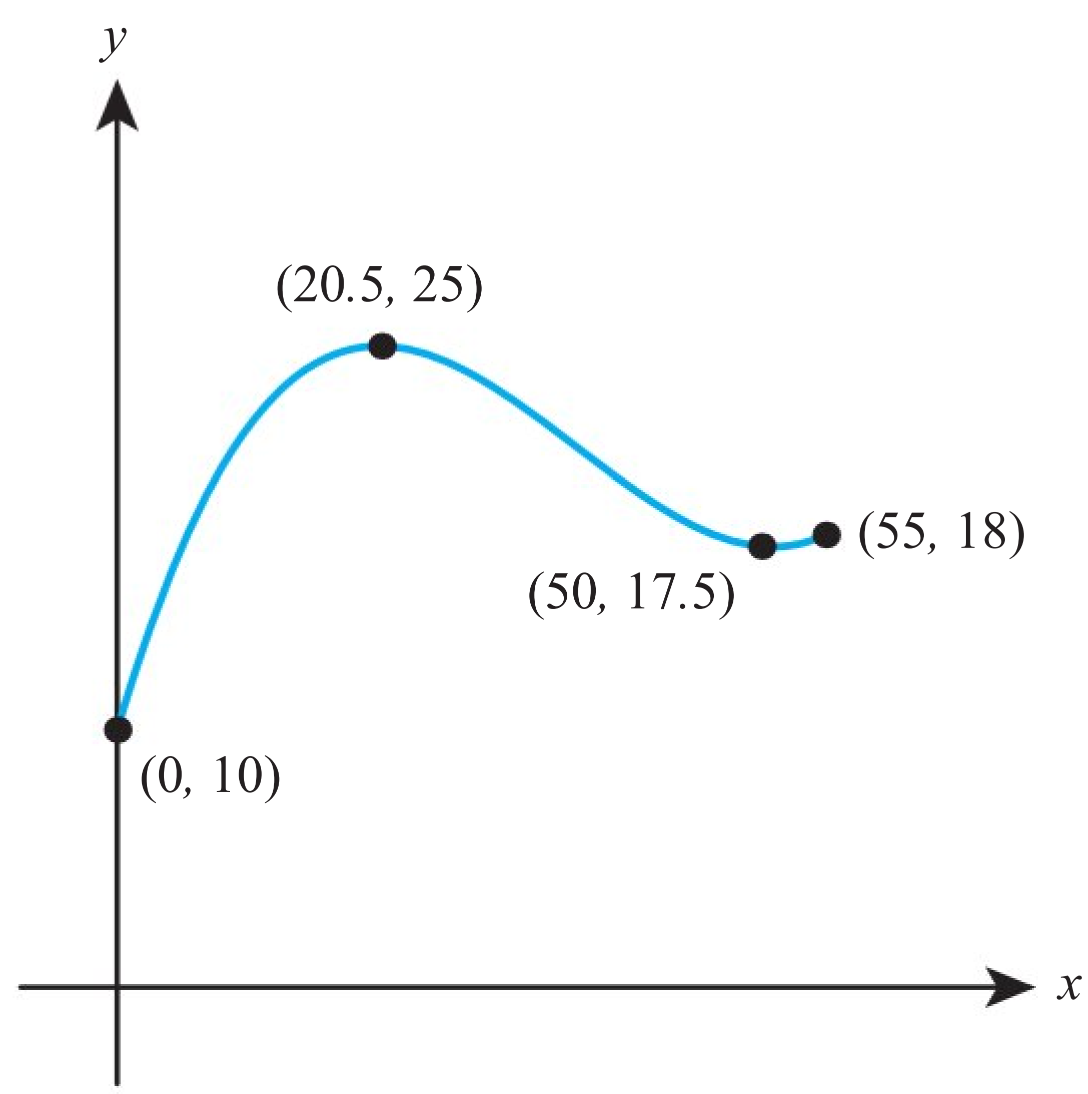
21 A large vase can be modelled by a solid of revolution formed when the cubic curve, shown in the diagram, is rotated about the x-axis. The units of length are centimetres.

a Find the equation of the curve in the form $y = ax^3 + bx^2 + cx + d$.

b Hence find the volume of the vase in litres.



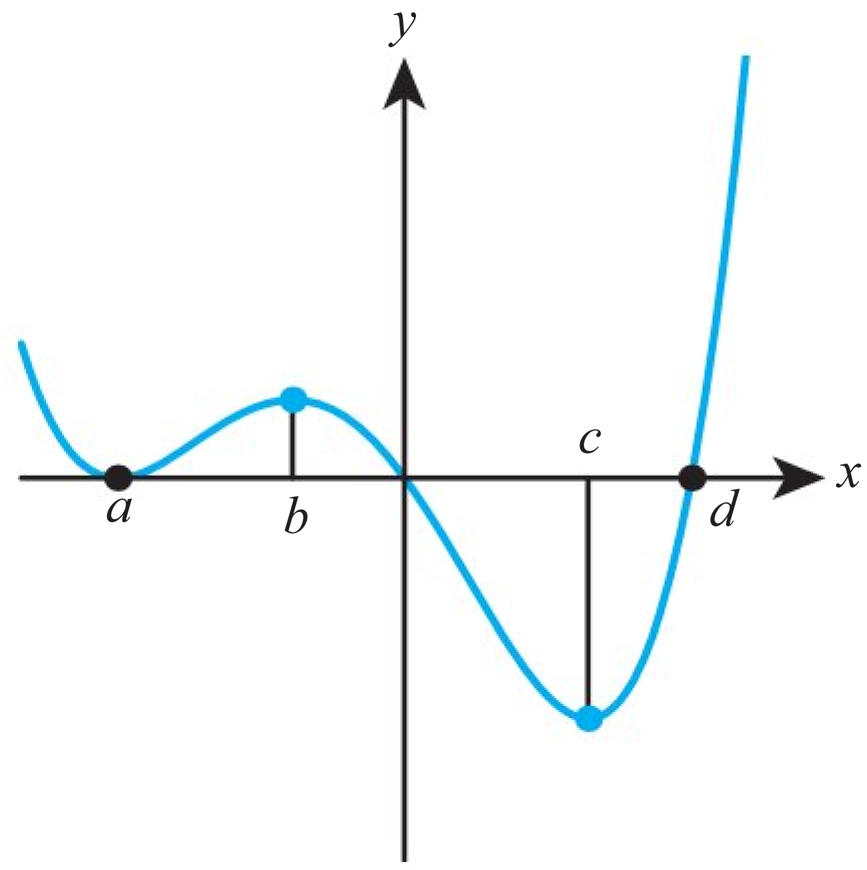
22 Given that $\int_2^5 \mathrm{f}(x)\,\mathrm{d}x = 3$, find $\int_5^8 2\mathrm{f}(x-3)\,\mathrm{d}x$.

23 The diagram shows the graph of the derivative, $y = \mathrm{f}'(x)$, of a function f.

a Find the range of values of x for which f is decreasing.

b Find the set of values of x for which f is concave-up.

c The total area enclosed by the graph of $y = \mathrm{f}'(x)$ and the x-axis is 20. Given that $\mathrm{f}(a) = 8$ and $\mathrm{f}(d) = 2$, find the value of $\mathrm{f}(0)$.



24 **a** Find the equation of the line passing through points $(a, 0)$ and (b, h), where $a, b, h > 0$.

b The part of this line, between $y = 0$ and $y = h$, is rotated about the y-axis. Prove that the volume of the resulting solid is $\frac{\pi h}{3}(a^2 + ab + b^2)$.

25 A parabola has equation $y = r^2 - x^2$ and a circle has radius r and centre at the origin.

a Sketch the parts of both curves with $y \geqslant 0$.

b When those parts of the curves are rotated about the y-axis, the resulting volumes are equal. Find the value of r.

26 **a** Find the area between the y-axis, the curve $y = \ln x$ and the lines $y = 0$ and $y = \ln a$.

b Hence evaluate $\int_1^a \ln x\,\mathrm{d}x$.

12 Kinematics

ESSENTIAL UNDERSTANDINGS

- Understanding rates of change allows us to model, interpret and analyse real-world problems and situations.
- Geometry and trigonometry allow us to quantify the physical world, enhancing our spatial awareness in two and three dimensions.
- This branch provides us with tools for analysis, measurement and transformation of quantities, movements and relationships.

In this chapter you will learn…

- how to apply calculus to kinematics, linking displacement, velocity, acceleration and distance travelled
- how to extend those ideas to describe motion in two dimensions, combining calculus with vectors.

CONCEPTS

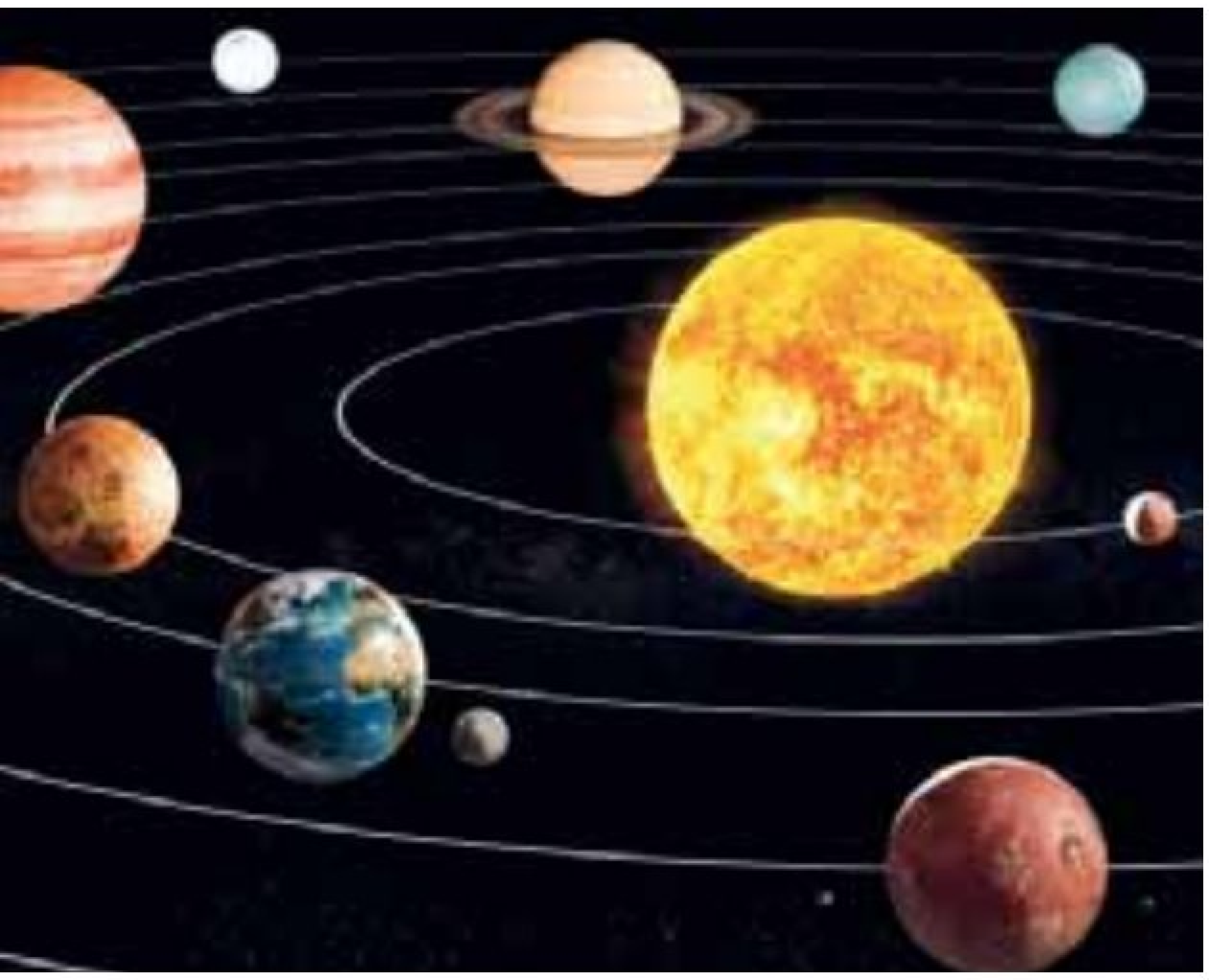
The following concepts will be addressed in this chapter:

- Kinematics allows us to describe the motion and direction of objects in closed **systems** in terms of displacement, velocity and acceleration in the physical world.
- Vectors allow us to determine position, **change** of position (movement) and force in two and three-dimensional **space**.

LEARNER PROFILE – Knowledgeable

What are the links between mathematics and other subjects? Most people see mathematics used in science, but did you know that in Ancient Greece music was considered a branch of mathematics, just like geometry or statistics are now. At the highest levels, philosophy and mathematics are increasingly intertwined. See if you can find any surprising applications of mathematics in some of your other favourite subjects.

Figure 12.1 How can we model the trajectories in each picture?



PRIOR KNOWLEDGE

Before starting this chapter, you should already be able to complete the following:

1. Differentiate with respect to t
   a. e^{-3t}
   b. $\sin\left(2t - \frac{\pi}{6}\right)$
2. If $x = \ln(2t)$, find $\frac{d^2x}{dt^2}$.
3. Find $\int (3t - \sin(3t))\, dt$.
4. Find the magnitude of the vector $\begin{pmatrix} -3 \\ 4 \end{pmatrix}$.
5. Find the angle that the vector $4\mathbf{i} + 7\mathbf{j}$ makes with the direction of $\mathbf{i}$.

Kinematics is the study of motion over time. It uses vectors to describe position, velocity and acceleration in two and three dimensions, and calculus to study how those quantities change with time. Combining the two sets of techniques provides us with powerful tools to describe and analyse motion.

Starter Activity

Look at the images in Figure 12.1. Which situations can be modelled by an object moving in a circle? What determines whether or not the path followed by an object is circular?

Now look at this problem:

For each of the situations in Figure 12.1:

a. For several points along the path, draw the displacement vector from a suitable origin.
b. Add the velocity and acceleration vectors at those points. What can you say about their directions?
c. For an object moving around a circle, draw velocity vectors at two nearby points and hence determine the direction of the acceleration vector.

Tip

If no reference point is mentioned, displacement is conventionally taken as relative to the initial position of an object.

In this section, we only look at motion in one dimension (along a straight line). In two or three dimensions, we need to use vectors and the symbol for the displacement will be **r** or **x**.

Tip

An alternative notation which you should be aware of has displacement called x and then velocity is $\dot{x}$ and the acceleration is $\ddot{x}$.

You are the Researcher

The derivative of acceleration with respect to time is called jerk. The derivative of jerk with respect to time is called jounce. You might like to find out more about how these quantities are used in physical applications, such as in the design of roller coasters.

12A Derivatives and integrals in kinematics

The basic quantity studied in kinematics is **displacement**. Displacement is the position of an object relative to a reference point and it is conventionally given the symbol s. Time is given the symbol t.

The letter 's' is used because it comes from the German word 'Strecke', which translates as 'route'. For much of the nineteenth and early twentieth centuries, Germany was at the heart of the development of mathematics and physics, so many terms studied in that era have names originating from German.

Derivatives in kinematics

The rate of change of displacement with respect to time is called **velocity**. It is given the symbol v. Based on this definition we can write the following.

KEY POINT 12.1

$$v = \frac{ds}{dt}$$

The rate of change of velocity with respect to time is called **acceleration**. It is given the symbol a.

KEY POINT 12.2

$$a = \frac{dv}{dt} = \frac{d^2s}{dt^2}$$

The dot and double dot notation mentioned in the tip is due to Isaac Newton, an English mathematician. He was very influential in applying calculus to mechanics, so his notation is often used in physics. See if you can find out about the notations used by Gottfried Leibniz (a German mathematician), Joseph Lagrange (an Italian mathematician) and Leonhard Euler (a Swiss mathematician). Where are the different notations still used? Does mathematics transcend geographical boundaries?

CONCEPTS – SPACE

The quantity of displacement is often confused with distance. Imagine a footballer running from one end of the pitch to the other and back. Her total distance travelled is approximately 200 m, but her final displacement in **space** is zero. Her speed while running might always be $8\,\mathrm{m\,s^{-1}}$ but her velocity one way will be $8\,\mathrm{m\,s^{-1}}$ and the other way will be $-8\,\mathrm{m\,s^{-1}}$.

Another vital idea when dealing with these quantities is realising that you have freedom to choose many starting points. You can often pick $t = 0$ to be any convenient time.

The fact that you can freely choose $t = 0$ and $s = 0$ turns out to be remarkably powerful. The German mathematician Emily Noether proved in 1915 that these observations actually lead to the conservation of momentum and energy.

WORKED EXAMPLE 12.1

If $s = \sin 3t$, find an expression for a.

First find v using the rule for differentiating composite functions (the chain rule) $v = \frac{ds}{dt} = 3\cos 3t$

Differentiate again to find a $a = \frac{dv}{dt} = -9\sin 3t$

Key Point 12.1 works if you know s or v as a function of t. However, if you only know v as a function of s you can use the chain rule to find a:

$$a = \frac{dv}{dt} = \frac{ds}{dt} \times \frac{dv}{ds}$$

Since $v = \frac{ds}{dt}$, we can simplify this to:

$$a = v\frac{dv}{ds}$$

As an alternative to this, we can use the chain rule again:

$$\frac{d}{ds}\left(\frac{1}{2}v^2\right) = \frac{d}{dv}\left(\frac{1}{2}v^2\right) \times \frac{dv}{ds} = v\frac{dv}{ds}$$

KEY POINT 12.3

$$a = v\frac{dv}{ds} = \frac{d}{ds}\left(\frac{1}{2}v^2\right)$$

WORKED EXAMPLE 12.2

If $v = 1 + s^2$, find an expression for the acceleration in terms of s.

First find $\frac{dv}{ds}$ $\frac{dv}{ds} = 2s$

Substitute into Key Point 12.3 $a = v\frac{dv}{ds} = 2s + 2s^3$

Links to: Physics

If you multiply the formula in Key Point 12.3 through by the mass of the object, m, you get an interesting insight:

$$ma = \frac{d}{ds}\left(\frac{1}{2}mv^2\right)$$

This can be interpreted as meaning that force is the rate of change of energy with respect to position. This is a very fundamental way of looking at force in physics.

Integrals in kinematics

You can use the fact that integration reverses differentiation.

Tip

Remember that definite integrals are often best done on the calculator.

KEY POINT 12.4

- Change in velocity from $t = t_1$ to $t = t_2$ is given by $\int_{t_1}^{t_2} a \, dt$.
- Change in displacement from $t = t_1$ to $t = t_2$ is given by $\int_{t_1}^{t_2} v \, dt$.

WORKED EXAMPLE 12.3

An object moves in a straight line. When $t = 4$ seconds, $s = 1$ metre and for the next 5 seconds the velocity is given by $\frac{1}{\sqrt{t}}$ m s^{-1}. Show that the displacement when $t = 9$ is 3 metres.

We can use a definite integral from Key Point 12.4 to find the difference in displacement

Change in displacment $= \int_4^9 v \, dt$

We substitute in what we know about the velocity, putting it in a form which is useful for applying the rules of integration

$= \int_4^9 t^{-0.5} \, dt$

We can find the indefinite integral

$= \left[\frac{t^{0.5}}{0.5}\right]_4^9$

Before substituting in limits, it is worth simplifying

$= \left[2\sqrt{t}\right]_4^9$

$= 2\sqrt{9} - 2\sqrt{4}$

$= 2$

Remember to take into account the initial displacement

So, the final displacement is $1 + 2 = 3$ m.

The net displacement between $t = t_1$ and $t = t_2$ can include some forward and backward motion cancelling each other out. If we want to just find the distance travelled, we have to ignore the fact that some of the motion is forward and some is backward. We can do this by just considering the modulus of the velocity.

KEY POINT 12.5

The distance travelled between $t = t_1$ and $t = t_2$ is given by $\int_{t_1}^{t_2} |v| \, dt$.

The distance covered is analogous to the area between the velocity function and the x-axis, which is why we use a similar method to Key Point 11.10.

WORKED EXAMPLE 12.4

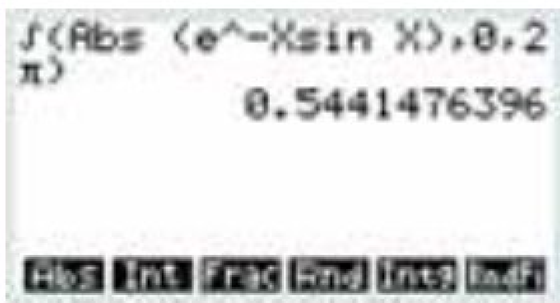
The motion of a particle is described by $v = e^{-t}\sin t$. Find the distance travelled in the first 2π seconds of the motion.

Write down an expression for the distance travelled using Key Point 12.5

$$\text{Distance} = \int_0^{2\pi} \left|e^{-t}\sin t\right| \, dt$$

Evaluate this integral on your GDC

$$= 0.544 \ (3 \text{ s.f.})$$



The distinction between speed and velocity

The speed is given by the modulus of the velocity.

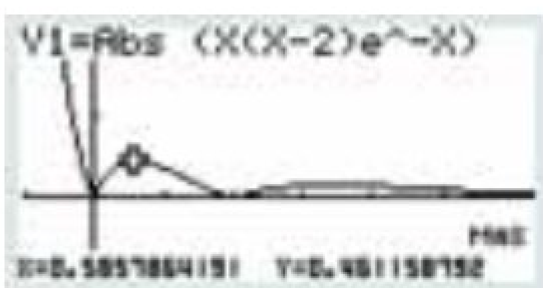
WORKED EXAMPLE 12.5

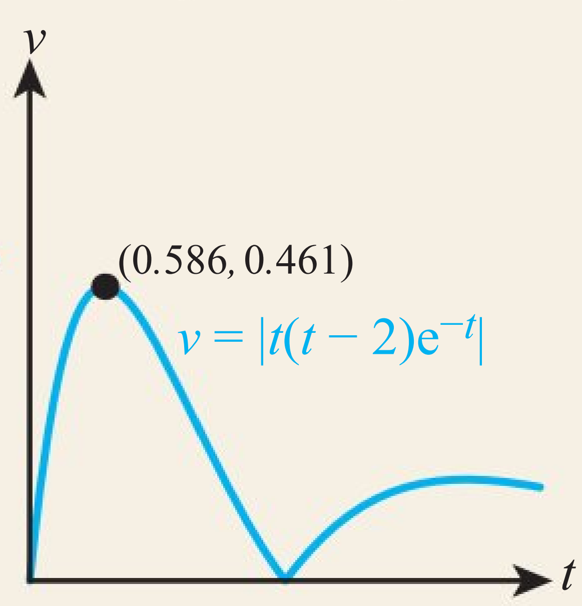
The velocity of a particle is given by $t(t-2)e^{-t}$ for $0 < t < 4$. Find its maximum speed.

Use the fact that speed is $|v|$

$$\text{Speed} = \left|t(t-2)e^{-t}\right|$$

Use your calculator to sketch this graph and find the maximum. You should show a sketch of this graph in your working





So, the maximum speed is 0.461

Distinguishing between speed and velocity becomes even more important in two and three dimensions, where velocity is a vector and speed is a scalar.

Exercise 12A

Note: In this exercise all displacements are in metres, times are in seconds and velocities are in metres per second.

For questions 1 to 4, use the method demonstrated in Worked Example 12.1 to find an expression for the velocity and acceleration.

1 **a** $s = t^3 + 3t^2 + 1$
b $s = t^4 - 5t + 2$

2 **a** $s = \cos 2t$
b $s = \sin \frac{t}{2}$

3 **a** $s = e^{3t}$
b $s = e^{-4t}$

4 **a** $s = \ln \frac{t}{3}$
b $s = \ln 2t$

For questions 5 and 6, use the method demonstrated in Worked Example 12.2 to find an expression for the acceleration.

5 **a** $v = s^3 - 1$
b $v = s^2 + s$

6 **a** $v = \frac{1}{s}$
b $v = e^{-s}$

For questions 7 and 8, use the method demonstrated in Worked Example 12.3 to find the change in displacement between the given times.

7 **a** $v = 4\sqrt[3]{t}$ between $t = 1$ and $t = 8$
b $v = 5\sqrt[4]{t}$ between $t = 1$ and $t = 16$

8 **a** $v = e^{\frac{t}{2}}$ between $t = 0$ and $t = \ln 9$
b $v = e^{-t}$ between $t = 0$ and $t = \ln 5$

For questions 9 and 10, use the information given in Key Point 12.4 to find the change in velocity between the given times.

9 **a** $a = 2\cos \frac{t}{3}$ between $t = 0$ and $t = \pi$
b $a = 3\sin 2t$ between $t = 0$ and $t = \frac{\pi}{2}$

10 **a** $a = \frac{1}{2t+1}$ between $t = 4$ and $t = 12$
b $a = \frac{1}{3t-2}$ between $t = 1$ and $t = 6$

For questions 11 to 13, use the method demonstrated in Worked Example 12.4 to find the distance travelled between the given times.

11 **a** $v = \frac{\sin t}{t}$ between $t = 1$ and $t = 5$
b $v = \frac{\cos t}{t}$ between $t = 1$ and $t = 6$

12 **a** $v = 2e^{-t^2} - 1$ between $t = 0$ and $t = 2$
b $v = 2e^{-t^3} - 1$ between $t = 0$ and $t = 1.5$

13 **a** $v = 5(\ln t)^3$ between $t = 0.5$ and $t = 2$
b $v = \frac{3\ln t}{t}$ between $t = 0.5$ and $t = 3$

For questions 14 and 15, use the method demonstrated in Worked Example 12.5 to find the maximum speed if:

14 **a** $v = 3e^{-t}\sin 2t$ between $t = 0$ and $t = 5$
b $v = 4e^{-t}\cos t$ between $t = 1$ and $t = 4$

15 **a** $v = \sqrt{t}(t-1)(t-2)$ between $t = 0$ and $t = 2$
b $v = \sqrt{t}(t-1)(t-3)$ between $t = 0$ and $t = 3$

16 The displacement of a particle, s, at time t is modelled by

$$s = 10t - t^2$$

a Find the displacement after 2 seconds.
b Find the velocity after 3 seconds.
c Find the acceleration after 4 seconds.

17 The velocity of a particle is given by $v = e^{-0.5t}$. Find the displacement of the particle after 3 seconds relative to its initial position.

18 The velocity of a particle is given by $v = 4 - x^2$.

a Find the displacement after 6 seconds.
b Find the distance travelled in the first 6 seconds.

19 The acceleration of a particle is modelled by $2t+1$. If the initial velocity is $3\,\mathrm{m\,s^{-1}}$, find the velocity after 4 seconds.

20 A bullet is fired through a viscous fluid. Its velocity is modelled by $v = 256 - t^4$ from $t = 0$ until it stops.

- **a** What is the initial speed of the bullet?
- **b** What is the acceleration after 2 seconds?
- **c** How long does it take to stop?
- **d** What is the distance travelled by the bullet?

21 Find the distance travelled by a particle in its first 2 seconds of travel if the velocity is modelled by $\dot{x} = e^t - 2$.

22 The displacement of a particle, s metres, at time t seconds is modelled by

$$s = 6t^2 - t^3 \text{ for } 0 \leqslant t \leqslant 6$$

- **a** Find the times at which the displacement is zero.
- **b** Find the time at which the velocity is zero.
- **c** Find the time at which the displacement is maximum.

23 The displacement of a particle is modelled by $s = t\sin t$ for $0 \leqslant t \leqslant 2\pi$.

- **a** Find an expression for v.
- **b** Hence find the initial velocity.
- **c** Find an exact expression for the velocity when $t = \frac{\pi}{4}$.
- **d** What is the initial acceleration?
- **e** At what times is the velocity equal to 1?
- **f** What is the total distance travelled by the particle?

24 If a particle has a constant acceleration, a, initial speed u and initial displacement zero, show that

- **a** $v = u + at$
- **b** $s = ut + \frac{1}{2}at^2$

25 If $x = \sin\omega t$, where ω is a constant, show that $\ddot{x} = -\omega^2 x$.

26 The velocity of a ball thrown off a cliff, v metres per second, after a time t seconds is modelled by $v = 5 - 10t$. The ball is initially 60 m above sea level.

- **a** What is the vertical acceleration of the ball?
- **b** When does the ball reach its maximum height above sea level? What is that maximum height?
- **c** When does the ball hit the sea?
- **d** What is the vertical distance travelled by the ball?
- **e** State one assumption being made in this model.

27 The velocity, in metres per second, of a bicycle t seconds after passing a time check point is modelled by $v = 18 - 2t^2$ for $0 < t < 6$.

- **a** What is the initial speed of the bicycle?
- **b** When is the velocity zero?
- **c** What is the acceleration of the bicycle after 2 seconds?
- **d** What is the distance travelled by the bicycle between $t = 0$ and $t = 6$?
- **e** Find the displacement after 6 seconds .
- **f** Without any further calculations, deduce the distance travelled in the first 3 seconds.

28 The displacement of a particle is modelled by $s = -t^3 + 6t^2 - 2t$. Find the maximum velocity of the particle.

29 A cyclist accelerates from rest. For the first two seconds his acceleration is modelled by $2t$. For the next two seconds his acceleration is modelled by $\frac{16}{t^2}$. Find the cyclists's velocity four seconds after starting.

30 Given that a particle travels with velocity $\frac{1}{t} - 1$, find the distance travelled between $t = 0.5$ and $t = 2$.

31 The velocity of a racing car as it travels along a long straight road is modelled by $v = 50 - 40e^{-0.02s}$, where s is the distance along the straight road.

a What is the car's speed at the beginning of the straight road?

b The straight road is 100 m long. What does the model predict is the car's speed at the end of the straight road?

c Find the acceleration of the car when $s = 10$.

32 The displacement of a particle, s metres after t seconds, is given by $s(t) = t(10 - t)$.

a Find the displacement after 10 seconds.

b Find the distance travelled in the first 10 seconds.

33 If $v = t(t-1)(t-4)$ for $0 \leqslant t \leqslant 4$ find

a the maximum speed

b the minimum speed

c the average speed.

34 a If a particle has acceleration $a = At$, initial velocity $u > 0$ and initial displacement zero, find expressions for

i its velocity in terms of A, t and u

ii its displacement in terms of A, t and u.

b Show that the average velocity up to time t is given by $\frac{v+2u}{3}$.

35 In a race, Jane starts 42 m ahead of Aisla. Jane has velocity given by $t + 2$. Aisla has velocity given by t^2. How long does it take Aisla to overtake Jane?

36 A juggler throws a ball vertically upward. The ball's speed is modelled by $v = 5 - 10t$.

Half a second later the juggler throws a second ball vertically upward in exactly the same way from the same position. At what height above the release position do the two balls collide?

37 The acceleration of a car is modelled by $a = s^3 + s$, where $s > 0$. Find the speed of the car as a function of s.

38 By considering kinematic quantities, prove that if a, b, c and d are positive and $\frac{a}{b} < \frac{c}{d}$, then $\frac{a}{b} < \frac{a+c}{b+d} < \frac{c}{d}$.

In Section 2C, you saw how to use vectors to describe motion with constant velocity in two and three dimensions.

12B Motion with variable velocity in two dimensions

In the previous section, you learnt about displacement, velocity and acceleration of an object moving in a straight line. But the real world has three dimensions so those quantities need to be described using vectors. The same rules about using differentiation and integration still apply.

We will use **r** to denote the displacement vector, **v** for the velocity vector and **a** for the acceleration vector. The displacement is usually measured from the origin, so **r** is in fact the position vector of the object.

Tip

In this section, the motion will be restricted to two dimensions, but the same ideas apply to three-dimensional motion.

KEY POINT 12.6

$$\frac{d\mathbf{r}}{dt} = \mathbf{v} \text{ and } \frac{d\mathbf{v}}{dt} = \mathbf{a}$$

To differentiate or integrate a vector, differentiate or integrate each component separately.

WORKED EXAMPLE 12.6

The velocity of an object is given by $\mathbf{v} = \begin{pmatrix} 5-2t \\ 3t^2+1 \end{pmatrix}$. The object is at the origin when $t = 0$. Find

a the acceleration vector when $t = 3$

b the displacement of the object from the origin when $t = 5$.

To find acceleration, differentiate each component of the velocity vector

a $\mathbf{a} = \frac{d\mathbf{v}}{dt} = \begin{pmatrix} -2 \\ 6t \end{pmatrix}$

When $t = 3$,

$$\mathbf{a} = \begin{pmatrix} -2 \\ 18 \end{pmatrix}$$

To find displacement, integrate each component of the velocity vector. Remember the constants of integration (one for each component)

b $\mathbf{r} = \int \mathbf{v}\, dt$

$$= \begin{pmatrix} 5t - t^2 + c_1 \\ t^3 + t + c_2 \end{pmatrix}$$

When $t = 0$, both components of the displacement are zero

When $t = 0$,

$5t - t^2 + c_1 = 0$, so $c_1 = 0$

$t^3 + t + c_2 = 0$, so $c_2 = 0$

Now use $t = 5$

When $t = 5$,

$$\mathbf{r} = \begin{pmatrix} 5(5) - 5^2 \\ 5^3 + 5 \end{pmatrix} = \begin{pmatrix} 0 \\ 130 \end{pmatrix}$$

Although displacement and velocity are vectors, distance and speed are still scalars.

KEY POINT 12.7

Speed is the magnitude of the velocity vector.

CONCEPTS – SYSTEMS

When studying the motion of an object you are free to choose the origin at any convenient place, but often you will choose the point that corresponds to $t = 0$. A common application of calculus with vectors is analysing the motion of **systems** of objects, such as two bodies moving under each other's gravitational attraction (for example, the Earth and the Moon). In such cases, it is often helpful to place one of the objects at the origin and consider the displacement of the second object relative to the first one.

WORKED EXAMPLE 12.7

The velocity of an object at time t seconds is given by $\mathbf{v} = \begin{pmatrix} 4-3t \\ 2t+1 \end{pmatrix}$. The distances are measured in metres. Find the speed of the object after 10 seconds.

Find the velocity vector first ……… When $t = 10$,

$$\mathbf{v} = \begin{pmatrix} 4-30 \\ 20+1 \end{pmatrix} = \begin{pmatrix} -26 \\ 21 \end{pmatrix}$$

Speed is the magnitude of velocity ……… $|\mathbf{v}| = \sqrt{26^2 + 21^2}$

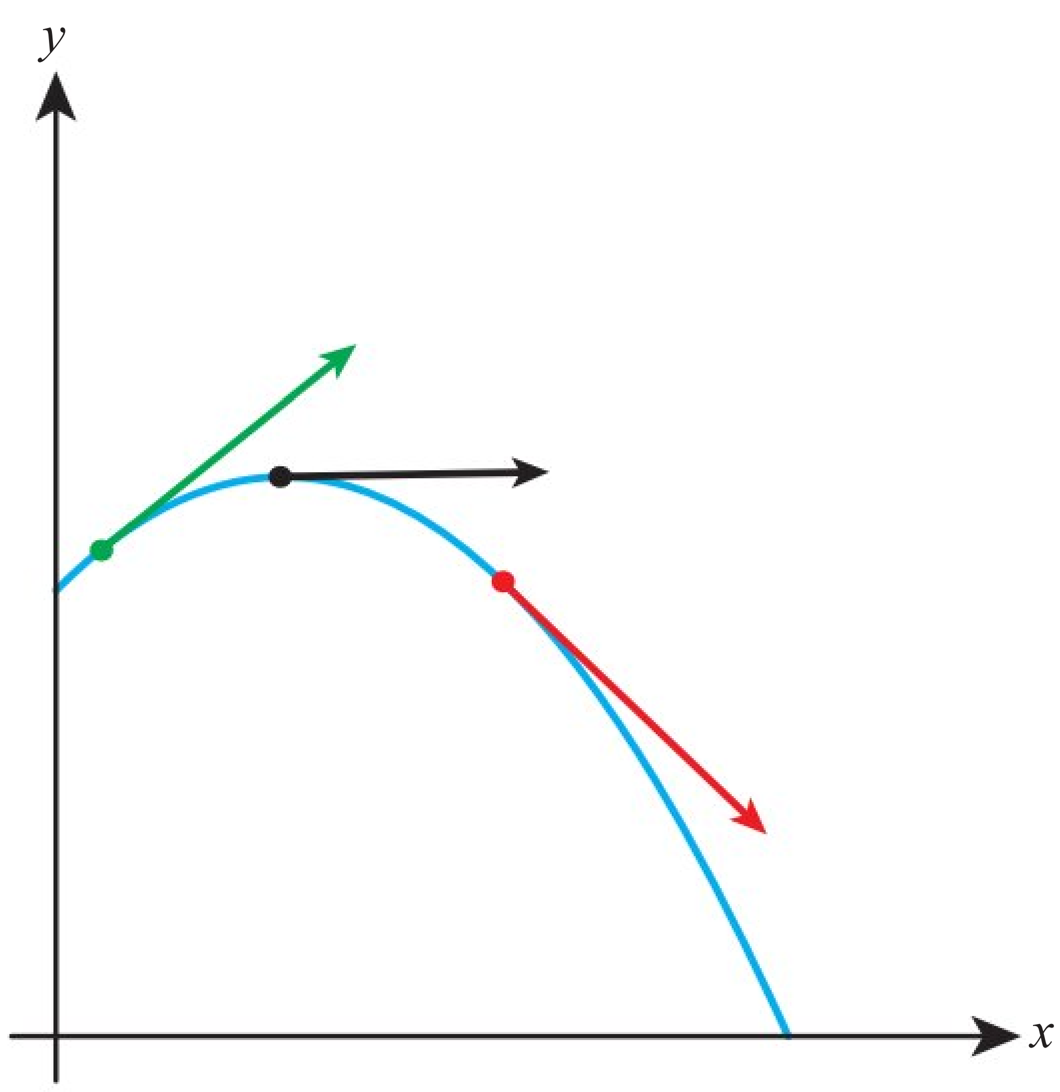
The units are metres per second ……… $= 33.4\,\mathrm{m\,s^{-1}}$

Two important examples of motion in two dimensions are projectile motion and circular motion.

Projectile motion

A projectile is a model for an object moving under the influence of gravity only. If the object is projected at an upward angle, it will move in a vertical plane. If the unit vector $\mathbf{i} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ is horizontal and the unit vector $\mathbf{j} = \begin{pmatrix} 0 \\ -1 \end{pmatrix}$ points vertically upwards, then the acceleration vector will be constant and equal to $\begin{pmatrix} 0 \\ -g \end{pmatrix}$, where $g = 9.8\,\mathrm{m\,s^{-1}}$ and is acceleration due to gravity. You can integrate the acceleration vector twice to obtain the velocity and displacement vectors:

$$\mathbf{v}(t) = \begin{pmatrix} u_x \\ u_y - gt \end{pmatrix} \text{ and } \mathbf{r} = \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x_0 + u_x t \\ y_0 + u_y t - \frac{1}{2}gt^2 \end{pmatrix}$$



In these equations, $\begin{pmatrix} u_x \\ u_y \end{pmatrix}$ is the initial velocity and $\begin{pmatrix} x_0 \\ y_0 \end{pmatrix}$ is the initial displacement. You can see that the horizontal component of the velocity is constant and the vertical component decreases.

You can find an equation for y in terms of x by rearranging the first component of $\mathbf{r}$ to get $t = \dfrac{x - x_0}{u_x}$ and substituting it into the second component. The result is $y = y_0 + u_y\left(\dfrac{x-x_0}{u_x}\right) - \dfrac{1}{2}g\left(\dfrac{x-x_0}{u_x}\right)^2$, which is a quadratic expression in x. Therefore the graph of y against x is a parabola. This is shown in the diagram alongside, with the arrows representing the velocity at various points along the parabola.

TOOLKIT: Modelling

In this model of a projectile, the only force acting on the object is gravity. List some forces that are being ignored. How do you think the path of the projectile would change if those forces were included?

WORKED EXAMPLE 12.8

The velocity of a projectile is given by $\mathbf{v} = \begin{pmatrix} 15 \\ 25 - 9.8t \end{pmatrix}$. The initial position is at the origin.

a At what time does the projectile reach its maximum height?

b Find the displacement from the origin at time t. Hence find the maximum height the projectile reaches.

c How far from the origin does the projectile land?

At the maximum point, the vertical component of the velocity is zero

a $25 - 9.8t = 0$

$t = 2.55$ seconds

Integrate the velocity vector and use the initial position to find the constants

b $\mathbf{r} = \int \mathbf{v}\,\mathrm{d}t = \begin{pmatrix} 15t + c_1 \\ 25t - 4.9t^2 + c_2 \end{pmatrix}$

When $t = 0$, $\mathbf{r} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$ so $c_1 = c_2 = 0$

Use the time when the maximum height is reached. The height is given by the second component of the displacement vector

Maximum height:

$h = 25(2.55) - 4.9(2.55^2)$

$= 31.9\,\text{m}$

The projectile lands when the height is zero

Lands when

$25t - 4.9t^2 = 0$

This quadratic equation has two solutions, but $t = 0$ corresponds to the starting position

$t = 5.10$

When the projectile lands, the distance from the origin is given by the first component of the displacement vector

When $t = 5.10$,

$\mathbf{r} = \begin{pmatrix} 15 \times 5.10 \\ 0 \end{pmatrix} = \begin{pmatrix} 76.5 \\ 0 \end{pmatrix}$

The distance from the origin is 76.5 m.

Tip

The time to landing is twice the time to reach maximum height.

Circular motion

If an object moves in a circle around the origin, its velocity changes direction all the time but its distance from the origin remains constant.

WORKED EXAMPLE 12.9

The velocity of an object at time t is given by $\mathbf{v} = \begin{pmatrix} -8\sin 2t \\ 8\cos 2t \end{pmatrix}$. When $t = 0$, the position vector of the object is $\begin{pmatrix} 4 \\ 0 \end{pmatrix}$.

a Find the position vector of the object at time t.

b Find the distance of the object from the origin, and hence describe the path of the object.

To get the position vector (which is the displacement from the origin), integrate the velocity vector

a $\mathbf{r} = \int \mathbf{v}\, \mathrm{d}t$

Remember that cos integrates to sin and sin integrates to −cos. You need to divide by 2 (because of the $2t$).

$$= \begin{pmatrix} 4\cos 2t + c_1 \\ 4\sin 2t + c_2 \end{pmatrix}$$

Use the initial position to find c_1 and c_2

When $t = 0$,

$$\begin{pmatrix} 4\cos 0 + c_1 \\ 4\sin 0 + c_2 \end{pmatrix} = \begin{pmatrix} 4 \\ 0 \end{pmatrix}$$

$$c_1 = c_2 = 0$$

The position at time t is

$$\mathbf{r} = \begin{pmatrix} 4\cos 2t \\ 4\sin 2t \end{pmatrix}$$

The distance from the origin is the magnitude of the position vector

b $|\mathbf{r}| = \sqrt{(4\cos 2t)^2 + (4\sin 2t)^2}$

$$= \sqrt{16(\cos^2 2t + \sin^2 2t)}$$

Remember that $\sin^2\theta + \cos^2\theta = 1$

$$= \sqrt{16} = 4$$

The object remains at the same distance from the origin at all times

Hence, the object moves in a circle of radius 4 around the origin.

Can you explain the role that the numbers 8 and 2, and the minus sign, play in the equation for circular motion? This is explored further in Questions 30 and 31 at the end of this section.

CONCEPTS – CHANGE

We use differentiation to study rates of **change** of quantities. But it is just as important to consider what remains the same. In the circular motion example above, you can check that, although the displacement, velocity and acceleration vectors all change with time, the magnitudes of all three vectors remain constant.

Modelling time-shift

When we consider the motion of two objects, it is possible that they did not start moving at the same time. We can incorporate this time-shift into the equations.

Compare this to Key Point 5.5. What effect does this transformation have on the graph of the trajectory?

KEY POINT 12.8

If an object started moving at $t = a$ rather than $t = 0$, replace t by $(t - a)$ in the equations for displacement, velocity and acceleration.

WORKED EXAMPLE 12.10

A small stone is projected from the origin so that its displacement at time t is given by

$\mathbf{r}_1 = \begin{pmatrix} 5t \\ 12t - 4.9t^2 \end{pmatrix}$, where the distance in measured in metres and the time in seconds.

A second stone is projected along the same path 0.5 seconds later. Find the distance between the two stones 1 second after the projection of the second stone.

To find the position of the second stone, use the equation for the displacement but with t replaced by $(t - 0.5)$

The position of the second stone is

$$\mathbf{r}_2 = \begin{pmatrix} 5(t - 0.5) \\ 12(t - 0.5) - 4.9(t - 0.5)^2 \end{pmatrix}$$

1 second after the projection of the second stone corresponds to $t = 1.5$

When $t = 1.5$,

$$\mathbf{r}_2 = \begin{pmatrix} 5 \\ 7.1 \end{pmatrix}$$

$$\mathbf{r}_1 = \begin{pmatrix} 7.5 \\ 6.975 \end{pmatrix}$$

The distance is the magnitude of the relative displacement

$$\mathbf{r}_2 - \mathbf{r}_1 = \begin{pmatrix} -2.5 \\ 0.125 \end{pmatrix}$$

So, distance $= \sqrt{2.5^2 + 0.125^2}$

$= 2.50\,\text{m}$

A special case of time-shift is phase shift in periodic functions, which you met when working with complex numbers in Section 6B.

Exercise 12B

In this exercise, time is measured in seconds and distance in metres, unless stated otherwise. The gravitational acceleration should be taken to be $g = 9.8\,\text{m}\,\text{s}^{-2}$.

For questions 1 to 4, use the method demonstrated in Worked Example 12.6 to find expressions for the acceleration and the displacement of the object at time t. You are given an expression for the velocity and the initial position of the object.

1 a $\mathbf{v} = \begin{pmatrix} 4t + 1 \\ 1 - 3t^2 \end{pmatrix}$, $\mathbf{r}_0 = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$

b $\mathbf{v} = \begin{pmatrix} 6t^2 - 1 \\ 6t + 3 \end{pmatrix}$, $\mathbf{r}_0 = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$

2 a $\mathbf{v} = \begin{pmatrix} 4e^t \\ 3e^t \end{pmatrix}$, $\mathbf{r}_0 = \begin{pmatrix} 4 \\ 3 \end{pmatrix}$

b $\mathbf{v} = \begin{pmatrix} 3e^t \\ -5e^t \end{pmatrix}$, $\mathbf{r}_0 = \begin{pmatrix} 3 \\ -5 \end{pmatrix}$

3 a $\mathbf{v} = \begin{pmatrix} 12e^{4t} \\ 6e^{-3t} \end{pmatrix}$, $\mathbf{r}_0 = \begin{pmatrix} 3 \\ -2 \end{pmatrix}$

b $\mathbf{v} = \begin{pmatrix} 5e^{-t} \\ 10e^{5t} \end{pmatrix}$, $\mathbf{r}_0 = \begin{pmatrix} -5 \\ 2 \end{pmatrix}$

4 a $\mathbf{v} = \begin{pmatrix} 3\cos 2t \\ 4\sin 2t \end{pmatrix}$, $\mathbf{r}_0 = \begin{pmatrix} 0 \\ -2 \end{pmatrix}$

b $\mathbf{v} = \begin{pmatrix} -5\sin 5t \\ 10\cos 5t \end{pmatrix}$, $\mathbf{r}_0 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$

For questions 5 to 8, you are given an expression for the velocity of an object at time t, in m s^{-1}. Use the method demonstrated in Worked Example 12.7 to find the speed of the object at the given time.

5 **a** $\mathbf{v} = \begin{pmatrix} 4t+1 \\ 1-3t^2 \end{pmatrix}$, $t = 2$ seconds

b $\mathbf{v} = \begin{pmatrix} 6t^2-1 \\ 6t+3 \end{pmatrix}$, $t = 3$ seconds

6 **a** $\mathbf{v} = \begin{pmatrix} 4e^t \\ 3e^t \end{pmatrix}$, $t = 0.5$ seconds

b $\mathbf{v} = \begin{pmatrix} 3e^t \\ -5e^t \end{pmatrix}$, $t = 1.2$ seconds

7 **a** $\mathbf{v} = \begin{pmatrix} 12e^{4t} \\ 6e^{-3t} \end{pmatrix}$, $t = 0.4$ seconds

b $\mathbf{v} = \begin{pmatrix} 5e^{-t} \\ 10e^{5t} \end{pmatrix}$, $t = 1.5$ seconds

8 **a** $\mathbf{v} = \begin{pmatrix} 3\cos 2t \\ 4\sin 2t \end{pmatrix}$, $t = 4$ seconds

b $\mathbf{v} = \begin{pmatrix} -5\sin 5t \\ 10\cos 5t \end{pmatrix}$, $t = 3$ seconds

Tip

Your calculator should be in radians.

9 An object moves in two dimensions so that its position vector at time t seconds is given by $\mathbf{r} = \begin{pmatrix} 0.5\sin 3t \\ 4t \end{pmatrix}$.

a Find an expression for the velocity of the object at time t.

b Find the speed of the object when $t = 4$.

10 The velocity of an object is given by $\mathbf{v} = (3t+4)\mathbf{i} + (7-2t)\mathbf{j}$.

a Find the angle that the initial velocity makes with the direction of $\mathbf{i}$.

b Show that the acceleration of the object is constant and find its magnitude.

11 A particle is at the origin when $t = 0$. Its velocity at time t is $\mathbf{v} = \begin{pmatrix} 3e^{-t} \\ 4e^{-t} \end{pmatrix}$.

a Find the speed of the particle when $t = 3$.

b Find an expression for the displacement of the particle from the origin at time t.

c Find the distance of the particle from the origin when $t = 3$.

12 An object moves with velocity $\mathbf{v} = \begin{pmatrix} 3t+1 \\ 10-0.1t^2 \end{pmatrix}$. At time $t = 2$, find

a the angle that the velocity makes with the direction of $\mathbf{i}$

b the speed of the object

c the magnitude of acceleration.

13 The velocity of an object at time t is given by $\mathbf{v} = \begin{pmatrix} 5-2t \\ 1+3t \end{pmatrix}$.

a Find the speed of the object when $t = 5$.

b Show that the acceleration is constant and find its magnitude.

14 The velocity of a projectile is modelled by $\mathbf{v} = 16\mathbf{i} + (12-9.8t)\mathbf{j}$. The vector $\mathbf{i}$ is horizontal and the vector $\mathbf{j}$ points upwards.

a State the magnitude and direction of acceleration.

b Find the initial speed of the projectile.

c Find an expression for the speed of the projectile at time t, and hence determine its minimum speed.

15 A particle is projected from the origin and moves in the vertical plane. Its velocity at time t is given by

$$\mathbf{v} = \begin{pmatrix} 9 \\ 7-9.8t \end{pmatrix}$$

a Find an expression for the displacement of the particle at time t.

b Show that the particle lands approximately 1.43 seconds after projection and find its distance from the origin at that time.

c Show that the particle lands with the same speed with which it was projected.

16 A projectile starts at the origin at $t = 0$ and its velocity at time t is $\mathbf{v} = \begin{pmatrix} 20 \\ 10 - 9.8t \end{pmatrix}$.

- **a** Find the angle that the velocity makes with the horizontal
  - **i** initially
  - **ii** after 2 seconds.
- **b** Find the time at which the projectile reaches is maximum height.
- **c** Find an expression for the displacement from the origin at time t. Hence find the maximum height the projectile reaches.

17 The velocity of a projectile, launched at $t = 0$ from the origin, is given by $\mathbf{v} = \begin{pmatrix} 12 \\ 9 - 9.8t \end{pmatrix}$.

- **a** Find an expression for the displacement from the origin at time t.
- **b** Find the distance of the projectile from the origin when it reaches the ground.

18 The position vector, relative to the origin, of an object at time t seconds is given by $\mathbf{r} = \begin{pmatrix} 5\cos 3t \\ 5\sin 3t \end{pmatrix}$.

- **a** By finding the distance of the object from the origin at time t, show that the object moves in a circle. State the radius of the circle.
- **b** Show that the velocity vector is always perpendicular to the position vector.
- **c** Show that the object moves with constant speed.

19 An object moves in a circle around the origin, so that its position at time t is given by $\mathbf{r} = \begin{pmatrix} 12\cos\left(\frac{t}{2}\right) \\ 12\sin\left(\frac{t}{2}\right) \end{pmatrix}$.

- **a** Find the velocity vector at time t.
- **b** Show that the speed of the object is constant and find its value.
- **c** Show that the acceleration vector is always parallel to the position vector.
- **d** Show that the magnitude of the acceleration is constant and find its value.

20 An object moves with velocity $\mathbf{v} = \begin{pmatrix} 0.9\cos 3t \\ 0.9\sin 3t \end{pmatrix}$.

When $t = 0$, the displacement of the object from the origin is $\begin{pmatrix} 0 \\ -0.3 \end{pmatrix}$.

- **a** Find the speed of the object.
- **b** Find the displacement of the object at time t, and show that the object moves in a circle around the origin. State the radius of the circle.
- **c** Find the acceleration vector and show that $\mathbf{a} = k\mathbf{r}$, where k is a constant to be found.
- **d** Hence state the direction of the acceleration vector.

21 An object moves in a circle around the origin so that its position vector is given by $\mathbf{r} = \begin{pmatrix} 0.5\cos 4t \\ 0.5\sin 4t \end{pmatrix}$.

- **a** Find the radius of the circle, r.
- **b** Write down the initial position of the object.
- **c** Find the time, T, when the object first returns to its starting position.
- **d** Find the velocity vector at time t. Hence verify that the speed of the object is given by $\frac{2\pi r}{T}$.

22 An object starts from the origin when $t = 0$. Its velocity at time t is given by $\mathbf{v} = \begin{pmatrix} 4 - 1.5t \\ 0.3t^2 + 5 \end{pmatrix}$.

- **a** Find an expression for the displacement of the object from the origin at time t.
- **b** Another object follows the same path with the same velocity, but leaves the origin 2 seconds later. Find the distance between the two objects 3 seconds after the second object left the origin.

23 An object moves with constant acceleration $\mathbf{a} = \begin{pmatrix} 3 \\ -4 \end{pmatrix}$. It is initially at rest at the origin.

a Find the expressions for the velocity and displacement at time t.

b How long does it take for the speed of the object to reach $20\,\text{m s}^{-1}$?

c Plot the position of the object in the x–y plane at times $t = 0, 1, 2, 3$. Hence describe the path of the object.

Tip

The velocity vector can be written as $\mathbf{v} = t\begin{pmatrix} 3 \\ -4 \end{pmatrix}$, so the velocity does not change direction.

24 An object moves with constant acceleration $\mathbf{a} = \begin{pmatrix} -2 \\ 1 \end{pmatrix}$. When $t = 0$, the object passes through the origin with velocity $\begin{pmatrix} 1 \\ 3 \end{pmatrix}$.

a Find the velocity and displacement vectors at time t.

b Find the angle between the initial velocity of the object and its velocity after 2 seconds.

c Plot the positions of the object at times $t = 0, 0.5, 1, 1.5, 2$. Does the object move in a straight line?

25 A projectile on Planet X is launched from the origin with initial velocity $\begin{pmatrix} 12 \\ 9 \end{pmatrix}$. Given that the maximum height reached by the projectile is 8.6 m, find, to two significant figures, the gravitational acceleration on Planet X.

26 The velocity of an object at time t seconds is given by $\mathbf{v} = \begin{pmatrix} 3t-5 \\ 4-t \end{pmatrix}$. Show that the object never returns to its starting position.

27 A particle is projected from the origin and its velocity at time t is given by $\mathbf{v} = \begin{pmatrix} 26 \\ 21-4.9t \end{pmatrix}$. A second particle is also projected from the origin and its velocity at time t is given by $\mathbf{v} = \begin{pmatrix} 30 \\ b-4.9t \end{pmatrix}$. Given that the two projectiles land in the same place, find the value of b.

28 A particle is projected from the origin with initial velocity $\begin{pmatrix} 12 \\ 30 \end{pmatrix}$ and moves under gravity, so that its acceleration is $\begin{pmatrix} 0 \\ -9.8 \end{pmatrix}$.

a Find an expression for the displacement of the particle from the origin at time t.

A second particle is projected from the origin 1 second later with initial velocity $\begin{pmatrix} 20 \\ b \end{pmatrix}$.

b Find an expression for the displacement of the second particle from the origin at time t.

c Given that the two particles collide, find the value of b.

d For this value of b, find how far from the origin they collide.

29 A projectile is launched from the origin with the initial velocity $\begin{pmatrix} 120 \\ 90 \end{pmatrix}$. A second projectile is launched from the origin with the same initial velocity 10 seconds later.

a Find an expression for the distance between the two projectiles at time t seconds after the launch of the first projectile, for $t \geqslant 10$.

b Find the time after the initial launch at which the distance between the projectiles is minimal, during the time they are both in flight.

30 The position vector of a particle is given by $\mathbf{r} = \begin{pmatrix} r\cos\omega t \\ r\sin\omega t \end{pmatrix}$.

- **a** Show that the particle moves in a circle.
- **b** Show that the speed of the particle is $v = r\omega$.
- **c** Show that the time taken for the particle to complete a full circle is $T = \frac{2\pi}{\omega}$.
- **d** Show that the magnitude of the acceleration is $\frac{v^2}{r}$.

31 A particle moves around a circle of radius r such that $\mathbf{r} = \begin{pmatrix} r\cos\omega t \\ r\sin\omega t \end{pmatrix}$. Show that

- **a** the velocity vector is perpendicular to the displacement vector
- **b** the acceleration vector is directed towards the centre of the circle.

Links to: Physics

In physics, ω is called **angular speed**. If the angular speed is constant, the acceleration has constant magnitude and is directed towards the centre of the circle. If the acceleration has a component in the direction tangential to the circle, the angular velocity will change.

Checklist

- You should be able to apply calculus to kinematics, linking displacement, velocity, acceleration and distance travelled in one dimension (an object moving along a straight line):
  - $v = \frac{\mathrm{d}s}{\mathrm{d}t}$
  - $a = \frac{\mathrm{d}v}{\mathrm{d}t} = \frac{\mathrm{d}^2 s}{\mathrm{d}t^2} = v\frac{\mathrm{d}v}{\mathrm{d}s} = \frac{\mathrm{d}}{\mathrm{d}s}\left(\frac{1}{2}v^2\right)$
  - Change in velocity from $t = a$ to $t = b$ is given by $\int_a^b a \, \mathrm{d}t$.
  - Change in displacement from $t = a$ to $t = b$ is given by $\int_a^b v \, \mathrm{d}t$.
  - The distance travelled between $t = a$ and $t = b$ is given by $\int_a^b |v| \, \mathrm{d}t$.
- You should be able to combine these calculus techniques with vectors:
  - In two dimensions, displacement, velocity and acceleration are all vectors.
  - Speed is the magnitude of the velocity vector.
  - Distance travelled is not necessarily the magnitude of the displacement vector.
  - To differentiate or integrate a vector, differentiate or integrate each component separately.
- You should recognize that projectile motion and circular motion are examples of motion with variable velocity.
- You can model a time-shift (e.g. one object starting motion a seconds after another) by replacing t by $(t - a)$ in expressions for displacement, velocity and acceleration.

Mixed Practice

Distance is measured in metres and time in seconds, unless stated otherwise.

1 A particle moves in a straight line so that its displacement at time t seconds is s metres, where $s = 3t - 0.06t^3$. Find the velocity and acceleration of the particle after 8.6 seconds.

2 The velocity of an object, in metres per second, is given by $v = 4e^{-2t}$. Find

a the acceleration when $t = 2$

b the distance travelled in the first two seconds.

3 An object moves in a straight line, with velocity given by $v = 5\sin(3t)$.

a Sketch the graph of velocity against time for $0 \leqslant t \leqslant 5$.

b Hence find the times when the object changes direction.

c Find the distance travelled during the first 5 seconds of motion.

4 A particle moves in a circle, with displacement from the origin given by $\mathbf{r} = \begin{pmatrix} 7\cos 3t \\ 7\sin 3t \end{pmatrix}$, where time is measured in seconds and distance is in centimetres.

a Find the position vector of the particle when $t = 0$.

b Hence find the radius of the circle.

c Find the speed of the particle.

5 A projectile has velocity $\mathbf{v} = \begin{pmatrix} 16 \\ 14 - 9.8t \end{pmatrix}$.

a Find the initial speed of the projectile.

b Find the velocity when $t = 2$. Is the projectile on the way up or on the way down at this point?

c Find an expression for the displacement from the initial position at time t.

d Hence find the distance from the starting point when $t = 2$.

6 An object moves in the plane with velocity $\mathbf{v} = \begin{pmatrix} 3t \\ 2e^{-0.5t} \end{pmatrix}$. Time is measured in seconds and distance is in centimetres.

a Find the initial speed of the object.

b Find the acceleration vector when $t = 0$.

7 A particle moves in a straight line with velocity $v = 12t - 2t^3 - 1$, for $t \geqslant 0$, where v is in centimetres per second and t is in seconds.

a Find the acceleration of the particle after 2.7 seconds.

b Find the displacement of the particle after 1.3 seconds.

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8 The velocity v m s^{-1} of an object after t seconds is given by $v(t) = 15\sqrt{t} - 3t$ for $0 \leqslant t \leqslant 25$.

a Sketch the graph of v, clearly indicating the maximum point.

b Let d be the distance travelled in the first nine seconds.

i Write down an expression for d.

ii Hence write down the value of d.

Mathematics SL November 2010 Paper 2 Q2

9 A particle moves in a straight line with the velocity at time t seconds equal to $\dot{x} = \left(9t - 3t^2\right)\,\text{m s}^{-1}$.

a Find $\ddot{x}$ when $t = 3$.

b The initial displacement of the particle from the origin is 5 m. Find the displacement from the origin after 3 seconds.

10 An object moves in the plane with velocity $\mathbf{v} = \begin{pmatrix} 3t \\ 2e^{-0.5t} \end{pmatrix}$. Time is measured in seconds and distance is in centimetres.

a The object is initially at $\begin{pmatrix} 0 \\ 0 \end{pmatrix}$. Find its displacement from the initial position at time t.

b Hence find its distance from $\begin{pmatrix} 0 \\ 0 \end{pmatrix}$ when $t = 2$.

11 The velocity of an object moving in two dimensions is given by $\mathbf{v} = \begin{pmatrix} 3\sin t \\ 4\cos t \end{pmatrix}$. When $t = 0$, the object is at $\left(-3, 0\right)$.

a Find an expression for the position vector at time t.

b Show that the object returns to the starting point when $t = 2\pi$.

c Find the maximum distance from the starting point.

d Find the angle between the velocity and displacement vectors when $t = \frac{\pi}{4}$.

12 A projectile has initial velocity $\begin{pmatrix} 2 \\ 17 \end{pmatrix}\text{m s}^{-1}$.

a Write down the velocity vector at time t.

b Find the displacement from the origin at time t. Hence find the time when the projectile lands.

c Find an expression for the distance of the projectile from the origin at time t. You do not need to simplify your expression.

d Hence find the largest distance of the projectile from the origin between launch and landing.

13 A particle moves in a plane so that its position vector at time t is given by $\mathbf{r} = \begin{pmatrix} 4\cos 2t \\ 5\sin 4t \end{pmatrix}$.

a Write down the initial position vector.

b Find an expression for the velocity at time t.

c Hence find the maximum speed of the particle.

d Show that the particle passes through the origin.

e Find the time when the particle first returns to the starting point.

14 **a** Sketch the graph of $y = 2\sin x + 1$ for $0 \leqslant x \leqslant 2\pi$.

The velocity of a particle at time t seconds is $v = (2\sin t + 1)\,\text{m s}^{-1}$.

b Find the speed of the particle after 2.5 seconds.

c Find the displacement of the particle from the initial position after 2π seconds.

d Find the distance travelled by the particle in the first 2π seconds.

15 If $v = se^{-s}$, find an expression for the acceleration in terms of s.

16 If $v^2 = 4s^2 + 1$, find the acceleration as a function of s.

17 The displacement of an object at time t is given by $s = -\frac{1}{20}t^5 + \frac{1}{8}t^4 - \frac{1}{30}t^3 + 2t^2$ for $0 \leqslant t \leqslant 4$.

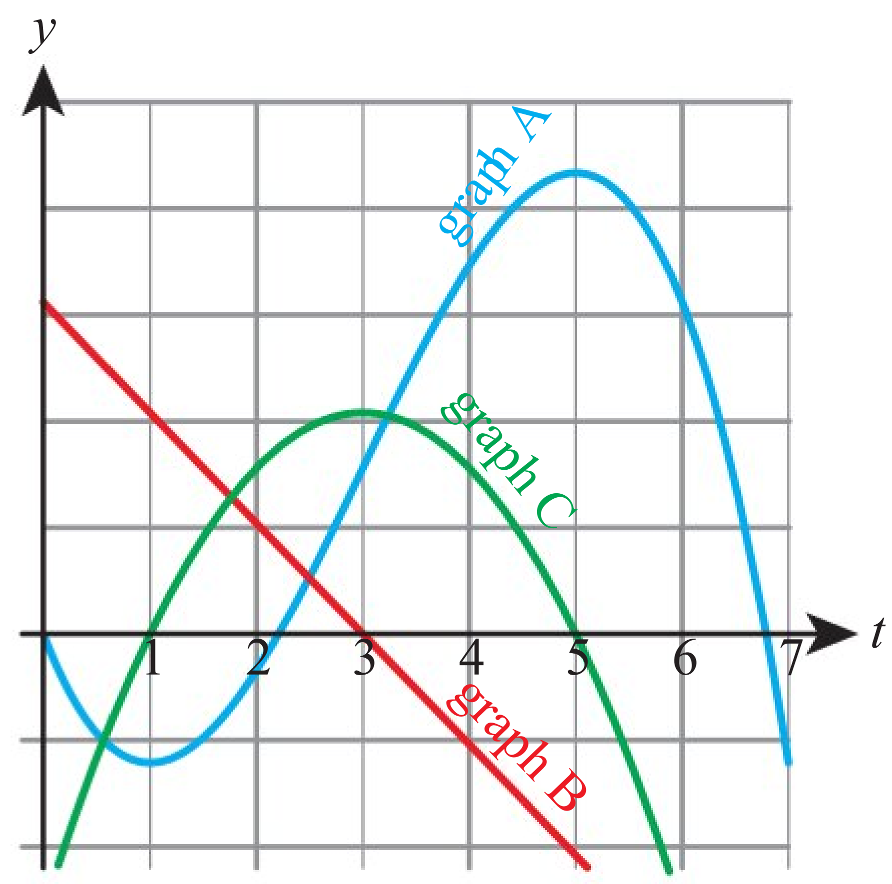
Find the maximum **speed** of the object during this time.

18 The diagram shows the graphs of the **displacement**, **velocity** and **acceleration** of a moving object as functions of time, t.

a Complete the following table by noting which graph A, B or C, corresponds to each function.

Function	Graph
Displacement	
Acceleration	

b Write down the value of t when the velocity is greatest.



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19 A particle, A, is moving along a straight line. The velocity, v_A m s^{-1}, of A, t seconds after its motion begins, is given by $v_A = t^3 - 5t^2 + 6t$.

a Sketch the graph of $v_A = t^3 - 5t^2 + 6t$ for $t \geqslant 0$, with v_A on the vertical axis and t on the horizontal. Show on your sketch the local maximum and minimum points, and the intercepts with the t-axis.

b Write down the times for which the velocity of the particle is increasing.

c Write down the times for which the magnitude of the velocity of the particle is increasing.

At $t = 0$ the particle is at point O on the line.

d Find an expression for the particle's displacement, x_A m, from O at time t.

A second particle, B, moving along the same line, has position x_B m, velocity v_B m s^{-1} and acceleration, a_B m s^{-2}, where $a_B = -2v_B$ for $t \geqslant 0$. At $t = 0$, $x_B = 20$ and $v_B = -20$.

e Find an expression for v_B in terms of t.

f Find the value of t when the two particles meet.

Mathematics HL May 2013 Paper 2 TZ1 Q12

20 Two particles leave the origin at the same time. Their velocities at time t are given by

$$\mathbf{v}_1 = \begin{pmatrix} 4t - 2 \\ 5 - 2t \end{pmatrix} \text{ and } \mathbf{v}_2 = \begin{pmatrix} \frac{8}{3}t \\ 2t - 1 \end{pmatrix}.$$

a Show that the two particles meet again.

b Find the angle between their velocities when they meet.

21 In this question, the displacement is measured upwards from ground level.

A particle is projected vertically upwards with speed 8 m s^{-1}. The acceleration is constant at -9.8 m s^{-2}.

a Find expressions for its velocity and displacement from the origin at time t.

b A second particle is projected with the same speed, 0.7 seconds later. Find the time when the two particles are at the same height.

22 A particle is projected from the origin with initial velocity $\begin{pmatrix} 8 \\ 5 \end{pmatrix}$ m s^{-1} and moves in a vertical plane.

The acceleration due to gravity is $\begin{pmatrix} 0 \\ -9.8 \end{pmatrix}$ m s^{-2}.

a Find expressions for the velocity and position vectors of the particle at time t.

A second particle is projected from the origin with the same initial speed, 0.5 seconds later.

b How long after the launch of the *second* projectile do the two particles have the same speed?

23 A particle moves in a vertical circle with velocity $\mathbf{v} = \begin{pmatrix} 16\cos 2t \\ 16\sin 2t \end{pmatrix}$. Vector **i** is horizontal, vector **j** is vertical, the origin is at the centre of the circle, distance is measured in centimetres and time is in seconds. Initially the particle is at the lowest point.

a Find an expression for the position vector of the particle at time t.

Another particle moves around the same circle, with the same velocity, but starts from the lowest point 1.2 seconds later.

b Find the first time when the two particles are at the same height. Find this height, relative to the starting point.

24 Richard, a marine soldier, steps out of a stationary helicopter, 1000 m above the ground, at time $t = 0$. Let his height, in metres, above the ground be given by $s(t)$. For the first 10 seconds, his velocity, $v(t)\,\text{m s}^{-1}$, is given by $v(t) = -10t$.

a **i** Find his acceleration $a(t)$ for $t < 10$.

ii Calculate $v(10)$.

iii Show that $s(10) = 500$.

At $t = 10$, his parachute opens and his acceleration $a(t)$ is subsequently given by $a(t) = -10 - 5v$, $t \geqslant 10$.

b Given that $\dfrac{dt}{dv} = \dfrac{1}{\frac{dv}{dt}}$, write down $\dfrac{dt}{dv}$ in terms of v.

You are told that Richard's acceleration, $a(t) = -10 - 5v$, is always positive for $t \geqslant 10$.

c Hence show that $t = 10 + \dfrac{1}{5}\ln\left(\dfrac{98}{-2 - v}\right)$.

d Hence find an expression for the velocity, v, for $t \geqslant 10$.

e Find an expression for his height, s, above the ground for $t \geqslant 10$.

f Find the value of t when Richard lands on the ground.

Mathematics HL May 2015 Paper 2 TZ1 Q13

13 Differential equations

ESSENTIAL UNDERSTANDINGS

- Calculus describes rates of change between two variables and the accumulation of limiting areas.
- Understanding these rates of change and accumulations helps us to model, interpret and analyse real-world problems.
- Calculus helps us to understand the behaviour of functions and allows us to interpret the features of their graphs.

In this chapter you will learn…

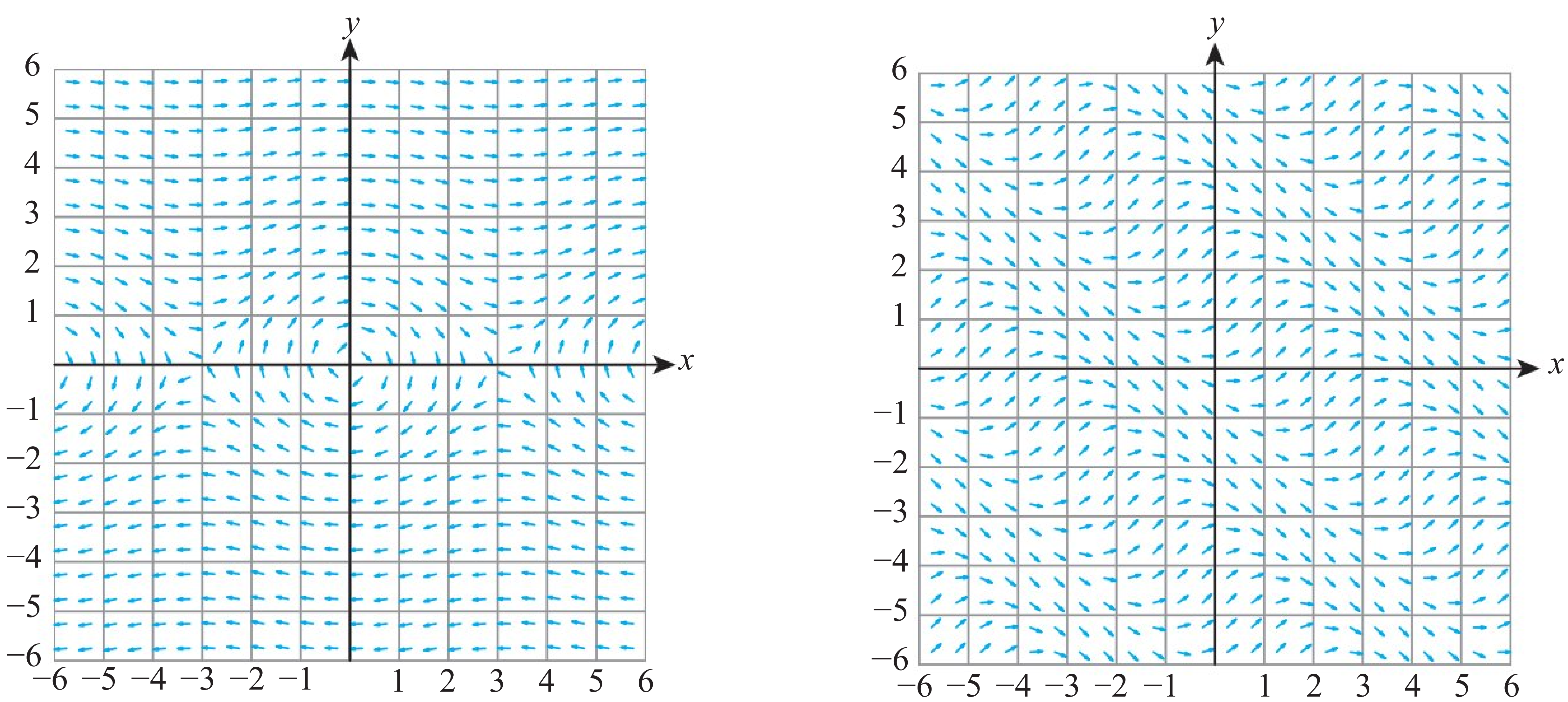
- how to create differential equations from a context
- when and how to solve differential equations by separating variables
- how to sketch solutions to differential equations, even without solving them
- how to find approximate solutions to first order differential equations
- how to analyse systems of differential equations both approximately and analytically
- how to represent systems of differential equations graphically
- how to analyse second order differential equations both approximately and analytically.

CONCEPTS

The following concepts will be addressed in this chapter:

- The derivative may be represented physically as a rate of **change** and geometrically as the gradient or slope function.
- Many physical phenomena can be modelled using differential equations and analytical and numerical methods can be used to calculate optimum **quantities**.
- Phase portraits enable us to visualize the behaviour of dynamic **systems**.

Figure 13.1 What do these slope fields represent?



PRIOR KNOWLEDGE

Before starting this chapter, you should already be able to complete the following:

1. Find $\int e^{2x}\,dx$.
2. If $f(x) = \sin 2x$, evaluate $f'(0)$.
3. Simplify $e^{2\ln x}$ if $x > 0$.
4. For the sequence $u_{n+1} = 2u_n - n$ with $u_0 = 2$, use technology to find u_{10}.

TOK Links

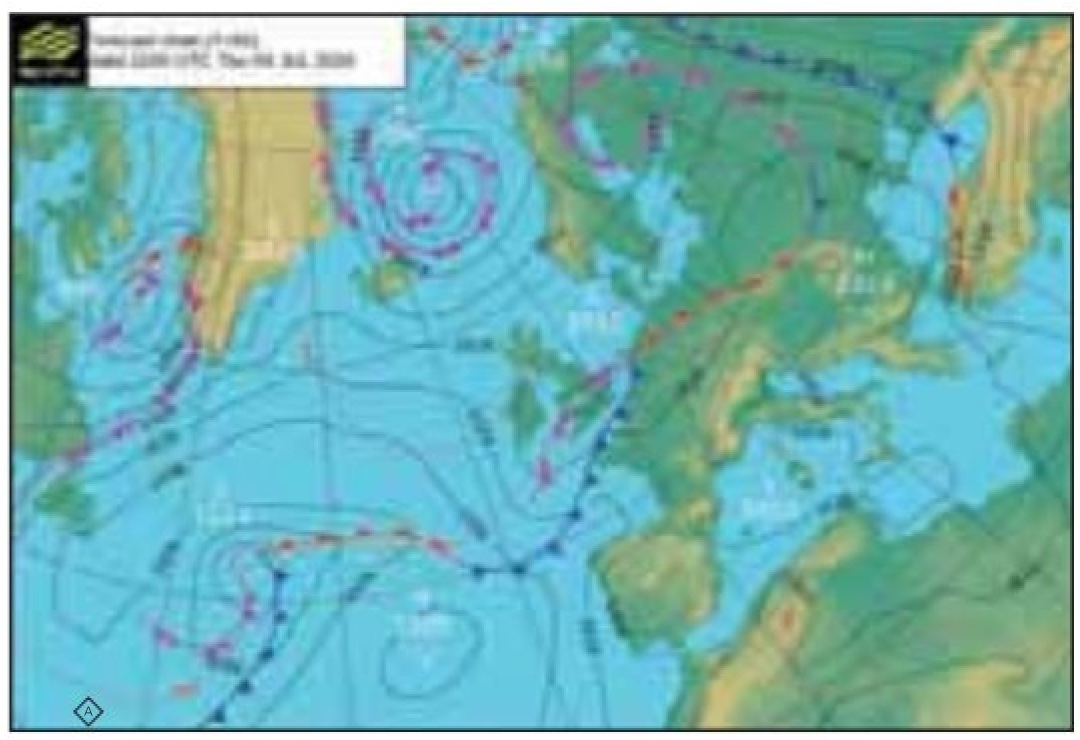
How can you decide if an approximate solution to a problem is correct? How far away from the correct value is acceptable? What criteria can you use?

In many real-life situations we know information about how a quantity changes. We can use this to create a model called a differential equation. Often there is more than one factor influencing a situation, and these can be described using systems of coupled differential equations. In this chapter, you will see how some of these differential equations can be solved exactly. However, not every differential equation can be solved in terms of well-known functions. Such is their importance in many real-world situations that even if they cannot be solved exactly, there are methods which are used to find approximate solutions.

Starter Activity

Look at the diagrams in Figure 13.1. They represent differential equations, and each small line shows what the gradient is at each point. Imagine that they represent streamlines in flowing water. If you were to drop a floating object it at one point, see if you can trace out its path.

Very complicated versions of these plots are used to help predict the weather, as shown in the diagram alongside.



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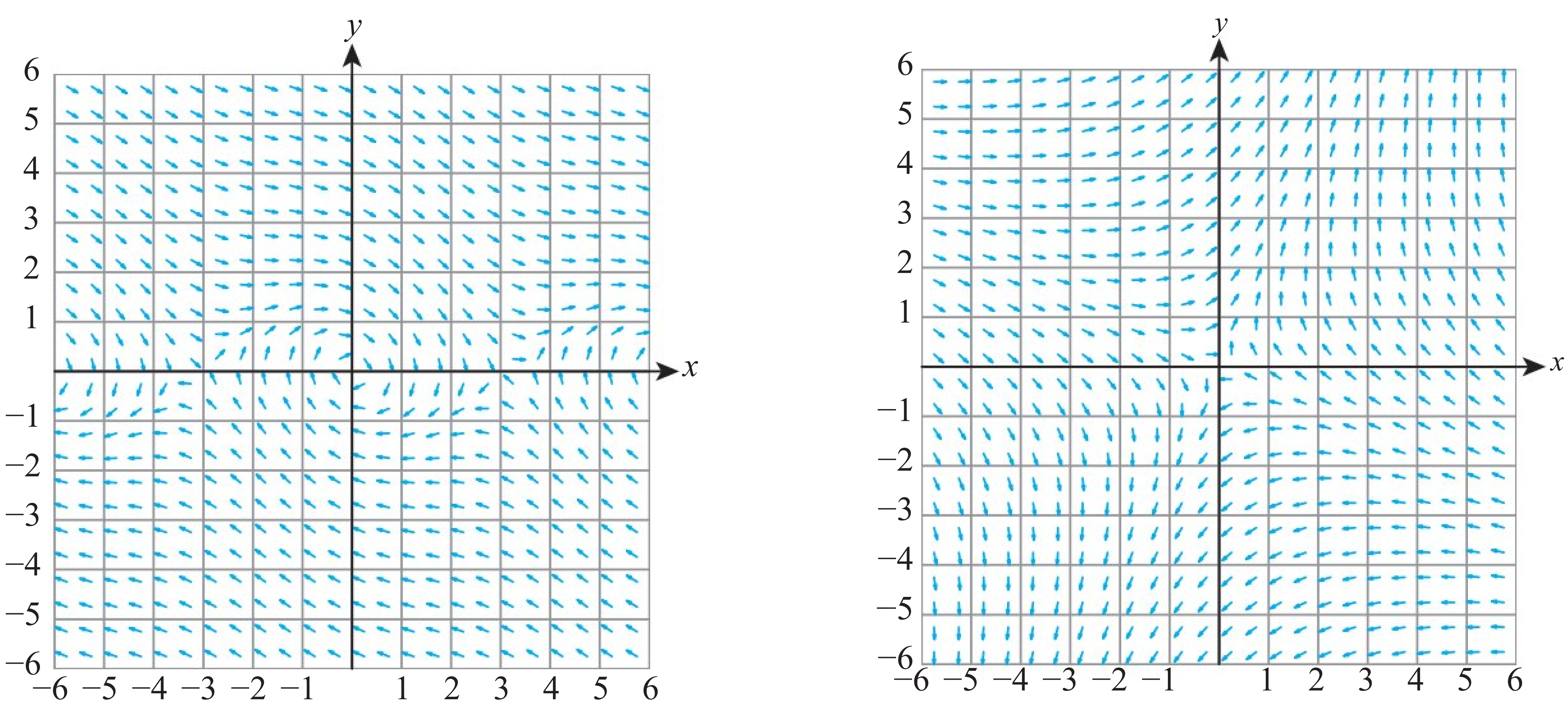
Now look at this problem:

Try to sketch a curve which has each of the following properties:

a. the gradient at every point is the same
b. the gradient at every point equals the y-coordinate
c. the gradient at every point equals the x-coordinate
d. the gradient at every point is perpendicular to the line connecting the point to the origin
e. the gradient at every point equals the distance from the origin.



13A Separation of variables

Setting up differential equations

A differential equation tells you something about the rate of change of a quantity; in other words, it is an equation relating $\frac{dy}{dx}$ to x and y.

WORKED EXAMPLE 13.1

The rate at which water flows out of a bucket is proportional to the square root of the volume remaining. Write this as a differential equation.

Define the variables

Let V = volume of water remaining (in litres), and t = time (in seconds).

'Proportional' means 'multiplied by a constant'

Remember that if a quantity is decreasing, the derivative is negative

$$\frac{dV}{dt} = -k\sqrt{V}$$

General solution of a differential equation

You already know how to solve some differential equations. For example, in Section 12A you learnt that the velocity is the rate of change of displacement.

WORKED EXAMPLE 13.2

Given that $\frac{ds}{dt} = e^{-t}$, find an expression for s in terms of t.

You can 'undo' the differentiation by integrating with respect to t

$$s = \int e^{-t}\,dt$$

Remember the constant of integration

$$= -e^{-t} + c$$

Notice that your expression for s in terms of t contains an unknown constant c. This is called a **general solution** – it is an expression which represents all the possible solutions you could get by changing the value of c.

You can find the value of c in any particular situation if you know a pair of values of t and s. For example, if you know that, initially (i.e. when $t = 0$), $s = 0$, you would find that $c = s + e^{-t} = 0 + e^0 = 1$.

Separation of variables

So far, the only types of differential equation you can solve are of the form $\frac{dy}{dx} = f(x)$, which you did by integrating both sides with respect to x. However, there are many situations where the right-hand side can be a function of both x and y. In this situation, you cannot integrate the right-hand side with respect to x since y is not a constant. However, if the right-hand side can be separated into a function of x multiplied by a function of y, then a method called separation of variables can be used.

KEY POINT 13.1

If $\frac{dy}{dx} = f(x)g(y)$, then

$$\int \frac{1}{g(y)}dy = \int f(x)dx$$

Tip

This looks a lot like you are treating $\frac{dy}{dx}$ as a fraction, and at the moment it is not a problem if you think about it like that, but technically this is not the case. In more advanced work, you will see that you are actually going through the process of

$$\int \frac{1}{g(y)}\frac{dy}{dx}\,dx = \int f(x)dx$$

WORKED EXAMPLE 13.3

Find the general solution to $\frac{dy}{dx} = xy$, given that $y > 0$.

Get all the instances of x on one side, all the instances of y on the other and then integrate

$$\int \frac{1}{y}\,dy = \int x\,dx$$

Perform the integrals on both sides. We only need to put a constant of integration on one side

$$\ln|y| = \frac{x^2}{2} + c$$

We can use the fact that $y > 0$ to remove the modulus, then do e to the power of both sides to remove the natural logarithm

Since $y > 0$,

$$y = e^{\left(\frac{x^2}{2}+c\right)}$$

Technically we are already finished, but it can be very useful to write the solution in a slightly different form. Instead of having the arbitrary constant $+c$ in the exponent, we can replace it by a constant multiplier; normally A is used

$$= e^{\left(\frac{x^2}{2}\right)}e^{c}$$

$$= Ae^{\left(\frac{x^2}{2}\right)}$$

Tip

A common error when solving these differential equations is to just put a '$+c$' at the end of the solution, but as you can see in Worked Example 13.3, the answer is not $y = e^{\left(\frac{x^2}{2}\right)} + c$.

Be the Examiner 13.1

Find the general solution to $\frac{dy}{dx} = ky$, where k is a constant and $y > 0$.

Which is the correct solution? Identify the errors in the incorrect solutions.

Solution 1	Solution 2	Solution 3
$\int \frac{1}{y} dy = \int k\, dx$ $\ln y = kx + c$ $y = Ae^{kx}$	$\int \frac{1}{y} dy = \int k\, dx$ $-\frac{1}{y^2} = kx + c$ $y = \sqrt{\frac{1}{c - kx}}$	$\int \frac{1}{y} dy = \int k\, dx$ $\ln y = kx + c$ $y = e^{kx} + c$

Exercise 13A

For questions 1 to 4, use the method demonstrated in Worked Example 13.1 to write a differential equation to model the given situation.

1 a The growth rate of bacteria in a petri dish is proportional to the number of bacteria (b).

b The rate of growth of the area covered in moss is proportional to the square root of the area already covered (a).

2 a The rate of increase of the radius of an inflating balloon is inversely proportional to the square of the current radius (r).

b The rate of increase of height in a human is inversely proportional to the height (h).

3 a The rate of decrease of a population of fish is proportional to the square root of its current size (F).

b The rate of decrease of the velocity of a car is inversely proportional to its current velocity (v).

4 a The rate of increase of the number of people with a disease (I) in a population of size N is proportional to the number of people with the disease and proportional to the number of people without the disease.

b The rate of spread of a rumour is proportional to the number of people who know a rumour (R) in a group of size N and the number of people who do not know the rumour, and is inversely proportional to the time (t) since the rumour started.

For questions 5 to 8, use integration, as demonstrated in Worked Example 13.2, to find the general solution of the following differential equations.

5 a $\frac{dy}{dx} = x^2$

b $\frac{dy}{dx} = \frac{4}{\sqrt{x}}$

6 a $\frac{dy}{dt} = e^{2t}$

b $\frac{dy}{dt} = \frac{3}{t}$

7 a $\frac{ds}{dt} = \sin 3t$

b $\frac{ds}{dt} = \frac{2}{\cos^2 t}$

8 a $\frac{dF}{dm} + \frac{3}{m} = 1$

b $\frac{dF}{dm} + \frac{1}{m^2} = 3$

For questions 9 to 12, use separation of variables, as demonstrated in Worked Example 13.3, to find the general solution of the following differential equations.

9 a $\frac{dy}{dx} = 2y,\ y > 0$

b $\frac{dy}{dx} = -y,\ y > 0$

10 a $\frac{dy}{dx} = y + 1,\ y > -1$

b $\frac{dy}{dx} = 1 - y,\ y < 1$

11 a $\frac{dy}{dx} = x^2 y^2$

b $\frac{dy}{dx} = x^3 y^3,\ y > 0$

12 a $\frac{dy}{dx} = \frac{y}{x},\ x, y > 0$

b $\frac{dy}{dx} = \frac{x}{y},\ x, y > 0$

13 Find the general solution of the differential equation $\frac{dy}{dx} = y^2$.

14 Find the general solution of the differential equation $\frac{dy}{dx} = \frac{\cos x}{y^2}$.

15 Solve the differential equation $\frac{dy}{dx} = 2xe^{-y}$.

16 Find the general solution of the differential equation $x\frac{dy}{dx} = y$. Give your answer in the form $y = f(x)$.

17 Find the general solution of the differential equation $\frac{1}{y^2}\frac{dy}{dx} = 2x$.

18 Given that $\frac{dy}{dx} = \frac{y}{\cos^2 x}$,

a find the general solution of the differential equation.

b Given also that $y = 4$ when $x = 0$, find an expression for y in terms of x.

19 Find the solution of the differential equation $\frac{dy}{dx} = \frac{9x^2}{4y}$ for which $y = 3$ when $x = 0$. You may leave your answer in the form $f(y) = g(x)$.

20 For the differential equation $y^2\frac{dy}{dx} = 3x$, find the solution with $y = 3$ when $x = 2$.

21 **a** Find the general solution of the differential equation $\frac{dy}{dx} = 2(x+2)(y-1)$.

b Given that $y = 2$ when $x = 0$, show that $y = 1 + e^{x^2+4x}$.

22 Given that $\frac{dy}{dx} = \cos x \cos^2 y$, use separation of variables to show that $\sin x - \tan y = c$ for some constant c.

23 The mass (m grams) of a radioactive substance decays at a rate proportional to the current mass. Initially, the mass of the substance is 25 g and the rate of decay is 5 g per second.

a Find the constant k such that $\frac{dm}{dt} = -km$.

b Find an expression for the mass of the substance after t seconds.

c How long does it take for the mass to decay to half of its initial value?

24 The population of bacteria, N thousand, grows at a rate proportional to its size. The initial size of the population is 2000 and the initial rate of increase is 500 bacteria per minute.

a Find the constant k such that $\frac{dN}{dt} = kN$.

b Find the size of the population after 10 minutes, giving your answer to the nearest thousand.

25 A balloon is being inflated at a rate inversely proportional to its current volume. Initially, the volume of the balloon is 300 cm^3 and it is increasing at the rate of 10 cm^3 s^{-1}.

a Show that $\frac{dV}{dt} = \frac{3000}{V}$.

b Find the volume of the balloon after t seconds.

26 An object of mass 1 kg falls vertically through the air. Taking into account air resistance, the acceleration of the object is given by $\frac{dv}{dt} = 10 - 0.1v$, where v is the velocity in m s^{-1}.

a Given that the object starts from rest, find an expression for the velocity at time t seconds.

b Find the distance travelled by the object in the first three seconds.

27 Variables x and y satisfy the differential equation $y\frac{dy}{dx} = 4e^{-2x}$. When $x = 0$, $y = -2$. Find an expression for y in terms of x.

28 Find the general solution of the differential equation $\frac{dy}{dx} = e^{x+y}$.

29 Given that $\frac{dy}{dx} = 2e^{x-2y}$ and that $y = 0$ when $x = 0$, express y in terms of x.

30 The variables x and y satisfy the differential equation $\frac{dy}{dx} = \frac{\sin x}{y}$. When $x = 0$, $y = 10$. Find an expression for y in terms of x.

31 **a** Use separation of variables to show that the general solution of the differential equation $\frac{dy}{dx} = \frac{\cos x}{\sin y}$ can be written as $\sin x + \cos y = c$.

b A particular solution of the differential equation satisfies $0 \leqslant x \leqslant \pi$ and $-\frac{\pi}{2} \leqslant y \leqslant \frac{\pi}{2}$, and has $y = \frac{\pi}{3}$ when $x = \frac{\pi}{6}$. Find the two possible values of y when $x = \frac{\pi}{2}$.

32 A raindrop is modelled as a perfect sphere. Its volume decreases at a rate proportional to its surface area. When the volume is $0.5\,\text{cm}^3$, it is decreasing at a rate of $0.1\,\text{cm}^3$ per minute.

a Find an expression for $\frac{dr}{dt}$, where r is in cm and t is in minutes.

b Hence determine how long it takes to completely evaporate.

13B Slope fields and Euler's method

With many differential equations, you cannot find an exact expression for y in terms of x. However, that does not mean that the equation has no solution – it can still be explored graphically.

Slope fields

Given a differential equation

$$\frac{dy}{dx} = f(x, y)$$

we can find the gradient at any given point (a, b) by simply putting these values of $x = a$ and $y = b$ into the equation. Although we could pick any coordinates for this, it is most convenient to choose integer-valued points.

KEY POINT 13.2

A plot of the tangents at all points (x, y) is called the **slope field** of a differential equation.

From the slope field, we can then construct approximate solution curves that correspond to different initial conditions. To do so, we just observe two rules: solution curves

1 follow the direction of the tangents at each point

2 do not cross.

WORKED EXAMPLE 13.4

You are given the differential equation $\frac{dy}{dx} = x - y^2 + 2$.

a Construct a table showing the gradient of the slope field at the points with integer coordinates $-2 \leqslant x,\ y \leqslant 2$.

b Sketch the slope field of the differential equation.

c Hence sketch the solution curves passing through the points $(-2, 2)$, $(2, 1)$ and $(0, -2)$.

For each point, substitute x and y values into the differential equation to find $\frac{dy}{dx}$

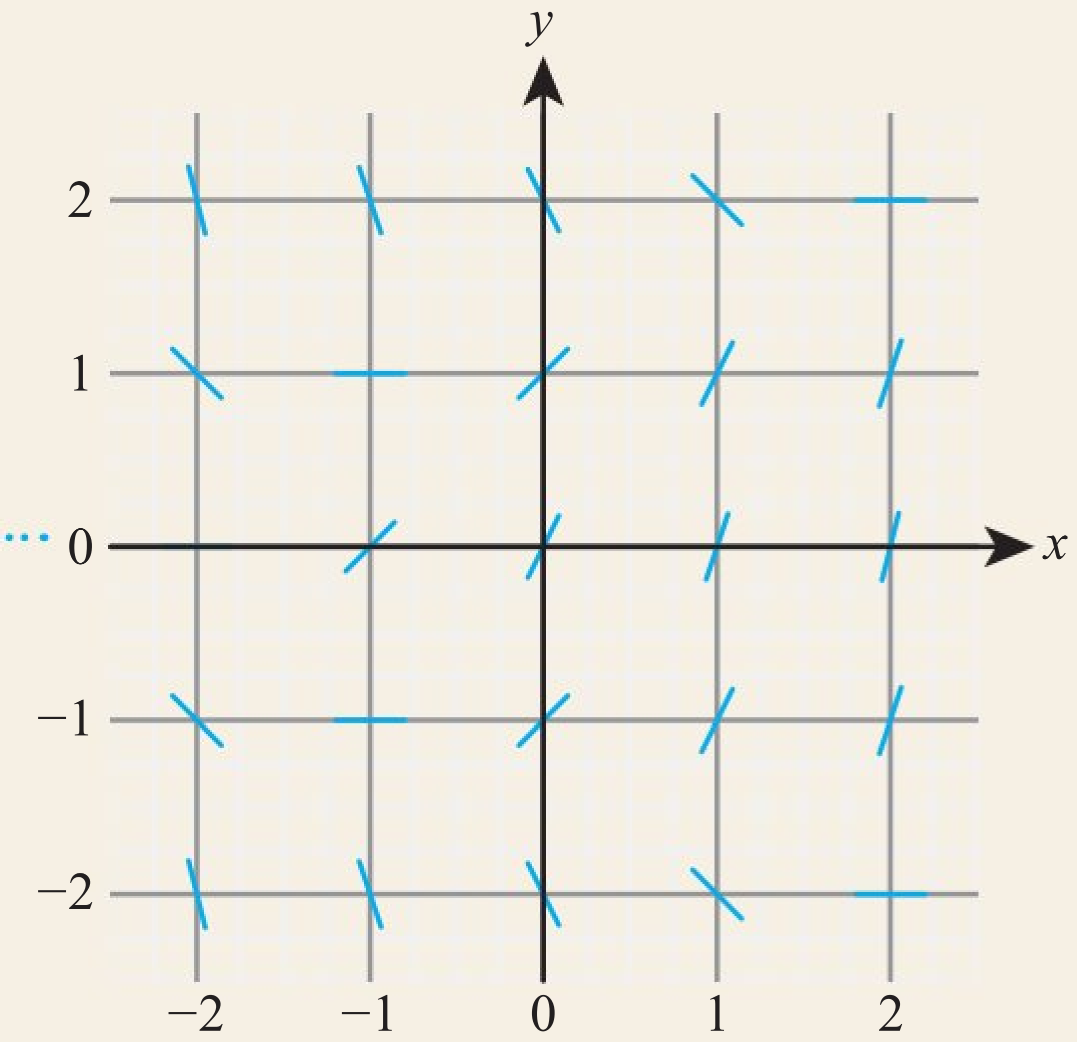
Show your calculation for at least one point

a e.g. at the point $(2, 2)$:

$$\frac{dy}{dx} = 2 - 2^2 + 2 = -4$$

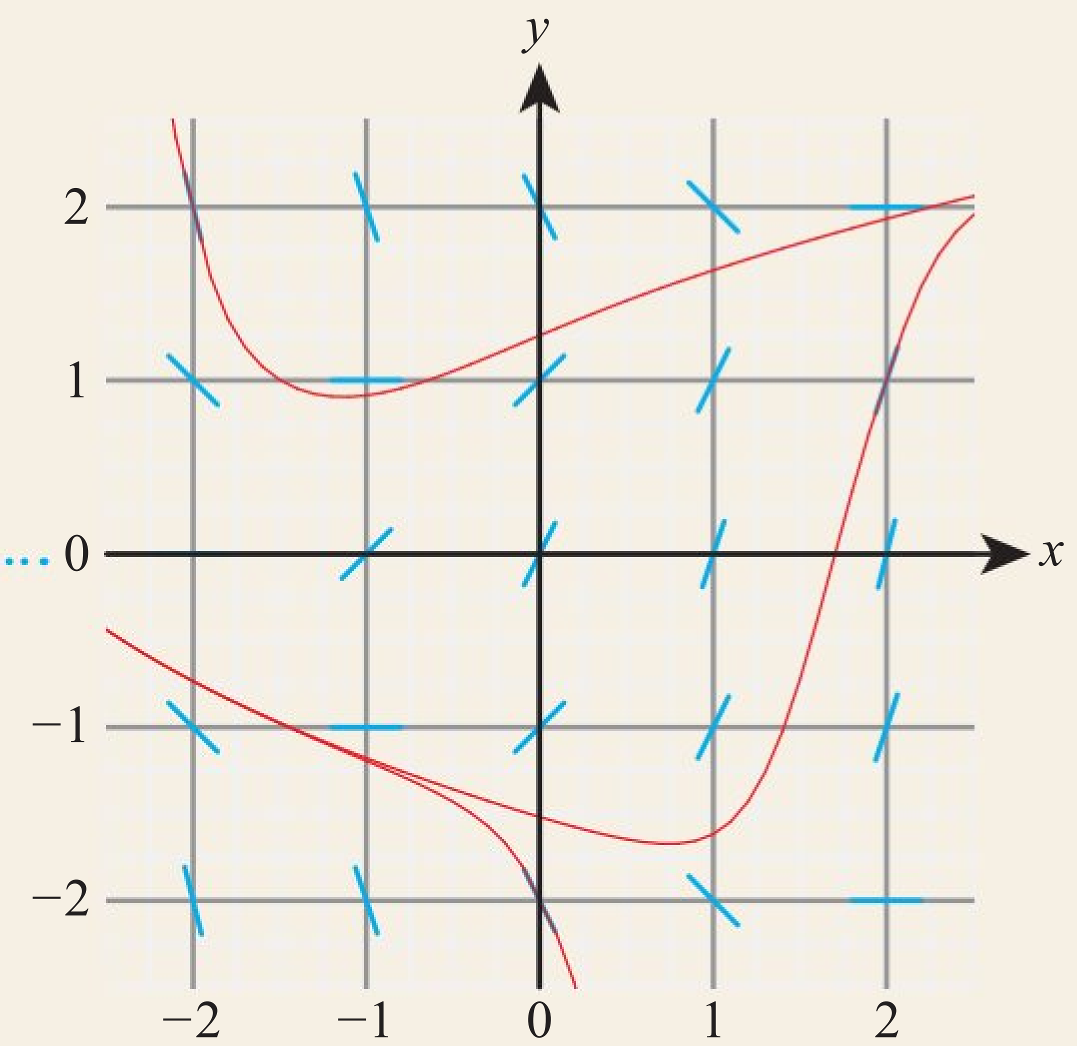
y \ x	−2	−1	0	1	2
−2	−4	−3	−2	−1	0
−1	−1	0	1	2	3
0	0	1	2	3	4
1	−1	0	1	2	2
2	−4	−3	−2	−1	0

b



You can represent the gradient at each point graphically by drawing the tangent at that point

c



Now follow the slope field from the three given points to draw solution curves, making sure they do not cross

Euler's method

We can calculate approximate y values for a sequence of x values by following the slope field.

To do this, we need to be given a differential equation in the form $\frac{dy}{dx} = f(x, y)$, and a starting pair of values, (x_0, y_0). We can use the differential equation to calculate the gradient at that point, $f'(x_0, y_0)$. We then use a fixed step length, h, to jump to the next value of x but use the gradient at the point we are leaving to determine how much y changes.

Since change in y is given by the change in x multiplied by the gradient, we find that

$$y_1 - y_0 \approx h \times f'(x_0, y_0)$$

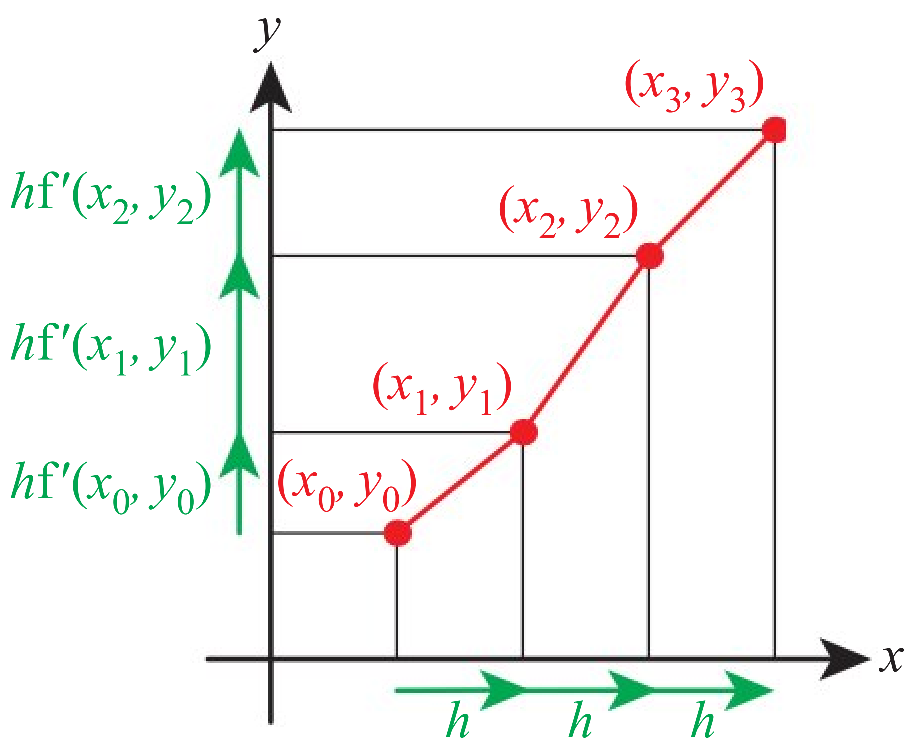
We can then repeat this process to get a general iteration formula.

KEY POINT 13.3

Euler's method:

- $x_{n+1} = x_n + h$
- $y_{n+1} = y_n + h \times f'(x_n, y_n)$

You can visualize this process graphically:



Tip

Make sure you are familiar with the sequence or iteration function on your calculator for this method. If you need to do many iterations to get to the required value, you can just record the first few and last few stages of the iteration.

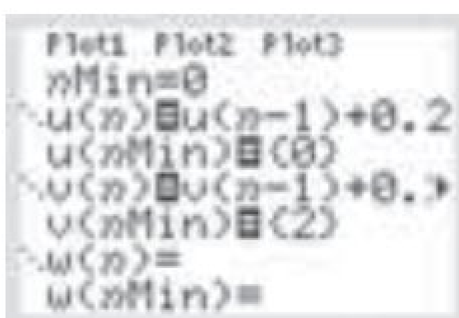
WORKED EXAMPLE 13.5

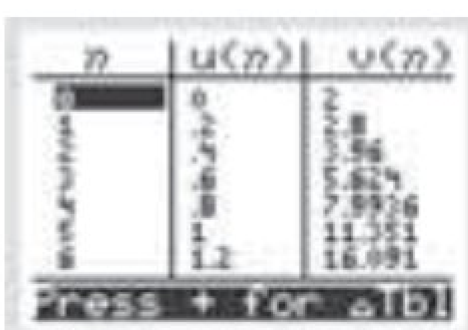
Use Euler's method with a step length of 0.2 to estimate the value of $y(1)$, given that $\frac{dy}{dx} = x + 2y$ and $y(0) = 2$.

Write down the iterative formula for the Euler method

$$y_{n+1} = y_n + 0.2(x_n + 2y_n)$$

Use your GDC to evaluate several iterations, recording the output at each stage





The process produces:

n	x_n	y_n
0	0	2
1	0.2	2.8
2	0.4	3.96
3	0.6	5.624
4	0.8	7.9936
5	1	11.35104

You can read off the value of y when $x = 1$

So, $y(1) \approx 11.4$

Leonard Euler (1707–1783) was born is Switzerland and was one of the most prolific and versatile mathematicians in history. His contributions ranged from celestial mechanics to ship building and music. He introduced the current mathematical meanings of the symbols $f(x)$, e, π and Σ and made some fundamental leaps in the understanding of complex numbers.

He continued writing even after going blind. He did much of the working in his head and instructed his sons to write down his results.

Leonard Euler (1707–1783)

You are the Researcher

There are various improvements that can be made to Euler's method. For example, if the derivative is just a function of x, then you can use the gradient at $\frac{(x_n + x_{n+1})}{2}$. There are further extensions to something called Runge–Kutta methods which are used in most modern computer programs to solve real-world differential equations. They will probably have been used by engineers to study the effect of wind on the next bridge you cross, by animators to make realistic looking hair in CGI graphics and by gaming programmers to make characters run, swim and jump correctly. This might be an interesting subject for an exploration, but remember that your use of mathematics has to be sophisticated and rigorous.

TOOLKIT: Problem Solving

Under what conditions will Euler's method underestimate the true value? When will it overestimate the true value?

What can you say about any predictions made by Euler's method in solving the differential equation $\frac{dy}{dx} = x^3$?

TOK Links

Which is better – a perfect solution to equations which vaguely model the real-world situation, or approximate solutions to equations which accurately model the real world situation? How should we decide when a model, method or solution is good enough?

Coupled systems

There are many situations where two variables are linked by coupled differential equations, where the rate of change of each variable depends on time as well as the value of the other variable. For example, the growth of two competing species could be modelled by such a system.

A coupled system of differential equations can be written as

$$\frac{dx}{dt} = f_1(x, y, t), \frac{dy}{dt} = f_2(x, y, t)$$

Euler's method can be extended to find how the x and y values change when t jumps in fixed amounts of size h.

KEY POINT 13.4

- $t_{n+1} = t_n + h$
- $x_{n+1} = x_n + h \times f_1(x_n, y_n, t_n)$
- $y_{n+1} = y_n + h \times f_2(x_n, y_n, t_n)$

WORKED EXAMPLE 13.6

Variables x and y are related by the coupled system

$$\frac{\mathrm{d}x}{\mathrm{d}t} = (x+y)\sin t,\ \frac{\mathrm{d}y}{\mathrm{d}t} = (x+y)\cos t$$

When $t = 0$, $x = 3$ and $y = 4$. Use Euler's method with step length 0.1 to find the values of x and y when $t = 0.5$.

Write down the iterative formula for Euler's method

$$x_{n+1} = x_n + 0.1(x_n + y_n)\sin t_n$$

$$y_{n+1} = y_n + 0.1(x_n + y_n)\cos t_n$$

Use your GDC to carry out the iteration and record the results in a table

t_n	x_n	y_n
0	3	4
0.1	3	4.70
0.2	3.08	5.47
0.3	3.25	6.30
0.4	3.53	7.22
0.5	3.95	8.21

So, $x(0.5) \approx 3.95$ and $y(0.5) \approx 8.21$

Exercise 13B

For questions 1 to 3, use the method demonstrated in Worked Example 13.4 to sketch the slope fields and solution curves for the following differential equations.

1 a $\frac{\mathrm{d}y}{\mathrm{d}x} = 2x - y$
b $\frac{\mathrm{d}y}{\mathrm{d}x} = xy$

2 a $\frac{\mathrm{d}y}{\mathrm{d}x} = 2x - y^2$
b $\frac{\mathrm{d}y}{\mathrm{d}x} = xy + 2x$

3 a $\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{x}{y}$
b $\frac{\mathrm{d}y}{\mathrm{d}x} = x^2 + y - 3$

For questions 4 to 7, use Euler's method, as demonstrated in Worked Example 13.5, with a step length of 0.2 to estimate $y(1)$, given that $y(0) = 1$, for each of the following differential equations.

4 a $\frac{\mathrm{d}y}{\mathrm{d}x} = y$
b $\frac{\mathrm{d}y}{\mathrm{d}x} = x$

5 a $\frac{\mathrm{d}y}{\mathrm{d}x} = x + y$
b $\frac{\mathrm{d}y}{\mathrm{d}x} = x - y$

6 a $\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{x}{y}$
b $\frac{\mathrm{d}y}{\mathrm{d}x} = xy$

7 a $\frac{\mathrm{d}y}{\mathrm{d}x} = x^2 + y^2$
b $\frac{\mathrm{d}y}{\mathrm{d}x} = ye^x$

For questions 8 to 10, use Euler's method, as demonstrated in Worked Example 13.6, with a step length of 0.1 to estimate the values of x and y when $t = 0.5$ for each of the following systems of differential equations.

8 a $\frac{\mathrm{d}x}{\mathrm{d}t} = 2y - t,\ \frac{\mathrm{d}y}{\mathrm{d}t} = -3x + t$ with $x(0) = 0.3$ and $y(0) = 0.2$
b $\frac{\mathrm{d}x}{\mathrm{d}t} = -4y + t,\ \frac{\mathrm{d}y}{\mathrm{d}t} = 5x - t$ with $x(0) = -1$ and $y(0) = 2$

9 a $\frac{\mathrm{d}x}{\mathrm{d}t} = 2xy,\ \frac{\mathrm{d}y}{\mathrm{d}t} = -3xy^2$ with $x(0) = 0.5$ and $y(0) = -0.5$
b $\frac{\mathrm{d}x}{\mathrm{d}t} = x + y^2,\ \frac{\mathrm{d}y}{\mathrm{d}t} = y - 3x^2$ with $x(0) = -1$ and $y(0) = 1$

10 a $\frac{\mathrm{d}x}{\mathrm{d}t} = \sin y \cos t,\ \frac{\mathrm{d}y}{\mathrm{d}t} = -\cos x \sin t$ with $x(0) = y(0) = 1$
b $\frac{\mathrm{d}x}{\mathrm{d}t} = y + 2e^{-2t},\ \frac{\mathrm{d}y}{\mathrm{d}t} = 3x - e^{-2t}$ with $x(0) = y(0) = 0$

11 Consider the differential equation

$$\frac{dy}{dx} = 2x$$

with the initial condition $y = 0$ when $x = 0$.

a Use Euler's method with step length 0.1 to estimate

i $y(1)$ **ii** $y(2)$

b Use Euler's method with step length 0.2 to estimate

i $y(1)$ **ii** $y(2)$

c Solve the differential equation exactly. Hence determine which one of your estimates in parts **a** and **b** are furthest away from the true value.

12 For the differential equation

$$\frac{dy}{dx} = (y+1)(x-3)$$

a construct the slope field at the points with integer coordinates $-2 \leqslant x \leqslant 4$, $-3 \leqslant y \leqslant 3$

b sketch the solution curves.

c What can you say about the solution curve corresponding to the initial condition $x = 0$, $y = -1$?

13 Consider the differential equation

$$\frac{dy}{dx} = \frac{xy}{x+y}$$

with the initial condition $y = 1$ when $x = 0$.

a Use Euler's method with step length 0.5 to estimate the value of y when $x = 10$.

b How could this estimate be improved?

14 The populations of rabbits (x) and foxes (y) on an island, at time t years, are modelled by the coupled system of equations

$$\frac{dx}{dt} = 0.005x(x-3y-20),\ \frac{dy}{dt} = 0.005y(3x-5y-300)$$

Initially, there are 150 rabbits and 40 foxes.

Use Euler's method with step length 0.1 to find the number of rabbits and foxes after six months.

15 Two water reservoirs are connected so that the water can flow between them. The depths of water in the two reservoirs, x m and y m, are modelled by the system of equations.

$$\frac{dx}{dt} = -0.2x\cos(\pi t),\ \frac{dy}{dt} = (-0.5y+0.2x)\cos(\pi t)$$

where t is the time in years. Initially, the heights were $x = 8.5$ and $y = 10.2$.

Use Euler's method with step length 0.01 to find the depth of water in each of the two reservoirs after three months.

16 For the differential equation $\dfrac{dy}{dx} = \dfrac{(x+y)}{xy+2}$,

a sketch the slope field for $-2 \leqslant x \leqslant 2$, $-2 \leqslant y \leqslant 2$

b sketch the solution curve passing through (0, 1).

c Given that $y = 1$ when $x = 0$, use Euler's method with step length $h = 0.2$ to find an approximate value of y when $x = 1$. Give your answer correct to three decimal places.

17 Use Euler's method with step length of 0.1 to approximately sketch the solution for $1 \leqslant x \leqslant 5$ to

$$\frac{dy}{dx} = y^2 - x^2$$

with $y(1) = 1$.

18 **a** Use Euler's method with step length of 0.1 to approximately sketch the solution for $0 \leqslant x \leqslant 4$ to

$$\frac{dy}{dx} = \sin(x+y)$$

with x and y in radians and $y(0) = 0$.

b Hence estimate, to one decimal place, the largest value of y in this range.

19 The height of a piece of ash (h metres) falling vertically into a fire is modelled by

$$\frac{dh}{dt} = -0.1h^2 - 0.5t$$

where t is in seconds.

Initially, the ash is 2 metres above the fire. Use the Euler's method to

a estimate the height of the piece of ash after 1 second

b estimate the time (to the nearest second) it takes to reach the fire.

20 Consider the differential equation $\frac{dy}{dx} = xe^y$ with $y = 0.3$ when $x = 1$.

a Use Euler's method with $h = 0.1$ to find an approximate value of y when $x = 1.3$.

b Solve the differential equation.

c i Find the percentage error in your approximation from part a.

ii How can this error be decreased?

21 The function $y = f(x)$ satisfies the differential eqation $f'(x) = x^2y$ with $f(0) = 0.5$.

a i Use Euler's method with step length $h = 0.25$ to find an approximate value of f(1).

ii How can your approximation be made more accurate?

b Solve the differential equation and hence find the actual value of f(1).

c Sketch the graph of your solution and use it to explain why your approximation from part a is smaller than the actual value of f(1).

22 The following table shows the speed of a car as it is accelerating, measured at 3 second intervals.

Time (s)	0	3	6	9	12	15
Speed (ms^{-1})	0	6	12	19	24	27

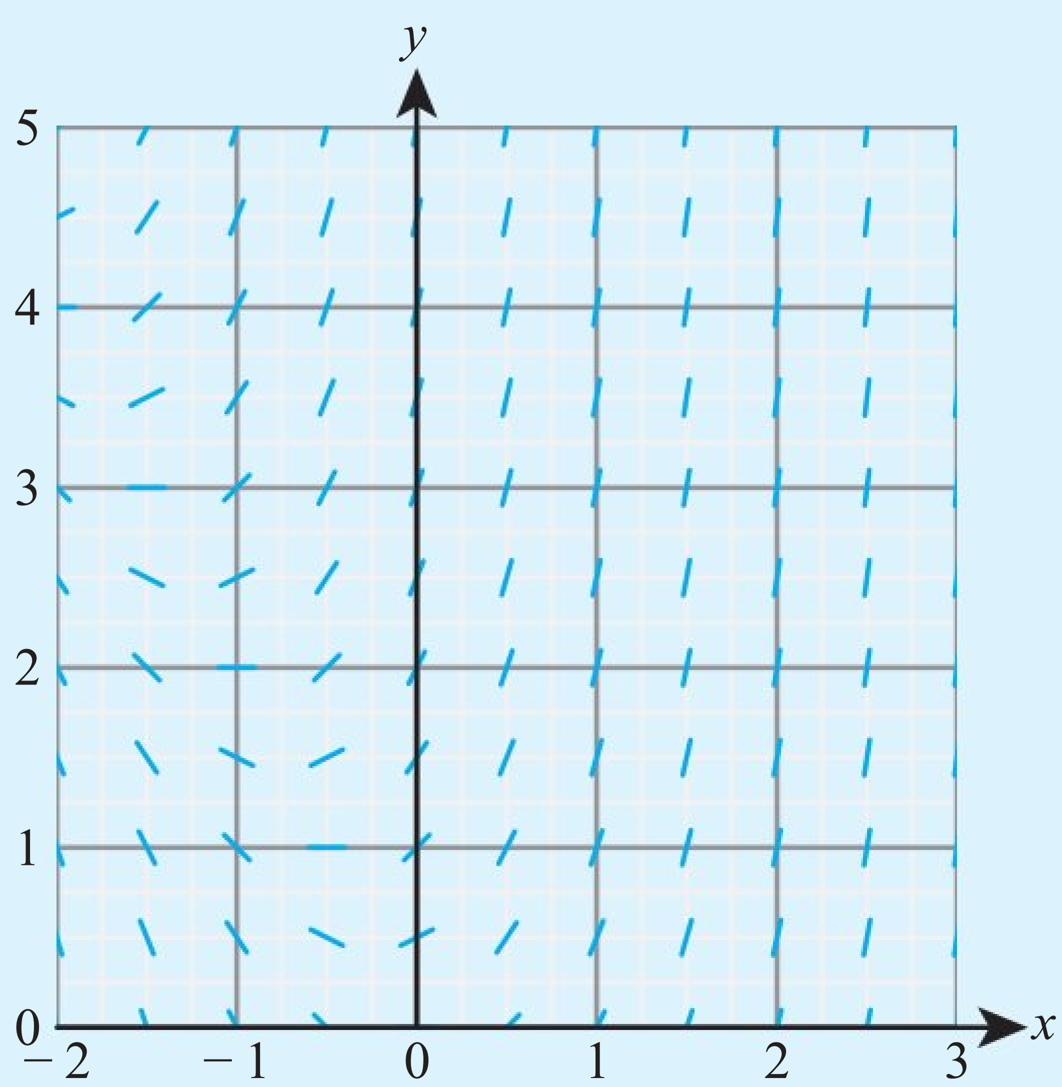
Use the Euler method to estimate the distance travelled by the car in the first 15 seconds.

23 The following table shows the speed of a car as it is decelerating, measured at 3 second intervals.

Time (s)	0	3	6	9	12	15
Speed (ms^{-1})	20	15	10	8	6	5

Use the Euler method to estimate the distance travelled by the car in the first 15 seconds.

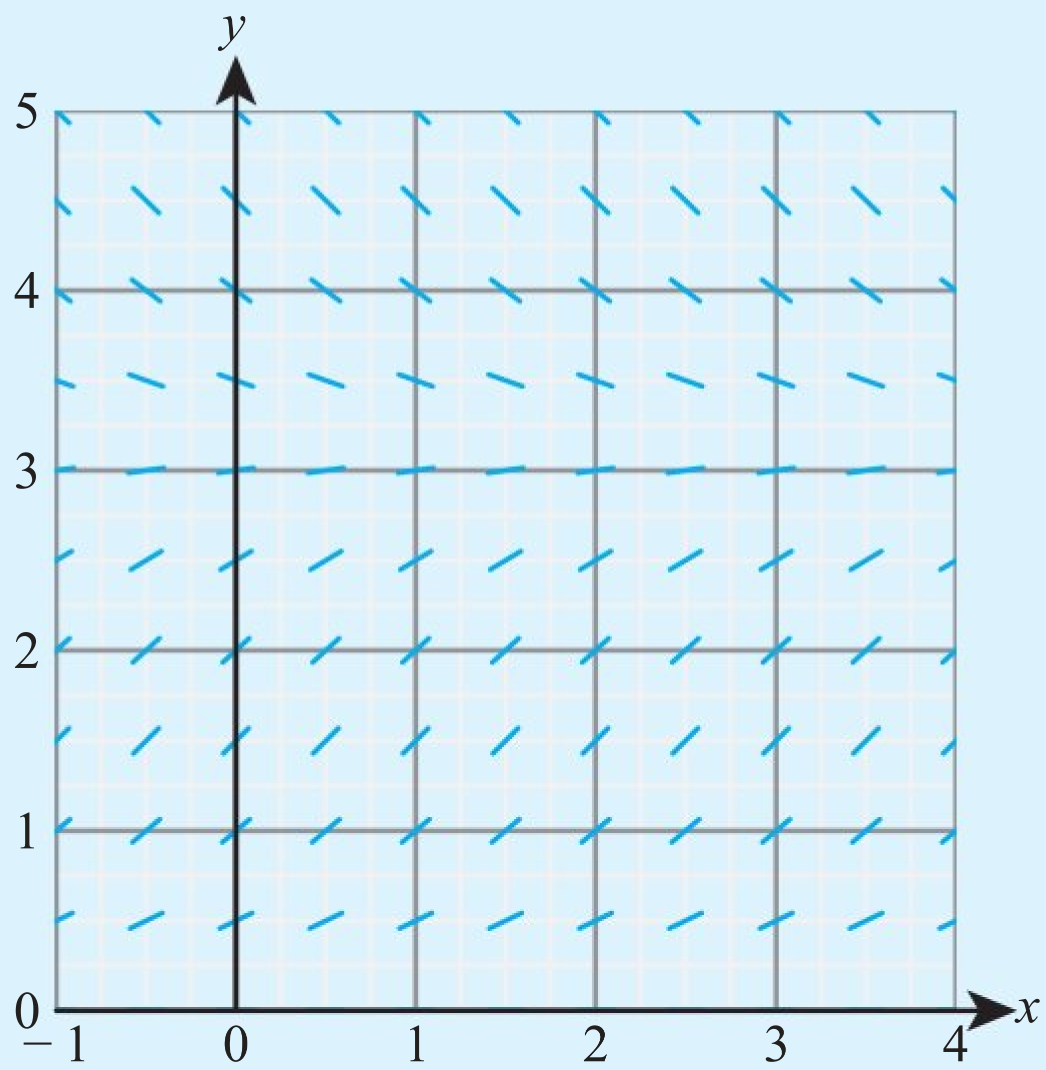
24 The diagram shows the slope field for a differential equation.



a Draw a solution curve passing through (0, 1).

b Given that $y(0) = 1$, estimate the value of $y(1)$.

25 The diagram shows the slope field for a differential equation.



a Draw a solution curve which has $y = 3$ when $x = 1$.

b For the above solution curve, estimate the value of y when $x = 3$.

13C Coupled systems and phase portraits

In Section 13B, you met coupled systems of differential equations, which can be used to model two related variables changing with time.

Exact solution of linear systems with real distinct eigenvalues

In the special case when the functions f_1 and f_2 are linear functions of x and y, it is possible to solve the coupled equations exactly. The system

$$\begin{cases} \frac{\mathrm{d}x}{\mathrm{d}t} = ax + by \\ \frac{\mathrm{d}y}{\mathrm{d}t} = cx + dy \end{cases}$$

can be written in matrix form as

$$\begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = \mathbf{M}\begin{pmatrix} x \\ y \end{pmatrix}, \text{ where } \mathbf{M} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

Tip

Remember that $\dot{x}$ means $\frac{\mathrm{d}x}{\mathrm{d}t}$.

The exact form of the solution depends on the eigenvalues of **M**. You only need to be able to write down the general solution when the eigenvalues of **M** are real and distinct.

KEY POINT 13.5

If **M** has real and distinct eigenvalues λ_1 and λ_2 with corresponding eigenvectors $\mathbf{p}_1$ and $\mathbf{p}_2$, then the general solution of the system is

$$\begin{pmatrix} x \\ y \end{pmatrix} = A\mathrm{e}^{\lambda_1 t}\mathbf{p}_1 + B\mathrm{e}^{\lambda_2 t}\mathbf{p}_2$$

You learnt about eigenvalues and eigenvectors in Section 3D.

Tip

Note that the general solution contains two arbitrary constants, A and B.

In Section 3D, you learnt how to diagonalize a matrix using its eigenvalues and eigenvectors. You can use the diagonalized form to derive the general solution of the differential equation $\begin{pmatrix}\dot{x}\\ \dot{y}\end{pmatrix} = \mathbf{M}\begin{pmatrix}x\\ y\end{pmatrix}$. However, you are not required to learn this derivation, and can just use the result given in the Mathematics: applications and interpretation formula book.

Proof 13.1

Prove that $\begin{pmatrix}x\\ y\end{pmatrix} = Ae^{\lambda_1 t}\mathbf{p}_1 + Be^{\lambda_2 t}\mathbf{p}_2$ satisfies the equation $\begin{pmatrix}\dot{x}\\ \dot{y}\end{pmatrix} = \mathbf{M}\begin{pmatrix}x\\ y\end{pmatrix}$.

Differentiate $\begin{pmatrix}x\\ y\end{pmatrix}$ with respect to t, remembering that the vectors $\mathbf{p}_1$ and $\mathbf{p}_2$ are constant

$$\begin{pmatrix}\dot{x}\\ \dot{y}\end{pmatrix} = A\lambda_1 e^{\lambda_1 t}\mathbf{p}_1 + B\lambda_2 e^{\lambda_2 t}\mathbf{p}_2$$

Now calculate $\mathbf{M}\begin{pmatrix}x\\ y\end{pmatrix}$

$$\mathbf{M}\begin{pmatrix}x\\ y\end{pmatrix} = \mathbf{M}\left(Ae^{\lambda_1 t}\mathbf{p}_1 + Be^{\lambda_2 t}\mathbf{p}_2\right)$$

$Ae^{\lambda_1 t}$ and $Be^{\lambda_2 t}$ are scalar multiples, so they can be taken out of the matrix multiplication

$$= \left(Ae^{\lambda_1 t}\right)\mathbf{M}\mathbf{p}_1 + \left(Be^{\lambda_2 t}\right)\mathbf{M}\mathbf{p}_2$$

Use the fact that $\mathbf{p}_1$ and $\mathbf{p}_2$ are eigenvectors of $\mathbf{M}$, so $\mathbf{M}\mathbf{p}_1 = \lambda_1\mathbf{p}_1$ and $\mathbf{M}\mathbf{p}_2 = \lambda_2\mathbf{p}_2$

$$= \left(Ae^{\lambda_1 t}\right)\lambda_1\mathbf{p}_1 + \left(Be^{\lambda_2 t}\right)\lambda_2\mathbf{p}_2$$

$$= A\lambda_1 e^{\lambda_1 t}\mathbf{p}_1 + B\lambda_2 e^{\lambda_2 t}\mathbf{p}_2$$

Hence, $\begin{pmatrix}\dot{x}\\ \dot{y}\end{pmatrix} = \mathbf{M}\begin{pmatrix}x\\ y\end{pmatrix}$, as required

TOOLKIT: Problem Solving

Proof 13.1 shows that the given solution works, but not that it is the only possible solution. It is in fact possible to prove that the solution is unique, but this goes far beyond this syllabus.

WORKED EXAMPLE 13.7

Find the general solution of the coupled differential equations

$$\begin{cases}\dfrac{dx}{dt} = x + 2y\\[2mm] \dfrac{dy}{dt} = 2x + y\end{cases}$$

Write down the matrix for the system

$$\mathbf{M} = \begin{pmatrix}1 & 2\\ 2 & 1\end{pmatrix}$$

Find the eigenvalues

$$\det\begin{pmatrix}1-\lambda & 2\\ 2 & 1-\lambda\end{pmatrix} = 0$$

$$(1-\lambda)^2 - 4 = 0$$

$$\lambda = -1 \text{ or } 3$$

Find the associated eigenvectors

When $\lambda = -1$,

$$\begin{pmatrix} x + 2y \\ 2x + y \end{pmatrix} = -\begin{pmatrix} x \\ y \end{pmatrix}$$

$$2x = -2y$$

So,

$$\mathbf{p}_1 = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$$

When $\lambda = 3$,

$$\begin{pmatrix} x + 2y \\ 2x + y \end{pmatrix} = 3\begin{pmatrix} x \\ y \end{pmatrix}$$

$$2x = 2y$$

So,

$$\mathbf{p}_2 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

Write down the general solution: $\begin{pmatrix} x \\ y \end{pmatrix} = Ae^{\lambda_1 t}\mathbf{p}_1 + Be^{\lambda_2 t}\mathbf{p}_2$

The general solution is

$$\begin{pmatrix} x \\ y \end{pmatrix} = Ae^{-t}\begin{pmatrix} 1 \\ -1 \end{pmatrix} + Be^{3t}\begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

Tip

If you only need to find one of the variables x or y, then you don't need to find the eigenvectors. You can just write $x = Ce^{\lambda_1 t} + De^{\lambda_2 t}$.

Phase portraits

A diagram showing how x and y values change over time is called a **phase portrait**.

In the case of real eigenvalues, the solution is of the form $\begin{pmatrix} x \\ y \end{pmatrix} = Ae^{\lambda_1 t}\mathbf{p}_1 + Be^{\lambda_2 t}\mathbf{p}_2$.

The shape of the phase portrait depends on whether the eigenvalues are positive or negative. For example, in Worked Example 13.7 above, $\begin{pmatrix} x \\ y \end{pmatrix} = Ae^{-t}\begin{pmatrix} 1 \\ -1 \end{pmatrix} + Be^{3t}\begin{pmatrix} 1 \\ 1 \end{pmatrix}$.

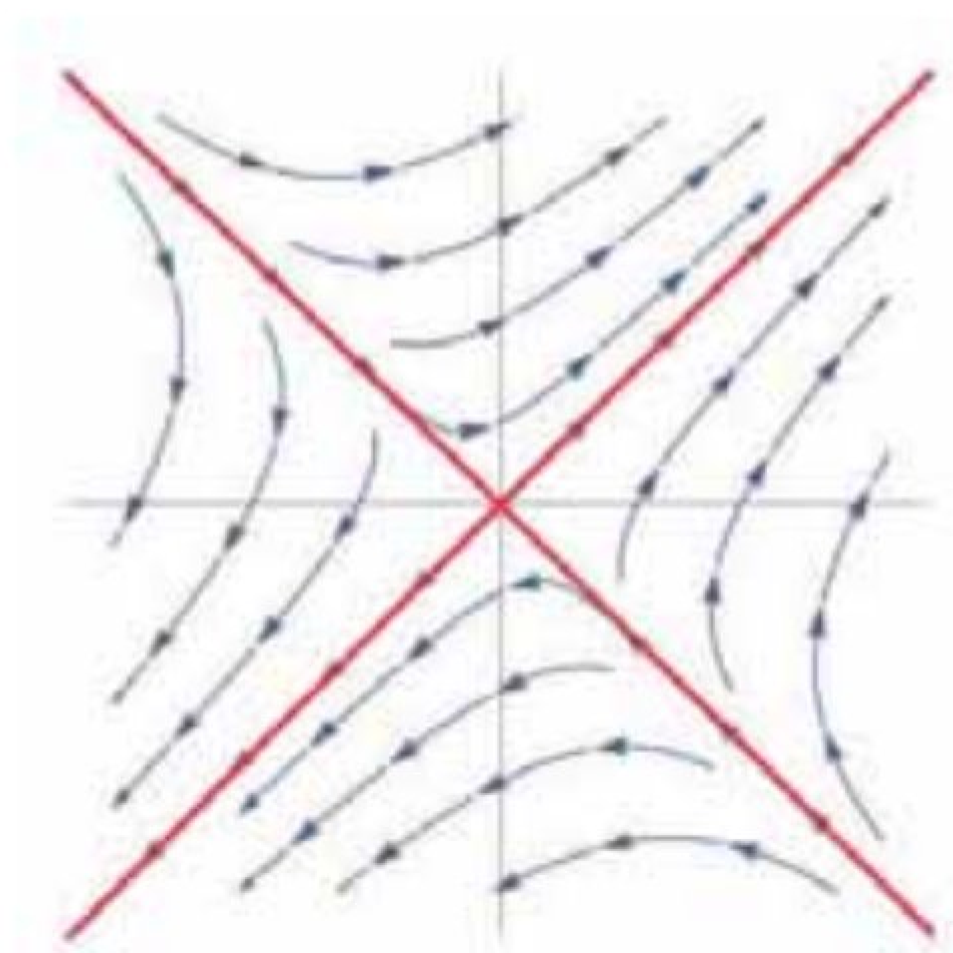
For large positive values of t, the first term tends to zero, so the solution curves approach the direction $\begin{pmatrix} 1 \\ 1 \end{pmatrix}$. On the other hand, when $t \to -\infty$, the solution curves approach the direction $\begin{pmatrix} 1 \\ -1 \end{pmatrix}$. The diagram below shows the phase portrait for this system.

Tip

You would not be expected to give this much detail when sketching a phase portrait.

The two red lines show the directions of the eigenvectors. The blue lines have been generated by technology to show trajectories for different initial conditions. The arrows indicate the direction of travel (as t increases, x and y change in the direction of the arrows). You could visualize this as an ocean current carrying a small object.

The two red lines are the eigenvectors. The trajectories along the eigenvector with the positive eigenvalue move away from the origin, and vice versa.



Tip

'Saddle point' is official IB terminology and may be used in examinations. There is no official IB terminology for a 'node' and you may see different terms used in different textbooks.

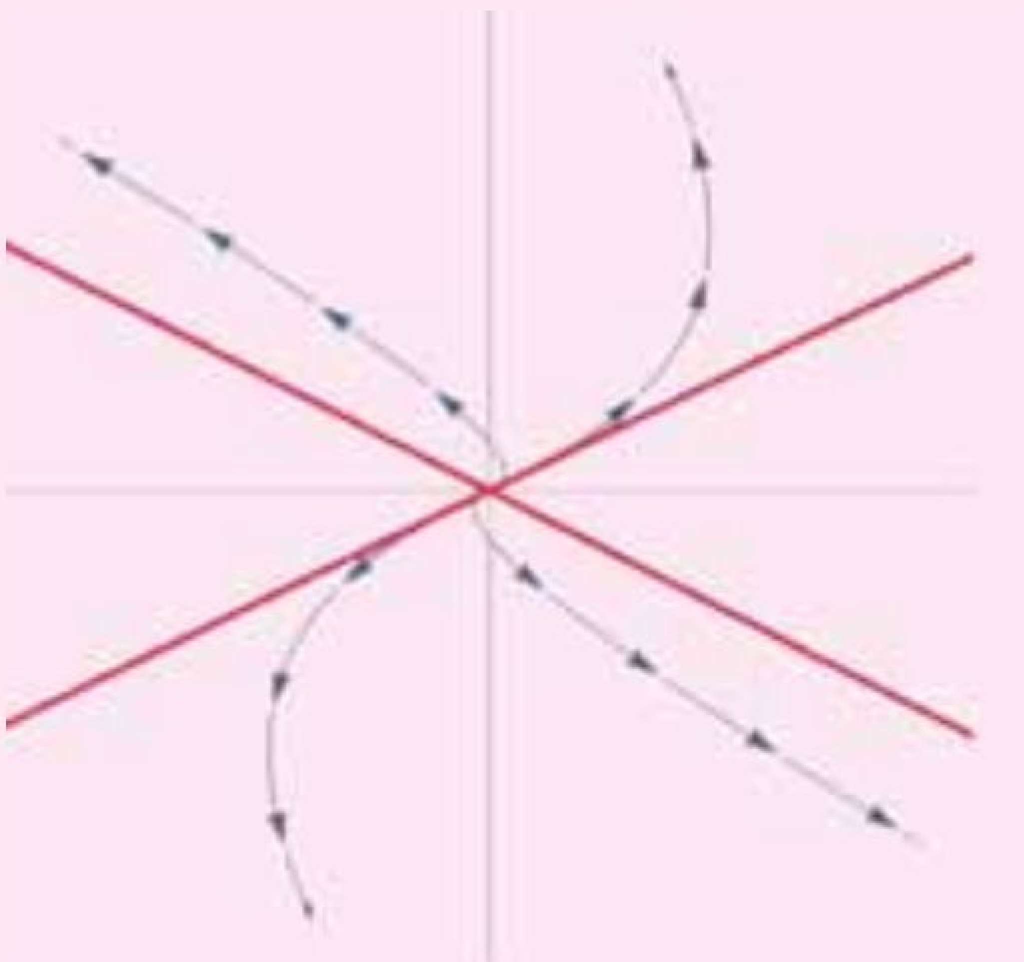
If both eigenvalues are negative, all solution curves will move towards the origin. If both eigenvalues are positive, both x and y move away from the origin.

If you start at the origin, you will stay there (since x' and y' are zero there); the origin is called an **equilibrium point**. If, in the long term, all solution curves move towards the equilibrium point, we say that the equilibrium point is **stable**; otherwise it is **unstable**. The equilibrium point with one positive and one negative eigenvalue is called a **saddle point**, otherwise it is called a (stable or unstable) **node**.

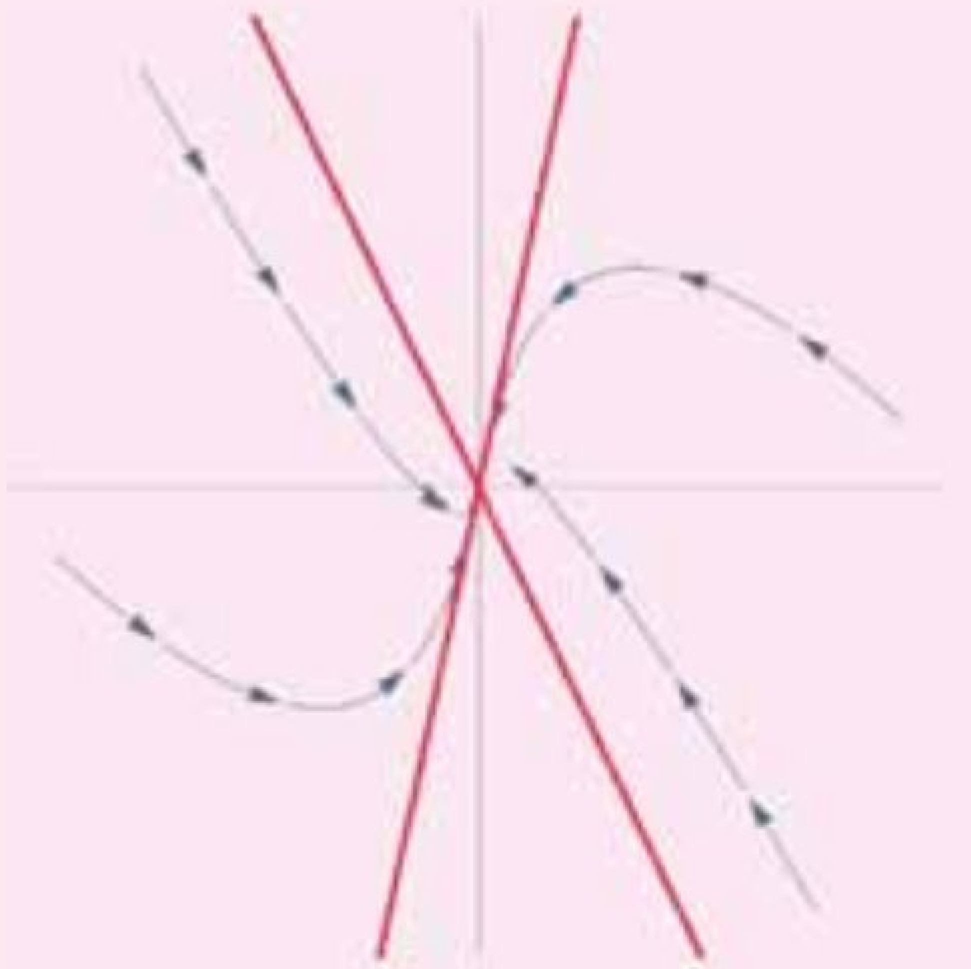
Sketching phase portraits can seem intimidating at first. Thankfully, when the eigenvalues are real and distinct, there are only three options.

KEY POINT 13.6

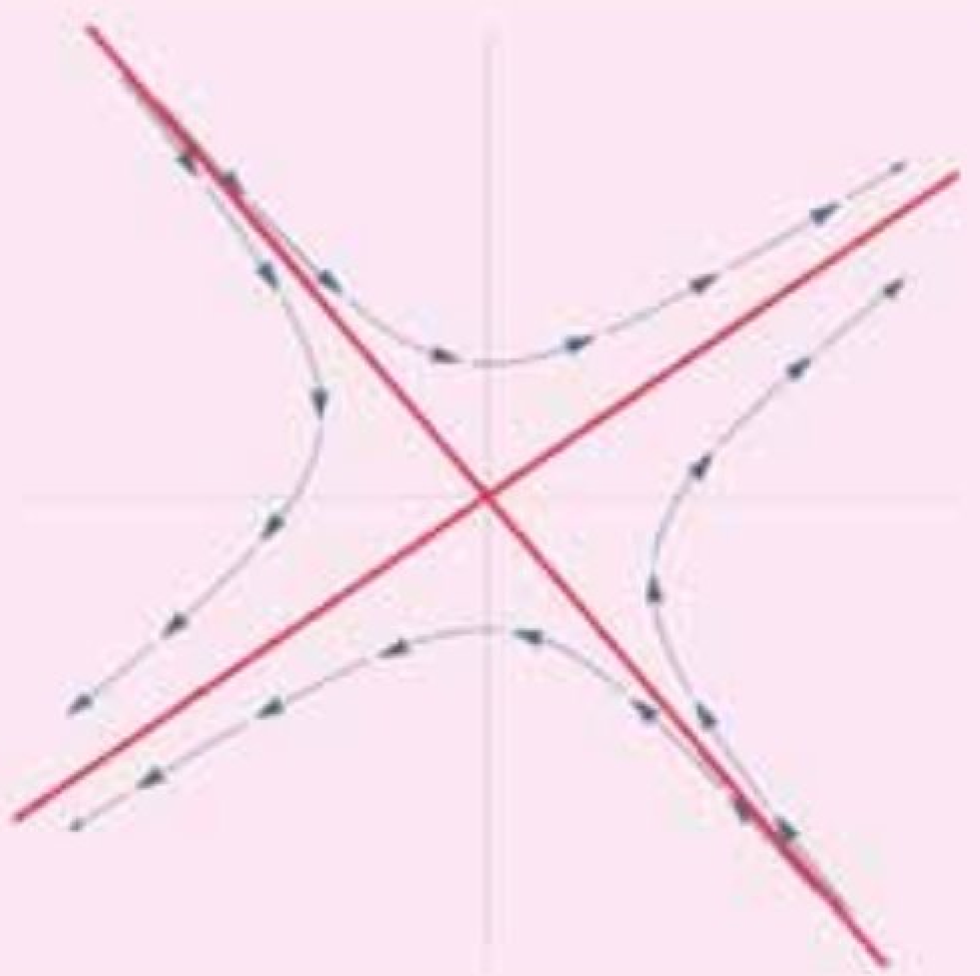
Phase portraits for real distinct eigenvalues.



■ Unstable node: $\lambda_1, \lambda_2 > 0$



■ Stable node: $\lambda_1, \lambda_2 < 0$



■ Saddle point (unstable): $\lambda_1 < 0 < \lambda_2$

Tip

When sketching a phase portrait, it is enough to show four trajectories. In the case of a saddle point, you should start by drawing the eigenvectors (shown as red lines). Sometimes eigenvectors can be useful for nodes as well.

WORKED EXAMPLE 13.8

a Find the eigenvalues for the matrix $\begin{pmatrix} 5 & -2 \\ 1 & 2 \end{pmatrix}$.

b Hence sketch the phase portrait for the system of differential equations $\begin{cases} \dfrac{dx}{dt} = 5x - 2y \\ \dfrac{dy}{dt} = x + 2y \end{cases}$

Solve the equation to find the eigenvalues

a $\det\begin{pmatrix} 5-\lambda & -2 \\ 1 & 2-\lambda \end{pmatrix} = 0$

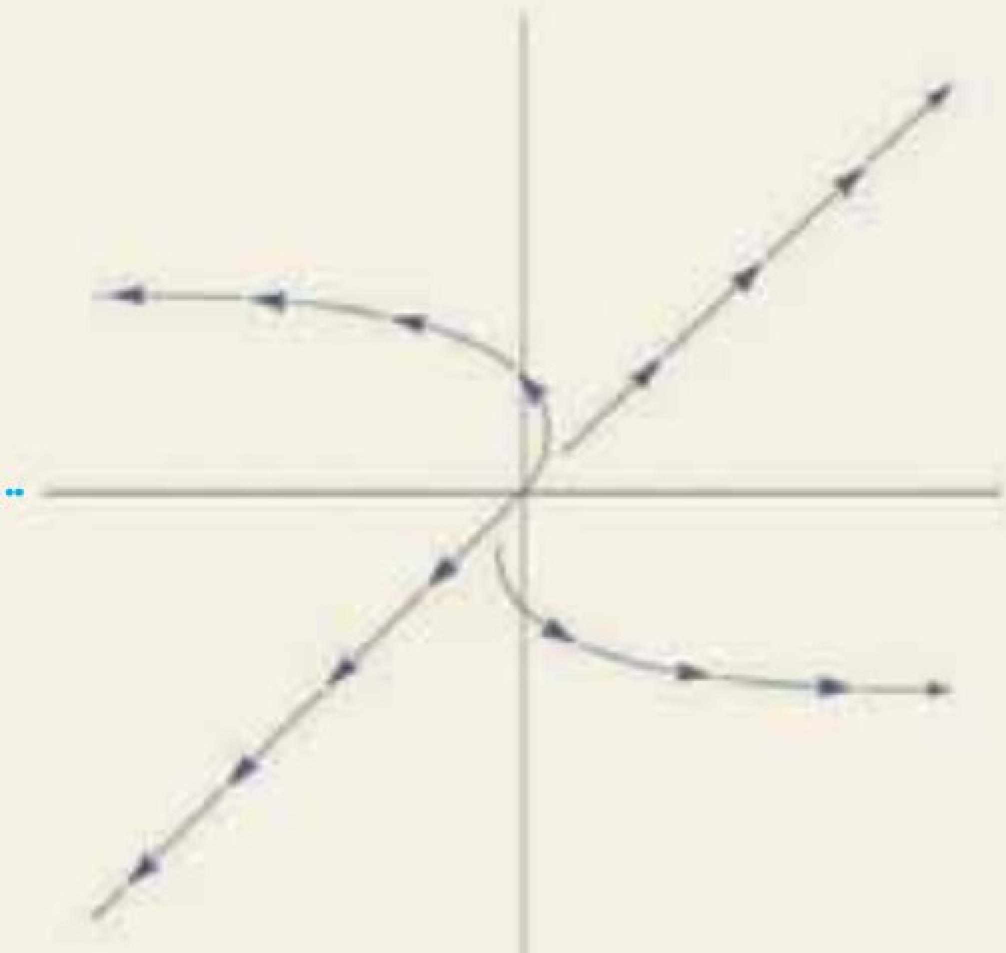
$\lambda^2 - 7\lambda + 12 = 0$

$\lambda = 3$ or 4

Both eigenvalues are positive, so the origin is an unstable node

b $\lambda_1, \lambda_2 > 0$

Draw several trajectories moving away from the origin. The exact shape of trajectories is unimportant.



Different mathematical communities have developed their own terminology of phase portraits. For example, an equilibrium can be described as a sink, source, node, focus, centre, Jordan node, nilpotent singularity and more.

When the eigenvalues are complex, the solution can still be written in the form $\begin{pmatrix} x \\ y \end{pmatrix} = Ae^{\lambda_1 t}\mathbf{p}_1 + Be^{\lambda_2 t}\mathbf{p}_2$. You do not need to be able to find the eigenvectors in this case, but you can still sketch the phase portrait. The equilibrium point in this case is called a **focus**.

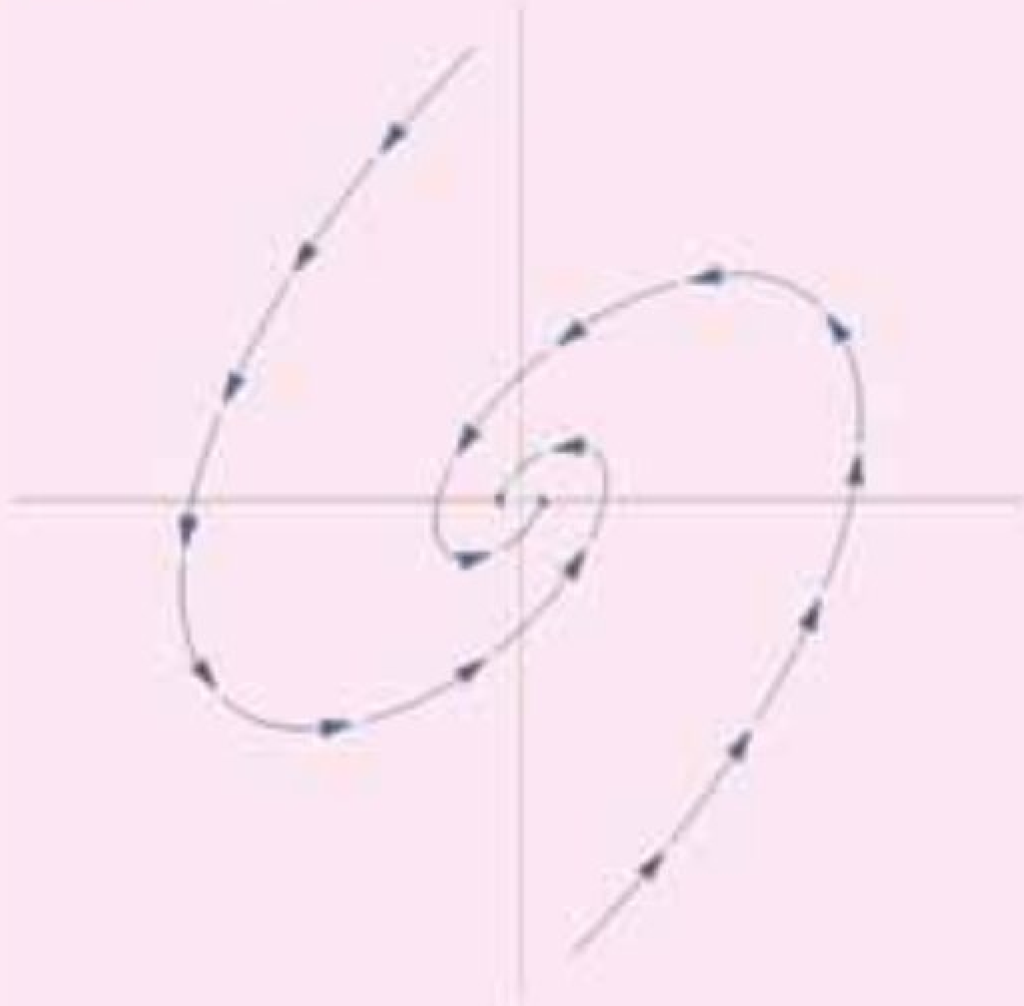
The complex eigenvalues are of the form $\lambda = p \pm iq$. The imaginary part introduces the factor $e^{\pm iqt}$ into the solution, and this can be written in terms of $\sin qt$ and $\cos qt$. This means that x and y vary periodically with t, creating a spiral on the phase portrait. The sign of the real part of λ determines whether the radius of the spiral increases or decreases.

KEY POINT 13.7

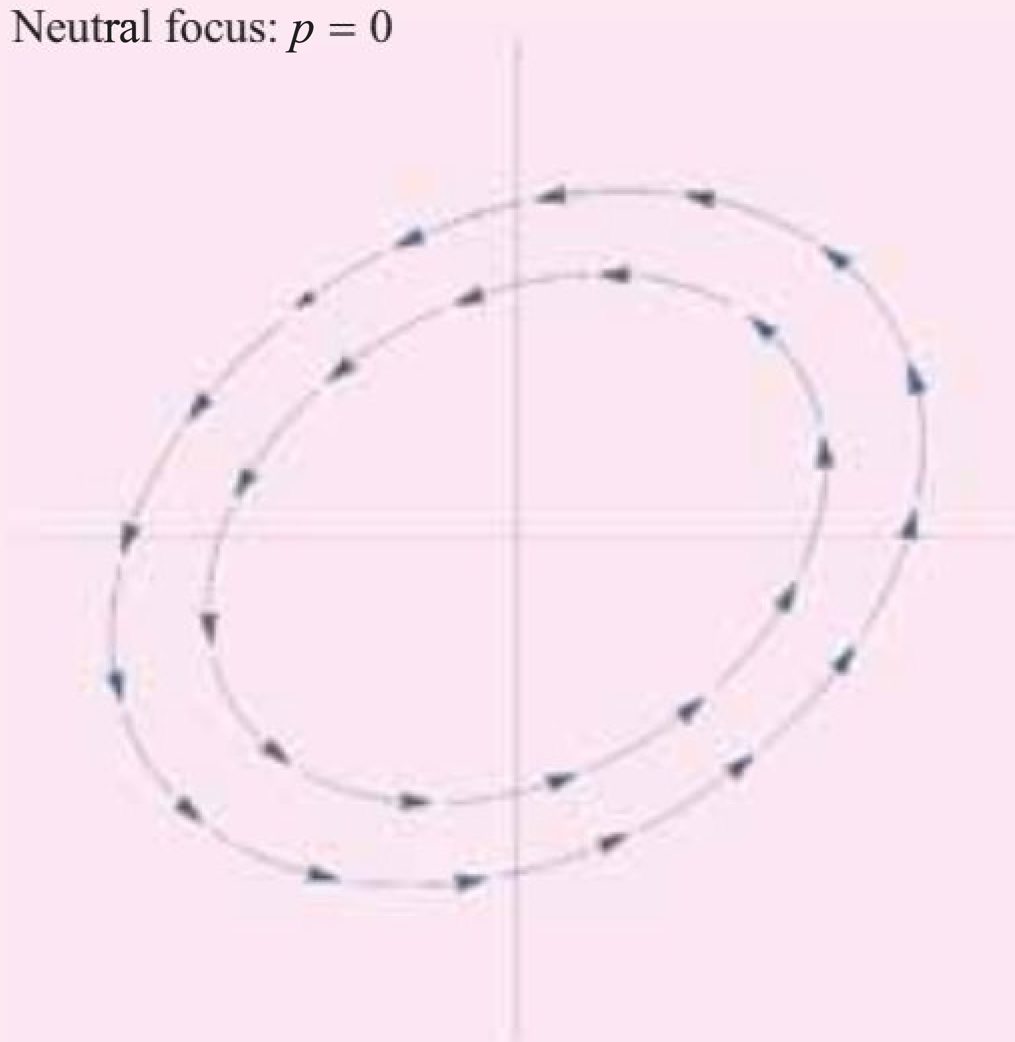
Phase portraits for complex eigenvalues $\lambda = p \pm \mathrm{i}q$.

Unstable focus: $p > 0$

Stable focus: $p < 0$



Neutral focus: $p = 0$



Tip

You can determine in which direction the solution curves spiral (clockwise or anticlockwise) by finding $\frac{\mathrm{d}x}{\mathrm{d}t}$ at the point (0, 1) on the y-axis.

WORKED EXAMPLE 13.9

Sketch the phase portrait for the system of differential equations

$$\begin{cases} \dot{x} = -3x + 4y \\ \dot{y} = -2x + y \end{cases}$$

Find the eigenvalues

$$\det\begin{pmatrix} -3-\lambda & 4 \\ -2 & 1-\lambda \end{pmatrix} = 0$$

$$\lambda^2 + 2\lambda + 5 = 0$$

$$\lambda = -1 \pm 2\mathrm{i}$$

The eigenvalues are complex with a negative imaginary part, so the solution curves spiral towards the origin

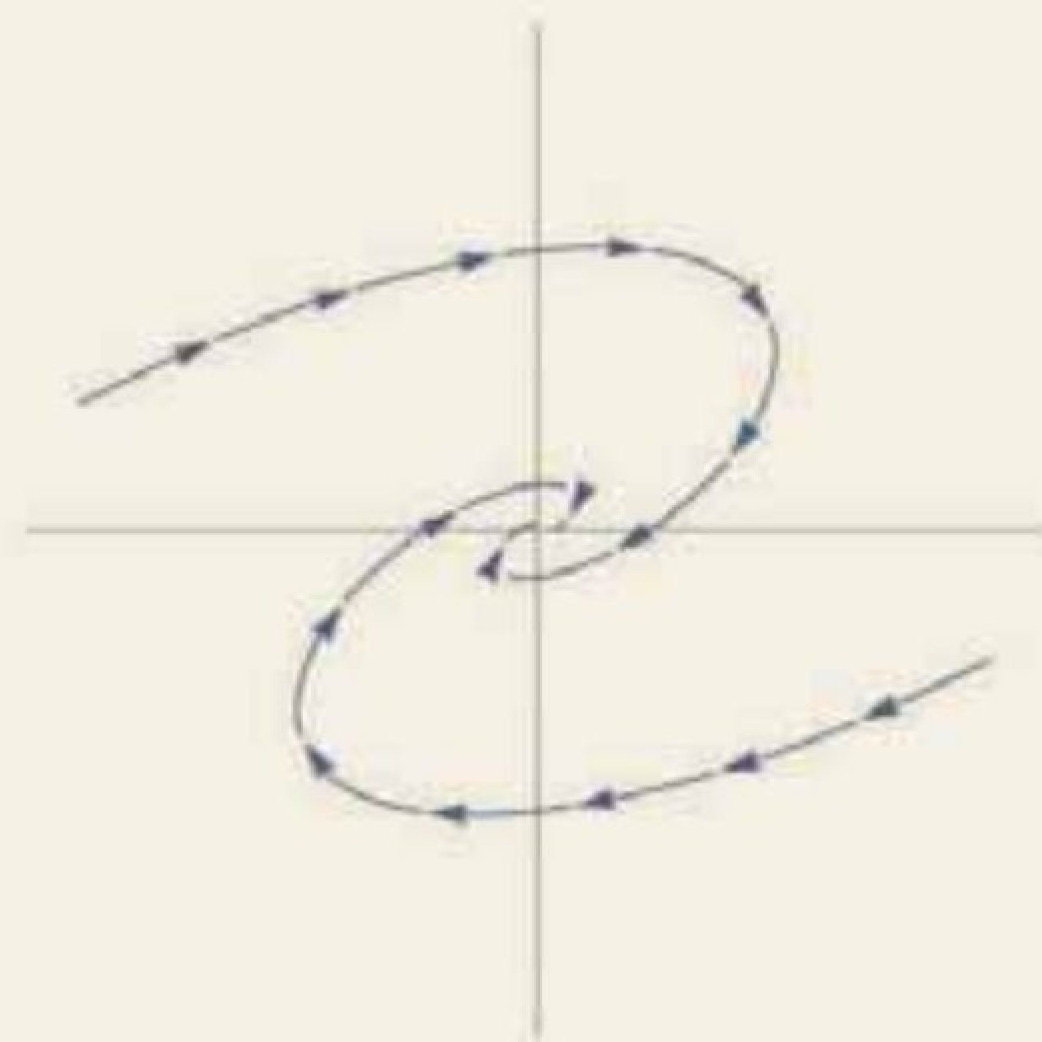
$\mathrm{Re}(\lambda) < 0$, so the origin is stable.

Find $\frac{\mathrm{d}x}{\mathrm{d}t}$ at (0, 1) to determine the direction of the spiral

When $x = 0,\ y = 1,\ \dot{x} = 0 + 4 = 1 > 0$

This means that x increases as the curve crosses the y-axis, so the spiral is clockwise

So, the spiral is clockwise.



You are the Researcher

With more complicated differential equations, there may be more than one equilibrium point. To investigate these, you need to know about techniques such as Taylor series and partial differentiation.

Phase portraits with technology

You can also sketch a phase portrait approximately by using Euler's method. You must be able to use your calculator to do this.

WORKED EXAMPLE 13.10

The differential equations

$$\frac{dx}{dt} = x - 3y$$

$$\frac{dy}{dt} = 2x + 2y$$

have initial conditions $x = 1$ and $y = 0$ when $t = 0$.

Use Euler's method with a step length of 0.1 to sketch

a a graph of x against t for $0 < t < 20$

b a phase portrait of y against x.

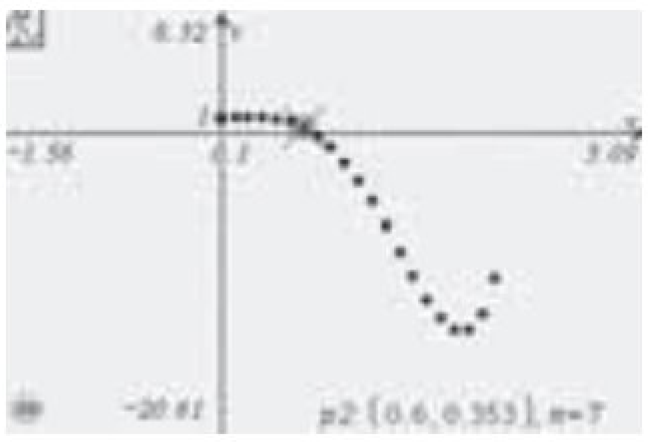
It is always a good idea to write down the Euler recurrence relations used

$$x_{n+1} = x_n + 0.1(x_n - 3y_n)$$

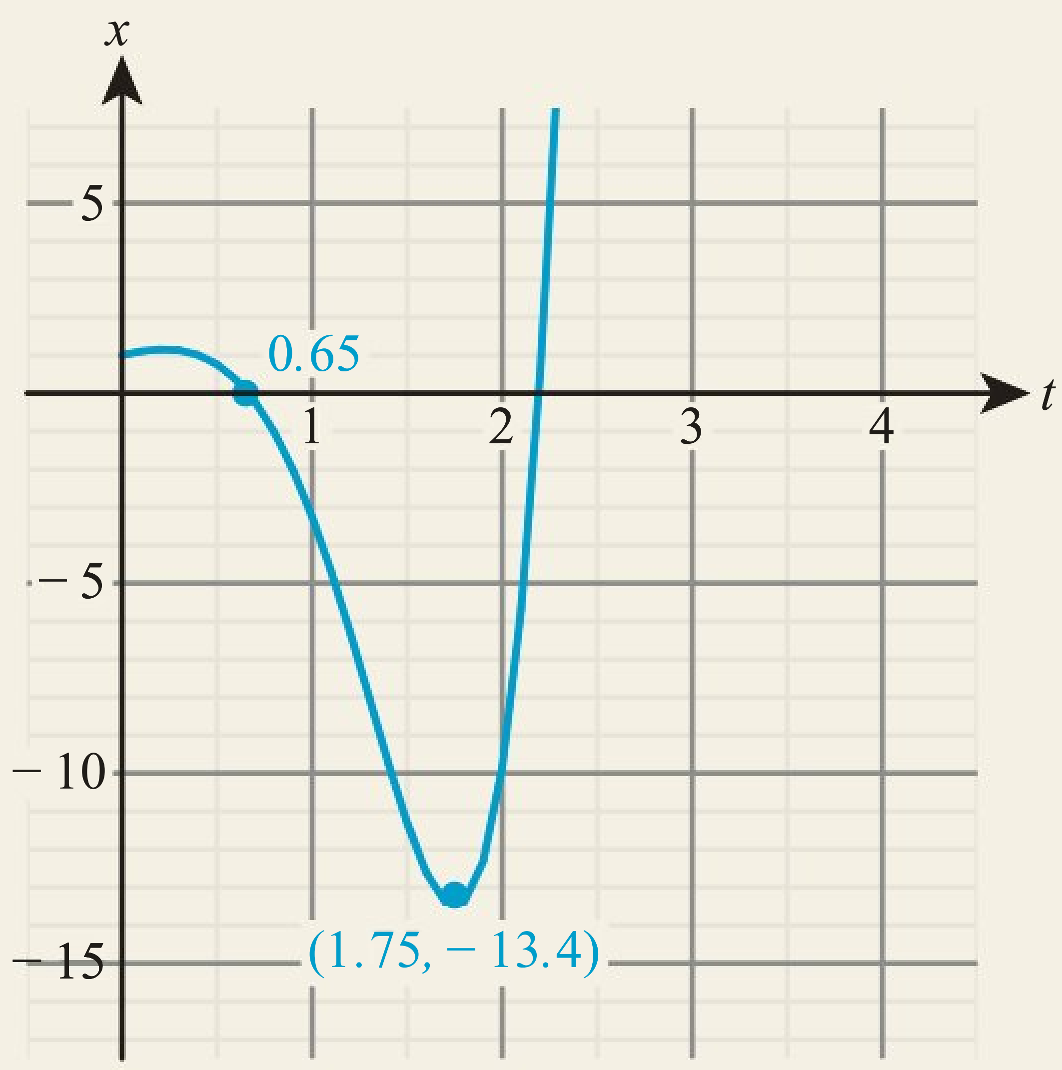
$$y_{n+1} = y_n + 0.1(2x_n + 2y_n)$$

$$t_n = 0.1n$$

From the calculator, you find that it is an oscillation with growing amplitude



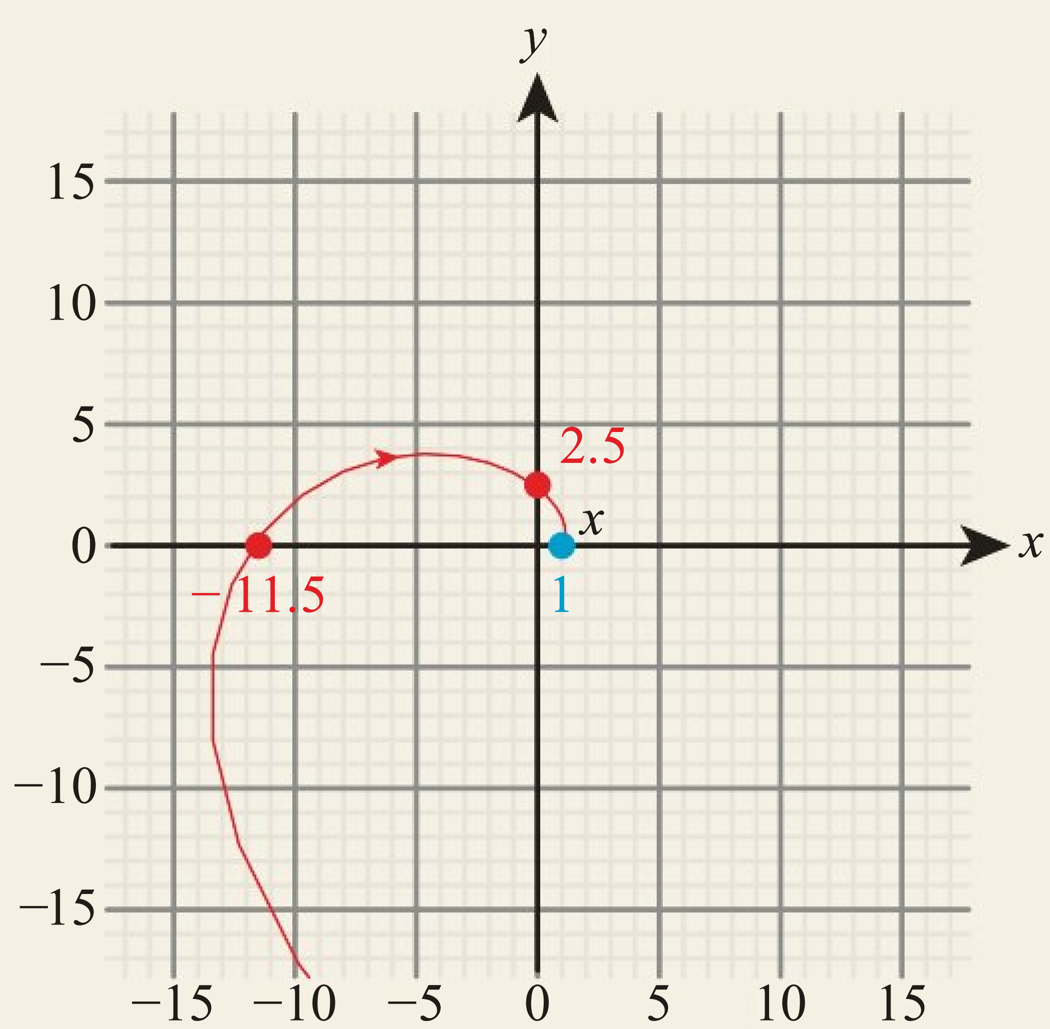
a



You should label the intercepts and the minimum point

From your calculator, you find that it is a spiral

b



TOOLKIT: Modelling

Systems of differential equations are vital in modelling the spread of disease. One fundamental model is called the SIR model. In this model, S represents the number of susceptible individuals in a population, I represents the number of infected individuals and R represents the number of recovered, resilient individuals.

The system is modelled by

$$\frac{dS}{dt} = -\beta SI$$

$$\frac{dI}{dt} = \beta SI - \nu I$$

$$\frac{dR}{dt} = \nu I$$

where $S + I + R = N$, the total number in the population.

a What do the parameters β and ν represent?

b At time $t = 0$, a single infected individual is inserted into a susceptible population (so when $t = 0$, $S = N - 1$, $I = 1$ and $R = 0$).

 i Use technology to explore phase portraits with different values of β, N and ν.

 ii An epidemic occurs when the number of infected individuals initially increases. Use your phase portraits to conjecture a condition (on β, N and ν) for an epidemic to be possible. You might like to research this further by exploring the epidemiological constant R_0.

c What effects are missing from the SIR model? How could you adapt it to include these effects?

Exercise 13C

For questions 1 to 3, use the method demonstrated in Worked Example 13.7 to find the general solution of the following coupled systems of differential equations.

1 a $\dot{x} = 4x - 2y,\ \dot{y} = x + y$
 b $\dot{x} = 7x + 3y,\ \dot{y} = -5x - y$

2 a $\dot{x} = -5x + 8y,\ \dot{y} = 3x - 7y$
 b $\dot{x} = 4y - 5x,\ \dot{y} = -3x + 2y$

3 a $\dot{x} = 4x - 5y,\ \dot{y} = x - 2y$
 b $\dot{x} = 5x + 15y,\ \dot{y} = -2x - 8y$

For questions 4 to 6, use the method demonstrated in Worked Example 13.8 to draw the phase portrait for the following coupled systems of differential equations (from Questions 1 to 3).

4 a $\dot{x} = 4x - 2y,\ \dot{y} = x + y$
 b $\dot{x} = 7x + 3y,\ \dot{y} = -5x - y$

5 a $\dot{x} = -5x + 8y,\ \dot{y} = 3x - 7y$
 b $\dot{x} = 4y - 5x,\ \dot{y} = -3x + 2y$

6 a $\dot{x} = 4x - 5y,\ \dot{y} = x - 2y$
 b $\dot{x} = 5x + 15y,\ \dot{y} = -2x - 8y$

For questions 7 to 9, use the method demonstrated in Worked Example 13.9 to draw the phase portrait for the following coupled systems of differential equations. Remember to check the direction of the spiral.

7 a $\dot{x} = 3x - y,\ \dot{y} = 8x - y$
 b $\dot{x} = x + 5y,\ \dot{y} = -5x + y$

8 a $\dot{x} = -3x + 4y,\ \dot{y} = -2x + y$
 b $\dot{x} = -2x + 4y,\ \dot{y} = -9x - y$

9 a $\dot{x} = -y,\ \dot{y} = 16x$
 b $\dot{x} = 3y,\ \dot{y} = -4x$

For questions 10 to 12, use the method demonstrated in Worked Example 13.10 (using technology) to sketch i x vs t and ii y vs x for $0 < t < 2$, given that $x(0) = 1$, $y(0) = 2$. Use a step length of 0.1.

10 a $x' = x + 2y$
 $y' = x - y$
 b $x' = 2x + 2y$
 $y' = x - y$

11 a $x' = -2x + y$
 $y' = -6x$
 b $x' = -2x + y$
 $y' = -x$

12 a $x' = 2x + y$
 $y' = -4x$
 b $x' = -y$
 $y' = 4x$

Tip

x' is another notation for $\dot{x}$ or $\frac{dx}{dt}$.

13 a Find the eigenvalues and eigenvectors of the matrix $\begin{pmatrix} 1 & 2 \\ 8 & 1 \end{pmatrix}$.

b Hence write down the general solution of the coupled equations

$$\frac{dx}{dt} = x + 2y, \ \frac{dy}{dt} = 8x + y$$

c Given that $x(0) = y(0) = 4$, find an expression for x and y in terms of t.

14 a Find the eigenvalues of the matrix $\begin{pmatrix} 1 & 3 \\ 2 & 2 \end{pmatrix}$.

b Hence find $x(t)$ given that

$$x' = x + 3y$$
$$y' = 2x + 2y$$

and that $x(0) = 4, \ x'(0) = 1$.

15 a Find the eigenvalues of the matrix $\begin{pmatrix} 0 & 0.9 \\ -0.4 & 0 \end{pmatrix}$.

b For the following system of differential equations

$$\dot{x} = 0.9y, \ \dot{y} = -0.4x$$

i determine whether x is increasing or decreasing when $x = 1$ and $y = 2$

ii sketch the phase portrait.

16 a Find the eigenvalues and eigenvectors of the matrix $\begin{pmatrix} 4 & 2 \\ 3 & 3 \end{pmatrix}$.

b Hence sketch the phase portrait for the system of equations

$$\dot{x} = 4x + 2y, \ \dot{y} = 3x + 3y$$

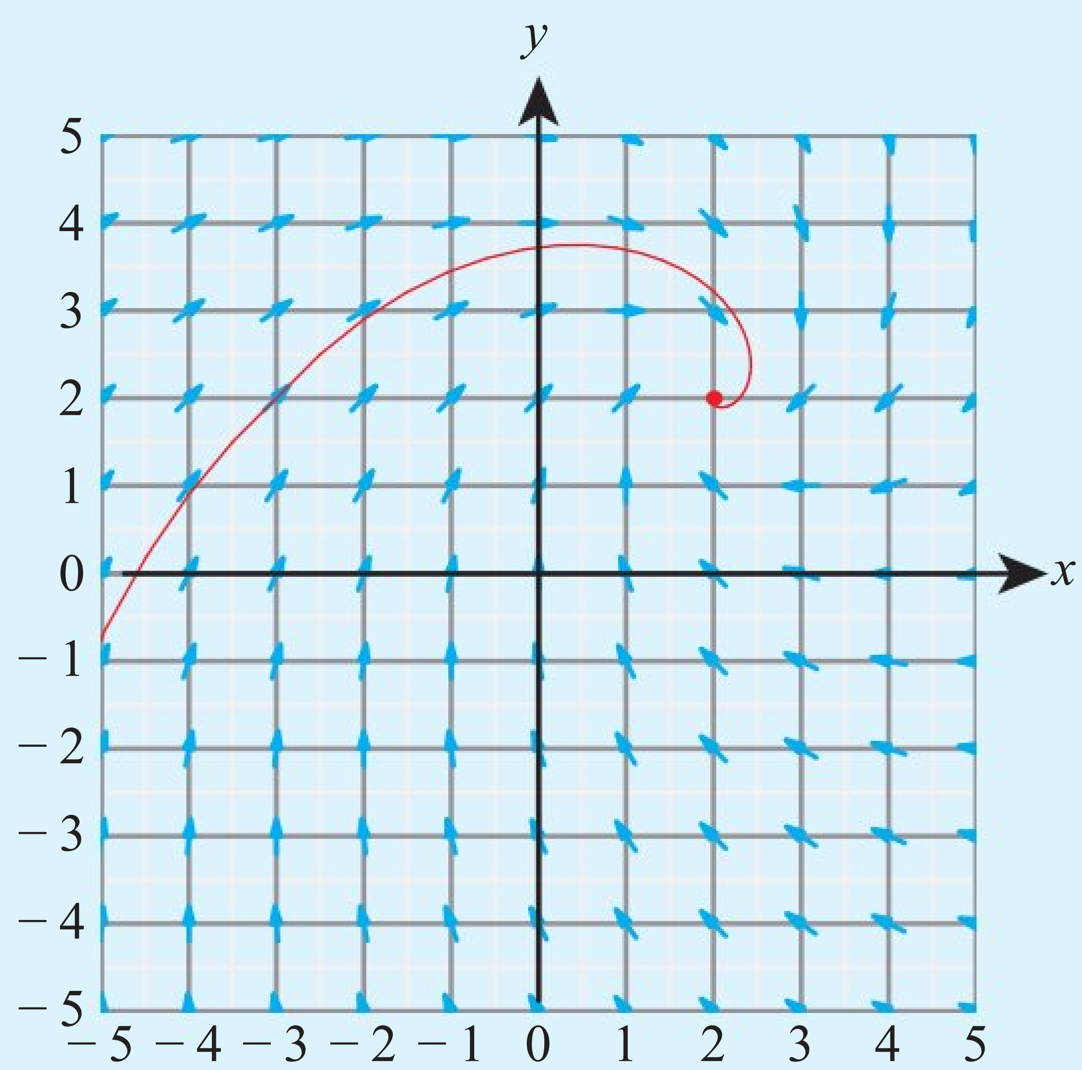
c Write down the general solution of the system. Hence find the expressions for x and y in terms of t for the solution curve with $x(0) = 6, \ y(0) = 1$.

d The equations are used to model the size of the populations of algae and fish in a pond. Use your answer to part **c** to comment on the suitability of the model in the long term.

17 Jenny used an online phase portrait sketcher to investigate the differential equations

$$\frac{dy}{dt} = y - x$$

$$\frac{dy}{dt} = 4 - x - y$$

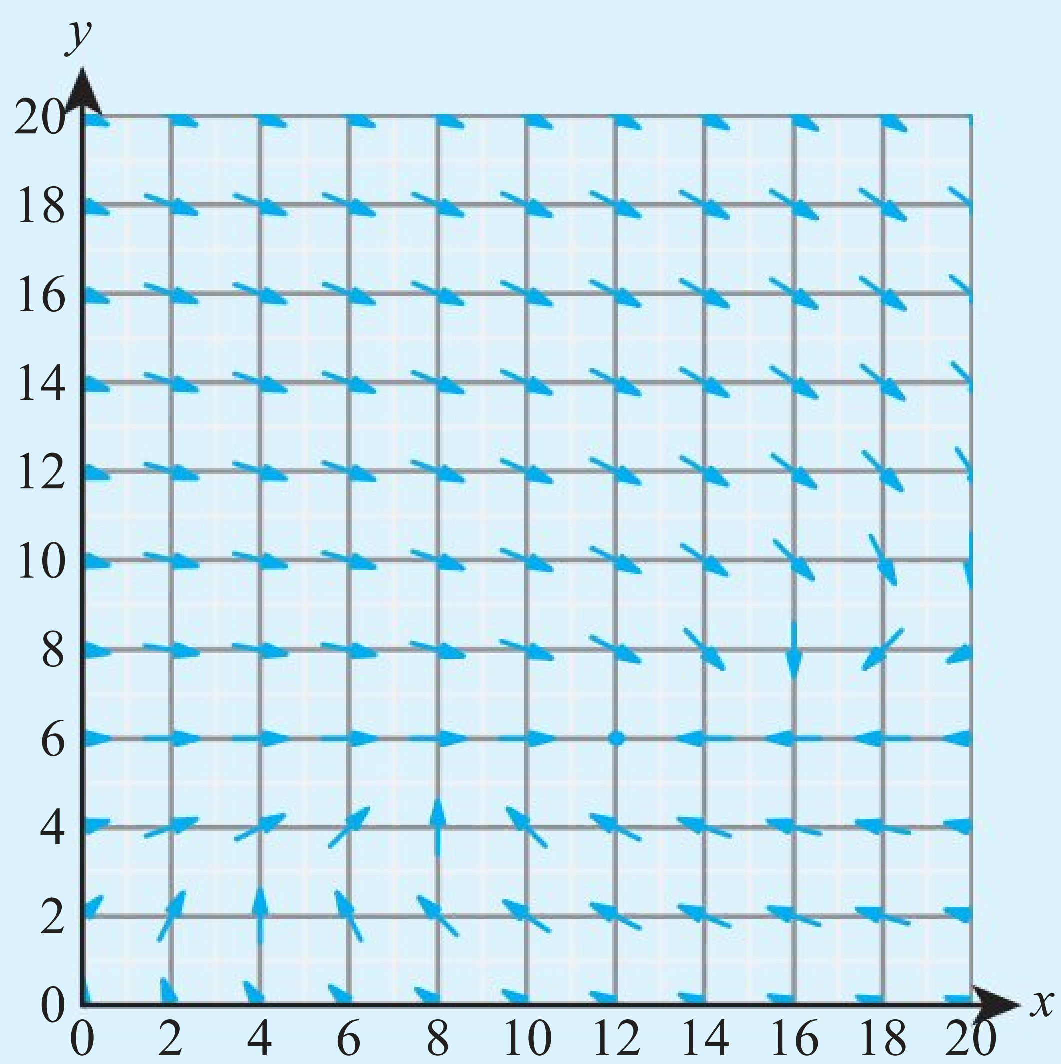


The red line shows the solution curve passing through $(-3, 2)$.

a Estimate the maximum y value along this solution curve.

b Estimate the long-term state of the system.

18 The phase portrait below was produced using an online sketching tool. The variables are x on the horizontal axis and y on the vertical axis.



Which of the statements about the system are correct?

a x always increases, independently of its initial value.

b Regardless of the initial values, x approaches 12 and y approaches 6.

c If x and y values are initially both large, they will decrease.

d If y is close to 6, it will decrease.

19 For the system of equations

$$\frac{dx}{dt} = -3x - 2y, \quad \frac{dy}{dt} = 2x - y$$

with initial values $x(0) = 4$, $y(0) = 9$, use technology to

a sketch the solution curve in the x-y plane

b sketch the graphs of x and y against t on separate diagrams

c describe how x and y change with t.

20 a Find the eigenvalues and eigenvectors of the matrix $\begin{pmatrix} -8 & 6 \\ -9 & 13 \end{pmatrix}$.

A population of and foxes (x) and rabbits (y) are introduced to an island. Initially there are 15 foxes and 8 rabbits. The sizes of the two populations are modelled by the system of differential equations

$$\dot{x} = 6y - 8x, \quad \dot{y} = 13y - 9x$$

where time is measured in years.

b Find expressions for $x(t)$ and $y(t)$.

c Show this solution curve on a phase portrait for $0 \leqslant x,\ y \leqslant 50$. Also show the directions of the two eigenvectors.

d What is the ratio of foxes to rabbits in the long term?

21 A population of birds has size $(2+x)$ hundred and a population of insects has size $(80+y)$ hundred. The time (t) is measured in months. The variables x and y are modelled by the system of differential equations

$$\dot{x} = 0.8y, \; \dot{y} = -0.6x$$

a Find the eigenvalues of the matrix $\begin{pmatrix} 0 & 0.8 \\ -0.6 & 0 \end{pmatrix}$.

b Write down the sizes of the two populations when $x = 0$ and $y = 0.5$, and determine whether the population of birds is increasing or decreasing at that point.

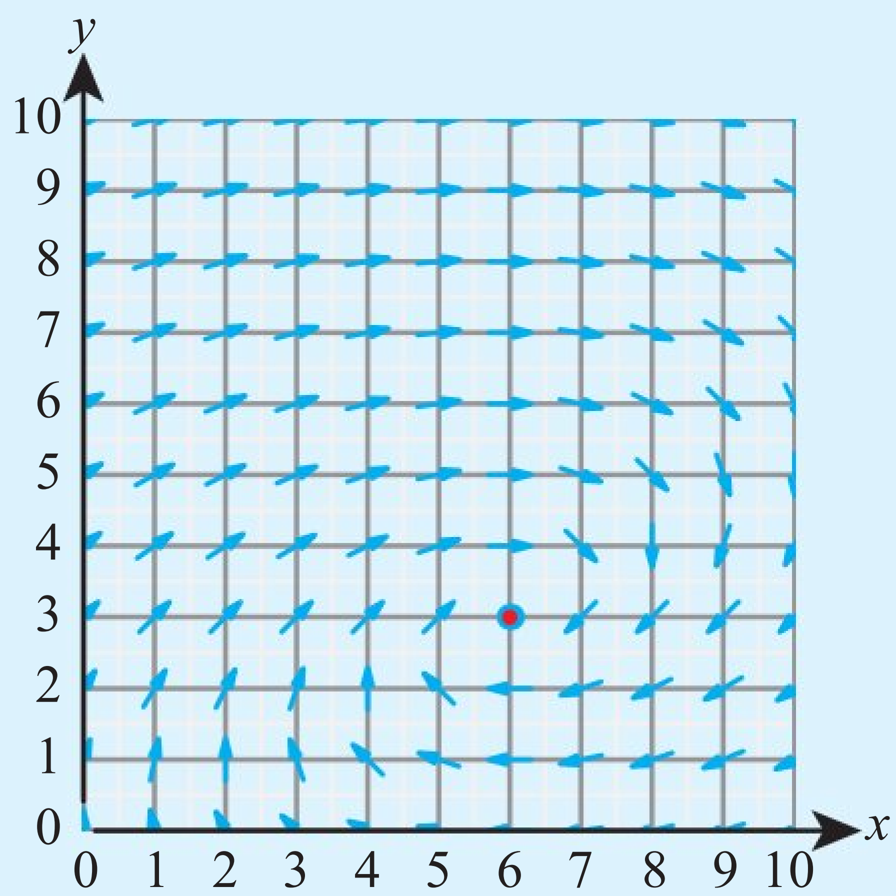
c Sketch the phase portrait for x and y.

d Hence sketch the phase portrait showing the populations of birds and insects. Describe how the two populations change over time.

22 Ishan models the populations of spiders (x hundred) and flies (y hundred) using the system of equations

$$\frac{dx}{dt} = y - 0.5x, \; \frac{dy}{dt} = 3 - 0.5x$$

He uses an online phase portrait plotter to produce the following diagram.



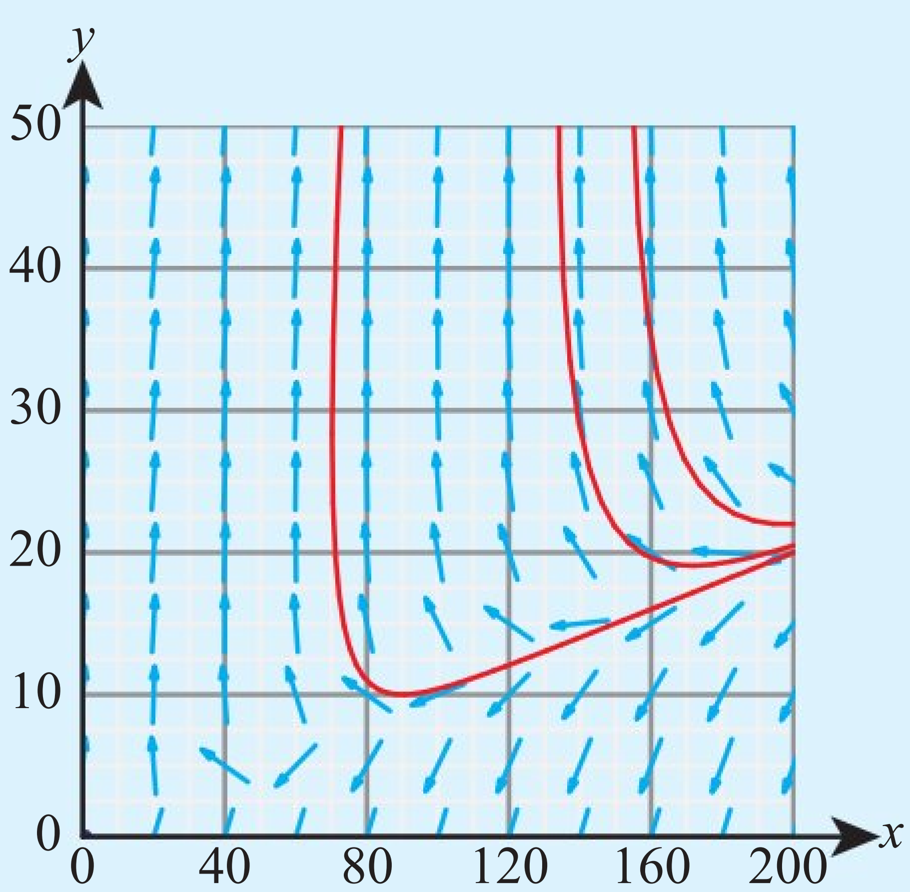
a What does Ishan's model predict about the long-term number of spiders and flies?

b Sketch a graph showing how the number of spiders varies with time.

23 Seojung is investigating the system of coupled differential equations

$$x' = ax + by, \; y' = cx + dy$$

She uses technology to produce this phase portrait for the system.



a Three possible solution curves are shown in red, corresponding to initial values near $x = 200, \; y = 21$. Describe the long-term behaviour of x and y for those initial values.

b Which of the following could be the eigenvalues and eigenvectors of the matrix $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$?

i Eigenvectors $\begin{pmatrix}1\\1\end{pmatrix}$ and $\begin{pmatrix}-1\\1\end{pmatrix}$ with eigenvalues 2 and 5

ii Eigenvectors $\begin{pmatrix}1\\1\end{pmatrix}$ and $\begin{pmatrix}-1\\1\end{pmatrix}$ with eigenvalues −2 and −5

iii Eigenvectors $\begin{pmatrix}1\\3\end{pmatrix}$ and $\begin{pmatrix}10\\1\end{pmatrix}$ with eigenvalues 3 and −0.3

iv Eigenvectors $\begin{pmatrix}1\\3\end{pmatrix}$ and $\begin{pmatrix}10\\1\end{pmatrix}$ with eigenvalues −3 and 0.3

24 Populations of sharks (x) and fish (y) are modelled by the following system of differential equations:

$$\dot{x} = -3x + 4y - 14, \quad \dot{y} = -2x + y - 1$$

Here time is measured in years and the number of animals in hundreds.

An *equilibrium state* of the system is when $\dot{x} = \dot{y} = 0$.

a Find the numbers of shark and fish at the equilibrium state.

b Make a substitution $u = x - 2$, $v = y - 5$ to transform the system into the form

$$\begin{pmatrix}\dot{u}\\\dot{v}\end{pmatrix} = \mathbf{A}\begin{pmatrix}u\\v\end{pmatrix}$$

where **A** is a matrix to be found.

c Find the eigenvalues of **A**.

d Hence sketch the phase portrait for the original system (in x and y).

e Describe the long-term behaviour of the population.

The model described in question 24 does not capture all the observed behaviour found in predator–prey ecosystems. In the 1920s, the American mathematician Alfred Lotka and the Italian mathematician Vito Volterra independently published an improved predator–prey model which explained an unexpected observation from studies of fishing during and after World War 1. During the war, when fishing was limited, predatory fish populations increased. After the war, when there was lots of dragnet fishing, the population of prey fish went up. This is explained by the Lotka–Volterra model which includes a predation term depending on both the predator and prey population.

13D Second order differential equations

The acceleration of a particle with displacement x is given by $\frac{d^2x}{dt^2}$. Many physical situations are best described in terms of the acceleration.

Links to: Physics

One reason why so many physical situations are best described in terms of acceleration is because of Newton's second law which links forces and acceleration.

A second order differential equation is one that involves $\frac{d^2x}{dt^2}$. For example, $\frac{d^2x}{dt^2} + 2\frac{dx}{dt} + 4x = 12$.

To solve these differential equations, we can use the fact that $\frac{d^2x}{dt^2} = \frac{d}{dt}\left(\frac{dx}{dt}\right)$ to turn the equation into coupled first order equations.

KEY POINT 13.8

To investigate the solution to $\frac{d^2x}{dt^2} = f\left(x, \frac{dx}{dt}, t\right)$, write as a system of coupled first order equations

$$\begin{cases} \frac{dx}{dt} = y \\ \frac{dy}{dt} = f(x, y, t) \end{cases}$$

WORKED EXAMPLE 13.11

a Write $\frac{d^2x}{dt^2} - 2\frac{dx}{dt} + 2x = 0$ as a system of coupled first order equations.

b Hence determine whether the solution to the differential equation is stable (i.e. whether x tends to zero as t increases).

a $\frac{dx}{dt} = y$

Use $\frac{d^2x}{dt^2} = \frac{d}{dt}(y) = 2\frac{dx}{dt} - 2x$

$\frac{dy}{dt} = 2y - 2x$

Consider the system as

$\frac{dx}{dt} = 0x + 1y$

$\frac{dy}{dt} = -2x + 2y$

b The system is described by the matrix

$$\begin{pmatrix} 0 & 1 \\ -2 & 2 \end{pmatrix}$$

The eigenvalues of the system are found from the characteristic equation

$$-\lambda(2-\lambda) + 2 = 0$$
$$\lambda^2 - 2\lambda + 2 = 0$$

From the GDC, this has solutions

$$\lambda = 1 \pm i$$

Since the eigenvalues are complex with positive real part, the solutions are unstable.

The fact that the eigenvalues are complex actually implies that the solutions to the second order differential equation oscillate. Since the solutions are unstable, the amplitude of oscillation increases with time.

We can also use Euler's method to find numerical solutions to the differential equations.

WORKED EXAMPLE 13.12

$\frac{d^2x}{dt^2} - 2\frac{dx}{dt} - x = 0$ and when $t = 0$, $x = 1$ and $\frac{dx}{dt} = 2$. Estimate $x(2)$ using Euler's method with a step length of 0.1.

Write as a system of coupled linear equations by setting $y = \frac{dx}{dt}$

If $y = \frac{dx}{dt}$, then

$$\frac{d^2x}{dt^2} = \frac{dy}{dt} = 2\frac{dx}{dt} + x$$

$$= 2y + x$$

Writing this in the standard form for differential equations gives

$$\frac{dx}{dt} = y$$

$$\frac{dy}{dt} = x + 2y$$

We can then use the methods from Key Point 13.4 to turn this into Euler recurrence relationships

$$x_{n+1} = x_n + 0.1(y_n)$$

$$y_{n+1} = y_n + 0.1(x_n + 2y_n)$$

From the initial conditions given, you have to set the initial value of the first variable (x in the equation, but a on the calculator screenshot above) to 1, but the second variable (y in the equation, b on the screenshot) to 2

From the calculator:

n	x_n	y_n
0	1	2
1	1.2	2.5

The required answer is $x_{20} = 64.571 \approx 64.6$.

Tip

You will use your calculator a lot when applying Euler's method. Make sure that you write down the recurrence relations you are using. It is also a good idea to write out a couple of lines from the table output from your calculator, but you do not have to write them all out.

TOOLKIT: Problem Solving

Consider the differential equation

$$\frac{d^2x}{dt^2} + b\frac{dx}{dt} + cx = 0$$

Determine a condition for the solution to be stable.

You might want to start off by investigating different values of b and c.

Once you have a conjecture, see if you can prove it.

Links to: Physics

A model for a mass bouncing on a spring says that the acceleration is proportional to the extension, x. What differential equation does this lead to? What is its solution?
This behaviour is called simple harmonic motion (SHM) and it is fundamental to many aspects of physics. You might want to see how Newton's and Hooke's laws lead to this equation for a mass on a spring. It is also applied to a pendulum, but is only valid for small oscillations. See if you can find the real equation for the oscillation of a pendulum and use numerical methods to work out how much error is introduced by modelling a pendulum using SHM.

Exercise 13D

For questions 1 to 5, use the method of Worked Example 13.11 to determine whether the solutions to the given differential equations are stable or unstable.

1 **a** $\frac{d^2x}{dt^2} - 5\frac{dx}{dt} + 6x = 0$

b $\frac{d^2x}{dt^2} - 9\frac{dx}{dt} + 20x = 0$

2 **a** $\frac{d^2x}{dt^2} + 5\frac{dx}{dt} + 4x = 0$

b $\frac{d^2x}{dt^2} + 6\frac{dx}{dt} + 8x = 0$

3 **a** $\frac{d^2x}{dt^2} - 4\frac{dx}{dt} + 5x = 0$

b $\frac{d^2x}{dt^2} - 4\frac{dx}{dt} + 8x = 0$

4 **a** $\frac{d^2x}{dt^2} + 6\frac{dx}{dt} + 13x = 0$

b $\frac{d^2x}{dt^2} + 6\frac{dx}{dt} + 10x = 0$

5 **a** $\frac{d^2x}{dt^2} + 2\frac{dx}{dt} - 3x = 0$

b $\frac{d^2x}{dt^2} - 16x = 0$

For questions 6 to 8, use the method of Worked Example 13.12 to estimate the value of $x(2)$ using Euler's method with a step length of 0.1.

6 **a** $\frac{d^2x}{dt^2} + 3\frac{dx}{dt} - 4x = 0,\ x(0) = 2,\ x'(0) = -1$

b $\frac{d^2x}{dt^2} + 4\frac{dx}{dt} + 3x = 0,\ x(0) = 0,\ x'(0) = 2$

7 **a** $\frac{d^2x}{dt^2} + 4x = 0,\ x(0) = 1,\ x'(0) = 1$

b $\frac{d^2x}{dt^2} + 4x = 0,\ x(0) = 2,\ x'(0) = -1$

8 **a** $\frac{d^2x}{dt^2} + 4\frac{dx}{dt} + 4x = 0,\ x(0) = 5,\ x'(0) = 0$

b $\frac{d^2x}{dt^2} + 4\frac{dx}{dt} + 4x = 0,\ x(0) = 0,\ x'(0) = 5$

9 **a** Show that the differential equation

$$\frac{d^2x}{dt^2} - \frac{dx}{dt} - 2x = 0$$

can be written as

$$\frac{dx}{dt} = y$$

$$\frac{dy}{dt} = y + 2x$$

b If the initial conditions are $x(0) = 0,\ \frac{dx}{dt}(0) = 2$ estimate the value of $x(10)$ using Euler's method with a step length of

i 0.5 **ii** 1

c Is your answer to **bi** or **bii** a more accurate estimate of the true value? Justify your answer.

10 a Show that the differential equation

$$\frac{d^2x}{dt^2} - \frac{dx}{dt} + x = 1 - t$$

can be written as

$$\frac{dx}{dt} = y$$

$$\frac{dy}{dt} = y - x + 1 - t$$

b When $t = 0$, $x = 0$ and $\dot{x} = 2$. Use Euler's method with a step length of 0.2 to estimate the value of x when $t = 4$.

11 The displacement, x in mm, of a bell t milliseconds after it has been hit is modelled by

$$\frac{d^2x}{dt^2} + 4\frac{dx}{dt} + 9x = e^{-t}$$

with $x(0) = 0$, $x'(0) = 1$.

a Show that the differential equation can be written as

$$\frac{dx}{dt} = y$$

$$\frac{dy}{dt} = -4y - 9x + e^{-t}$$

b Use Euler's method with a step length of 0.1 to estimate $x(1)$.

c Use technology to sketch the output from Euler's method for $0 < t < 2$.

d Hence estimate the maximum displacement, giving your answer to two significant figures.

12 A financial analyst predicts that the price of shares in a company (a financial product) can be modelled by

$$\frac{d^2p}{dt^2} + 1.5\frac{dp}{dt} + 5p = 10 - e^{-t}$$

where p is the price in dollars and t is the time in years.

The initial price is \$1 and it is expected to initially fall at a rate of \$2 per year. Use Euler's method with a step length of 0.1 to determine to 1 decimal place, according the model,

a when the analyst should purchase the shares

b how long the analyst should keep the shares

c how much profit the analyst will make per share

d the long-term price of the shares.

e By considering the observed behaviour, suggest why the model is unlikely to be accurate.

13 a In a predator–prey model, the population of foxes (f thousand) at a time t years is modelled by

$$\frac{d^2f}{dt^2} + 2\frac{df}{dt} + 8f = 4$$

Initially, $f = 1$ and $\frac{df}{dt} = 1$. Use Euler's method with step length 0.1 to describe the behaviour for $0 \leqslant t \leqslant 7$.

b The population of rabbits in the model, r thousand, is modelled by

$$\frac{d^2r}{dt^2} + 3\frac{dr}{dt} + 10r = 12$$

Initially, $r = 10$ and $\frac{dr}{dt} = 1$.

Use Euler's method with step length 0.1 to sketch the behaviour over $0 \leqslant t \leqslant 7$. Hence explain why this model is not satisfactory for predicting the rabbit population during these seven years.

14 A guitar string vibrates with displacement x mm at a time t milliseconds.

a The motion is modelled by the differential equation $\frac{d^2x}{dt^2} + x = 0$ subject to $x = 0$, $\frac{dx}{dt} = 1$ when $t = 0$.

i Use Euler's method with step length 0.1 to sketch the behaviour for $0 < t < 7$.

ii Hence estimate the amplitude and time period of the oscillation, giving your answer to two significant figures.

b The motion is then driven by two different tuning forks. The string is initially still, so that $x = 0$, $\frac{dx}{dt} = 0$ when $t = 0$.

i When the first tuning fork is applied to the string, the motion is modelled by $\frac{d^2x}{dt^2} + x = -\sin t$

Use Euler's method with step length 0.1 to estimate the maximum displacement in the first 7 millliseconds.

ii When the second tuning fork is applied to the string, the motion is modelled by $\frac{d^2x}{dt^2} + x = -\sin 4t$

Use Euler's method with step length 0.1 to estimate the maximum displacement in the first 7 millliseconds.

You are the Researcher

Question 14 illustrates the very important concept of resonance, which is hugely important in music, optics and engineering. One very spectacular example of what happens if resonance is not considered is the Tacoma Narrows bridge collapse.

15 The differential equation

$$\frac{d^2y}{dt^2} - 0.7\frac{dy}{dt} + 0.12y = 0$$

is subject to $y = 1$, $\dot{y} = 0.2$ when $t = 0$.

a Estimate the value of $y(3)$ using Euler's method with a step length of

i 0.5

ii 0.1

b By writing as a system of coupled differential equations, find the exact solution to the differential equation.

c Hence find the percentage error in each of your estimates in part **a**.

16 The differential equation

$$\frac{d^2y}{dt^2} + \left(\frac{dy}{dt}\right)^2 + y^2 = 1 - \sin t$$

has initial conditions $y = 0$ and $\frac{dy}{dt} = 1$ when $t = 0$.

a Use Euler's method with step length 0.1 to estimate y when

i $t = 1$

ii $t = 2$

b Show that $y = \sin t$ satisfies the differential equations and the initial conditions.

c Hence find the percentage errors in each of your answers to part **a**.

Checklist

- You should know how to set up a differential equation from a context.
- You should be able to solve a differential equation using separation of variables. Differential equations of the form $\frac{dy}{dx} = f(x)g(y)$ can be solved using

$$\int \frac{1}{g(y)}\,dy = \int f(x)\,dx$$

- You should know how to use and interpret slope fields.
- You should know how to use Euler's method to find approximate solutions to first order differential equations of the form $\frac{dy}{dx} = f(x, y)$:
  - $x_{n+1} = x_n + h$
  - $y_{n+1} = y_n + h \times f'(x_n, y_n)$
- You should know how to use Euler's method to find approximate solutions to the coupled systems of differential equations $\frac{dx}{dt} = f_1(x, y, t)$, $\frac{dy}{dt} = f_2(x, y, t)$:
  - $t_{n+1} = t_n + h$
  - $x_{n+1} = x_n + h \times f_1(x_n, y_n, t_n)$
  - $y_{n+1} = y_n + h \times f_2(x_n, y_n, t_n)$
- You should be able to solve coupled systems of the form $\begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = \mathbf{M}\begin{pmatrix} x \\ y \end{pmatrix}$:
  - If **M** has real and distinct eigenvalues λ_1 and λ_2 with corresponding eigenvectors $\mathbf{p}_1$ and $\mathbf{p}_2$, then the general solution of the system is

$$\begin{pmatrix} x \\ y \end{pmatrix} = Ae^{\lambda_1 t}\mathbf{p}_1 + Be^{\lambda_2 t}\mathbf{p}_2$$

- You should be able to sketch and use phase portraits for systems of differential equations by finding the eigenvalues of **M** and by using technology.
- You should be able to solve second order differential equations of the form $\frac{d^2x}{dt^2} = f\left(x, \frac{dx}{dt}, t\right)$ by writing as a system of coupled first order equations:

$$\begin{cases} \frac{dx}{dt} = y \\ \frac{dy}{dt} = f(x, y, t) \end{cases}$$

Mixed Practice

1 a Use separation of variables to find the general solution of the differential equation $\frac{dy}{dx} = 3y\cos 2x$ for $y > 0$.
b Find the particular solution given that $y = 5$ when $x = 0$.

2 Find the general solution of the differential equation $\frac{dy}{dx} + 2y = 3x^2 y$ by separating variables.

3 Consider the differential equation $\frac{dy}{dx} + y^2 = e^x$ with initial condition $y(0) = 2$.

Use Euler's method with step length 0.1 to estimate the value of $y(2)$.

4 a Sketch the slope field for the differential equation $\frac{dy}{dx} = y - x$ using the points with $x \in \{0, 1, 2, 3, 4\}$ and $y \in \{0, 1, 2, 3, 4\}$.
b Add to your sketch the solution curve that passes through the point $(2, 2)$.
c For this solution curve, use Euler's method with step length 0.1 to estimate the value of y when $x = 2.5$.

5 Variables x and y satisfy the differential equation $\frac{dy}{dx} = \frac{x}{y^2}$ when $x = 1$, $y = 5$.

a Use Euler's method with step length 0.05 to estimate the value of y when $x = 2$, giving your answer correct to 3 decimal places.
b Use separation of variables to find the exact solution of the equation.
c Hence find the percentage error when using Euler's method.

6 a Use separation of variables to find the general solution of the differential equation $\frac{dy}{dx} = 4x\sqrt{y}$.
b Sketch the solution which satisfies $y = 4$ when $x = 0$.

7 a Sketch the slope field for the differential equation $\frac{dy}{dx} = 2x - y$ for $x, y \in \{-2, -1, 0, 1, 2\}$.
b Add to your sketch the solution curve which passes through $(0, 0)$.

8 The rate of increase of the volume of a balloon is proportional to the square root of the current volume. When the volume is 225 cm^3, it is increasing at the rate of 90 $cm^3\,s^{-1}$.

a Show that $\frac{dV}{dt} = 6\sqrt{V}$.
b Use separation of variables to find the general solution of this differential equation.
c Given that the volume is 225 cm^3 when $t = 0$, find how long it takes for the volume to increase to 2500 cm^3.

9 Consider the system of coupled differential equations

$$\frac{dx}{dt} = 3x + y,\ \frac{dy}{dt} = 4x + 3y$$

a Show that the eigenvalues of the matrix $\begin{pmatrix} 3 & 1 \\ 4 & 3 \end{pmatrix}$ are 1 and 5.

b The corresponding eigenvectors are $\begin{pmatrix} 1 \\ -2 \end{pmatrix}$ and $\begin{pmatrix} 1 \\ 2 \end{pmatrix}$. Write down the general solution of the system.

10 Variables x and y satisfy the system of differential equations

$$\dot{x} = 3x - 2y,\ \dot{y} = -x + 5y$$

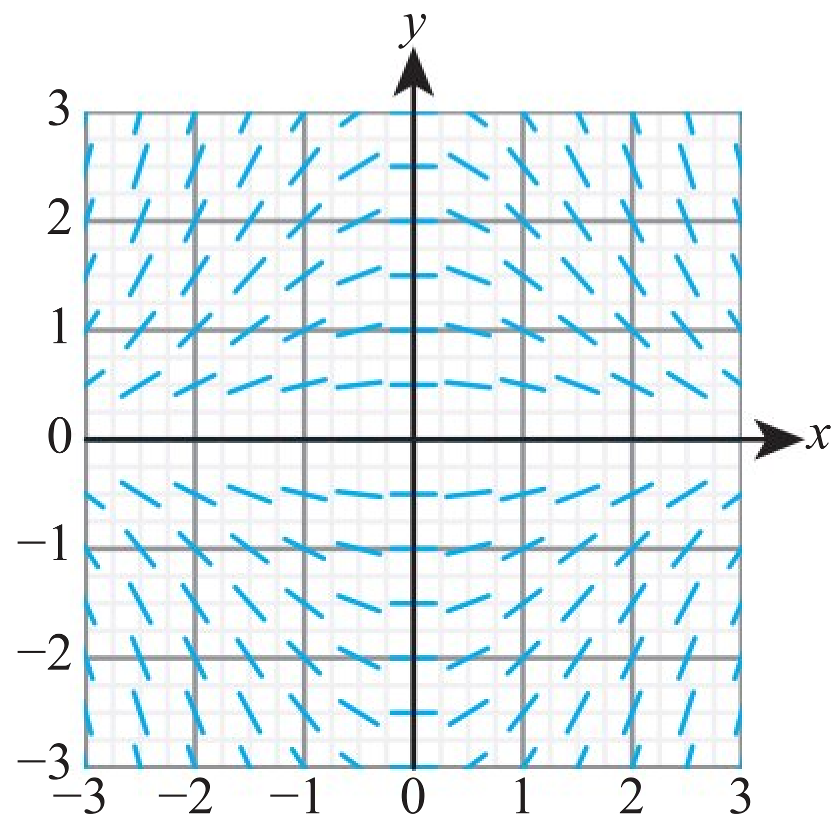
When $t = 0$, $x = 0$ and $y = 2$. Use Euler's method with step length 0.1 to estimate the values of x and y when $t = 0.6$.

11 A curve passes through $(-1, 1)$ on the slope field shown.

Estimate

a the maximum value of y on the curve, giving your answer to two significant figures

b the value of y as x tends to infinity.



12 The phase portrait shown illustrates the differential equations

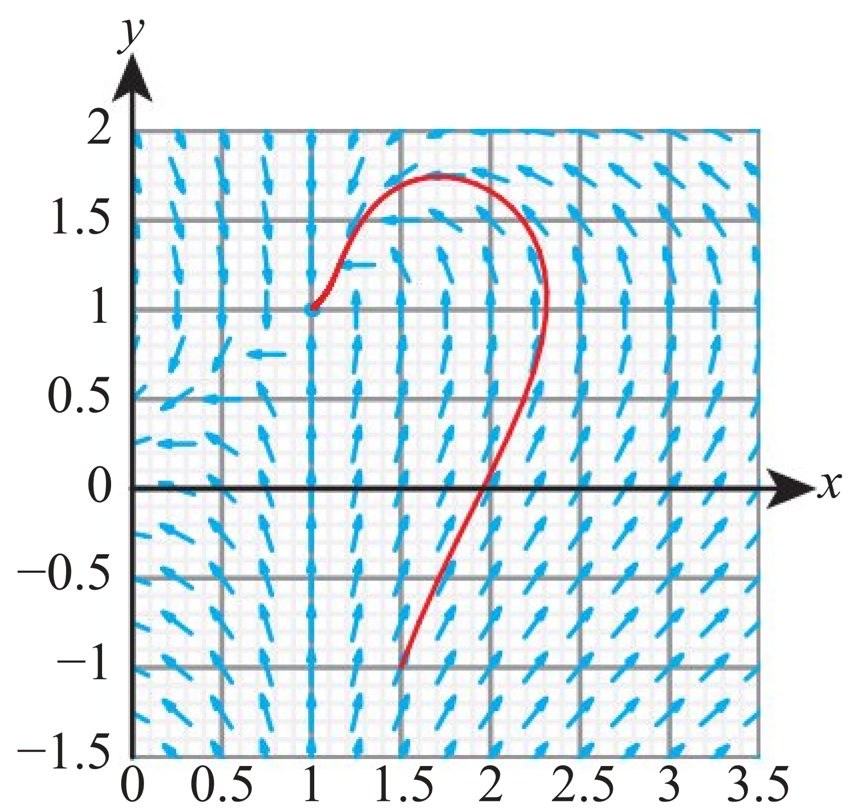
$$\frac{dx}{dt} = (x-1)(1-y), \quad \frac{dy}{dt} = x - y$$

The displayed solution curve starts at the point (1.5, –1).
Estimate, to one decimal place,

a the maximum value of x on the solution curve

b the maximum value of y on the solution curve

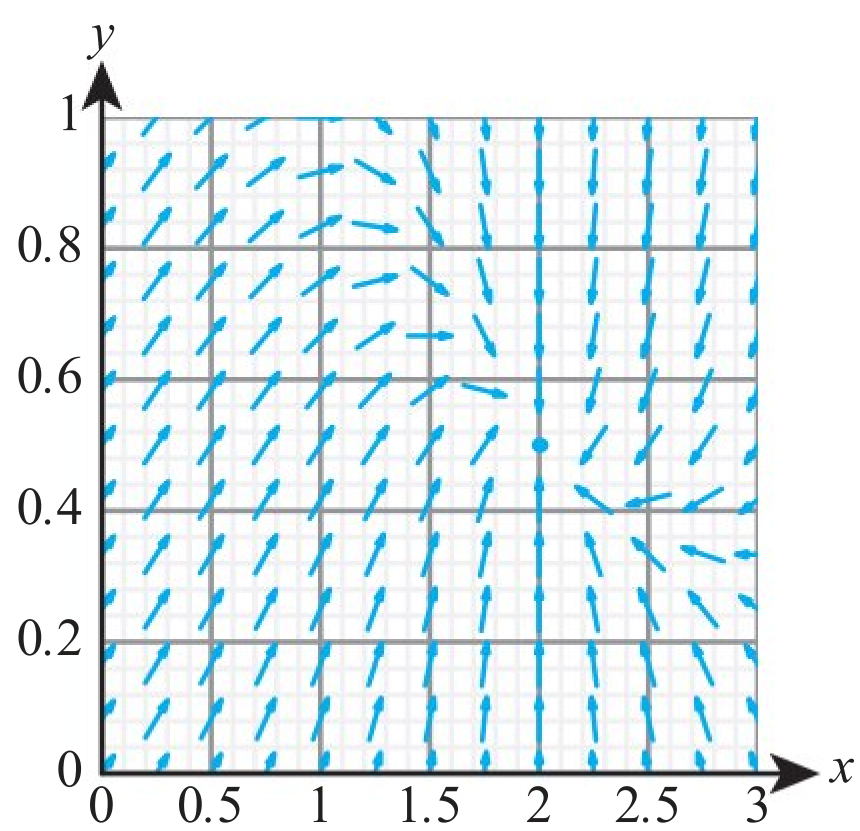
c the long term behaviour of the solution curve.



13 For a solution curve starting at the origin

a estimate the maximum value of y, giving your answer to one decimal place

b estimate the long term value of x.



14 Consider the differential equation $\frac{\mathrm{d}y}{\mathrm{d}x} = x^2 + y^2$, where $y = 1$ when $x = 0$.

a Use Euler's method with step length 0.1 to find an approximate value of y when $x = 0.4$.
b Write down, giving a reason, whether your approximate value for y is greater than or less than the actual value of y.

Mathematics HL May 2011 Paper 3 Series and differential equations Q2

15 The rate of change of mass (R) of a radioactive substance is proportional to the amount of substance remaining. This can be written as the differential equation

$$\frac{\mathrm{d}R}{\mathrm{d}t} = -kR$$

a Solve this differential equation, given that the initial mass of the substance is R_0.
b Find the time taken for the mass of the substance to halve from its initial mass.
c What does the fact that your answer to part **b** is independent of R_0 tell you?

16 **a** Use Euler's method with step length 0.25 to sketch the solution in $0 < x < 10$ to $\frac{\mathrm{d}y}{\mathrm{d}x} + \sqrt{y} = \sqrt{x}$, where $y = 1$ when $x = 0$.
b Hence estimate, to one decimal place, the minimum value of y on this curve.

17 Consider the differential equation $\frac{\mathrm{d}y}{\mathrm{d}x} = \mathrm{e}^{y-x}$ with the initial condition $y(0) = -1$.

a Use Euler's method with step length 0.1 to estimate the value of $y(2)$.
b Solve the equation exactly to find the error in your estimate in part **a**.
c How could you decrease the error in your estimate?

18 The growth rate of a population of insects depends on its current size but also varies according to the time of year. This can be modelled by the differential equation $\frac{\mathrm{d}N}{\mathrm{d}t} = 0.2N\left(1 + 2\sin\left(\frac{\pi t}{6}\right)\right)$, where N thousand is the population size and t is the time in months since the measurements began. The initial population is 2000. Solve the differential equation to find the size of the population at time t.

19 Two variables satisfy the differential equation $\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{3y}{x^2}$. When $x = 1$, $y = 2$.

a Use Euler's method with step length 0.1 to find an approximate value of y when $x = 1.3$. Give your answer to two decimal places.
b Solve the differential equation.
c Hence find the percentage error in your approximation from part **a**.
d How can the accuracy of your approximation be improved?

20 An economic model is formed to predict the retail price ($\$R$) of a commodity and the cost to manufacture ($\$C$).

The model is of the form

$$\frac{\mathrm{d}R}{\mathrm{d}t} = 0.02R + 0.01C$$

$$\frac{\mathrm{d}C}{\mathrm{d}t} = 0.03R + 0.02C$$

where t is the time in years.

a The profit, $\$P$, is given by $P = R - C$. Show that the profit is a decreasing function.

Initially, the commodity has a retail price of \$1 and the cost to manufacture is \$0.6.

b Use Euler's method with a step length of 0.5 years to
  i estimate the retail price after 10 years
  ii sketch the profit as a function of time for $0 \leqslant t \leqslant 25$
  iii determine for how long, to the nearest year, the product will have a positive profit.

21 **a** Find the general solution of the differential equation $\frac{dm}{dt} = -km$, where k is a constant.

The differential equation is used to model radioactive decay, where m is the amount of radioactive substance (in milligrams) and t is time in years.

b Given that the half-life of the radioactive substance is seven years, find the value of k.

c Find, to the nearest year, the time required for the amount of the substance to decrease to one-tenth of its original value.

22 **a** Use separation of variables to find the general solution of the differential equation

$$\frac{dy}{dx} = (x-1)(y+2)$$

b Given that $y = 5$ when $x = 1$, find the value of y when $x = 3$.

23 Use separation of variables to find the general solution of the differential equation

$$\frac{dy}{dx} = 6e^{-y}\cos 2x$$

24 Consider the system of coupled differential equations

$$\frac{dx}{dt} = x + 2y$$

$$\frac{dy}{dt} = x - y$$

When $t = 0$, $x = 1$ and $y = 2$

Use Euler's method with a step length in t of 0.1 to estimate the values of x and y when $t = 1$.

25 Consider the differential equation $\frac{d^2x}{dt^2} + 4x = e^{-t}$.

a Show that the equation can be written in the form $\frac{dx}{dt} = y$, $\frac{dy}{dt} = f(x, t)$ where f is a function to be found.

When $t = 0$, $x = 5$ and $\frac{dx}{dt} = -1$.

b Use Euler's method with step length 0.1 to estimate the value of x when $t = 0.8$.

26 A lever makes an angle θ with the vertical. It is being turned slowly so that the rate of change of θ is proportional to $\cos\theta$.

Initially, the lever makes a 60° angle with the vertical and the angle is changing at the rate of 0.1 radians per second.

a Show that $\frac{d\theta}{dt} = 0.2\cos\theta$.

b Use Euler's method with step length 0.1 to estimate the angle (in degrees) that the lever makes with the vertical after 2 seconds.

27 The acceleration of a ball falling through viscous liquid is proportional to $(10 - 0.2v^2)$, where v is the velocity. The ball starts from rest when $t = 0$ and its initial acceleration is $0.5\,\text{m s}^{-2}$.

a Write down a differential equation to model this situation.

b Use Euler's method with step length 0.05 to estimate the velocity of the ball after 2 seconds.

28 The number of mobile phones, Y hundred thousand, in a country t years after a new network is installed is modelled by $\frac{\mathrm{d}Y}{\mathrm{d}t} = 2Ye^{-t}$

a Sketch the slope field for this differential equation for $t, Y \in \{0, 1, 2, 3, 4, 5\}$.

When $t = 0$, $y = 1$.

b Solve the differential equation.
c How long will it take for the number of mobile phones to double from its initial value?
d How many mobile phones will eventually be in the country?

> This model is named after Benjamin Gompertz (1779–1865), a British mathematician. It has been applied to human lifespan, tumour growth and, as in this example, uptake of mobile phones.

29 Let the differential equation $\frac{\mathrm{d}y}{\mathrm{d}x} = \sqrt{x+y}\ (x + y \geqslant 0)$ satisfy the initial conditions $y = 1$ when $x = 1$. Also let $y = c$ when $x = 2$.

a Use Euler's method to find an approximation for the value of c, using a step length of $h = 0.1$. Give your answer to four decimal places.

You are told that if Euler's method is used with $h = 0.05$ then $c = 2.7921$, if it is used with $h = 0.01$ then $c = 2.8099$ and if it is used with $h = 0.005$ then $c = 2.8121$.

b Plot on graph paper, with h on the horizontal axis and the approximation for c on the vertical axis, the four points (one of which you have calculated and three of which have been given). Use a scale of 1cm = 0.01 on both axes. Take the horizontal axis from 0 to 0.12 and the vertical axis from 2.76 to 2.82.
c Draw, by eye, the straight line that best fits these four points, using a ruler.
d Use your graph to give the best possible estimate for c, giving your answer to three decimal places.

Mathematics HL November 2012 Paper 3 Series and differential equations Q2

30 Consider the differential equation $\frac{\mathrm{d}^2y}{\mathrm{d}x^2} + xe^{-x^2} = 0$ subject to the initial conditions that when $x = 0$, $y = 0$ and $\frac{\mathrm{d}y}{\mathrm{d}x} = 1$.

Use Euler's method with step length 0.1 to estimate y when $x = 1$.

31 Consider the differential equation $\frac{\mathrm{d}^2y}{\mathrm{d}x^2} = 2x + y$, subject to initial conditions that when $x = 0$, $y = 1$ and $\frac{\mathrm{d}y}{\mathrm{d}x} = 2$. Consider an Euler's method approach (applied to second order differential equation) with step length 0.1.

a Show that, when $x = 0.1$, this method predicts that $y = 1.2$ and $\frac{\mathrm{d}y}{\mathrm{d}x} = 2.1$.
b Use Euler's method with step length 0.1 to predict $y(1)$.

32 **a** Show that $\frac{1}{x} + \frac{1}{1-x} = \frac{1}{x(1-x)}$.

A model for the proportion of people who know a rumour (p) is modelled by $\frac{\mathrm{d}p}{\mathrm{d}t} = 2p(1-p)$ where t is the time in weeks.

b Sketch a slope field for $0 \leqslant p \leqslant 1$ and $0 \leqslant t \leqslant 2.5$, using intervals of 0.1 for p and 0.25 for t. Add the solution curves for which, initially, $p = 0.1$ and $p = 0.6$.
c When $t = 0$, $p = 0.1$.
i Solve the differential equation.
ii Hence estimate the time taken for half of the people to know the rumour.

33 **a** Write the differential equation $\frac{d^2x}{dt^2} = x$ as a system of coupled differential equations.

b **i** Sketch the phase plane for the system of differential equations.

ii State the nature of the solution.

c Jamilia wants to find $x(2)$ given that $x(0) = 0,\ x'(0) = 2$.

i Explain why Jamilia cannot use the phase portrait to estimate $x(2)$.

ii Use Euler's method with step length 0.2 to estimate the value of $x(2)$.

iii Explain why this is an underestimate of the true value.

d **i** By finding the eigenvalues of an appropriate matrix, find the exact solution to the differential equation, given that $x(0) = 0,\ x'(0) = 2$.

ii Hence estimate the percentage error in the estimate found in part **c ii**.

iii How could the percentage error in the estimate be reduced?

34 The number of plants, Y thousands, infected by a fungal infection is modelled by a monomolecular model $\frac{dY}{dt} = 6 - 3Y$

a **i** Sketch a slope field for positive Y and t, showing slopes for $0 \leqslant t, Y \leqslant 3$ at intervals of 0.25.

ii Add a solution curve to your slope field illustrating the situation where no plants are initially infected.

iii How many plants will eventually be infected?

b Solve the model, given that no plants are initially infected.

c An updated model includes the fact that the fungal spores slowly die. The new model states that $\frac{dY}{dt} = e^{-2t}(6 - 3Y)$.

i Without calculating specific values for the slopes, sketch a slope field for the new model for $0 \leqslant t,\ Y \leqslant 3$ at intervals of 0.25, adding a solution curve illustrating the situation where no plants are initially infected.

ii Solve the new model, given that no plants are initially infected.

iii According to your solution to part **ii**, how many plants will eventually be infected?

Links to: Chemistry

This model is called the monomolecular model because it is also used to model the amount of reactant remaining when a reaction is a single molecule decomposing into two molecules.

35 In a simple model for the spread of a disease, people are categorized into susceptibles (S million) and infected (I million).

The infection rate is 0.2 per susceptible per week.

The recovery rate is 0.4 per infected per week. When someone recovers from this disease, they become susceptible again.

a Write down a system of coupled differential equations for $\frac{dS}{dt}$ and $\frac{dI}{dt}$.

b **i** Show that the total population, $S + I$, remains constant according to this model.

ii Hence, or otherwise, find $\frac{dS}{dI}$.

c If $\frac{ds}{dt} = 0$, find the relationship between S and I. Show that $\frac{dI}{dt} = 0$ if the same relationship holds.

d Hence sketch a phase portrait for I against S.

e Initially, $I = 1$ and $S = 14$.

i Find an expression for S as a function of t.

ii Find the long-term number of infected individuals.

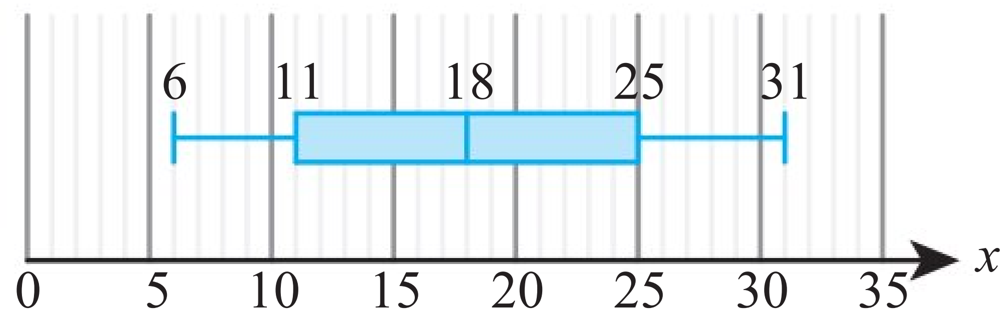
f Suggest one way in which the model is too simplistic.

Applications and interpretation HL:

Practice Paper 1

2 hours, maximum mark for the paper [110 marks].

1 The box plot shows the distribution of lengths of leaves, in cm, of a certain plant.



a Given that a leaf is longer than 11 cm, find the probability that it is longer than 25 cm.

b Five leaves are selected at random. Find the probability that exactly two of them have length between 11 cm and 25 cm.

[5]

2 If $x = \log_{10} a$ and $y = \log_{10} b$, determine an expression in terms of x and y for

a $\log_{10}(100ab^2)$

b $\log_b a$

[5]

3 A company models its profit, y dollars, in month x, by the equation

$$y = x^3 - 25x^2 + 150x + 200$$

a According to this model, what is the maximum profit in a single month

i during the first 12 months

ii during the first 18 months?

b Is the model suitable in the long term? Explain your answer.

[5]

4 An arithmetic sequence has 5th term 5 and 10th term −15.

a Find the first term and the common difference.

b If the sum of the first n terms is n, find the value of n.

[6]

5 The function f is given by $f(t) = 16e^{-3t}$ for $t \geqslant 0$

a Find the range of f.

b Find the inverse function, $f^{-1}(t)$.

c The function $f(t)$ is used to model the amount of a radioactive substance, where t is the time in years and $f(t)$ is the mass of the remaining substance. Find the half-life of the substance.

[6]

6 You are given that $z = 1 + 2i$ and $w = 2 - i$.

a Calculate $\frac{z}{w^*}$.

b Find the real values of p and q such that $pz + qw = i$.

[6]

7 **a** Write $\dfrac{1}{3x\sqrt{x}}$ in the form ax^n, where a and n are rational numbers.

b Differentiate $\dfrac{1}{3x\sqrt{x}}$.

c Find $\displaystyle\int \frac{1}{3x\sqrt{x}}\,\mathrm{d}x$.

[6]

8 Starting on his 18th birthday, Morgan puts \$100 into a savings account earning 4% interest annually. On each birthday he adds \$100 to his account.

u_n is the amount in dollars in the account on his nth birthday, so $u_{18} = 100$ and $u_{19} = 204$.

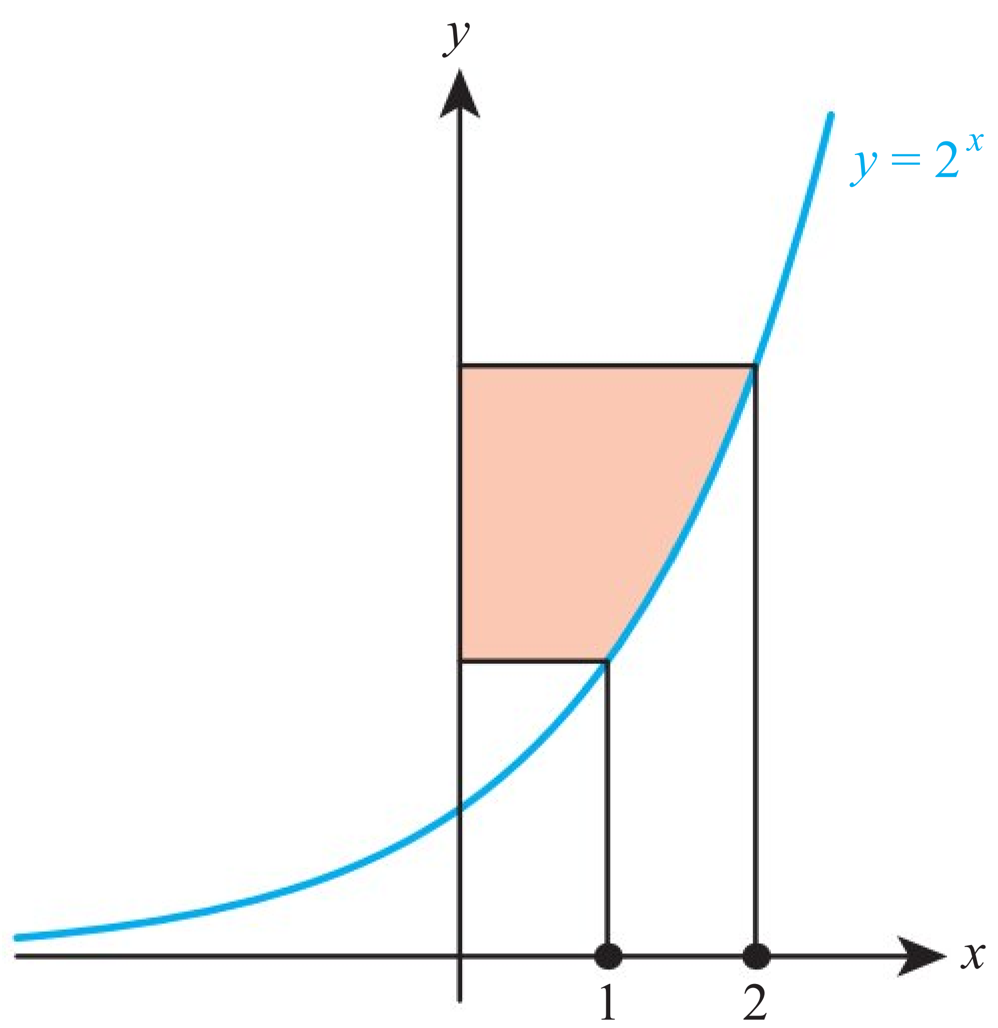
a Write an expression for u_{n+1} in terms of u_n.

b On what birthday will Morgan first have more than \$2000 in his account?

c What percentage of the amount in the account at the birthday in part **b** is due to interest?

[6]

9 The diagram shows the area enclosed by $y = 2^x$, the lines $y = 2$ and $y = 4$ and the y-axis.



The inside of a toy cup is modelled as a solid obtained by rotating this area about the y-axis. The distances are measured in cm. Find the volume of the cup.

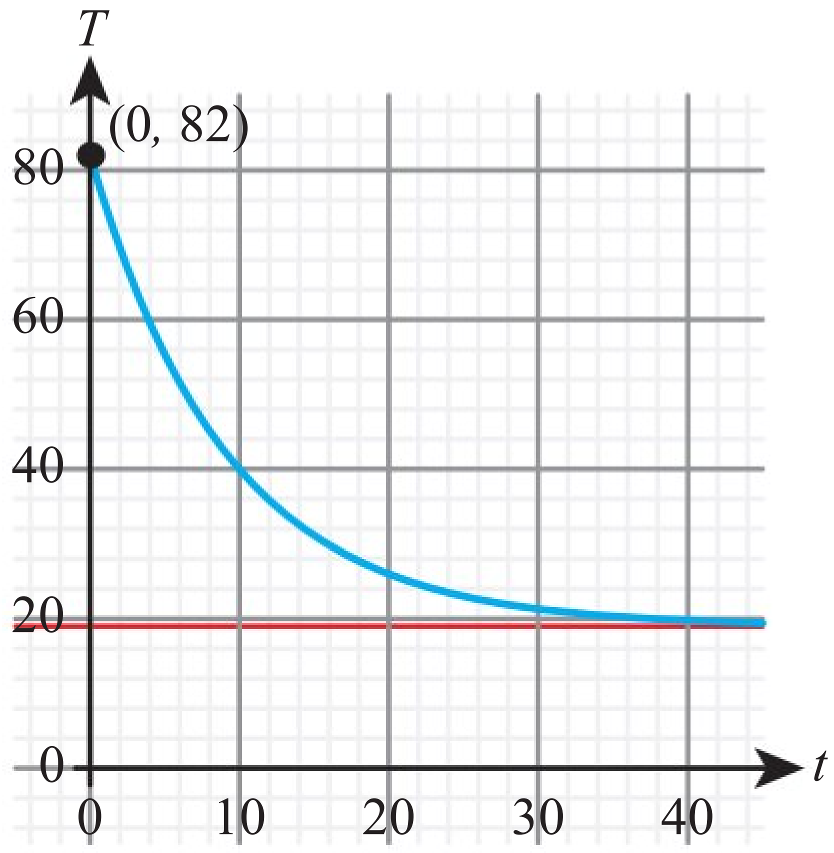
[5]

10 **a** Find the exact solutions of the equation $3x^2 + x + 4 = 0$.

b The solutions are represented by points A and B on an Argand diagram. Find, in radians, the size of the angle AOB (where O is the origin).

[5]

11 A bowl of soup is served at a temperature of 82°C and left to cool. The diagram shows the temperature of the soup against time.



a Use the graph to estimate the room temperature.

The equation of the graph is $T = 19 + Ae^{-kt}$.

b Find the value of A.

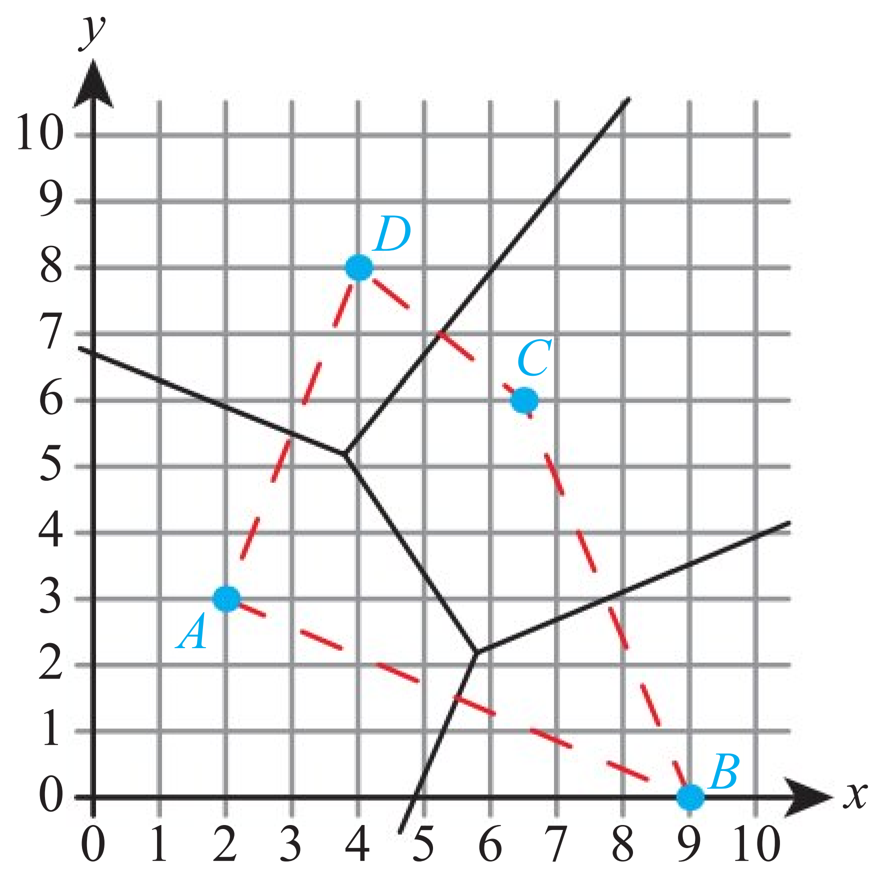
It takes 8 minutes for the soup to cool down to 45°C.

c Find the value of k, correct to two decimal places.

d Find, to the nearest degree, the temperature of the soup after *another* 8 minutes.

[6]

12 The sites of the Voronoi diagram below represent four weather stations.



The table shows the temperature recorded at each station at a particular time.

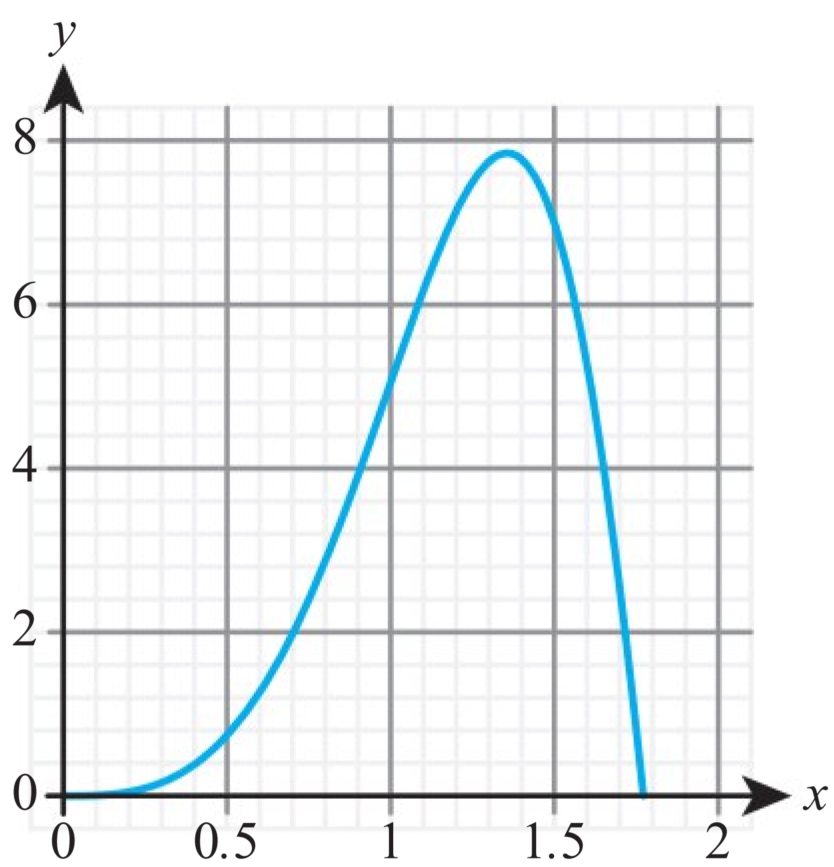
Station	A	B	C	D
Temperature °C	17	18	19	21

a Estimate the temperature at the location with coordinates (8, 2).

b A new weather station is to be built within the area enclosed by the dotted lines, so that it is as far as possible from any existing station. Write down the approximate coordinates of the new station.

[4]

13 The diagram shows the part of the graph of $y = 6x\sin\left(x^2\right)$ for $0 \leqslant x \leqslant \sqrt{\pi}$.



a Use the trapezium rule with four strips to estimate the area enclosed by the graph and the x-axis, giving your answer correct to four decimal places.

b Use integration to find the exact value of the area.

c Find the percentage error in the estimate from part **a**.

[8]

14 The displacement of an object from its starting position at time t is given by $s = t\ln(t+1)$. Find expressions for the velocity and acceleration at time t.

[6]

15 A zoologist is investigating a species of fish living in very cold water. She estimates the number of fish, y hundred, found in several locations, and wants to model how this depends on the temperature of the water, x°C. The proposed model has the form

$$y = A - Be^{0.2x} - Ce^{-1.5x}$$

Some values of x and y are given in the table.

Location	Temperature °C	Number of fish
1	1.2	614
2	3.5	570
3	6.3	295

a Estimate the values of A, B and C, giving the values correct to one decimal place.

b According to this model, what is the highest water temperature that supports this species of fish? Give your answer to the nearest degree Celsius.

[5]

16 An astronomer observes three stars, A, B and C, with position vectors

$$\mathbf{a} = \begin{pmatrix} 2 \\ -3 \\ 5 \end{pmatrix}, \mathbf{b} = \begin{pmatrix} -2 \\ 3 \\ 6 \end{pmatrix} \text{ and } \mathbf{c} = \begin{pmatrix} 10 \\ 1 \\ h \end{pmatrix}$$

a Given that AC is perpendicular to AB, find the value of h.

b Find the area of the triangle formed by the points A, B and C.

[7]

17 Consider the differential equation $(1+x)^2 \frac{dy}{dx} = e^x - y^2$.

a Sketch the slope field for the integer values of x and y with $0 \leqslant x, y \leqslant 3$. Add to your sketch the solution curve through $(0, 2)$.

b Use Euler's method with step length 0.1 to estimate the value of y when $x = 0.3$, given that $y = 2$ when $x = 0$. Give your answer to two decimal places.

[7]

18 The number of birds in a forest, y hundred, at time t years, is modelled by the differential equation

$$\frac{dy}{dt} = -0.06(t-1)(y-5)$$

At time $t = 1$ there are 1500 birds.

a Find an expression for the number of birds at time t years.

b According to this model, describe the long-term behaviour of the population.

[6]

19 Two voltage sources are connected in a circuit. The voltages at time t are given by

$$V_1 = 15\sin(30t) \quad \text{and} \quad V_2 = 20\sin(30t+5)$$

Use complex numbers to find the total voltage in the form $V = A\sin(30t + B)$.

[6]

Applications and interpretation HL:

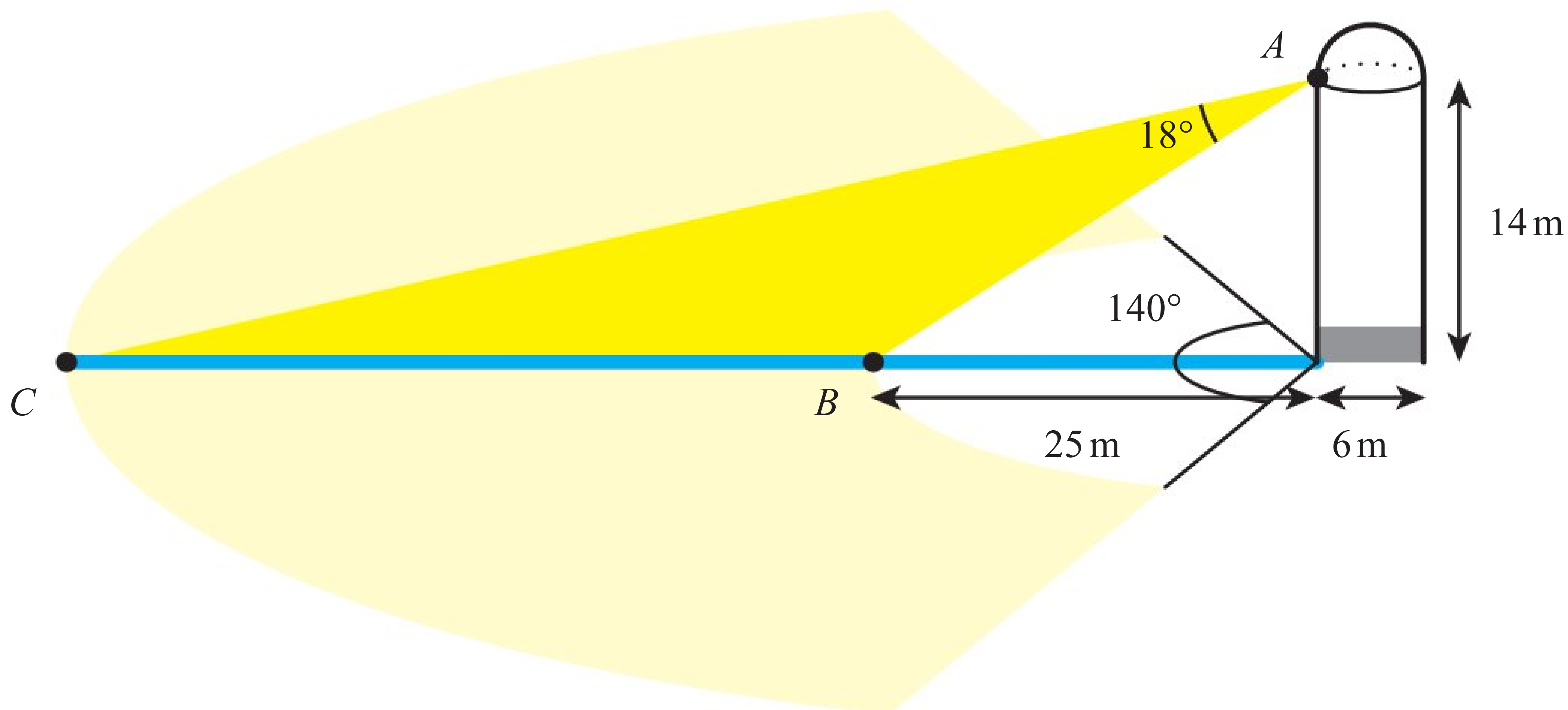
Practice Paper 2

2 hours, maximum mark for the paper [110 marks].

1 *[13 marks]*

A lighthouse is formed of a cylindrical tower and a hemispherical dome. The tower is 14 m high and has a 6 m diameter.

The exterior of the lighthouse needs to be treated to preserve it from the salt air of the sea.



a Find the surface area of the lighthouse. [3]

The light at point A forms a beam that makes a fixed 18° angle such that it extends from a point, B, 25 m from the coast to a point C. The light then rotates through 140° in a plane parallel to the ground, illuminating a band of water. Find

b the length AB [1]

c the angle ABC [2]

d the distance BC [3]

e the area of water the light can cover. [4]

2 *[15 marks]*

Two schools, Jomer Tree and St Atistics, compete in a race.

Out of 50 students from Jomer Tree, 30% take between 7 and 10 minutes to complete the race and 40% take between 12 and 15 minutes.

a Estimate the mean and standard deviation of the times for all 50 students from Jomer Tree. [3]

Students who take less than 10 minutes score 5 points. Students who take between 10 minutes and 12 minutes score 3 points. Students who take more than 12 minutes score 0 points.

b Find the expected value of the number of points scored by the 50 students from Jomer Tree. [2]

Jomer Tree enters two students in the race. The outcome for each student is independent of the outcome of any other student.

c Find the probability that Jomer Tree scores less than 6 points in total. [3]

The times for students from St Atistics are thought to be normally distributed. The mean is 11 minutes and the standard deviation is 3 minutes. In the race, St Atistics enters one student.

d If a student scores 0 points, find the probability that the student comes from Jomer Tree. [3]

e The team with most points wins. Find the probability that Jomer Tree wins. [4]

3 *[20 marks]*

Analysis has shown that, after students graduate from university, the probability that they donate money to the alumni fund in a given year is 0.72 if they donated the previous year and 0.16 if they did not donate the previous year. It is assumed that final year students do not donate.

a Write down the transition matrix, $\mathbf{T}$, for this information. [2]

b Find the probability of a student donating in the third year after graduating. [3]

c Find the eigenvalues and corresponding eigenvectors of $\mathbf{T}$. [6]

d Hence write down matrices $\mathbf{P}$ and $\mathbf{D}$ such that $\mathbf{T} = \mathbf{PDP}^{-1}$. [3]

e Find an expression for the probability of donating in year n. [4]

f Hence find the long-term probability of donating. [2]

4 *[20 marks]*

Breakdowns of a photocopier in an office occur at a rate of 0.8 per month. It is assumed that the number of breakdowns follows a Poisson distribution.

a State two conditions that need to be met for the Poisson distribution to be appropriate. [2]

b Find the probability of

i no breakdowns in a given month

ii at most 2 breakdowns in a given month

iii at least 6 breakdowns in a three-month period. [4]

If the photocopier breaks down at least 6 times in at least two 3-month periods in a given year, the manufacturer must replace the photocopier free of charge.

c Find the probability that the manufacturer has to replace the machine free of charge in a given year. [3]

Let X be the number of breakdowns in a 6-month period.

d Find $\mathrm{E}(X)$ and $\mathrm{Var}(X)$. [2]

The office manager believes that the photocopier breaks down at a rate of more than 0.8 per month. He records the number of breakdowns each month for 6 months and carries out a hypothesis test to investigate his suspicion. He finds that there are 9 breakdowns in the 6 months he is monitoring.

e Carry out a test at the 5% level of significance to investigate his suspicion. [5]

f Find the probability that he makes a Type I error. [2]

In reality the mean number of breakdowns per month is 1.1.

g Find the probability he makes a Type II error. [2]

5 *[13 marks]*

An air traffic control tower is situated at $\begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$.

At 12 noon, the controller observes the trajectories of two planes.

Plane A follows the path $\mathbf{r} = \begin{pmatrix} 1 \\ 4 \\ 4 \end{pmatrix} + t\begin{pmatrix} 1 \\ 2 \\ 2 \end{pmatrix}$

and plane B follows the path $\mathbf{r} = \begin{pmatrix} -5 \\ 1 \\ 22 \end{pmatrix} + t\begin{pmatrix} 1.5 \\ 1.5 \\ -2 \end{pmatrix}$

where t is the time in minutes and the units are in kilometres.

a Find the speed of plane A. [2]

The two planes both pass through the same point, but do not collide.

b Find the position of that point and the time at which each plane passes through it. [6]

c Find an expression for the distance between the two planes at time t. [2]

If the two planes will come within 5 km of each other, the controller issues an alert.

d Determine whether the controller should issue an alert. [3]

6 *[17 marks]*

The differential equation

$$\frac{d^2x}{dt^2} + 2\frac{dx}{dt} - 3x = 0$$

has initial conditions $x = 1$, $\dot{x} = -2$ when $t = 0$.

a Write the second order differential equation as a system of coupled first order equations. [2]

b Hence, using Euler's method with a step length of 0.2 [6]

i estimate the value of $x(2)$

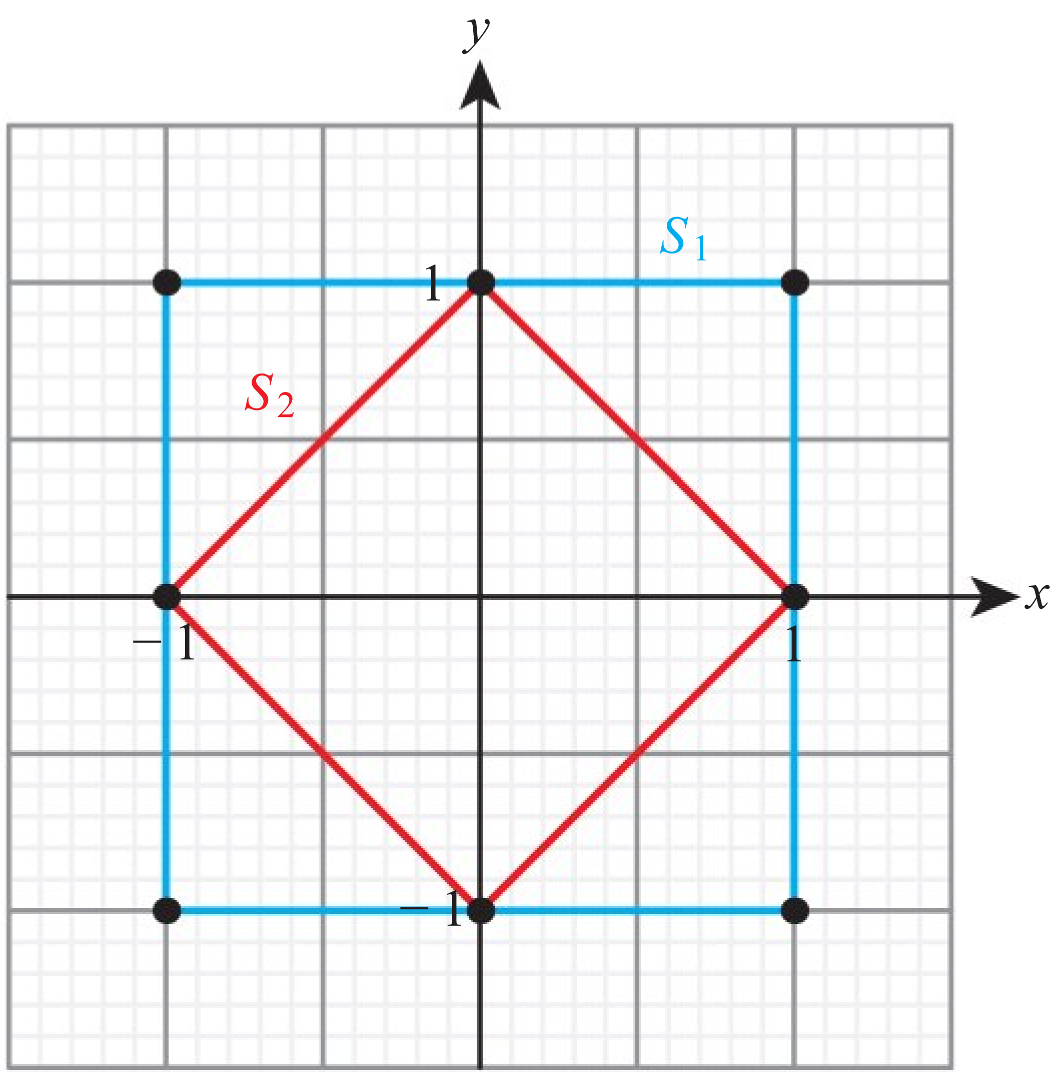
ii sketch a graph of x against t for $0 < t < 3$.

c Find the exact solution to the differential equation and explain whether it is stable or unstable. [7]

d Find the percentage error in your estimate in part **b i**. [2]

7 *[12 marks]*

The square S_1 with vertices at $(1, 1)$, $(-1, 1)$, $(-1, -1)$, $(1, -1)$ is mapped to the square S_2 with vertices at $(0, 1)$, $(-1, 0)$, $(0, -1)$, $(1, 0)$.



The matrix **M** that represents the transformation which maps S_1 to S_2 is the composition of an enlargement and a rotation anticlockwise around the origin.

a Find the matrices that represent the enlargement and the rotation. [3]

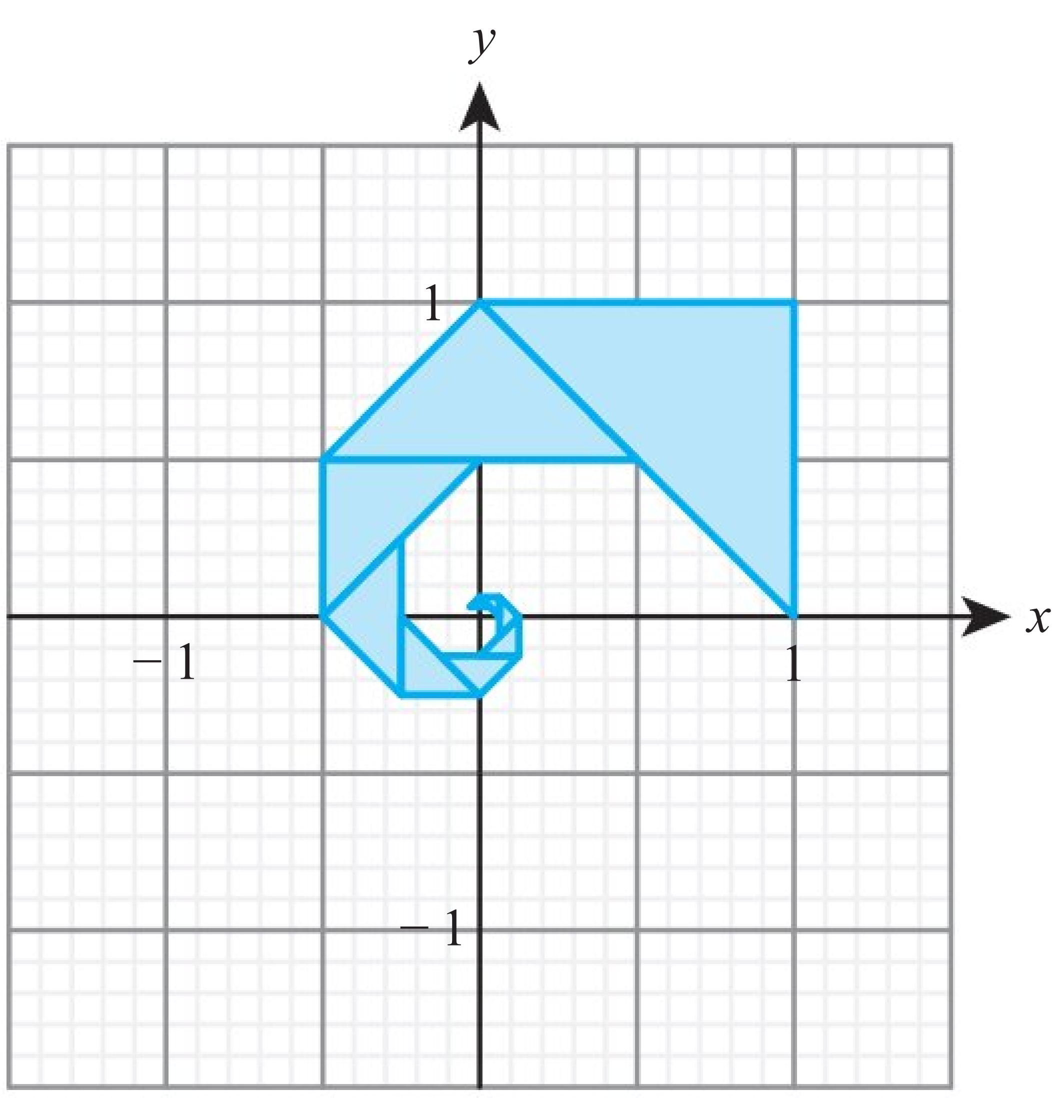
b Hence find the matrix **M**. [2]

The triangle T_1 has vertices $(1, 0)$, $(1, 1)$, $(0, 1)$. T_2 is the image of T_1 under **M**.

c Find the area of T_1. [1]

d Hence use the matrix **M** to find the area of T_2. [2]

Repeated applications of **M** yield the fractal shown below.



e Find the area of the fractal. [4]

Guidance for Paper 3

Paper 3 is a new type of examination to the IB. It will include long, problem-solving questions. Do not be intimidated by these questions – they look unfamiliar, but they will be structured to guide you through the process. Although each question will look very different, it might help you to think about how these questions are likely to work:

- There might be some 'data collection', which will likely involve working with your calculator or simple cases to generate some ideas.
- There might be a conjecturing phase where you reflect on your data and suggest a general rule.
- There might be a learning phase where you practise a technique on a specific case.
- There might be a proving phase where you try to form a proof. It is likely that the method for this proof will be related to the techniques suggest earlier in the question.
- There might be an extension phase where you apply the introduced ideas to a slightly different situation.

All of these phases have their own challenges, so it is not always the case that questions get harder as you go on (although there might be a trend in that direction). Do not assume that just because you could not do one part you should give up – there might be later parts that you can still do. Some parts might look unfamiliar, and it is easy to panic and think that you have just not been taught about something. However, one of the skills being tested is mathematical comprehension so it is possible that a new idea is being introduced. Stay calm, read the information carefully and be confident that you do have the tools required to answer the question, it might just be hidden in a new context.

You are likely to have a lot of data so be very systematic in how you record it. This will help you to spot patterns. Then when you are suggesting general rules, always go back to the specific cases and check that your suggestion works for them.

These questions are meant to be interlinked, so if you are stuck on one part try to look back for inspiration. This might be looking at the answers you have found, or it might be trying to reuse a method suggested in an earlier part. Similarly, even more than in other examinations, it is vital in Paper 3 that you read the whole question. Sometimes later parts will clarify how far you need to go in earlier parts, or give you ideas about what types of method are useful in the question.

These questions are meant to model the thinking process of mathematicians. Perhaps the best way to get better at them is to imitate the mathematical process at every opportunity. So the next time you do a question, see if you can spot underlying patterns, generalize them and then prove your conjecture. The more you do this, the better you will become.

Applications and interpretation HL:

Practice Paper 3

1 hour, maximum mark for the paper [55 marks].

1 *[27 marks]*

This question is about investigating the validity and reliability of a psychologist's methods and results.

Rushi is a psychologist exploring children's speech development. He wants to investigate whether being an eldest child has an impact on speech development.

He uses a vocabulary test to measure each child's speech development. The child can score between 0 and 10 on the vocabulary test with higher scores indicating better speech development.

a Rushi decides to only use children within one month of their third birthday in this study. Explain, in the context of statistical investigations, why this is appropriate. [1]

b Rushi collects data for 10 children. His results are shown below.

Participant	Vocabulary test result	Eldest child
A	6.4	Yes
B	8.1	Yes
C	7.2	Yes
D	9.4	Yes
E	2.2	Yes
F	3.1	No
G	7.4	No
H	7.1	No
I	4.0	No
J	6.0	No

i Rushi has no prior belief about whether being an eldest child might improve speech development. He wants to see if the mean speech development differs between eldest and non-eldest children. Write down appropriate null and alterative hypotheses.

ii Conduct an appropriate hypothesis test at the 5% significance level.

iii Rushi's friend Aretha sees his analysis and makes the following statement:

'There is no difference between the speech development of eldest and non-eldest children.'

Make three criticisms about the validity of this statement. [8]

c Suggest one way in which Rushi can check

i the content validity of the vocabulary test

ii the criterion validity of the vocabulary test. [2]

d To assess the statistical validity of the test used in part **b**, Rushi decides to collect more data to understand the underlying distribution of scores. He collects the following data.

Score, v	$0 \leqslant v < 2$	$2 \leqslant v < 4$	$4 \leqslant v < 6$	$6 \leqslant v < 8$	$8 \leqslant v < 9$	$9 \leqslant v < 10$
Frequency	4	20	35	40	15	5

i Find unbiased estimates of the mean and variance of these data.

ii Based on the manufacturers of the test, for children in Rushi's sample age range, the mean should be 5 and the standard deviation 2. By combining groups suitably, test at the 5% significance level whether a normal distribution with these parameters fits the data.

iii Determine, using a 5% significance level, whether the data set fits any normal distribution.

iv Is the test used in part **b** valid? [12]

e Rushi wants to check the reliability of his test by using a test-retest. Two months after the original vocabulary test was conducted, he conducts a retest with the following results.

Participant	1st test result	2nd test result
A	6.4	6.8
B	8.1	8.3
C	7.2	6.9
D	9.4	9.2
E	2.2	3.3
F	3.1	2.8
G	7.4	7.0
H	7.1	8.1
I	4.0	6.9
J	6.0	5.4

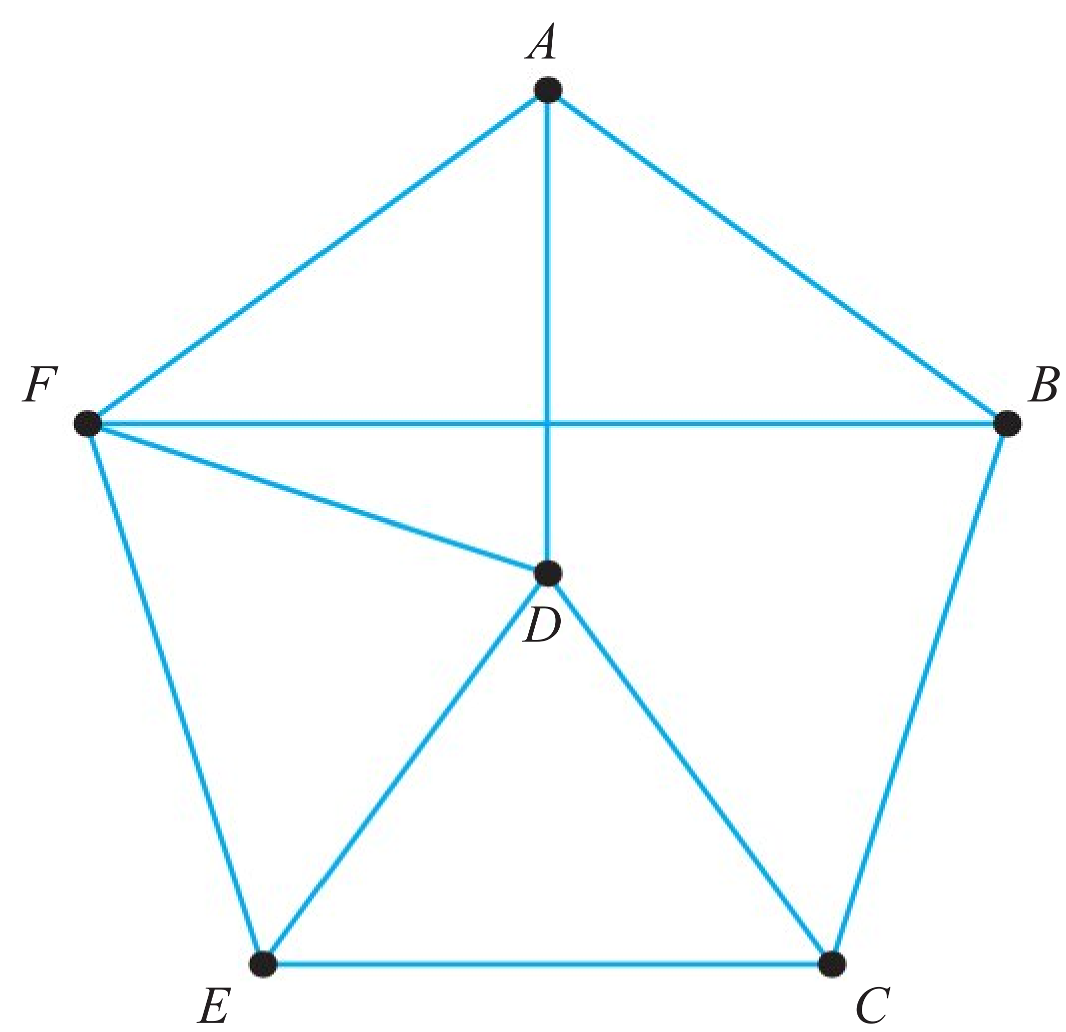
i Explain why a change in means between the two tests does not indicate that the test is unreliable.

ii Rushi decides to test the reliability by using a two-tailed Spearman's rank correlation test. The critical value for this test is 0.6485. Determine whether the test is reliable. [4]

2 *[28 marks]*

This question is about a council planning various facilities in a village.

The graph represents a network of roads between six junctions in a city. All roads are two-way.



a i Write down the adjacency matrix for this graph.

ii Hence find the number of different ways to drive from Junction B to Junction C using exactly five (not necessarily different) roads. [4]

The table shows the travel times for direct roads between different pairs of junctions. You can assume that, for each road, the travel time is the same in both directions. The sum of all the times in the table is 63 minutes.

	B	C	D	E	F
A	6	–	4	–	4
B	–	8	–	–	10
C		–	8	6	–
D			–	5	5
E				–	7

b The city council wants to send out a car to inspect each road. The car only needs to travel along each road in one direction.

i Explain whether it is possible for the car to travel along each road exactly once, starting and finishing at Junction D.

ii Find the shortest time required for the car to travel along each road at least once, starting and finishing at Junction D.

iii Instead, the car needs to travel along each road exactly twice, starting and finishing at D. Explain why this is possible, and state the time required to do this.

[9]

The table shows the shortest time required to travel between each pair of junctions.

	B	C	D	E	F
A	6	a	4	b	4
B	–	8	10	14	10
C		–	8	6	13
D			–	5	5
E				–	7

c A small shop is located at each junction. A delivery van needs to visit each shop at least once and return to its stating point. Let T denote the shortest amount of time required to complete this task.

i Find the missing values a and b in the table above.

ii Use the nearest neighbour algorithm, starting from vertex F, to find an upper bound for the travelling salesman problem.

iii By deleting vertex A, find the lower bound.

iv Another lower bound, found by deleting vertex F, is 30. Write down an inequality for T.

[10]

Roads AD and DF are now made one-way, so that it is only possible to drive from D to A and from F to D.

d The council decides to build a petrol station at one of the six junctions. They would like to choose a junction which is most likely to be visited by a car driving randomly between the junctions. Use the PageRank algorithm to determine at which junction the petrol station should be built. If there is more than one possible option, state all of them.

[5]

Answers

Chapter 1 Prior Knowledge

1 a $15x^9y^5$ b $4c^{-1}d^2$ c $9a^8b^{-4}$
2 100
3 $x = \frac{5}{3}$
4 $x = \ln 11 \approx 2.40$
5 64
6 $r = 0.894,\ y = 1.33x + 1.67$

Exercise 1A

1 a 2 b 3
2 a 4 b 5
3 a 7 b 5
4 a 4 b 8
5 a 125 b 25
6 a $\frac{1}{10}$ b $\frac{1}{10}$
7 a $\frac{1}{4}$ b $\frac{1}{9}$
8 a $\frac{1}{4}$ b $\frac{1}{1000}$
9 a $x^{\frac{5}{2}}$ b $x^{\frac{4}{3}}$
10 a $x^{\frac{5}{3}}$ b $x^{\frac{3}{2}}$
11 a $4x^{-\frac{3}{2}}$ b $5x^{-\frac{5}{2}}$
12 a $\frac{1}{5}x^{\frac{3}{2}}$ b $\frac{1}{3}x^{\frac{2}{3}}$
13 a $x^{\frac{5}{6}}$ b $x^{\frac{5}{3}}$
14 $\frac{1}{16}$
15 2
16 $\frac{8}{27}$
17 $x^{\frac{11}{12}}$
18 $3x^{-\frac{1}{3}} + 2x^{\frac{1}{2}}$
19 $\frac{1}{3}x^{-\frac{3}{2}}$
20 $\frac{1}{4}$
21 $\frac{25}{4}$
22 $x^{\frac{7}{6}}$
23 $x^{-\frac{3}{2}}$
24 $x^{\frac{8}{3}}$
25 $\frac{1}{8}x^{-\frac{3}{2}}$
26 $x^{\frac{1}{2}} + x^{-1}$
27 $x^{\frac{3}{2}} - x^{\frac{1}{2}}$
28 $\frac{1}{3}x^{-\frac{3}{2}}$
29 $\frac{1}{2}x^1 + \frac{3}{2}x^{-\frac{1}{2}}$
30 $y^4 = 16x^{\frac{8}{3}}$
31 $\sqrt[3]{y} = 3x^{\frac{1}{6}}$
32 $y = x^{\frac{3}{2}}$
33 $y^3 = \frac{8}{27}x^{-\frac{3}{2}}$
34 $\frac{4}{3}$
35 $-\frac{8}{3}$
36 $-\frac{9}{7}$
37 9
38 $x = y^{\frac{2}{3}}$
39 ± 27
40 64

Exercise 1B

1 a 0.01 b 0.1
2 a 1.1 b 2.01
3 a 98 b 997
4 a 2 b -2
5 a e^2 b e^5
6 a $\sqrt{e}$ b $\frac{1}{\sqrt[3]{e}}$
7 a $\frac{e^2-4}{2}$ b $\frac{e^4+1}{3}$
8 a $2p-q$ b $3q-p$
9 a $2p-3q$ b $4q-2p$
10 a $\frac{3}{2}p+\frac{1}{2}q$ b $2p+\frac{3}{2}q$
11 a $2+p-2q$ b $1+2p-5q$
12 a 500 b 2
13 a 47 b 3.5
14 a $\frac{7}{3}$ b $\frac{101}{99}$
15 a $\frac{3+5e^4}{1-e^4}$ b $\frac{e^3+2}{e^3-1}$
16 a $\frac{\ln 2+2\ln 3}{\ln 3-\ln 2}$ b $\frac{2\ln 2+\ln 3}{\ln 3-\ln 2}$
17 a $\frac{3\ln 2+5\ln 7}{2\ln 7-\ln 2}$ b $\frac{8\ln 2-\ln 7}{3\ln 7-\ln 2}$
18 a $x+4y$ b $2x+y-5z$ c $1+2x+3y$
19 a $2+\frac{1}{2}x$ b $y-1-5z$
20 $\ln(a^2b^6)$
21 $\ln\left(\frac{\sqrt[3]{x}}{\sqrt{y}}\right)$
22 997
23 7
24 a 1.43 b 1.77
25 19.9
26 a $-\frac{1}{2}$ b $\frac{1}{3}$
27 2
28 4
29 a 2.4 b 1.26
30 $-\frac{9}{7}$
31 $\frac{3\ln 5+5\ln 9}{\ln 9-2\ln 5}$
32 a 10 b 6.58
33 a i 1000 ii 1210
b 7.27
34 7.97
35 $x=100,\ y=10$
36 $x=0.1,\ y=e^2$
37 $\frac{\ln 96}{\ln 72}$
38 6.64
39 a $-a^b$ b 1

Exercise 1C

1 a $\frac{1}{3}$ b 1
2 a 1 b $\frac{5}{9}$
3 a $\frac{16}{3}$ b 9
4 a $\frac{1}{3}$ b $\frac{1}{4}$
5 a $\frac{75}{8}$ b $\frac{64}{7}$
6 a $1<x<3$ b $-4<x<-2$
7 a $|x|<\frac{1}{2}$ b $|x|<\frac{1}{3}$
8 a $|x|<2$ b $|x|<5$
9 a $-5<x<-3$ b $0<x<2$
10 a $|x|<\frac{2}{3}$ b $|x|<\frac{3}{4}$
11 4
12 4
13 $\frac{3}{2}$
14 $-\frac{1}{3}$
15 $\frac{1}{4}$
16 $2, \frac{2}{3}, \frac{2}{9}$
17 $\frac{1}{3}, \frac{2}{3}$
18 a $1, \frac{2}{5}, \frac{4}{25}$ b $\frac{5}{3}$
19 3
20 a $|x|<9$ b $\frac{27}{7}$
21 a $2<x<4$ b $\frac{5}{4-x}$

22 a $|x| < \frac{1}{2}$ b $\frac{2}{1-2x}$

23 6

24 a $0 < x < \frac{1}{2}$ b $\frac{x}{1-4x^2}$

25 a $\frac{6}{5}$ b $|x| < 2$

26 9

Exercise 1D

1 a $\log y$ against $\log x$ is a straight line with gradient 4 and y-intercept 0

b $\log y$ against $\log x$ is a straight line with gradient 5 and y-intercept 0

2 a $\log y$ against $\log x$ is a straight line with gradient 4 and y-intercept $\log 3$

b $\log y$ against $\log x$ is a straight line with gradient 2 and y-intercept $\log 5$

3 a $\log y$ against $\log x$ is a straight line with gradient 0.5 and y-intercept $\log 4$

b $\log y$ against $\log x$ is a straight line with gradient $\frac{1}{3}$ and y-intercept $\log 2$

4 a $\log y$ against $\log x$ is a straight line with gradient -1 and y-intercept 0

b $\log y$ against $\log x$ is a straight line with gradient -0.5 and y-intercept $\log 2$

5 a $\log y$ against x is a straight line with gradient $\log 2$ and y-intercept 0

b $\log y$ against x is a straight line with gradient $\log 5$ and y-intercept 0

6 a $\log y$ against x is a straight line with gradient $\log 2$ and y-intercept $\log 3$

b $\log y$ against x is a straight line with gradient $\log 3$ and y-intercept $\log 2$

7 a $\log y$ against x is a straight line with gradient $-\log 7$ and y-intercept $\log 4$

b $\log y$ against x is a straight line with gradient $-\log 4$ and y-intercept $\log 6$

8 a against x is a straight line with gradient 3 and y-intercept $\ln 2$

b $\ln y$ against x is a straight line with gradient -1 and y-intercept $\ln 10$

9 $y = 5\times x^4$

10 $y = \frac{10}{x^2}$

Answers to questions 11–13 are approximate.

11 $y = 4\times 2^x$

12 $y = 50\times \frac{1}{4}^n$

13 $y = 2\times 5^x$

14 a $\ln X = -kt + \ln(X_0)$

b $X_0 = 18.33,\ k = 0.472$

c 1.47 minutes

15 a $\log F = n\log d + \log k$ b $n = -2.0,\ k = 0.15$

16 a $a \approx 96.3,\ b \approx 1.40$ b 40%

c 42 000 miles

17 a $a = 0.2,\ n = 1.5$ b 84 years

18 2

19 a i -0.998 ii -0.965

b $P = aR^n$ c $a \approx 6.6,\ n \approx -1.0$

20 a i -0.990 ii -0.945

b $F = aR^n$

c $n = -0.72,\ a = 4600$ d 4990

21 $a = -4.88,\ n = 2$

22 a $A \approx 10.5,\ k \approx 0.416$ b 10:50 am

23 a 10

b $\ln\left(\frac{10}{P} - 1\right) = -\beta t + \ln J$

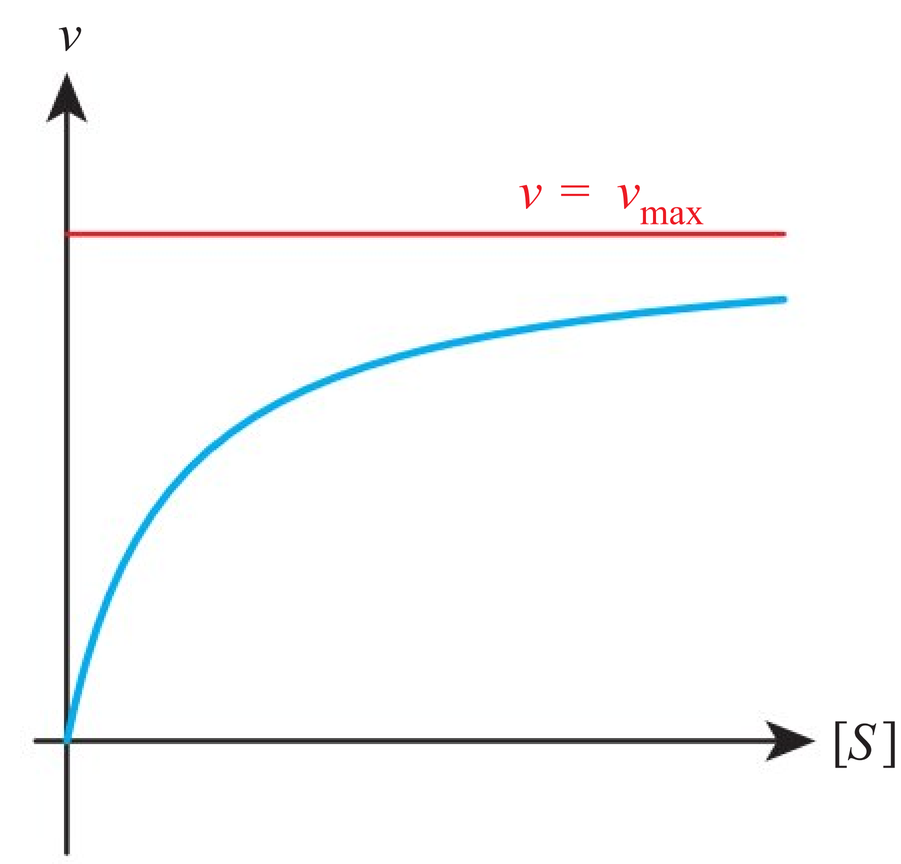
c $\beta = 0.195,\ J = 0.977$

d 11.2 years e 5060 rabbits

f Extrapolating from the observed data

24 a $\frac{v_{\max}}{2}$

b



c Gradient: $\frac{K_m}{v_{\max}}$, y-intercept: $\frac{1}{v_{\max}}$

d i $\frac{1}{v} = 0.313\frac{1}{[S]} + 0.185$

ii $v_{max} = 5.42,\ K_m = 1.69$

e i Gradient: $-K_m$, y-intercept: v_{max}

ii $v_{max} = 5.31,\ K_m = 1.61$

Chapter 1 Mixed Practice

1 a $\frac{27}{8}$ b -3

2 $\ln 36$

3 5

4 14.2

5 2

6 18

7 e^{-4}

8 a $2a + b$ b $a - b - 3c$ c $\frac{1}{2}c + \frac{3}{2}a$

9 $\frac{1}{2}$

10 $\frac{9}{2}$

11 3

12 $y = 6x^{-4}$

13 a $x - \frac{1}{2}y$ b $2x - 3y - 3$

14 a $x + y$ b $x + 2y$ c $x - 2y$

15 $\log 6250$

16 $\frac{1}{2}e^{\frac{5}{3}}$

17 $-\frac{13}{3}$

18 $x = 1000,\ y = 100$

19 $\sqrt{2}$

20 $\frac{\ln 2}{2\ln 3 - 1}$

21 $\frac{\ln 5 + 3\ln 7}{\ln 7 - 2\ln 5}$

22 $\frac{\log 12}{\log 48}$

23 a 150 b 3397 c 1.82 hours

24 ex^3

25 $p = 2,\ q = -1$

26 a 3 b -1

c 3.5

27 16th

28 a $\frac{1}{3} < x < 1$ b $\frac{5}{9}$

29 $\frac{2}{3}$

30 a i -0.999 ii -0.939

b $C = ab^y$

c $a = 3.1,\ b = 0.97$

d 100 billion tonnes

31 $\frac{19}{20}$

32 $-\frac{\ln 2}{\ln 3}$

33 2

34 a $\ln k = -\frac{E_a}{R} \times \frac{1}{T} + \ln A$ b $99.2\,\text{kJ mol}^{-1}$

35 $210 \ln x$

36 -4

37 $\frac{15}{2}$

38 b $\frac{e^{-x}}{1 - e^{-x}}$ c $\ln\left(\frac{3}{2}\right)$

39 $\frac{1}{3}$

40 a ii 2^a iii $2^a\left(2^d\right)^{n-1}$

b i $\frac{2^a\left(2^{nd} - 1\right)}{2^d - 1}$ ii $d < 0$

iii $\frac{2^a}{1 - 2^d}$ iv -1

c $k = p^n q^{\frac{n(n-1)}{2}}$

Chapter 2 Prior Knowledge

1 $\sqrt{70}$

2 $2x + 5y = 23$

3 0

4 $x = 4,\ y = -1$

Exercise 2A

1 $\mathbf{a} = \begin{pmatrix} 2 \\ 3 \end{pmatrix}, \mathbf{b} = \begin{pmatrix} 3 \\ 1 \end{pmatrix}$

2 $\mathbf{a} = \begin{pmatrix} 3 \\ -1 \end{pmatrix}, \mathbf{b} = \begin{pmatrix} 2 \\ -3 \end{pmatrix}$

3 $\mathbf{a} = \begin{pmatrix} -1 \\ 4 \end{pmatrix}, \mathbf{b} = \begin{pmatrix} -3 \\ 2 \end{pmatrix}$

4 $\mathbf{a} = \begin{pmatrix} -2 \\ -1 \end{pmatrix}, \mathbf{b} = \begin{pmatrix} -1 \\ -3 \end{pmatrix}$

5 a $\mathbf{i} + 2\mathbf{j} + 3\mathbf{k}$ b $4\mathbf{i} + 5\mathbf{j} + 6\mathbf{k}$

6 a $-3\mathbf{i} + \mathbf{j}$ b $2\mathbf{i} - 2\mathbf{j}$

7 a $3\mathbf{i}$ b $-5\mathbf{j}$

8 a $\begin{pmatrix} 2 \\ -3 \\ 1 \end{pmatrix}$ b $\begin{pmatrix} 3 \\ 1 \\ -4 \end{pmatrix}$

9 a $\begin{pmatrix} 1 \\ 0 \\ 3 \end{pmatrix}$ b $\begin{pmatrix} 0 \\ 2 \\ -1 \end{pmatrix}$

10 a $\begin{pmatrix} -1 \\ 4 \\ -2 \end{pmatrix}$ b $\begin{pmatrix} -3 \\ 0 \\ 1 \end{pmatrix}$

11 a $\mathbf{a} + \mathbf{b}$ b $\mathbf{b} + \mathbf{c}$

12 a $-\mathbf{a} - \mathbf{b} - \mathbf{c}$ b $-\mathbf{b} - \mathbf{c}$

13 a $\begin{pmatrix} -3 \\ 7 \\ -3 \end{pmatrix}$ b $\begin{pmatrix} 3 \\ -7 \\ 3 \end{pmatrix}$

14 a $\begin{pmatrix} 11 \\ -4 \\ 21 \end{pmatrix}$ b $\begin{pmatrix} 22 \\ -21 \\ 36 \end{pmatrix}$

15 a $\begin{pmatrix} 11 \\ -17 \\ 15 \end{pmatrix}$ b $\begin{pmatrix} 2 \\ -9 \\ 0 \end{pmatrix}$

16 a $\mathbf{a} + \frac{4}{3}\mathbf{b}$ b $\mathbf{a} + \frac{1}{2}\mathbf{b}$

17 a $-\frac{3}{2}\mathbf{a} + \mathbf{b}$ b $-\frac{1}{2}\mathbf{b} + \frac{1}{2}\mathbf{a}$

18 a $\frac{3}{2}\mathbf{a} - \mathbf{b}$ b $-\frac{4}{3}\mathbf{b} + \frac{1}{2}\mathbf{a}$

19 a $p = 3, q = 15$ b $p = 4, q = 16$

20 a $p = -6, q = 3$ b $p = 2 - 6, q = -8$

21 a $p = 4, q = 1$ b $p = 45, q = -1$

22 a $p = -3, q = -18$ b $p = 4, q = -2$

23 a $p = -2, q = -10$ b $p = -3, q = -3$

24 a $\mathbf{a} = \begin{pmatrix} 1/3 \\ 2/3 \\ 2/3 \end{pmatrix}$ b $\mathbf{a} = \begin{pmatrix} 3/5 \\ 0 \\ 4/5 \end{pmatrix}$

25 a $\mathbf{a} = \begin{pmatrix} -1/\sqrt{6} \\ 1/\sqrt{6} \\ 2/\sqrt{6} \end{pmatrix}$ b $\mathbf{a} = \begin{pmatrix} 1/\sqrt{14} \\ -2/\sqrt{14} \\ 3/\sqrt{14} \end{pmatrix}$

26 a $\mathbf{a} = \frac{3}{\sqrt{11}}\mathbf{i} + \frac{1}{\sqrt{11}}\mathbf{j} - \frac{1}{\sqrt{11}}\mathbf{k}$

b $\mathbf{a} = \frac{1}{\sqrt{6}}\mathbf{i} - \frac{2}{\sqrt{6}}\mathbf{j} + \frac{1}{\sqrt{6}}\mathbf{k}$

27 a $\mathbf{a} = \frac{1}{\sqrt{17}}\mathbf{i} - \frac{4}{\sqrt{17}}\mathbf{j}$ b $\mathbf{a} = \frac{2}{\sqrt{13}}\mathbf{j} - \frac{3}{\sqrt{13}}\mathbf{k}$

28 a $\begin{pmatrix} 2 \\ 3 \\ 9 \end{pmatrix}$ b $\begin{pmatrix} -6 \\ 3 \\ -1 \end{pmatrix}$

29 a $\begin{pmatrix} 8 \\ 0 \\ 10 \end{pmatrix}$ b $\begin{pmatrix} 6 \\ -3 \\ 1 \end{pmatrix}$

30 a $\begin{pmatrix} -2 \\ -3 \\ -9 \end{pmatrix}$ b $\begin{pmatrix} -8 \\ 0 \\ -10 \end{pmatrix}$

31 a $\sqrt{53}$ b $\sqrt{30}$

32 a $\sqrt{2}$ b $\sqrt{2}$

33 a $\sqrt{94}$ b $\sqrt{53}$

34 $x = -6, y = -13, z = 4$

35 a $\mathbf{a} + \frac{1}{2}\mathbf{b}$ b $\frac{1}{2}\mathbf{b} - \frac{1}{2}\mathbf{a}$ c $2\mathbf{a} - \mathbf{b}$

36 a $\mathbf{a} - \mathbf{b}$ b $\frac{1}{3}\mathbf{a} + \frac{2}{3}\mathbf{b}$ c $-\frac{1}{6}\mathbf{a} + \frac{2}{3}\mathbf{b}$

37 a $\begin{pmatrix} 8 \\ 0 \\ -19 \end{pmatrix}$ b $\sqrt{146}$

38 $\pm 2\sqrt{11}$

39 $\frac{1 \pm \sqrt{41}}{10}$

40 $2\sqrt{42}$ N

41 $\begin{pmatrix} 2 \\ 0 \\ -3/4 \end{pmatrix}$

42 -2

43 a $\frac{2}{3}\mathbf{i} + \frac{2}{3}\mathbf{j} - \frac{1}{3}\mathbf{k}$ b $\begin{pmatrix} 8 \\ -2 \\ 4\sqrt{2} \end{pmatrix}$

44 $-\frac{4}{3}$

45 -2

46 $\begin{pmatrix} 4\sqrt{2} \\ -\sqrt{2} \\ \sqrt{2} \end{pmatrix}$

47 $3, -\frac{5}{3}$

48 $-2, -\frac{23}{15}$

49 a $\frac{3}{2}\mathbf{i} + \frac{3}{2}\mathbf{j} - 2\mathbf{k}$ b $\left(\frac{1}{2}, \frac{13}{2}, 0\right)$

50 $\frac{\sqrt{65}}{3}$

Exercise 2B

1 a $\mathbf{r} = \begin{pmatrix} 2 \\ 1 \\ 5 \end{pmatrix} + t\begin{pmatrix} -1 \\ 2 \\ 2 \end{pmatrix}$, yes

b $\mathbf{r} = \begin{pmatrix} -1 \\ 0 \\ 3 \end{pmatrix} + t\begin{pmatrix} 4 \\ 1 \\ 5 \end{pmatrix}$, no

2 a $\mathbf{r} = \begin{pmatrix} 4 \\ 0 \\ 3 \end{pmatrix} + t\begin{pmatrix} 4 \\ 0 \\ 3 \end{pmatrix}$, no

b $\mathbf{r} = \begin{pmatrix} -1 \\ 5 \\ 1 \end{pmatrix} + t\begin{pmatrix} 0 \\ 0 \\ 7 \end{pmatrix}$, no

3 a $\mathbf{r} = \begin{pmatrix} 4 \\ 1 \end{pmatrix} + \lambda\begin{pmatrix} -3 \\ 1 \end{pmatrix}$, no

b $\mathbf{r} = \begin{pmatrix} 2 \\ 7 \end{pmatrix} + \lambda\begin{pmatrix} 2 \\ -9 \end{pmatrix}$, yes

4 a $\mathbf{r} = \begin{pmatrix} 1 \\ 0 \\ 5 \end{pmatrix} + \lambda\begin{pmatrix} 1 \\ 3 \\ -3 \end{pmatrix}$ b $\mathbf{r} = \begin{pmatrix} -1 \\ 1 \\ 5 \end{pmatrix} + \lambda\begin{pmatrix} 3 \\ -2 \\ 2 \end{pmatrix}$

5 a $\mathbf{r} = \begin{pmatrix} 0 \\ 2 \\ 3 \end{pmatrix} + \lambda\begin{pmatrix} 1 \\ 0 \\ -3 \end{pmatrix}$ b $\mathbf{r} = \begin{pmatrix} 4 \\ -3 \\ 0 \end{pmatrix} + \lambda\begin{pmatrix} 2 \\ 3 \\ -1 \end{pmatrix}$

6 a $\mathbf{r} = \begin{pmatrix} 4 \\ -1 \end{pmatrix} + \lambda\begin{pmatrix} 1 \\ 4 \end{pmatrix}$ b $\mathbf{r} = \begin{pmatrix} 4 \\ 1 \end{pmatrix} + \lambda\begin{pmatrix} 2 \\ -3 \end{pmatrix}$

7 a $x = 3 + 8\lambda,\ y = 5 + 2\lambda,\ z = 2 + 4\lambda$

b $x = 4 - 2\lambda,\ y = -1 + 3\lambda,\ z = 2 + 5\lambda$

8 a $x = 2 + \lambda,\ y = 3,\ z = -4\lambda$

b $x = 2\lambda,\ y = 3 + \lambda,\ z = -1$

9 a $x = -1,\ y = 3 + 5\lambda$

b $x = 4 - 3\lambda,\ y = 2\lambda$

10 a $(10, -7, -2)$ b $(-1, 1, 6)$

11 a $(0.5, 0, 1)$ b $(4.5, 0, 0)$

12 a $(3, 3, 1)$ b $(7, 2, 4)$

15 a $\mathbf{r} = \begin{pmatrix} 3 \\ -1 \\ 3 \end{pmatrix} + \lambda\begin{pmatrix} -4 \\ 2 \\ -3 \end{pmatrix}$ b No

16 $\begin{cases} x = -1 + 2\lambda \\ y = 1 - \lambda \\ z = 2 - 3\lambda \end{cases}$

17 No

18 a $\mathbf{r} = \begin{pmatrix} 3 \\ -1 \\ -\frac{1}{3} \end{pmatrix} + \lambda \begin{pmatrix} -2 \\ 0 \\ \frac{4}{3} \end{pmatrix}$ b $\frac{1}{3}$

19 $p = 12, q = 5$

20 $(3, -2, 1)$

21 No

23 b $(0, 3, 0)$

24 a $\mathbf{r} = \begin{pmatrix} 7 \\ 1 \\ 2 \end{pmatrix} + \lambda \begin{pmatrix} -4 \\ -2 \\ 3 \end{pmatrix}$ b $(-5, -5, -11)$

25 a $\mathbf{r} = \begin{pmatrix} 2 \\ 1 \\ 4 \end{pmatrix} + \lambda \begin{pmatrix} 2 \\ -3 \\ 6 \end{pmatrix}$ b 7

c $(-8, 16, -26), (12, -14, 34)$

26 $(4, 3, 3)$

27 a $(0, -22, 0)$

28 3; $(3, 7, 2)$

29 3

30 $\sqrt{\frac{6}{11}}$

Exercise 2C

1 a $51.3°$ b $33.7°$

2 a $351°$ b $159°$

3 a $13\,\mathrm{m\,s^{-1}}$ b $7.62\,\mathrm{m\,s^{-1}}$

4 a $9.11\,\mathrm{m\,s^{-1}}$ b $7\,\mathrm{m\,s^{-1}}$

5 a $7.2\mathbf{i} - 9.6\mathbf{j}$ b $6\mathbf{i} + 2.5\mathbf{j}$

6 a $\frac{8}{\sqrt{53}}\begin{pmatrix} 2 \\ 7 \end{pmatrix}$ b $\frac{5}{\sqrt{10}}\begin{pmatrix} -1 \\ 3 \end{pmatrix}$

7 a $\frac{10}{\sqrt{30}}(\mathbf{i} - 5\mathbf{j} + 2\mathbf{k})$ b $\frac{15}{\sqrt{41}}(2\mathbf{i} + \mathbf{j} - 6\mathbf{k})$

8 a $\begin{pmatrix} 4 \\ -3 \\ 12 \end{pmatrix}$ b $\begin{pmatrix} 1.8 \\ 0.6 \\ 0.9 \end{pmatrix}$

9 a $(5.6\mathbf{i} - 1.4\mathbf{j})\,\mathrm{m}$ b $(10\mathbf{i} + 7.2\mathbf{j})\,\mathrm{m}$

10 a $\begin{pmatrix} 32 \\ -53 \end{pmatrix}\mathrm{m}$ b $\begin{pmatrix} -7.8 \\ 20.8 \end{pmatrix}\mathrm{m}$

11 a $(57\mathbf{i} - 22.5\mathbf{j} + 32\mathbf{k})\,\mathrm{m}$

b $(44\mathbf{i} - 7\mathbf{j} + 1.8\mathbf{k})\,\mathrm{m}$

12 a $\begin{pmatrix} -2 \\ 11 \\ -13 \end{pmatrix}\mathrm{m}$ b $\begin{pmatrix} 20.8 \\ -12 \\ 32.8 \end{pmatrix}\mathrm{m}$

13 a $2.55\,\mathrm{m\,s^{-1}}$

b $\mathbf{r} = (12\mathbf{i} - 5\mathbf{j} + 11\mathbf{k}) + t(0.5\mathbf{i} + 2\mathbf{j} + 1.5\mathbf{k})$

c No

14 $16.8\,\mathrm{m}$

15 a $\begin{pmatrix} 3 \\ -1 \\ 4 \end{pmatrix}$ b $3.32\,\mathrm{m\,s^{-1}}$ c $14.9\,\mathrm{m}$

16 a $\mathbf{r}_A = 9\mathbf{i} + \mathbf{j} + t(4\mathbf{i} + 5\mathbf{j})$

$\mathbf{r}_B = -3\mathbf{i} + 2\mathbf{j} + t(6\mathbf{i} - 2\mathbf{j})$

b $15.3\,\mathrm{km}$

c 6 hours

17 $\mathbf{r} = \begin{pmatrix} 2 \\ 0 \\ 0 \end{pmatrix} + t\begin{pmatrix} 596 \\ -596 \\ 298 \end{pmatrix}$

18 b No

19 a $\sqrt{54}, 3$ b No

20 a $\mathbf{r}_P = -\mathbf{i} - \mathbf{j} + 2\mathbf{k} + t(\mathbf{i} + 4\mathbf{j} + 3\mathbf{k})$

$\mathbf{r}_Q = 7\mathbf{i} - 2\mathbf{j} + 5\mathbf{k} + t(2\mathbf{i} - 3\mathbf{j} - \mathbf{k})$

21 a $\mathbf{r}_1 = 3\mathbf{i} + t(-2\mathbf{i} + 5\mathbf{j}),\ \mathbf{r}_2 = 5\mathbf{j} + t(4\mathbf{i} + \mathbf{j})$

b $\sqrt{52t^2 - 76t + 34}$

c 2.50 miles

d 1.25 hours

22 a $\mathbf{r}_A = \begin{pmatrix} 12 \\ -10 \\ 1 \end{pmatrix} + t\begin{pmatrix} -4 \\ 3 \\ 1 \end{pmatrix}$

$\mathbf{r}_B = \begin{pmatrix} 5 \\ 1 \\ 4 \end{pmatrix} + t\begin{pmatrix} 2 \\ 0 \\ 1 \end{pmatrix}$

b $\sqrt{(6t-7)^2 + (11-3t)^2 + 9}$

c $7.35\,\mathrm{m}$

23 a ii $\begin{pmatrix} 4.4 \\ 7 \\ 1.8 \end{pmatrix}$ b 0.245

24 b $1.58\,\mathrm{m}$

25 a $c = 1$ b $75.5\,\mathrm{km\,h^{-1}}$

Exercise 2D

1 a 35 b 20
2 a 67.3 b 9.64
3 a −54.0 b −36.9
4 a 16 b −56
5 a −16 b 10
6 a 9 b 9
7 a 64° b 67°
8 a 108° b 101°
9 a 61° b 65°
10 a 96° b 111°
11 a $\frac{2}{7}$ b $-\frac{1}{2}$
12 a 3 b 2
13 a $\frac{17}{5}$ b $\frac{16}{7}$
14 a $\frac{4}{5}$ b −12
15 a 31.4° b 31.8°
16 a 38.8 b 80.0°
17 a 83.7° b 7.13°
18 a 60.6 b 34.6
19 a 251 b 11.5
20 a 64.3 b 25.8
21 a $\begin{pmatrix} -2 \\ 6 \\ -1 \end{pmatrix}$ b $\begin{pmatrix} -1 \\ -10 \\ 7 \end{pmatrix}$
22 a $\begin{pmatrix} -9 \\ -19 \\ 2 \end{pmatrix}$ b $\begin{pmatrix} -23 \\ 1 \\ 8 \end{pmatrix}$
23 a $-5\mathbf{i} - 11\mathbf{j} - 2\mathbf{k}$ b $12\mathbf{i} + 2\mathbf{j} + 6\mathbf{k}$
24 a $\frac{\sqrt{153}}{2}$ b $\sqrt{157}$
25 a $\frac{15\sqrt{3}}{2}$ b $\frac{9}{2}$
26 a $\frac{3\sqrt{182}}{2}$ b $\frac{\sqrt{707}}{2}$
27 a 0.312 b 4.11
28 a 3.05 b 7.16
29 a 4 b 1.15
30 a 4.01 b 0.234
31 a 5.92 b 4.23
32 a 6.15 b 4.54
33 141°
34 40.0°
35 48.2°
36 98.0°
37 3
38 81.8°
40 17.5
41 0.775
42 0.630
43 $-8\mathbf{i} - 5\mathbf{j} + \mathbf{k}$
44 a $\frac{10}{\sqrt{13}}\begin{pmatrix} 2 \\ 3 \end{pmatrix}$ b $7.40\,\text{km}\,\text{h}^{-1}$
45 61.0°, 74.5°, 44.5°
46 94.3°, 54.2°, 31.5°
47 b 41.8°, 48.2° c 161
48 6
49 $-\frac{3}{4}$
50 $0, \frac{3}{2}$
51 a $\mathbf{r} = \begin{pmatrix} 1/2 \\ 7 \\ 2 \end{pmatrix} + \lambda \begin{pmatrix} 3/2 \\ 0 \\ -4 \end{pmatrix}$ b 69.4°
52 11.0°
53 $\begin{pmatrix} 0 \\ -1/\sqrt{2} \\ 1/\sqrt{2} \end{pmatrix}$
54 $\frac{\sqrt{14}}{14}\begin{pmatrix} 3 \\ 1 \\ -2 \end{pmatrix}$
55 b 0
56 a $\begin{pmatrix} 18 \\ -12 \\ 72 \end{pmatrix}, \begin{pmatrix} -18 \\ 12 \\ -72 \end{pmatrix}$ b $\mathbf{p} = -\mathbf{q}$
57 b 42.6
58 $\frac{\sqrt{46}}{2}$
59 a (11, −2, 0) b 21.9
61 No
62 a $\mathbf{a} + \mathbf{b}, \mathbf{b} - \mathbf{a}$ b $|\mathbf{b}|^2 - |\mathbf{a}|^2$

63 b 2 c $4\sqrt{5}$

64 a $(9, -5, 8)$ c $(3, 4, -1)$

65 a $C(5, 4, 0)$, $F(5, 0, 2)$, $G(5, 4, 2)$, $H(0, 4, 2)$ b 11.9

Chapter 2 Mixed Practice

1 a $(-276\mathbf{i} + 62\mathbf{j} - 7.5\mathbf{k})\,\text{km}$
b $283\,\text{km}$

2 a $\frac{1}{2}\mathbf{b} - \mathbf{a}$ b $\frac{1}{2}\mathbf{a} + \frac{3}{4}\mathbf{b}$

3 a $\begin{pmatrix} 2 \\ 0 \\ k-7 \end{pmatrix}$ c $(3, 6, 1)$ d $\frac{-1}{\sqrt{10}}$

4 $(3, 3, 8)$

5 b No

7 a $\mathbf{r} = \begin{pmatrix} 1 \\ -3 \\ 2 \end{pmatrix} + \lambda \begin{pmatrix} 1 \\ 5 \\ -1 \end{pmatrix}$ b $88.5°$
c 24 d 23.6

8 a $\mathbf{r} = \begin{pmatrix} 1 \\ 4 \\ -1 \end{pmatrix} + \lambda \begin{pmatrix} 3 \\ -2 \\ 3 \end{pmatrix}$ b $\frac{1}{\sqrt{22}} \begin{pmatrix} 3 \\ -2 \\ 3 \end{pmatrix}$

9 a $128\,\text{m s}^{-1}$ b 83.3 seconds

10 a $\mathbf{r}_A = \begin{pmatrix} 2 \\ -8 \end{pmatrix} + \begin{pmatrix} -5 \\ 3 \end{pmatrix} t$, $\mathbf{r}_B = \begin{pmatrix} 3 \\ 4 \end{pmatrix} + \begin{pmatrix} 2 \\ -6 \end{pmatrix} t$
b $26.6\,\text{km}$
c 80 minutes

11 a $\mathbf{r} = 2\mathbf{i} - 3\mathbf{j} + t(10\mathbf{i} - 9\mathbf{j} + 7\mathbf{k})$
b $15.2\,\text{m s}^{-1}$
c $6.5\,\text{s}$

12 $\frac{3\pi}{4}$

13 a -14 b $\frac{19}{16}$

14 a $6\mathbf{i} - 5\mathbf{j} - 2\mathbf{k}$ b $\mathbf{i} + 2\mathbf{j} - 2\mathbf{k}$
c $\sqrt{65}$

15 a $25.9\,\text{m s}^{-1}$ b $25.8\,\text{m s}^{-1}$
c $2.09\,\text{m s}^{-1}$

16 a i $\frac{1}{2}(\mathbf{c} - \mathbf{a})$ b i $\frac{1}{3}(\mathbf{a} - \mathbf{b})$

17 $a = 9$, $b = -1$, $c = -14$

18 a $\mathbf{v} = (10\sqrt{6}\mathbf{i} - 5\sqrt{6}\mathbf{j} + 5\sqrt{6}\mathbf{k})$ mph
b $\left(\frac{5}{6}\sqrt{6}\mathbf{i} - \frac{5}{12}\sqrt{6}\mathbf{j} + \frac{5}{12}\sqrt{6}\mathbf{k}\right)$ miles

19 b $107°$, $73.2°$ c $\frac{5}{4}$

20 a $x = \frac{1}{2} + 2\lambda$, $y = -2 + 3\lambda$, $z = \frac{4}{3} - 2\lambda$
b Yes (at $\left(\frac{11}{6}, 0, 0\right)$)
c $61.0°$

21 a $51.7°$ b $(1, 1, -1)$
d 91.2

22 a $\mathbf{r}_A = \begin{pmatrix} 7 \\ -3 \\ 2 \end{pmatrix} + t \begin{pmatrix} -1 \\ 0 \\ 2 \end{pmatrix}$

$\mathbf{r}_B = \begin{pmatrix} 1 \\ 1 \\ 26 \end{pmatrix} + t \begin{pmatrix} 0 \\ 1 \\ 3 \end{pmatrix}$

b ii $(1, -3, 14)$

23 a $108°$
b $\mathbf{r} = -\mathbf{i} - 3\mathbf{j} + t(9\mathbf{i} - 3\mathbf{j})$ km
c $p = 43$, $q = -21$

24 a $15\mathbf{i} - 5\mathbf{j} + 10\mathbf{k}$ b $3.74\,\text{m s}^{-1}$
c No

25 a $\begin{pmatrix} 3t \\ 5-4t \\ t \end{pmatrix}$ d $5.48\,\text{km}$

26 a $\sqrt{2(1-\cos\alpha)}$, $\sqrt{2(1+\cos\alpha)}$
b $28.1°$

27 $\left(\frac{64}{9}, \frac{4}{9}, \frac{19}{9}\right)$

28 a $\mathbf{r} = \begin{pmatrix} 0 \\ -1 \\ 2 \end{pmatrix} + \lambda \begin{pmatrix} 4 \\ -3 \\ 3 \end{pmatrix}$ b $\left(\frac{30}{17}, -\frac{14}{17}, -\frac{3}{17}\right)$
c 2.81

29 a 1.6 b $68.7°$, $21.3°$, $90°$
c 88.7

30 $\frac{19}{3}$

31 $\frac{\sqrt{94}}{2}$

32 $p = \frac{3}{8}, q = \frac{1}{8}$

33 $d = \sqrt{\frac{14}{3}}$ m

34 a $(4, 1, -2)$ c $(1, 1, 2)$ d $\frac{5\sqrt{26}}{2}$

35 b $48.5°$

d $\frac{11\sqrt{11}}{6}$ $(= 6.08)$

e 4.55

36 b $(5, -7, 6), (-1, 5, -6)$

c $\mathbf{r} = \begin{pmatrix} 2 \\ -1 \\ 0 \end{pmatrix} + \lambda \begin{pmatrix} 3 \\ -1 \\ 4 \end{pmatrix}$, a second possible solution is $\mathbf{r} = \begin{pmatrix} 2 \\ -1 \\ 0 \end{pmatrix} + \lambda \begin{pmatrix} 1 \\ 3 \\ 0 \end{pmatrix}$

37 a $\left(\frac{64}{9}, \frac{4}{9}, \frac{19}{9}\right)$ b $\frac{\sqrt{29}}{3}$

c $\left(\frac{65}{9}, -\frac{10}{9}, \frac{11}{9}\right)$

38 a $7\lambda + 5\mu = 11$ b 3

39 a $\frac{1}{2}|\mathbf{a} \times \mathbf{b}|$ b $|\mathbf{c}|\cos\theta$

d $\frac{1}{3}$ e $\frac{2}{\sqrt{99}}$

f B $\left(h = \frac{2}{\sqrt{203}}\right)$

40 11:52

Chapter 3 Prior Knowledge

1 a $\begin{pmatrix} 1 \\ 9 \end{pmatrix}$ b $\begin{pmatrix} 3 \\ 1 \end{pmatrix}$ c $\begin{pmatrix} 8 \\ -12 \end{pmatrix}$

2 $\sqrt{34}$

Exercise 3A

1 a 2×2 b 3×3

2 a 3×2 b 2×4

3 a 3×1 b 1×3

4 a $x = 5, y = -5$ b $x = 7, y = -4$

5 a $x = -2, y = 1$ or $x = -2, y = -1$

b $x = 3, y = -3$ or $x = -3, y = 3$

6 a $\begin{pmatrix} 1 & 7 \\ -1 & 16 \end{pmatrix}$ b $\begin{pmatrix} 9 & -13 \\ 3 & 2 \end{pmatrix}$

7 a $\begin{pmatrix} 1 & 0 & 2 & -6 \\ -2 & -3 & 9 & 0 \end{pmatrix}$ b $\begin{pmatrix} 7 & 18 & -2 & -8 \\ 0 & 13 & -3 & 4 \end{pmatrix}$

8 a Not possible b Not possible

9 a $\begin{pmatrix} 8 & -3 \end{pmatrix}$ b $\begin{pmatrix} 14 & -7 \end{pmatrix}$

10 a $\begin{pmatrix} 6 & 0 \\ 2 & -6 \\ -4 & 8 \end{pmatrix}$ b $\begin{pmatrix} 15 & 0 \\ 5 & -15 \\ -10 & 20 \end{pmatrix}$

11 a $\begin{pmatrix} -2 & 4 & -12 \\ -8 & 0 & 4.8 \end{pmatrix}$ b $\begin{pmatrix} -1.5 & 3 & -9 \\ -6 & 0 & 3.6 \end{pmatrix}$

12 a $\begin{pmatrix} -0.4 & 4 & 0.6 \\ 0.2 & -0.8 & -1.3 \\ 0.1 & 1.2 & 0 \end{pmatrix}$ b $\begin{pmatrix} -1 & 10 & 1.5 \\ 0.5 & -2 & -3.25 \\ 0.25 & 3 & 0 \end{pmatrix}$

13 a $\begin{pmatrix} -1 & 10 \\ 5 & 2 \end{pmatrix}$ b $\begin{pmatrix} 12 & 20 \\ -4 & -11 \end{pmatrix}$

14 a $\begin{pmatrix} -7 & 16 & 28 \\ -5 & 14 & 23 \\ 5 & -4 & 7 \end{pmatrix}$ b $\begin{pmatrix} 11 & -8 & 3 \\ -5 & -8 & 4 \\ -2 & -18 & 11 \end{pmatrix}$

15 a $\begin{pmatrix} 4 & -1 & 13 \\ -2 & 17 & 21 \\ 3 & -6 & 1 \\ 0 & -9 & -15 \end{pmatrix}$ b Product doesn't exist

16 a Product doesn't exist b $\begin{pmatrix} -1 & -48 \end{pmatrix}$

17 a (-1) b $\begin{pmatrix} 10 & -15 & 20 \\ 2 & -3 & 4 \\ -4 & 6 & -8 \end{pmatrix}$

18 a 2×3 b Product doesn't exist

19 a Product doesn't exist **b** 4×3

20 a Product doesn't exist
b Product doesn't exist

21 a $\begin{pmatrix} 5 & 0 & 2 \\ 0 & -5 & 4 \\ 6 & -4 & 7 \end{pmatrix}$ **b** $\begin{pmatrix} -3 & 0 & -1 \\ 0 & 2 & -2 \\ -3 & 2 & -4 \end{pmatrix}$

22 a $\begin{pmatrix} 1 & 0 & 0.25 \\ 0 & -0.25 & 0.5 \\ 0.75 & -0.5 & 1.25 \end{pmatrix}$ **b** $\begin{pmatrix} -8 & 0 & -2 \\ 0 & 2 & -4 \\ -6 & 4 & -10 \end{pmatrix}$

23 $\begin{pmatrix} 9 & 3 & 8 \\ 8 & 2 & 10 \\ 11 & 5 & 4 \end{pmatrix}$

24 $\begin{pmatrix} 864.00 & 604.80 \\ 691.20 & 345.60 \\ 734.40 & 324.00 \end{pmatrix}$

25 $x = 5,\ y = -2$

26 $p = 4,\ q = -3$

27 a $\begin{pmatrix} -6 & 15 \\ 5+3k & 4-3k \end{pmatrix}$ **b** $\begin{pmatrix} -1 & -5 \\ 10-k & 12+k \end{pmatrix}$

28 a $\begin{pmatrix} 1+k & -k \\ 1 & 0 \end{pmatrix}$ **b** $\begin{pmatrix} 0 & 5k \\ 3-2k & 4k+9 \end{pmatrix}$

29 a $\begin{pmatrix} 7 & k+1 & 1 \\ 1 & k+4 & k+3 \\ 3 & 0 & k+5 \end{pmatrix}$

b $\begin{pmatrix} -k & k^2+1 & 4 \\ 2k+8 & 4k+2 & k^2+20 \\ 16 & 1 & 5k+4 \end{pmatrix}$

30 a $\mathbf{C} = \begin{pmatrix} 50 & 38 & 24 & 80 \\ 55 & 30 & 26 & 75 \end{pmatrix}\begin{pmatrix} 100 \\ 50 \\ 125 \\ 30 \end{pmatrix}$

b $\mathbf{C} = \begin{pmatrix} 12\,300 \\ 12\,500 \end{pmatrix}$

31 $x = 1,\ y = 4$

32 $p = -3,\ q = 2$

33 a $\mathbf{M} = \begin{pmatrix} 49\,310 & 47\,400 \\ 44\,350 & 43\,150 \end{pmatrix}$

b Both should use factory *H*.

34 a $\begin{pmatrix} 17\,557 & 18\,264 \\ 20\,311 & 21\,427 \\ 18\,381 & 19\,142 \end{pmatrix}$ **b** 37 523

35 $a = 3,\ b = -1.5,\ s = 2,\ t = 7$

36 $a = -1,\ b = -2,\ p = 5,\ q = 2$

37 $a = -\frac{2}{7},\ b = -\frac{1}{7},\ c = \frac{3}{11},\ d = \frac{2}{11}$

38 $k = -7$

39 $d = 2.5c$

40 a $\mathbf{AB} = \begin{pmatrix} a_1b_1 + a_2b_3 & a_1b_2 + a_2b_4 \\ a_3b_1 + a_4b_3 & a_3b_2 + a_4b_4 \end{pmatrix}$

$\mathbf{BC} = \begin{pmatrix} b_1c_1 + b_2c_3 & b_1c_2 + b_2c_4 \\ b_3c_1 + b_4c_3 & b_3c_2 + b_4c_4 \end{pmatrix}$

Exercise 3B

1 a 3 **b** 8
2 a 0 **b** 0
3 a -8 **b** -14
4 a 2 **b** 4
5 a -4 **b** 5
6 a 0 **b** 0

7 a $\frac{1}{3}\begin{pmatrix} 18 & 1 & -11 \\ -33 & -1 & 20 \\ 24 & 1 & -14 \end{pmatrix}$ **b** $\frac{1}{8}\begin{pmatrix} -2 & 6 & -4 \\ 3 & -1 & 2 \\ -5 & 15 & -14 \end{pmatrix}$

8 a Doesn't exist **b** Doesn't exist

9 a $\frac{1}{8}\begin{pmatrix} -20 & 2 & -8 \\ 10 & 1 & 4 \\ -14 & 1 & -4 \end{pmatrix}$

b $\frac{1}{14}\begin{pmatrix} 11 & -10 & -15 \\ -6 & 8 & 12 \\ 2 & 2 & -4 \end{pmatrix}$

10 a $\begin{pmatrix} 4 & 0 & -1 & -1 \\ 8 & -1 & -3 & -2 \\ -4 & 0.5 & 1 & 1 \\ 1 & -0.5 & -1 & 0 \end{pmatrix}$

b $\begin{pmatrix} 0.25 & 5 & 0.25 & -3.25 \\ 0.75 & 0 & -0.25 & 0.25 \\ -1.5 & 3 & 0.5 & -1.5 \\ 0 & -2 & 0 & 1 \end{pmatrix}$

11 a $\begin{pmatrix} -3 & -0.5 & 1.5 & 1 \\ 1 & 0.25 & -0.25 & -0.5 \\ 3 & 0.25 & -1.25 & -0.5 \\ -3 & 0 & 1 & 1 \end{pmatrix}$

b $\begin{pmatrix} 0.2 & 9.6 & -30 & 25.4 \\ 0 & -2 & 7 & -6 \\ 0.2 & -13.4 & 43 & 36.6 \\ 0 & -2 & 6 & -5 \end{pmatrix}$

12 a Doesn't exist **b** Doesn't exist

13 a $\mathbf{B} = \begin{pmatrix} -20 & 3 & 5 \\ -26.5 & 0 & 4 \end{pmatrix}$ **b** $\mathbf{B} = \begin{pmatrix} 5 & 3 \\ -4 & -1 \\ 2 & 1 \end{pmatrix}$

14 a $\mathbf{B} = \begin{pmatrix} 1 & 4 \end{pmatrix}$ **b** $\mathbf{B} = \begin{pmatrix} 3 \\ 5 \end{pmatrix}$

15 a $\mathbf{B} = \begin{pmatrix} 1 & 3 & -2 \\ 0 & 5 & -4 \\ -6 & -2 & 5 \end{pmatrix}$ **b** $\mathbf{B} = \begin{pmatrix} 2 & 1 & -1 \\ 0 & 3 & 1 \\ 2 & 5 & 0 \end{pmatrix}$

16 a $\mathbf{B} = \begin{pmatrix} 2 \\ 3 \\ 1 \end{pmatrix}$ **b** $\mathbf{B} = \begin{pmatrix} 2 & 3 & -4 \end{pmatrix}$

17 a 14 **b** 5

18 a −8 **b** −2

19 a 0 **b** 0

20 a $k = -\frac{3}{2}$ **b** $k = \frac{2}{5}$

21 a $k = \pm\frac{1}{2}$ **b** $k = \pm 3$

22 a $k = 5, -2$ **b** $k = 0, 4$

23 a $\frac{1}{14}\begin{pmatrix} 2 & 4 \\ -3 & 1 \end{pmatrix}$ **b** $\frac{1}{5}\begin{pmatrix} -1 & -4 \\ 2 & 3 \end{pmatrix}$

24 a $-\frac{1}{8}\begin{pmatrix} 3 & -2 \\ 5 & -6 \end{pmatrix}$ **b** $-\frac{1}{2}\begin{pmatrix} 8 & 2 \\ 1 & 0 \end{pmatrix}$

25 a Doesn't exist **b** Doesn't exist

26 a $k = \pm\sqrt{2}$ **b** $\frac{1}{2k^2 - 4}\begin{pmatrix} k & -1 \\ -4 & 2k \end{pmatrix}$

27 b $\frac{1}{3c^2 + 2}\begin{pmatrix} 2 & -3c \\ c & 1 \end{pmatrix}$

28 a $\mathbf{A}^{-1} = \begin{pmatrix} 0.5 & 1 \\ -0.75 & -2 \end{pmatrix}$ **b** $\mathbf{B} = \begin{pmatrix} 1 & 3 & 2 \\ 5 & -1 & 6 \end{pmatrix}$

29 a $\mathbf{B}^{-1} = \frac{1}{117}\begin{pmatrix} -7 & 19 & 16 \\ 19 & 32 & -10 \\ 25 & -1 & -7 \end{pmatrix}$

b $\mathbf{A} = \begin{pmatrix} -3 & 4 & -1 \\ 5 & 1 & 2 \end{pmatrix}$

30 $\begin{pmatrix} 7 & 2 \\ 4 & 4 \\ 3 & 9 \end{pmatrix}$

31 $\frac{1}{4}\begin{pmatrix} 7 & 9 \\ 15 & 1 \end{pmatrix}$

32 $\begin{pmatrix} -4.25 & 5.5 & -4.75 \\ 3.75 & -4.5 & 4.25 \\ -9.5 & 12.5 & -10.5 \end{pmatrix}$

36 $\mathbf{B}^{-1}$

Exercise 3C

1 a $x = 3,\ y = -2$ **b** $x = -4,\ y = 1$

2 a No solution **b** No solution

3 a $x = -5.5,\ y = -12.5$ **b** $x = -3.25,\ y = 7.75$

4 a $x = 2,\ y = 1,\ z = 0$ **b** $x = 6,\ y = 2,\ z = -5$

5 a $x = 2,\ y = 1,\ z = 2$ **b** $x = 4,\ y = 0,\ z = 1$

6 a No solution **b** No solution

7 a $\begin{pmatrix} 5 & -4 \\ 3 & -2 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 7 \\ -1 \end{pmatrix}$

b $\frac{1}{2}\begin{pmatrix} -2 & 4 \\ -3 & 5 \end{pmatrix}$

c $x = -9,\ y = -13$

8 a $\begin{pmatrix} 1 & -1 & 1 \\ 5 & 3 & 2 \\ 0 & 4 & -3 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ 4 \end{pmatrix}$

b $-\frac{1}{12}\begin{pmatrix} -17 & 1 & -5 \\ 15 & -3 & 3 \\ 20 & -4 & 8 \end{pmatrix}$

c $x = 3,\ y = -2,\ z = -4$

9 $R = 1.5,\ S = 2.5$

10 a $\begin{pmatrix} 1 & 1 & 1 \\ 1.052 & 1.031 & 1.065 \\ 1 & 0 & -1 \end{pmatrix}\begin{pmatrix} A \\ B \\ C \end{pmatrix} = \begin{pmatrix} 50\,000 \\ 52\,447.5 \\ 10\,000 \end{pmatrix}$

b $A = \$22\,500,\ B = \$15\,000,$
$C = \$12\,500$

11 a $\begin{pmatrix} k & 3 \\ k-2 & 4 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} -1 \\ 1 \end{pmatrix}$

b $k = -6$

c $x = -\frac{7}{k+6},\ y = \frac{2k-2}{k+6}$

12 a $\begin{pmatrix} k & 5 \\ 2 & k-3 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 2 \\ -1 \end{pmatrix}$

b $k = -2, 5$

c $x = \frac{2k-1}{k^2-3k-10}$

$y = \frac{-k-4}{k^2-3k-10}$

Exercise 3D

1 a $\lambda = 3, -1$

$\begin{pmatrix} 5 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \end{pmatrix}$

b $\lambda = 2, 3$

$\begin{pmatrix} 2 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \end{pmatrix}$

2 a $\lambda = 2, 4$

$\begin{pmatrix} 3 \\ -5 \end{pmatrix}, \begin{pmatrix} 1 \\ -1 \end{pmatrix}$

b $\lambda = 2, -5$

$\begin{pmatrix} -5 \\ 1 \end{pmatrix}, \begin{pmatrix} -3 \\ 2 \end{pmatrix}$

3 a $\lambda = 0.3, 1$

$\begin{pmatrix} 1 \\ -1 \end{pmatrix}, \begin{pmatrix} 3 \\ 4 \end{pmatrix}$

b $\lambda = 0.6, 1$

$\begin{pmatrix} 1 \\ -1 \end{pmatrix}, \begin{pmatrix} 5 \\ 3 \end{pmatrix}$

4 a $\begin{pmatrix} 5 & 1 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} 3 & 0 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} 5 & 1 \\ 1 & 1 \end{pmatrix}^{-1}$

b $\begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} 2 & 0 \\ 0 & 3 \end{pmatrix}\begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}^{-1}$

5 a $\begin{pmatrix} 3 & 1 \\ -5 & -1 \end{pmatrix}\begin{pmatrix} 2 & 0 \\ 0 & 4 \end{pmatrix}\begin{pmatrix} 3 & 1 \\ -5 & -1 \end{pmatrix}^{-1}$

b $\begin{pmatrix} -5 & -3 \\ 1 & 2 \end{pmatrix}\begin{pmatrix} 2 & 0 \\ 0 & -5 \end{pmatrix}\begin{pmatrix} -5 & -3 \\ 1 & 2 \end{pmatrix}^{-1}$

6 a $\begin{pmatrix} 1 & 3 \\ -1 & 4 \end{pmatrix}\begin{pmatrix} 0.3 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 3 \\ -1 & 4 \end{pmatrix}^{-1}$

b $\begin{pmatrix} 1 & 5 \\ -1 & 3 \end{pmatrix}\begin{pmatrix} 0.6 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 5 \\ -1 & 3 \end{pmatrix}^{-1}$

7 a $\frac{1}{4}\begin{pmatrix} 5(3^n) - (-1)^n & 5(-1)^n - 5(3^n) \\ 3^n - (-1)^n & 5(-1)^n - 3^n \end{pmatrix}$

b $\begin{pmatrix} 2^{n+1} - 3^n & 2(3^n) - 2^{n+1} \\ 2^n - 3^n & 2(3^n) - 2^n \end{pmatrix}$

8 a $\frac{1}{2}\begin{pmatrix} 5(4^n) - 3(2^n) & 3(4^n) - 3(2^n) \\ 5(2^n) - 5(4^n) & 5(2^n) - 3(4^n) \end{pmatrix}$

b $\frac{1}{7}\begin{pmatrix} 10(2^n) - 3(-5)^n & 15(2^n) - 15(-5)^n \\ 2(-5)^n - 2^{n+1} & 10(-5)^n - 3(2^n) \end{pmatrix}$

9 a $\frac{1}{7}\begin{pmatrix} 4(0.3^n) + 3 & 3 - 3(0.3^n) \\ 4 - 4(0.3^n) & 3(0.3^n) + 4 \end{pmatrix}$

b $\frac{1}{8}\begin{pmatrix} 3(0.6^n) + 5 & 5 - 5(0.6^n) \\ 3 - 3(0.6^n) & 5(0.6^n) + 3 \end{pmatrix}$

10 a i $\lambda^2 - 11\lambda + 24 = 0$

ii $\lambda = 3, 8$

b $\begin{pmatrix} 1 \\ -2 \end{pmatrix}, \begin{pmatrix} 2 \\ 1 \end{pmatrix}$

11 $\lambda = 2, -3$

$\begin{pmatrix} 1 \\ 2 \end{pmatrix}, \begin{pmatrix} -2 \\ 1 \end{pmatrix}$

12 $\lambda = -6, 1$

$\begin{pmatrix} 5 \\ 3 \end{pmatrix}, \begin{pmatrix} 1 \\ 2 \end{pmatrix}$

13 a $a = 3$

b **i** $\lambda_1 = 5$

ii $\lambda_2 = 1, \boldsymbol{v}_2 = \begin{pmatrix} 1 \\ -2 \end{pmatrix}$

14 **a** $\lambda = -1, -11$

$\begin{pmatrix} 2 \\ 1 \end{pmatrix}, \begin{pmatrix} 4 \\ -3 \end{pmatrix}$

b $\mathbf{C} = \begin{pmatrix} 2 & 4 \\ 1 & -3 \end{pmatrix}, \mathbf{D} = \begin{pmatrix} -1 & 0 \\ 0 & -11 \end{pmatrix}$

15 **a** $\mathbf{P} = \begin{pmatrix} -1 & -2 \\ 1 & 1 \end{pmatrix}, \mathbf{D} = \begin{pmatrix} 6 & 0 \\ 0 & 7 \end{pmatrix}$

b $\begin{pmatrix} 470 & 254 \\ -127 & 89 \end{pmatrix}$

16 **a** $\mathbf{P} = \begin{pmatrix} 1 & -2 \\ -1 & 1 \end{pmatrix}, \mathbf{D} = \begin{pmatrix} 3 & 0 \\ 0 & 5 \end{pmatrix}$

b $\begin{pmatrix} 2(5^n) - 3^n & 2(5^n) - 2(3^n) \\ 3^n - 5^n & 2(3^n) - 5^n \end{pmatrix}$

17 **a** $\mathbf{P} = \begin{pmatrix} 2 & 2 \\ 1 & 3 \end{pmatrix}, \mathbf{D} = \begin{pmatrix} 2 & 0 \\ 0 & -2 \end{pmatrix}$

b **i** $2^{n-1}\begin{pmatrix} 4 & -4 \\ 3 & -4 \end{pmatrix}$

ii $2^n\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$

18 **a** $\begin{pmatrix} 2.5 & -1.5 \\ 4.5 & -3.5 \end{pmatrix}$ and $\begin{pmatrix} -3.5 & 1.5 \\ -4.5 & 2.5 \end{pmatrix}$

b $\mathbf{M}_1\mathbf{M}_2 = -2\mathbf{I}$. All vectors in the plane can be expressed as $\mathbf{r} = a\begin{pmatrix} 1 \\ 3 \end{pmatrix} + b\begin{pmatrix} 1 \\ 1 \end{pmatrix}$, and each of the eigenvectors has eigenvalue $(-2) \times 1$ through $\mathbf{M}_1\mathbf{M}_2$.

19 **a** $\lambda = -0.25, 1$

$\begin{pmatrix} -1 \\ 1 \end{pmatrix}, \begin{pmatrix} 2 \\ 3 \end{pmatrix}$

b $\mathbf{P} = \begin{pmatrix} -1 & 2 \\ 1 & 3 \end{pmatrix}, \mathbf{D} = \begin{pmatrix} -0.25 & 0 \\ 0 & 1 \end{pmatrix}$

20 **a** $a + b, a - b$

b $\begin{pmatrix} 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ -1 \end{pmatrix}$

c $\mathbf{M} = \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}\begin{pmatrix} a+b & 0 \\ 0 & a-b \end{pmatrix}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}^{-1}$

d $\frac{1}{2}\begin{pmatrix} (a+b)^n + (a-b)^n & (a+b)^n - (a-b)^n \\ (a+b)^n - (a-b)^n & (a+b)^n + (a-b)^n \end{pmatrix}$

e $\begin{pmatrix} 0.5 & 0.5 \\ 0.5 & 0.5 \end{pmatrix}$

Chapter 3 Mixed Practice

1 **a** $\mathbf{P} = \begin{pmatrix} 68 & 55 \\ 89 & 48 \\ 42 & 69 \end{pmatrix}$

b **i** $\mathbf{Q} = 1.05\mathbf{R} - 1.02\mathbf{F}$

ii $\mathbf{Q} = \begin{pmatrix} 72.66 & 58.65 \\ 95.40 & 52.02 \\ 46.74 & 75.63 \end{pmatrix}$

2 $\begin{pmatrix} k & -2k-3 \\ 4k & 4 \end{pmatrix}$

3 **a** $k = -2$

b For example, $\mathbf{C} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \mathbf{D} = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$

4 $\begin{pmatrix} -2 & 1 \\ -5 & 3 \end{pmatrix}$

5 $\mathbf{C} = \frac{1}{4}\begin{pmatrix} 3k-2 & 6 \\ 5k-2 & 14 \end{pmatrix}$

6 **a** $\frac{1}{3}\begin{pmatrix} 1 & -k \\ 0 & 3 \end{pmatrix}$ **b** $\begin{pmatrix} 1 & 0 \\ 2 & 1-2k \end{pmatrix}$

7 5241 and 7392

8 **b** $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ **c** $x = 2, y = -3$

9 $A = 2\,\text{kg}, B = 3\,\text{kg}, C = 7\,\text{kg}$

10 **a** **i** $\lambda^2 + 5\lambda - 14 = 0$

ii $\lambda = -2, 7$

b $\begin{pmatrix} 1 \\ 1 \end{pmatrix}, \begin{pmatrix} -5 \\ 4 \end{pmatrix}$

11 $\lambda = 3, 4$

$\begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} -2 \\ 1 \end{pmatrix}$

12 $a = \frac{4}{3}, b = 6, c = \frac{7}{3}, k = -3$

13 a i $k = 4.8$

ii $\frac{1}{24-5k}\begin{pmatrix} 8 & -k \\ -5 & 3 \end{pmatrix}$

b $x = 9, y = -4$

14 a

b $\frac{1}{6}\begin{pmatrix} -27 & -3 \\ 20 & 2 \end{pmatrix}$

15 $x = 1, y = -0.5$

16 $\frac{1}{7}\begin{pmatrix} 15 & -2 \\ -3 & 20 \end{pmatrix}$

17 a $\mathbf{P} = \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}, \mathbf{D} = \begin{pmatrix} 5 & 0 \\ 0 & 7 \end{pmatrix}$

b $\begin{pmatrix} 2(5^n) - 7^n & 2(7^n) - 2(5^n) \\ 5^n - 7^n & 2(7^n) - 5^n \end{pmatrix}$

18 a i $\lambda = a + b - 1$ ii $\begin{pmatrix} b-1 \\ a-1 \end{pmatrix}, \begin{pmatrix} 1 \\ -1 \end{pmatrix}$

b $\mathbf{P} = \begin{pmatrix} b-1 & 1 \\ a-1 & -1 \end{pmatrix}$

$\mathbf{D} = \begin{pmatrix} 1 & 0 \\ 0 & a+b-1 \end{pmatrix}$

c

$\frac{1}{2-a-b}\begin{pmatrix} 1-b+(1-a)(a+b-1)^n & 1-b-(1-b)(a+b-1)^n \\ 1-a-(1-a)(a+b-1)^n & 1-a+(1-b)(a+b-1)^n \end{pmatrix}$

Chapter 4 Prior Knowledge

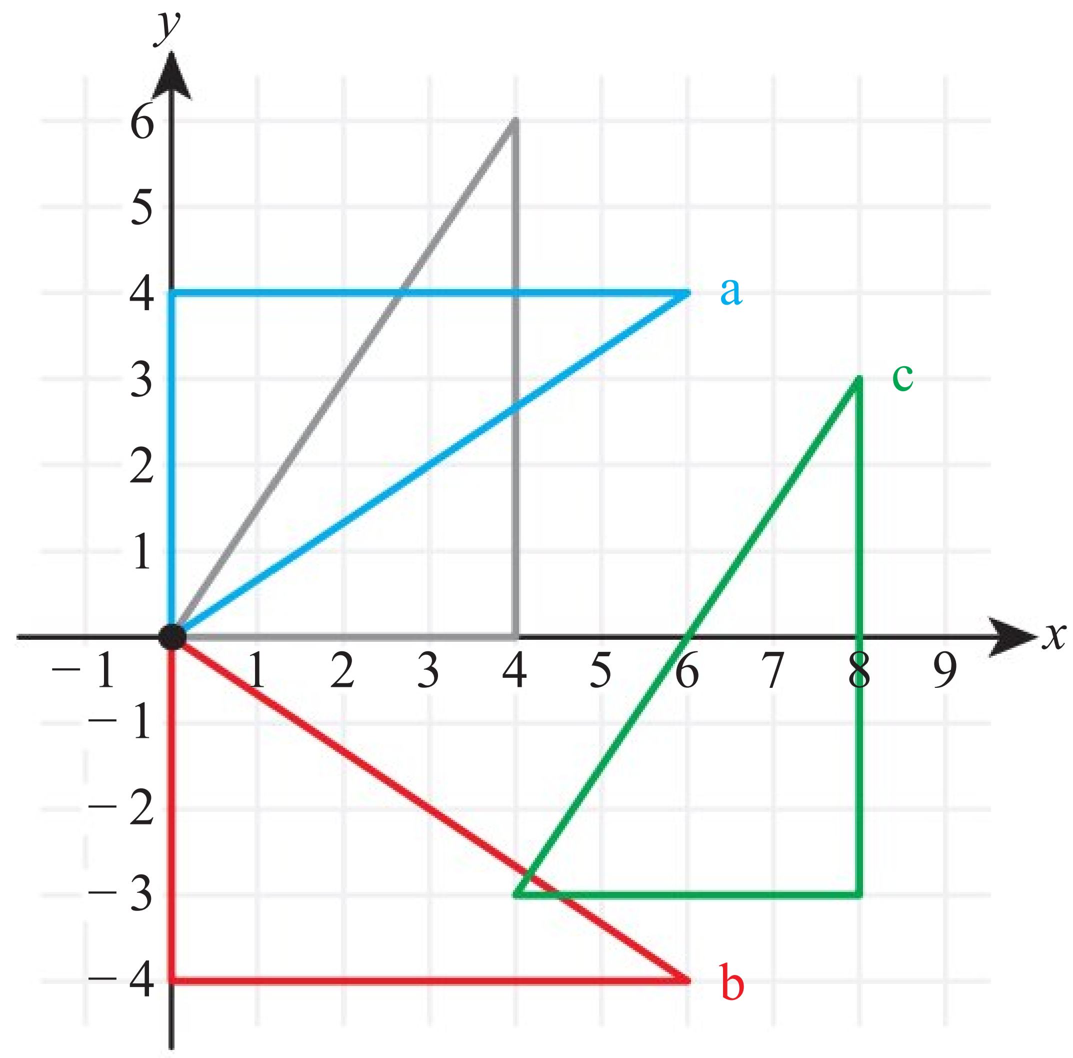
1 $\theta = 21°$

2 $-1.97, -0.218, 2.50, 9.83$

3 a $\begin{pmatrix} -1 \\ -21 \end{pmatrix}$ b $\begin{pmatrix} 5 & -2 \\ -4 & 7 \end{pmatrix}$

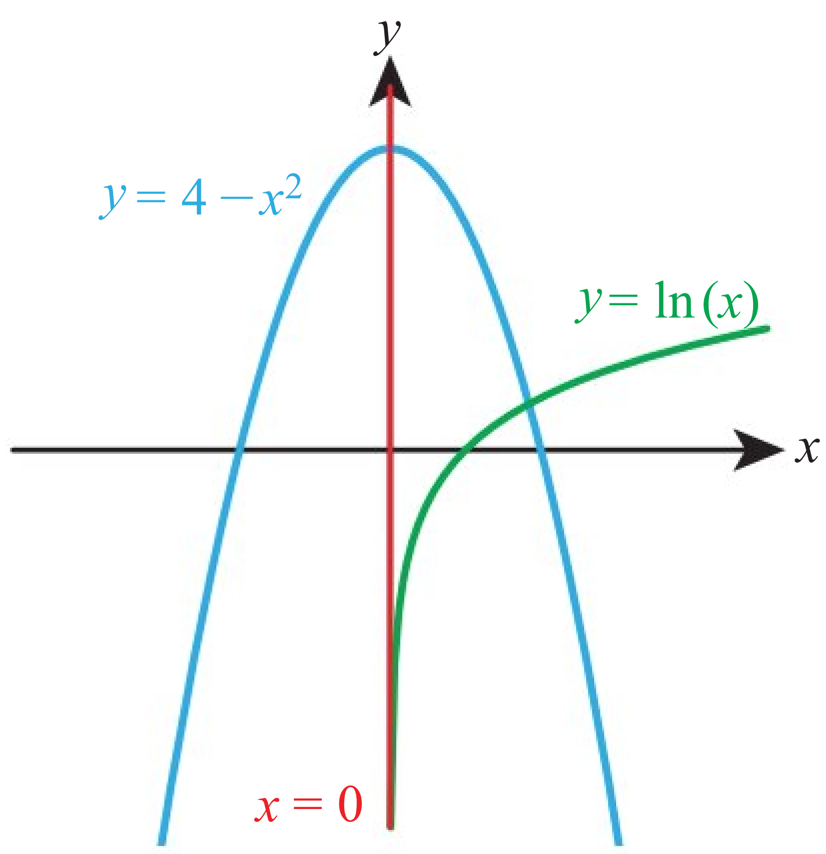
c $\begin{pmatrix} \frac{2}{3} & \frac{1}{3} \\ -\frac{1}{3} & \frac{1}{3} \end{pmatrix}$

4



Exercise 4A

1	a $\frac{\pi}{3}$	b $\frac{\pi}{4}$
2	a $\frac{5\pi}{6}$	b $\frac{2\pi}{3}$
3	a $\frac{\pi}{2}$	b $\frac{3\pi}{2}$
4	a 0.489	b 0.628
5	a 1.17	b 1.36
6	a 3.42	b 4.12
7	a 35.5°	b 47.6°
8	a 72.2°	b 77.3°
9	a 264.1°	b 300.2°
10	a 36°	b 22.5°
11	a 105°	b 48°
12	a 420°	
	b 330°	

13


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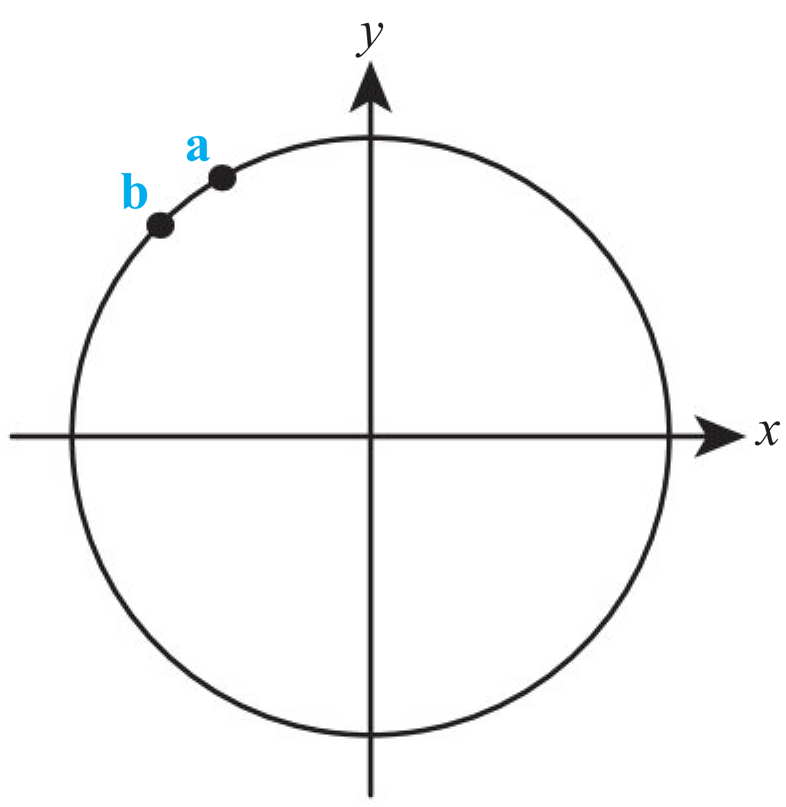

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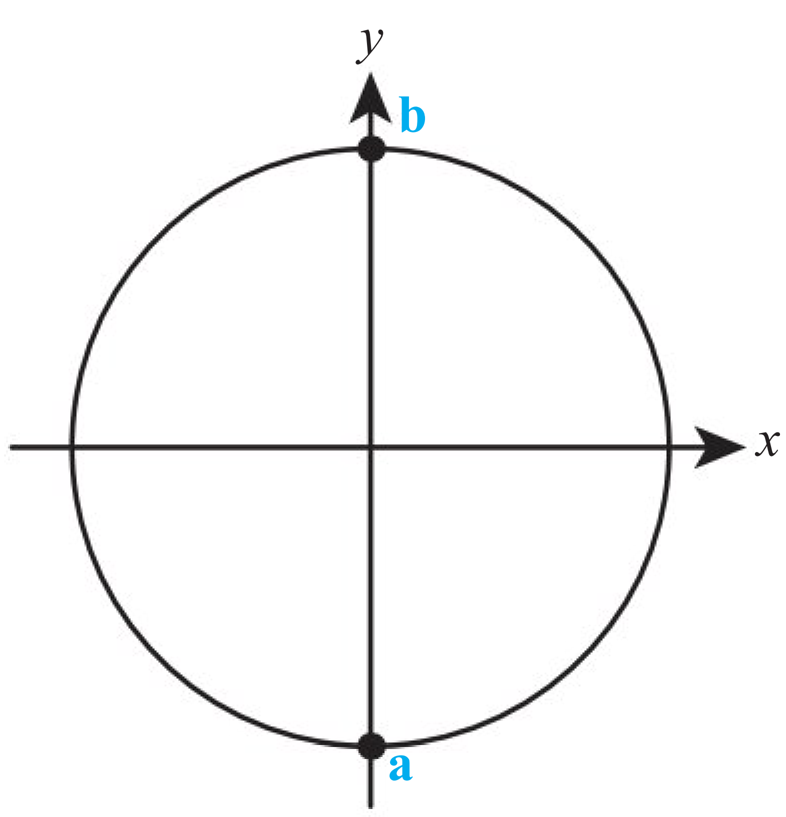

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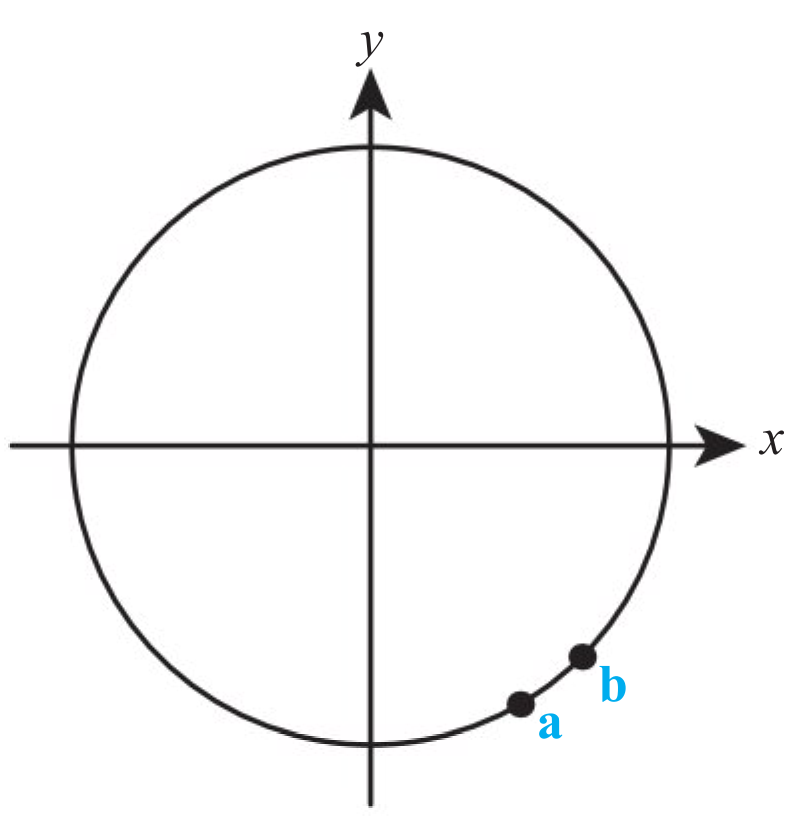

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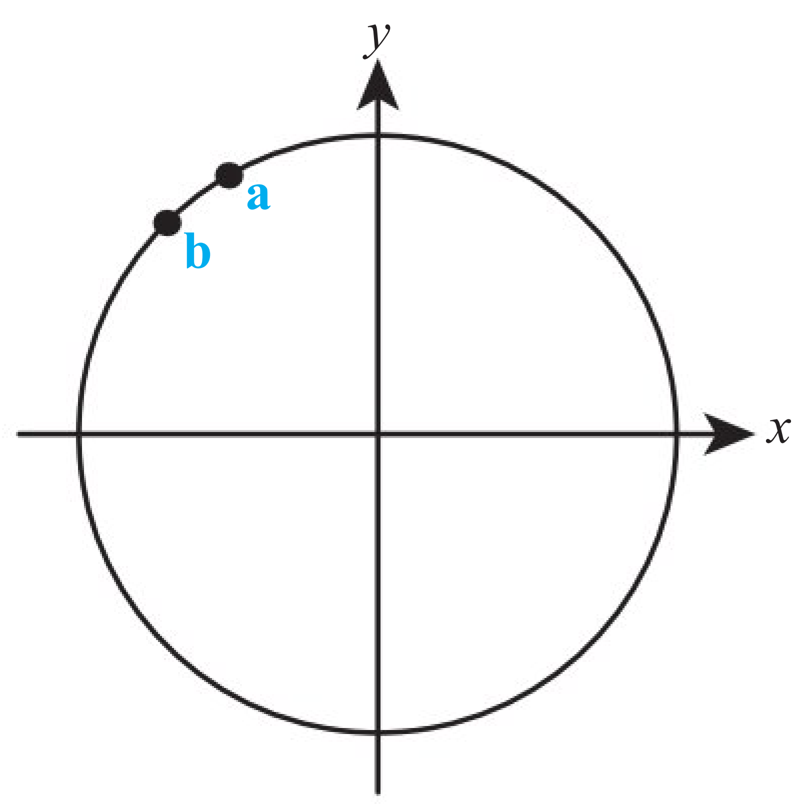

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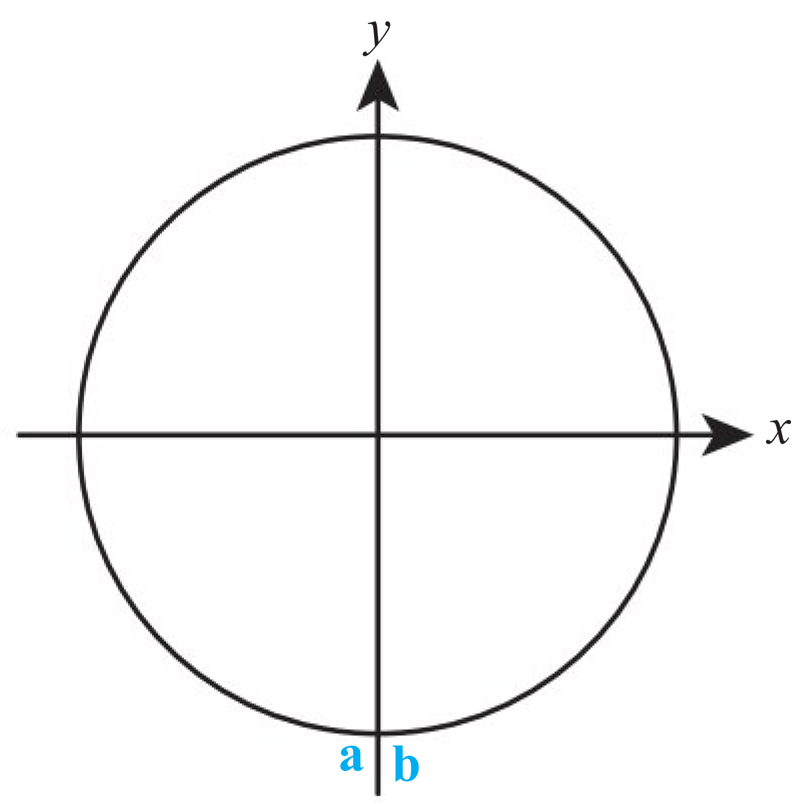


19 a 3.5 cm, 8.75 cm^2 b 8.8 cm, 35.2 cm^2

20 a 7.2 cm, 14.4 cm^2 b 14.7 cm, 51.5 cm^2

21 a 33.6 cm, 134 cm^2 b 25.5 cm, 63.8 cm^2

22 a 19.3 cm^2, 28.0 cm b 4.73 cm^2, 15.9 cm

23 a 41.2 cm^2, 33.7 cm b 15.5 cm^2, 19.2 cm

24 a 177 cm^2, 57.3 cm b 8.56 cm^2, 13.2 cm

25 Perimeter = 20.8 cm, area = 19.2 cm^2

26 Perimeter = 27.9 cm, area = 48.1 cm^2

27 10.3 cm

28 a 1.4 cm b 17.5 cm^2

29 0.699

30 15.5 cm

31 a 11.6 cm b 38.2 cm

32 Area = 57.1 cm^2, perimeter = 30.3 cm

33 6.84 cm

34 1.32, 13.7

35 12π

36 0.241 cm

37 Perimeter = 14.3 cm, area = 2.55 cm^2

38 b 1.74 c 13.1 cm

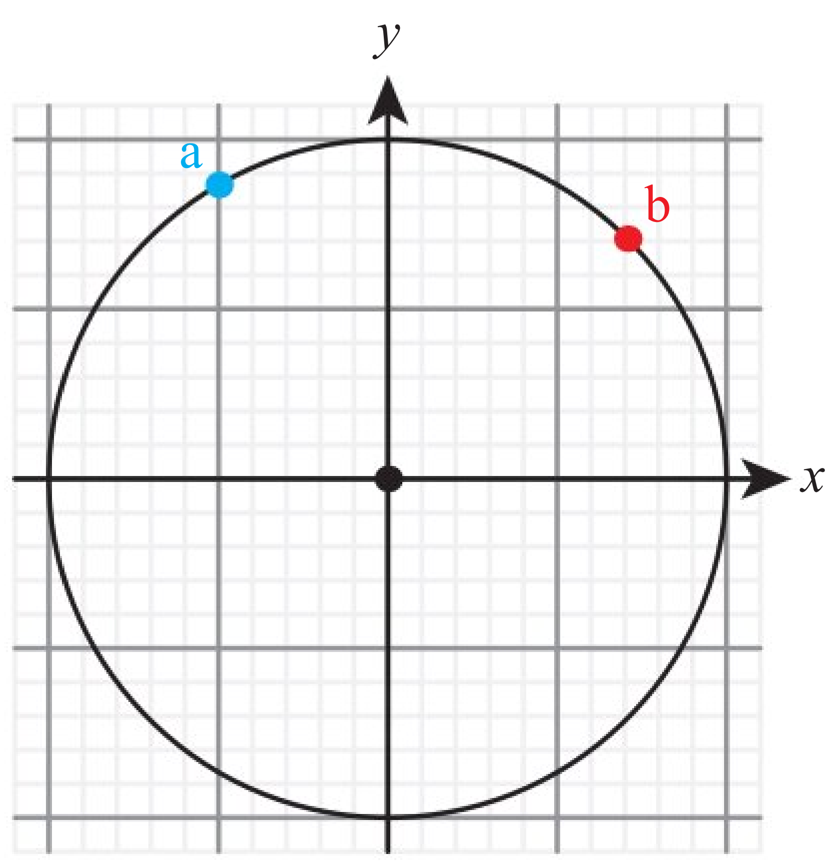
39 3

40 $r = 23.4\,\text{cm}$, $\theta = 2.15$

41 Perimeter $= 33.5\,\text{cm}$, area $= 78.6\,\text{cm}$

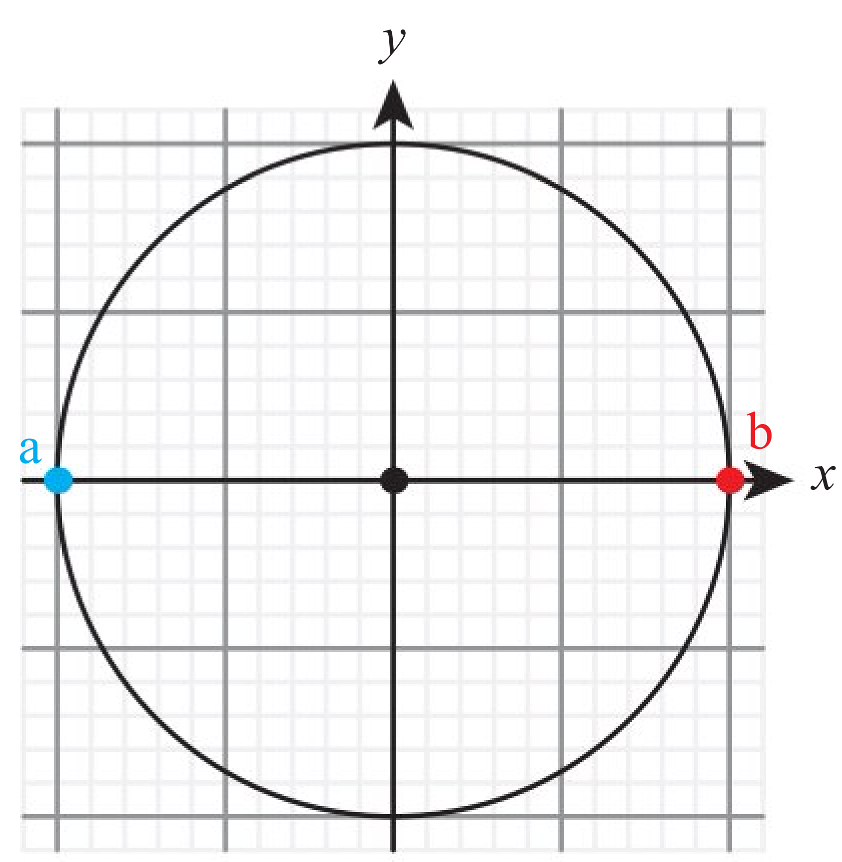
Exercise 4B

1



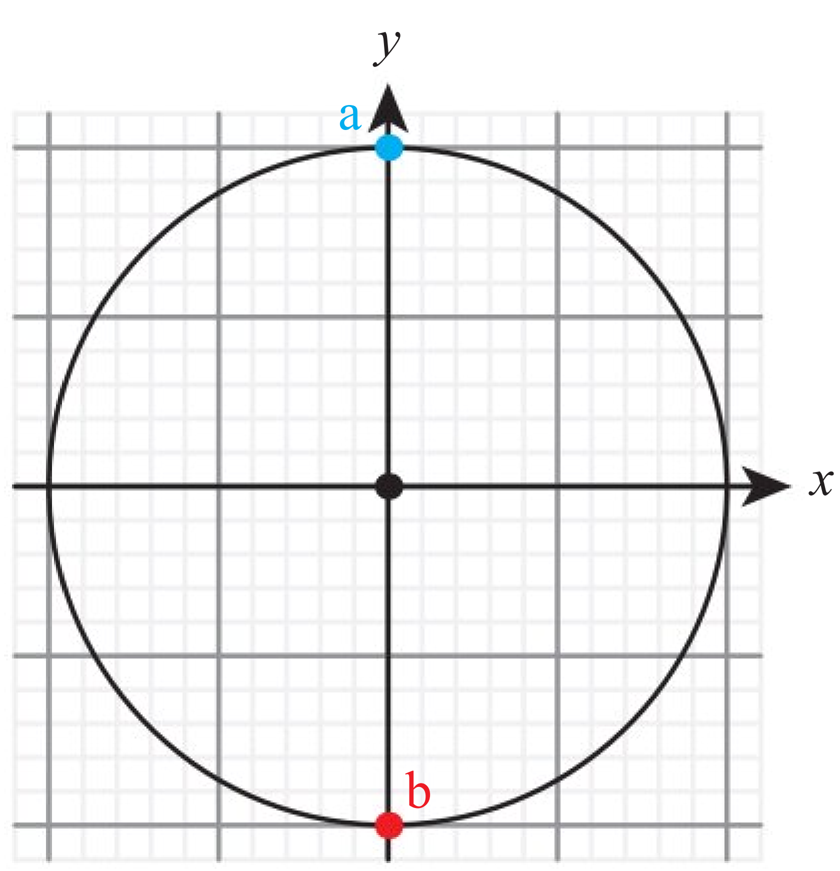
a $0.9, -0.5$ b $0.7, 0.7$

2



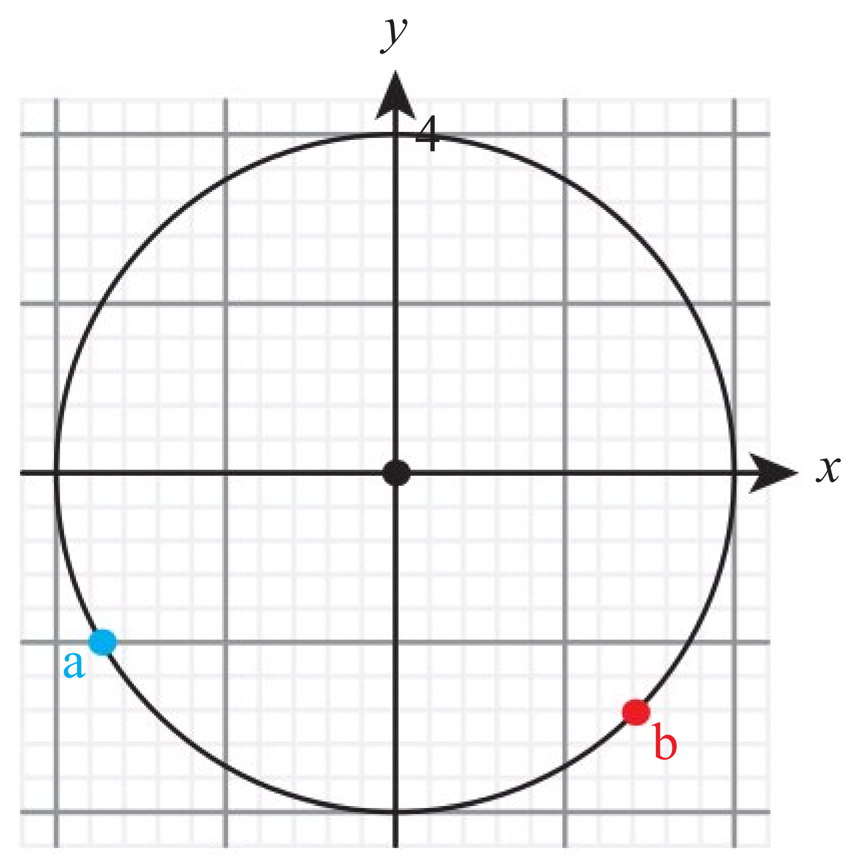
a $0, -1$ b $0, 1$

3



a $1, 0$ b $-1, 0$

4



a $-0.5, -0.9$ b $-0.7, 0.7$

5 a $64°, 116°$ b $49°, 131°$

6 a $55°$ b $57°$

7 a $59°, 121°$ b $66°, 114°$

8 a $57°$ b $50°$

9 a $\sin x = \dfrac{\sqrt{15}}{4}, \tan x = \sqrt{15}$

b $\sin x = \dfrac{\sqrt{5}}{3}, \tan x = \dfrac{\sqrt{5}}{2}$

10 a $\cos x = -0.925, \tan x = -0.412$

b $\cos x = 0.692, \tan x = 1.04$

11 a $\sin x = -0.998, \tan x = 16.1$

b $\sin x = 0.556, \tan x = -0.669$

12 a $\cos x = -\dfrac{2\sqrt{2}}{3}, \tan x = \dfrac{\sqrt{2}}{4}$

b $\cos x = \dfrac{4}{5}, \tan x = -\dfrac{3}{4}$

13 a $x = 0.167, 2.97, 6.45, 9.26$

b $x = 1.67, 4.61, 7.95, 10.9$

14 a $0.386, 1.71, 2.48$ b $0.912, 2.23$

15 a No solutions b No solutions

16 a $-0.795, 0.205$ b $5.20, 11.5$

17 $\pm\dfrac{\sqrt{65}}{9}$

18 a $-\dfrac{\sqrt{21}}{5}$ b $-\dfrac{\sqrt{21}}{2}$

19 a $-\dfrac{2\sqrt{10}}{7}$ b $-\dfrac{3\sqrt{10}}{20}$

20 a $6.6\,\text{m}$ b 6 am

21 a 8 minutes b $11\,\text{m}$ c $8.6\,\text{m}$

22 a $1.6\,\text{m}$ b 7 c 0.279 seconds

23 52.7°, 127°

24 4.42 cm, 6.37 cm

25 18.2 cm

26 $7 - 4\sin^2 x$

27 $9\cos^2 x - 5$

32 $\dfrac{1 \pm \sqrt{17 - 8k}}{4}$

Exercise 4C

1 a (7, 11) **b** (10, 12)

2 a (5, 0) **b** (3, 2)

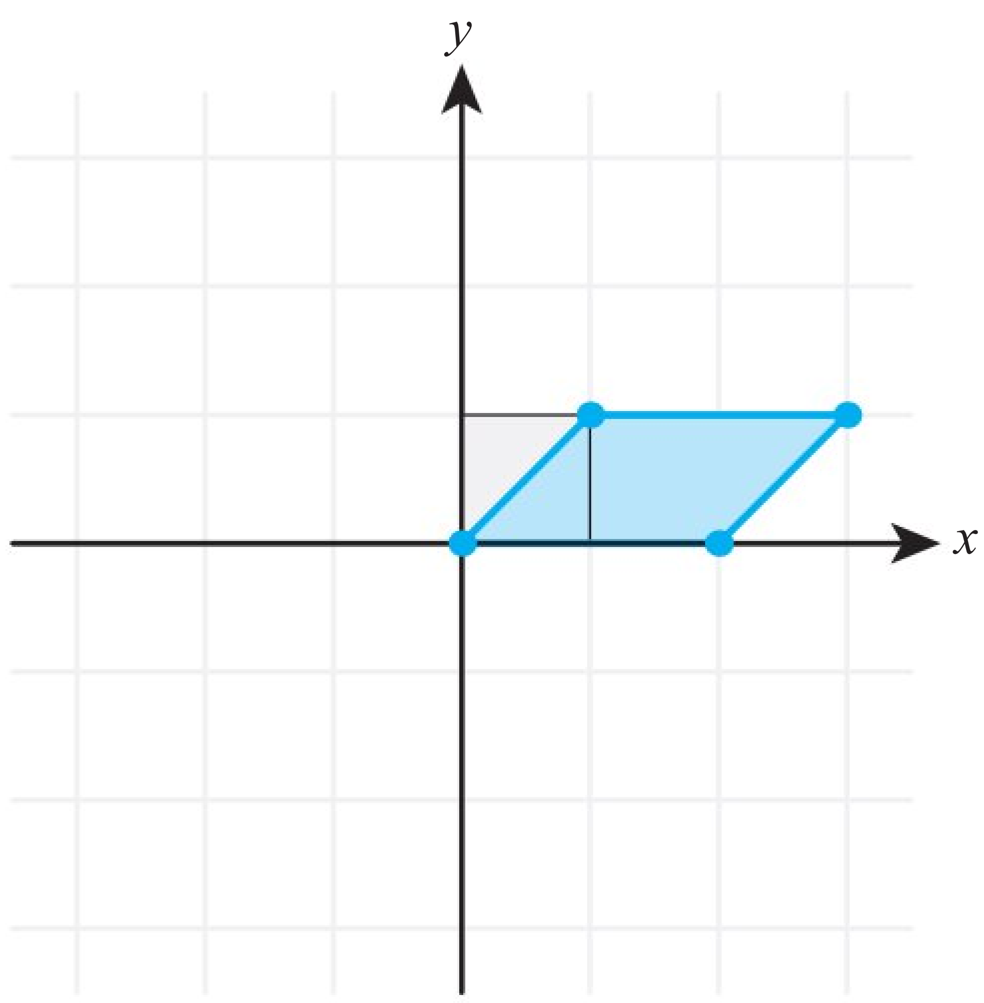
3 a (5, 13) **b** (1, –13)

4 a (2, 2) **b** (1, 3)

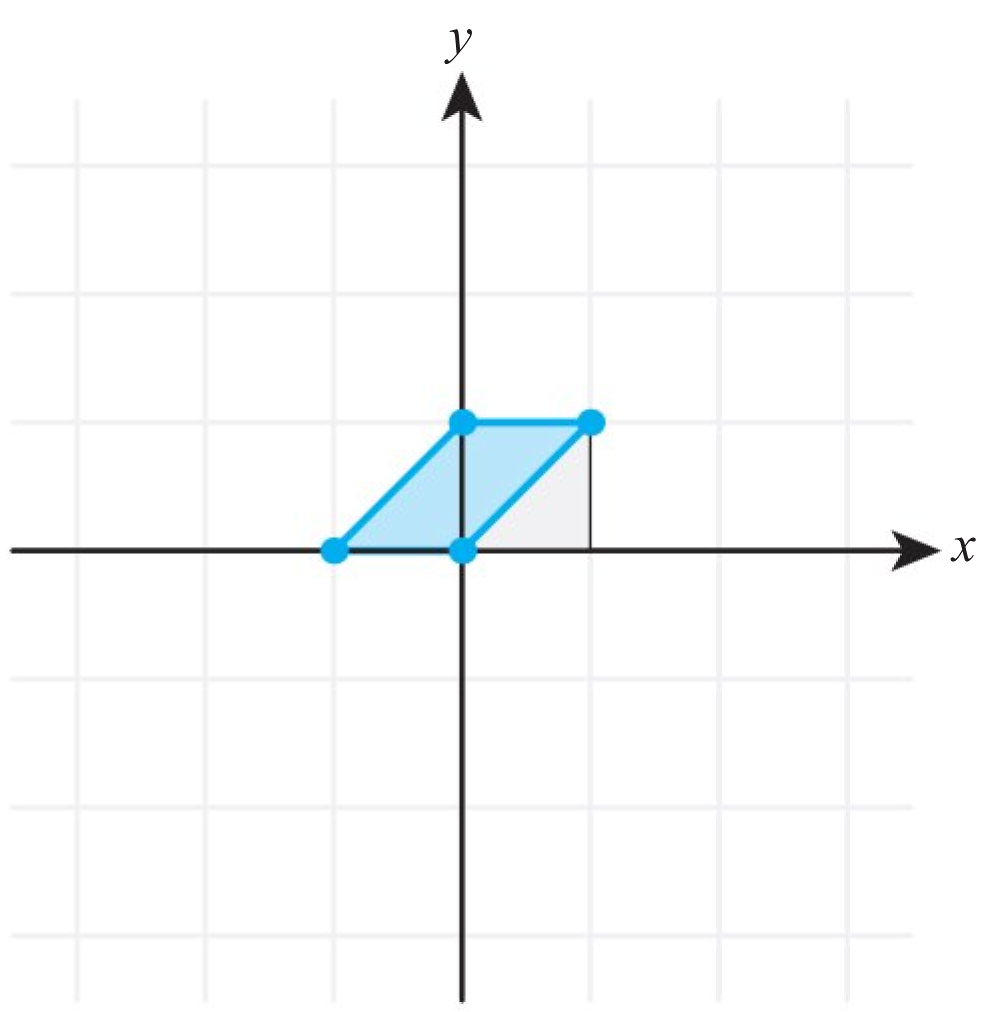
5 a (–1, 3) **b** (2, –1)

6 a (0, –2) **b** (3, 0)

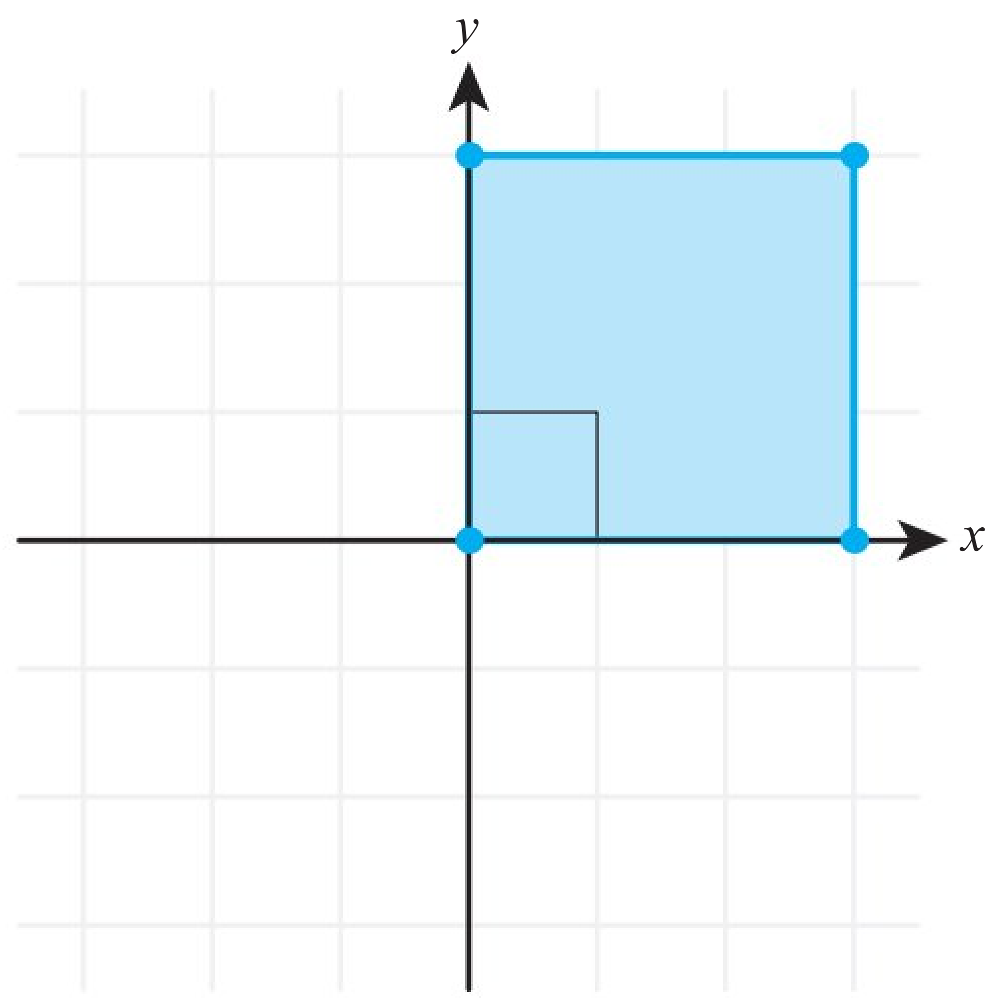
7 a



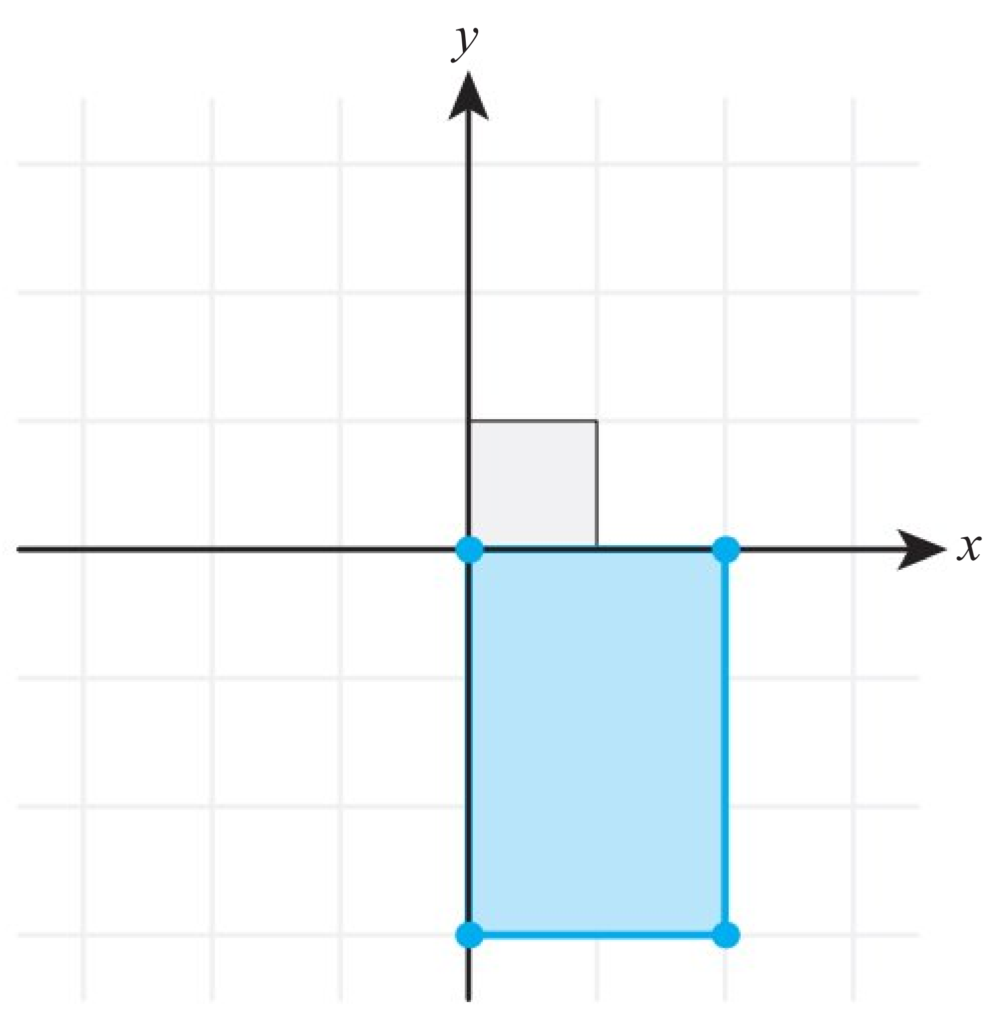
b



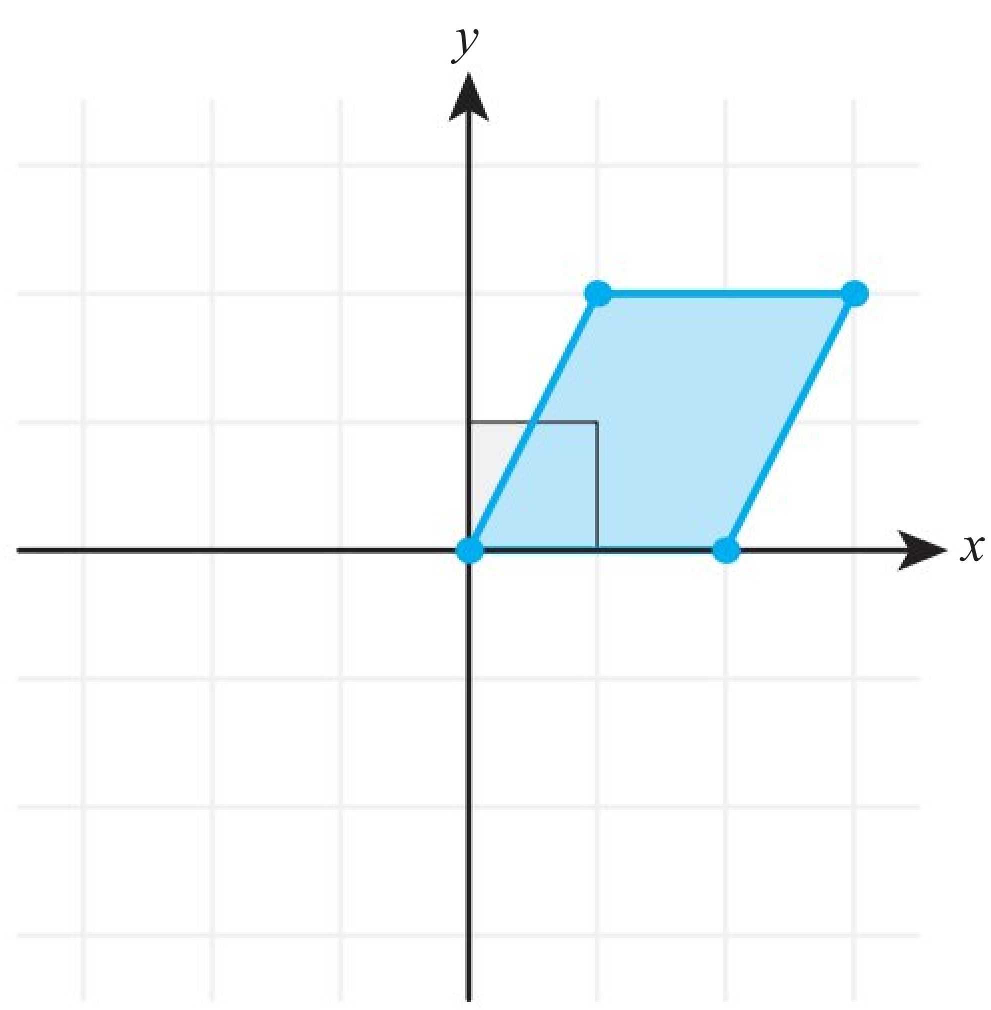
8 a



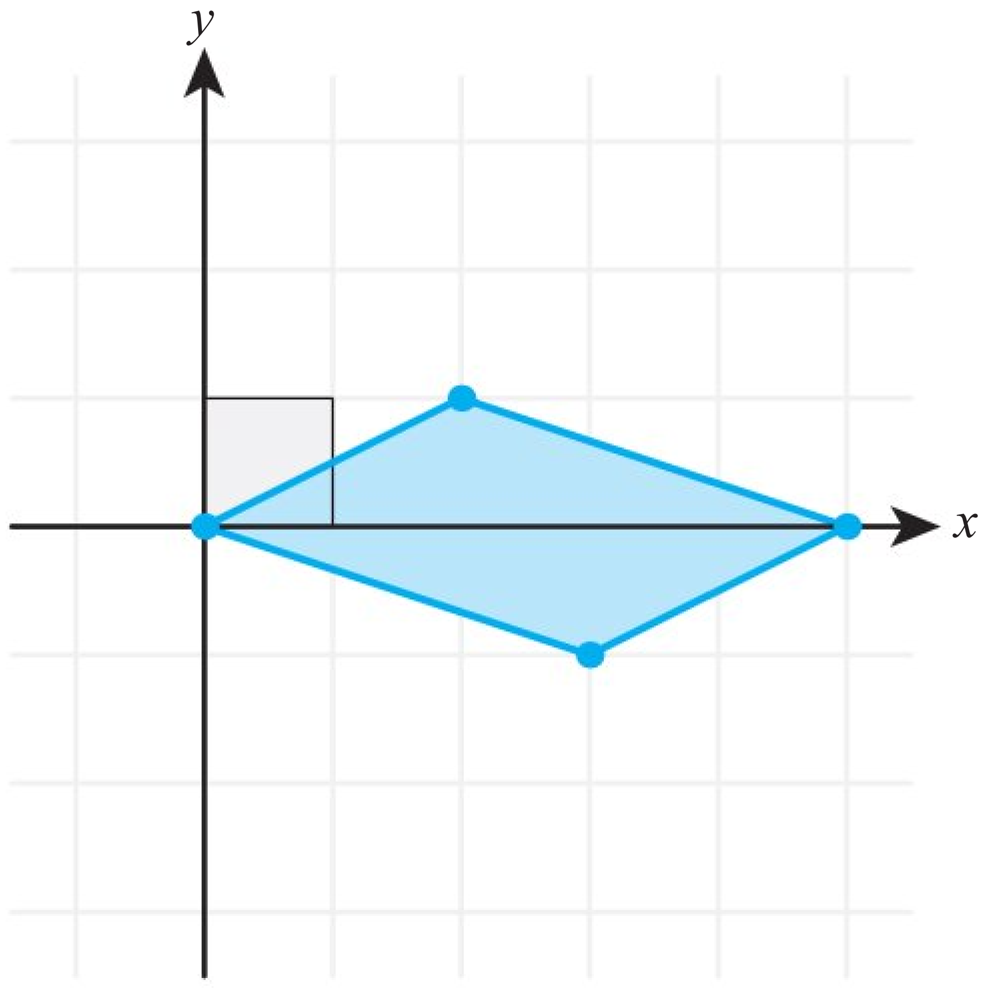
b



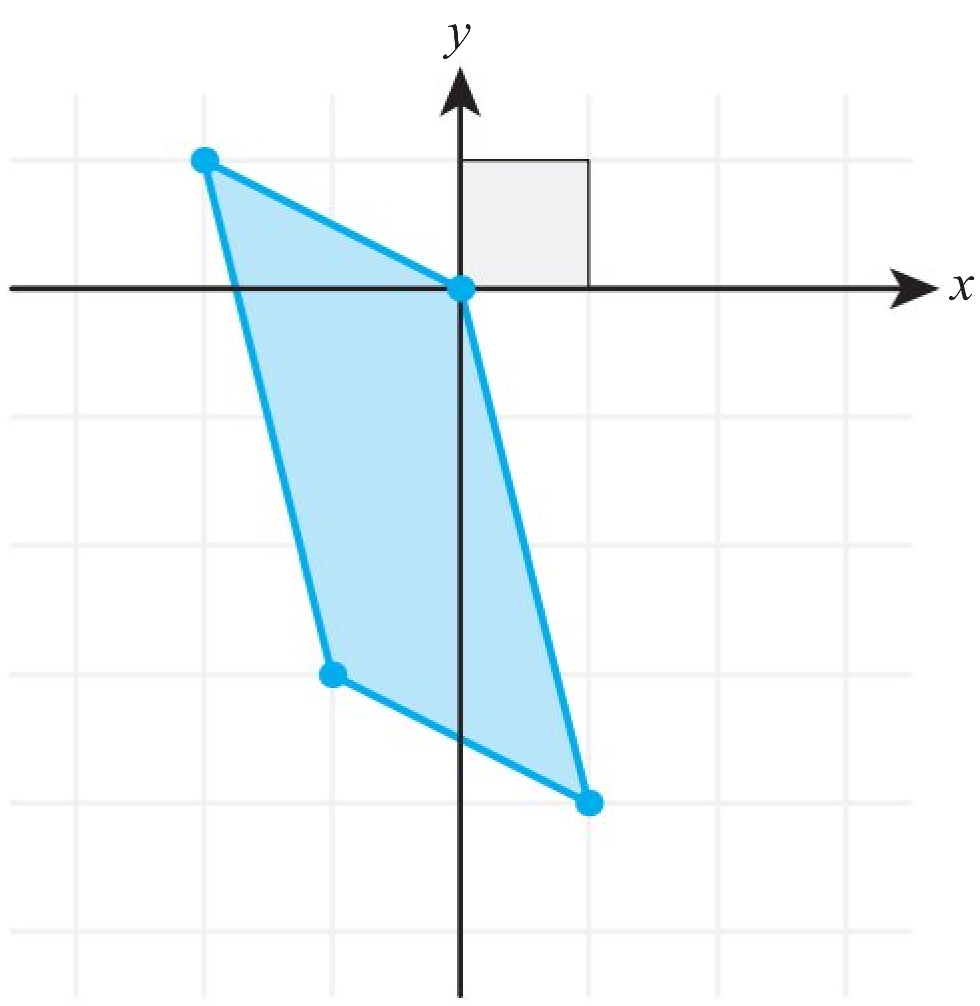
9 a



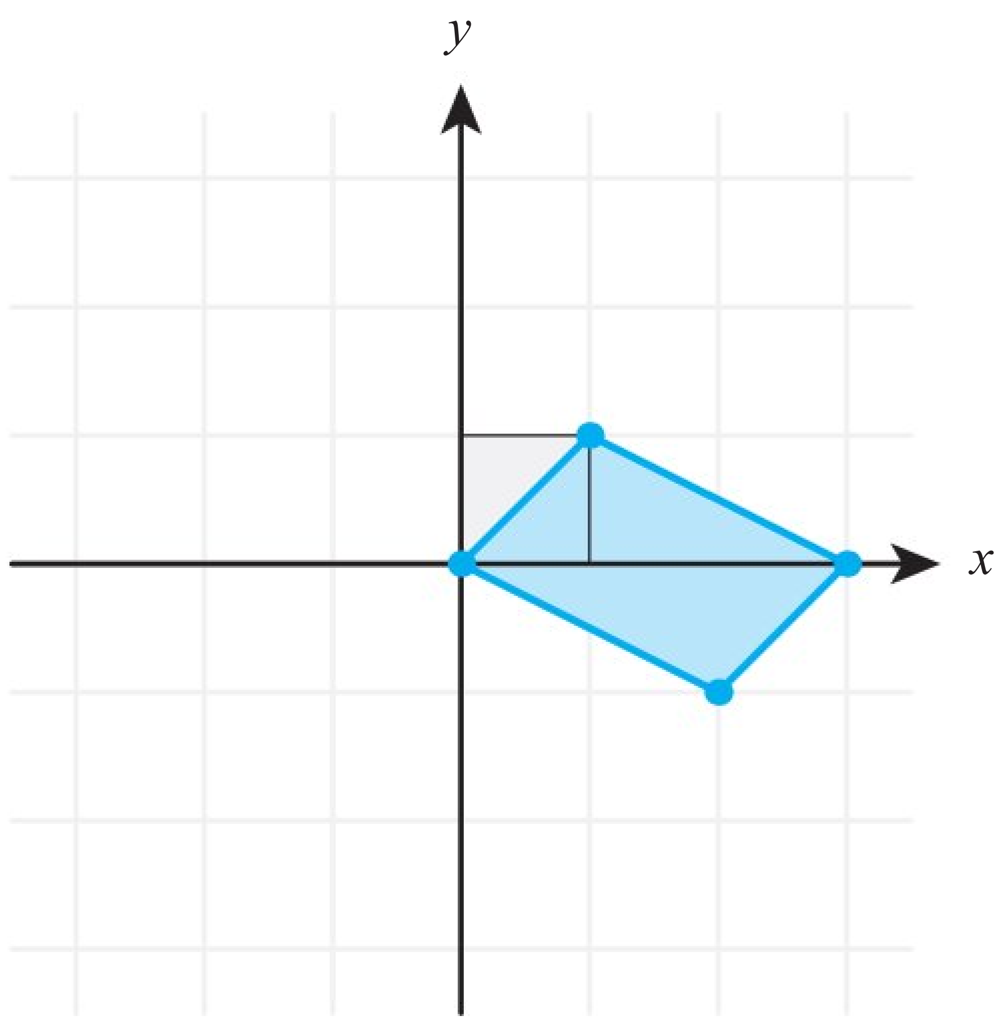
b



10 a



b



11 a $\begin{pmatrix} 2 & 2 \\ 1 & 3 \end{pmatrix}$ **b** $\begin{pmatrix} 3 & 1 \\ 1 & 2 \end{pmatrix}$

12 a $\begin{pmatrix} 0.5 & 1 \\ 0 & 5 \end{pmatrix}$ **b** $\begin{pmatrix} 4 & 0 \\ 2 & 0.5 \end{pmatrix}$

13 a $\begin{pmatrix} 2 & -1 \\ 1 & 2 \end{pmatrix}$ **b** $\begin{pmatrix} 2 & -2 \\ 2 & 1 \end{pmatrix}$

14 a $\begin{pmatrix} -3 & -1 \\ 0 & -1 \end{pmatrix}$ **b** $\begin{pmatrix} -2 & -3 \\ 0 & -2 \end{pmatrix}$

15 a $\begin{pmatrix} 0 & 2 \\ 3 & 0 \end{pmatrix}$ **b** $\begin{pmatrix} 0 & 4 \\ 2 & 0 \end{pmatrix}$

16 a $\begin{pmatrix} -1 & 4 \\ 1 & -1 \end{pmatrix}$ **b** $\begin{pmatrix} -2 & 1 \\ 3 & -1 \end{pmatrix}$

17 a $\begin{pmatrix} 3 & 0 \\ 0 & 1 \end{pmatrix}$ **b** $\begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix}$

18 a $\begin{pmatrix} 1 & 0 \\ 0 & 0.25 \end{pmatrix}$ **b** $\begin{pmatrix} 1/3 & 0 \\ 0 & 1 \end{pmatrix}$

19 a $\begin{pmatrix} 5 & 0 \\ 0 & 5 \end{pmatrix}$ **b** $\begin{pmatrix} 6 & 0 \\ 0 & 6 \end{pmatrix}$

20 a $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ **b** $\begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix}$

21 a $\begin{pmatrix} -1/2 & \sqrt{3}/2 \\ \sqrt{3}/2 & 1/2 \end{pmatrix}$ **b** $\begin{pmatrix} 1/2 & \sqrt{3}/2 \\ \sqrt{3}/2 & -1/2 \end{pmatrix}$

22 a $\begin{pmatrix} \sqrt{3}/2 & -1/2 \\ 1/2 & \sqrt{3}/2 \end{pmatrix}$ **b** $\begin{pmatrix} \sqrt{2}/2 & -\sqrt{2}/2 \\ \sqrt{2}/2 & \sqrt{2}/2 \end{pmatrix}$

23 a $\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$ **b** $\begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$

24 a $\begin{pmatrix} -1/2 & \sqrt{3}/2 \\ -\sqrt{3}/2 & -1/2 \end{pmatrix}$ **b** $\begin{pmatrix} -\sqrt{3}/2 & 1/2 \\ -1/2 & -\sqrt{3}/2 \end{pmatrix}$

25 a $(2, 6)$ **b** $(1, -1)$

26 a $(9, 0)$ **b** $(-4, 11)$

27 a $(-1, 0)$ **b** $(6, -6)$

28 a $\begin{pmatrix} 8 & 3 \\ 8 & 10 \end{pmatrix}$ **b** $\begin{pmatrix} -2 & 2 \\ 6 & 10 \end{pmatrix}$

29 a $\begin{pmatrix} 0 & -2 \\ 3 & 1 \end{pmatrix}$ **b** $\begin{pmatrix} 1 & 3 \\ -3 & 1 \end{pmatrix}$

30 a $\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$ **b** $\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$

31 a 9 **b** 1.2

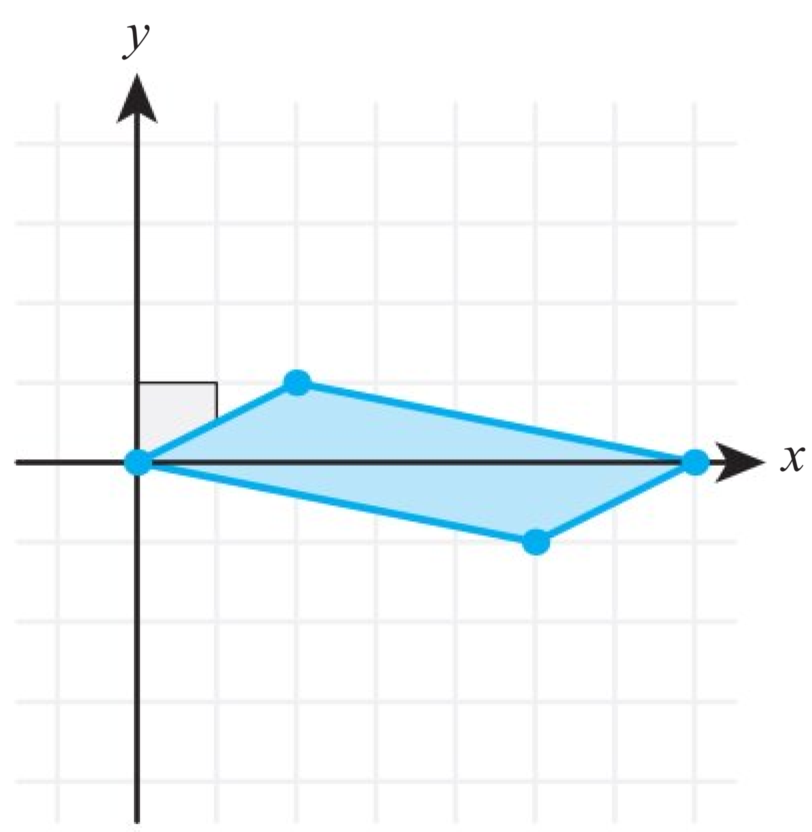
32 a 1.5 **b** 24

33 a 6 **b** 6

34 a 12 **b** 30

35 a $(8, -3)$

b



c 7

36 a $p = 5,\ q = 2$ **b** 15

c $(4, -9)$

37 a $(2, 3)$ **b** $(b-3, 1-a)$

38 a $\left(\frac{\sqrt{2}}{2}, \frac{5\sqrt{2}}{2}\right)$ **b** $(2\sqrt{2}, 10\sqrt{2})$

c 16

39 a $\begin{pmatrix} -0.8 & 0.6 \\ 0.6 & 0.8 \end{pmatrix}$ **b** $(-0.4, 2.8)$

40 a $(7, 20)$ **b** $(-2.4, 2.8)$

41 a $(2\sqrt{3}-1, 2+\sqrt{3})$ **b** $\begin{pmatrix} \sqrt{3} & -1 \\ 1 & \sqrt{3} \end{pmatrix}$

43 a $y = x$ **b** $y = -x$

44 a $\mathbf{T} = \begin{pmatrix} -1/4 & -\sqrt{3}/4 \\ \sqrt{3}/4 & -1/4 \end{pmatrix}$, $\mathbf{T}^2 = \begin{pmatrix} -1/8 & \sqrt{3}/8 \\ -\sqrt{3}/8 & -1/8 \end{pmatrix}$

b $D(-2\sqrt{3}, -2)$, $F(\sqrt{3}, -1)$

c 42

45 a A has coordinates $(4, 4)$.

b 64

46 a $\begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix}$

b $\lambda = 1, -1,\ v = \begin{pmatrix} -1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \end{pmatrix}$

c Points on the line $y = x$ are unaffected by the transformation; points in the line $y = -x$ are reflected through the origin.

47 a $\lambda_1 = -1,\ \mathbf{v}_1 = \begin{pmatrix} 1 \\ 3 \end{pmatrix}$; $\lambda_2 = -5,\ \mathbf{v}_2 = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$

b $y = 3x$ and $y = -x$

48 a $\frac{1}{5}\begin{pmatrix} -4 & -3 \\ -3 & 4 \end{pmatrix}$ **b** $y = -3x$ and $y = \frac{1}{3}x$

c $y = -3x$

49 a $\left(\frac{1}{2}x + 1, \frac{1}{2}y\right)$

b $P_2\left(\frac{1}{4}x + \frac{3}{2}, \frac{1}{4}y\right)$, $P_3\left(\frac{1}{8}x + \frac{7}{4}, \frac{1}{8}y\right)$,

$P_4\left(\frac{1}{16}x + \frac{15}{8}, \frac{1}{16}y\right)$

c $\left(\frac{1}{2^n}x + \frac{2^n - 1}{2^{n-1}}, \frac{1}{2^n}y\right)$ **d** $\frac{2}{3}$

50 a $\frac{1}{16}$ **b** $\frac{5}{2}$

Chapter 4 Mixed Practice

1 a 1.3m **b** 2.51m

2 55.2°, 125°

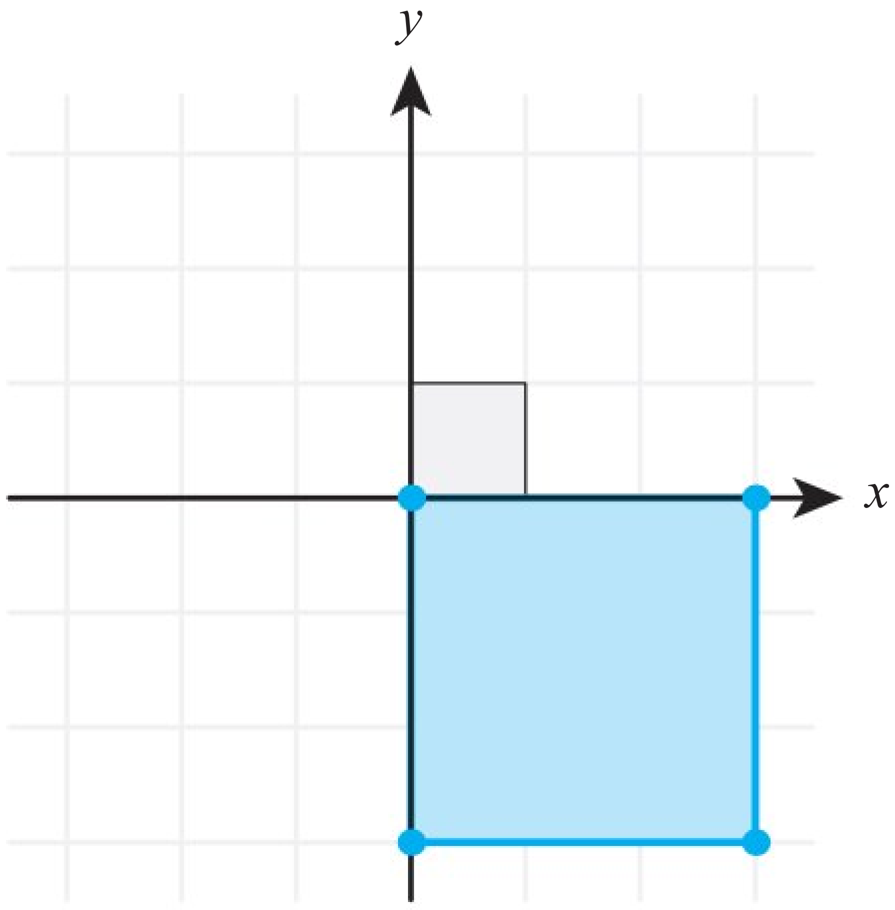
3 a $-\frac{12}{13}$ **b** $-\frac{5}{12}$

4 a $(4, 7)$ **b** $(-1, 4)$

c 1

5 a $\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$ **b** $\begin{pmatrix} 0 & 3 \\ -3 & 0 \end{pmatrix}$

c



6 **a** $\begin{pmatrix} 0 & 2 \\ 1 & 0 \end{pmatrix}$ **b** No

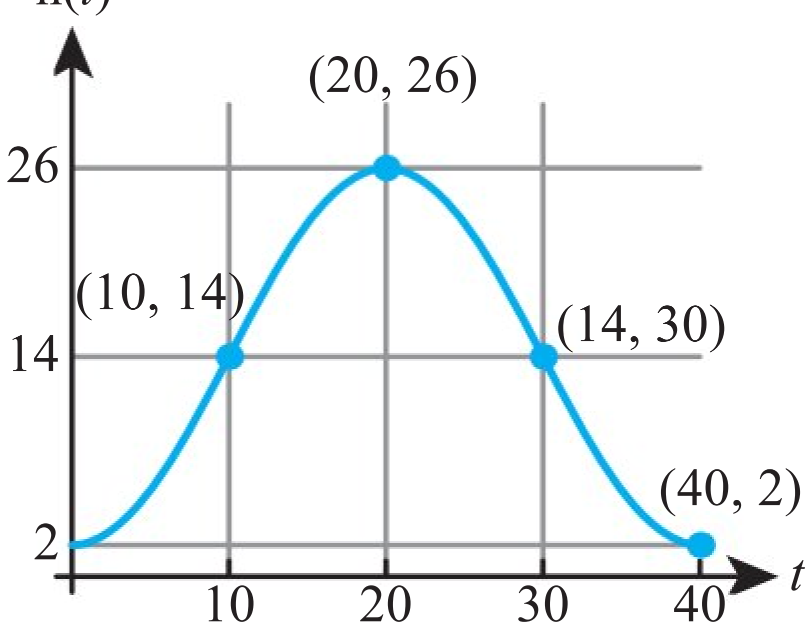
7 20π

8 $0 \leqslant y \leqslant 25$

9 **a** **i** 14 m **ii** 26 m

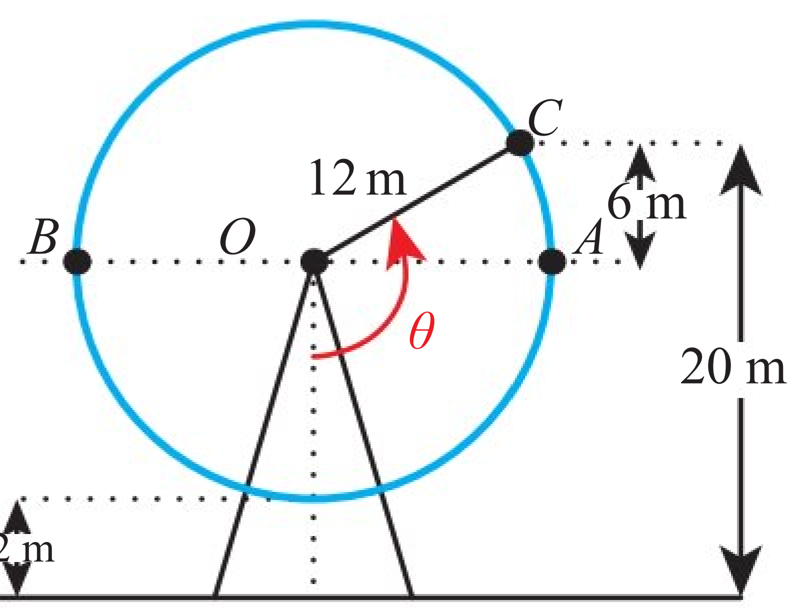
b 10 seconds, 30 seconds

c



d $a = 12,\ b = \frac{\pi}{20},\ c = 14$

e **i**



ii 120° **iii** 13.3 seconds

10 **a** $\begin{pmatrix} 1/2 & -\sqrt{3}/2 \\ \sqrt{3}/2 & 1/2 \end{pmatrix}$

b 60° rotation (anticlockwise) around the origin

c 60° rotation clockwise around the origin

11 **a** $A = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$ **b** $\begin{pmatrix} 0 & 1 \\ -3 & 0 \end{pmatrix}$ **c** $(-4, 6)$

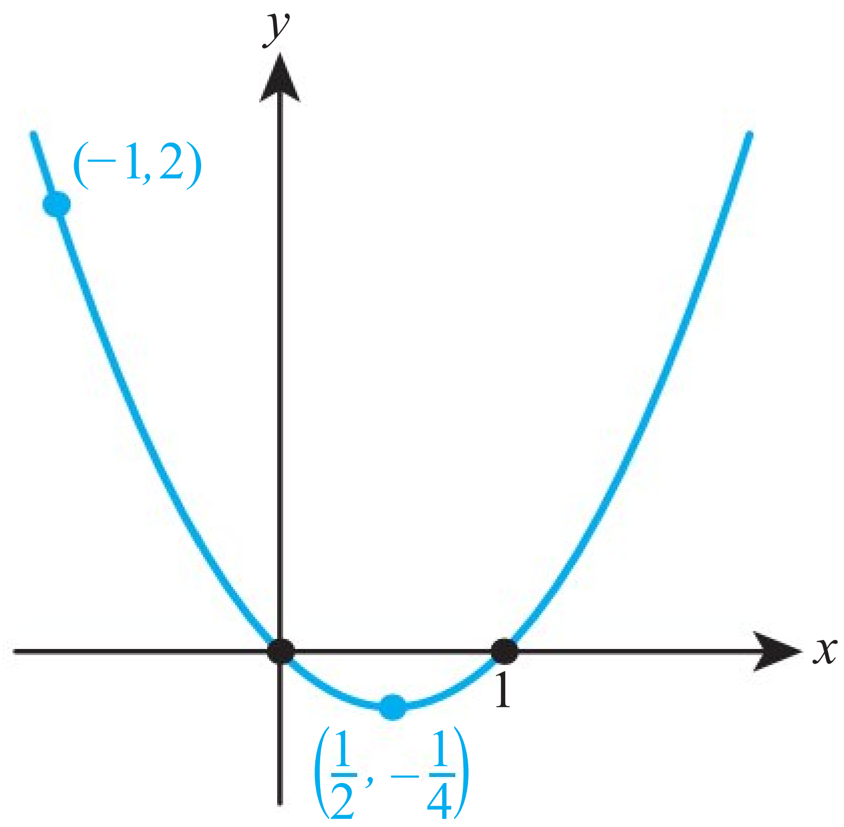
$B = \begin{pmatrix} 1 & 0 \\ 0 & 3 \end{pmatrix}$

12 $17.8\,\text{cm}^2$

13 3.7 cm and 3.6 cm

15 6π

16 **a**



b $-0.25 \leqslant k \leqslant 2$

17 **a** $\begin{pmatrix} -1/2 & \sqrt{3}/2 \\ \sqrt{3}/2 & 1/2 \end{pmatrix}$

b Reflection in $y = \sqrt{3}x$

18 Rotation 45° clockwise about the origin

19 **a** 82.3° (1.44 radians) **b** 111 cm

20 **a** 44.0° (0.769 radians) **b** $1.18\,\text{cm}^2$

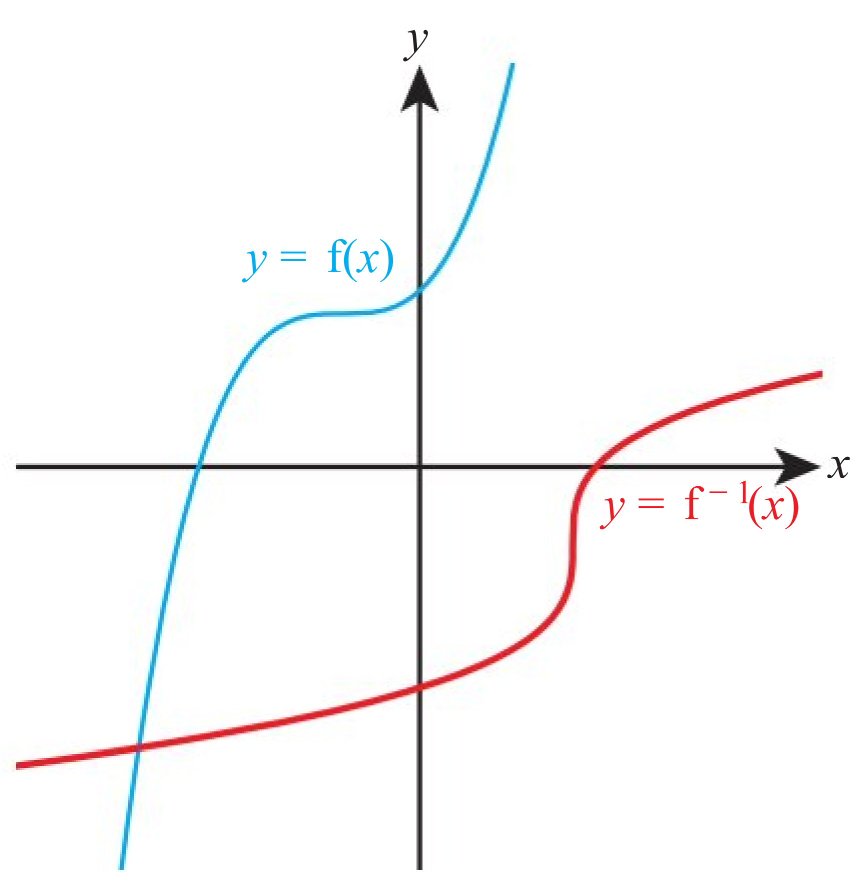
Chapter 5 Prior Knowledge

1 11

2 $x \neq -3$

3 **a** $f(x) \geqslant -3$ **b** Many-to-one

4



5 a $x = \frac{y+1}{y-3}$ b $x = \ln(y+2)$

Exercise 5A

1 a $3x^2 - 1$ b $4x^2 + 2$

2 a $(2x+1)^2$ b $(3x-2)^2$

3 a $12x^2 + 4x$ b $20x^2 - 6x$

4 a $3e^{2x+5}$ b $4e^{3x+1}$

5 a $12e^x + 1$ b $6e^x + 5$

6 a $e^{3x} - 2e^x$ b $e^{3x} + 4e^x$

7 a $\frac{1}{3x+3}$ b $\frac{1}{2x+3}$

8 a $\frac{3}{x} - \frac{2}{x^2}$ b $\frac{4}{x} + \frac{3}{x^2}$

9 a $x > -8$ b $x > -9$

10 a $x < 1$ b $x < 0$

11 a $x \geqslant \frac{1}{2}$ b $x \geqslant \frac{5}{3}$

12 a $x \leqslant \frac{7}{2}$ b $x \leqslant \frac{5}{3}$

13 a $x \neq \frac{1}{4}$ b $x \neq 1$

14 a $x \neq \ln 3$ b $x \neq \ln 7$

15 a $11 - 9x$ b $\frac{1}{3}$

16 a $x^4 + 2x^2 + 2$ b 1

17 $16x^9$

18 a $x \geqslant -2$ b $\frac{19}{3}$

19 a $fg(8) = \ln 3$, $gf(8) = \ln 8 - 5$

b $e^8 + 5$

20 a i 2 ii 1

b 4

21 a i 1 ii 1

b 1

22 a $x > 3$ b $e + 3$

c e^4 d $\frac{3e^3}{e^3 - 1}$

23 a $\frac{x-3}{2x-5}$ b $x \neq 2.5, 3$ c $\frac{7}{3}$

24 $k = 16$

25 a $x \neq -\frac{2}{3}, -\frac{7}{6}$ b $y \neq 0, \frac{1}{2}$ c $-\frac{5}{3}$

Exercise 5B

1 a $f^{-1}(x) = \frac{2x-1}{4}$ b $f^{-1}(x) = \frac{5x-4}{3}$

2 a $f^{-1}(x) = \frac{x+3}{4}$ b $f^{-1}(x) = \frac{x-1}{5}$

3 a $f^{-1}(x) = \frac{1}{4}\ln x$ b $f^{-1}(x) = \frac{1}{3}\ln x$

4 a $f^{-1}(x) = \ln\left(\frac{x}{3}\right) + 2$ b $f^{-1}(x) = \ln\left(\frac{x}{2}\right) - 3$

5 a $f^{-1}(x) = \frac{1}{3}(2^x - 1)$ b $f^{-1}(x) = \frac{1}{4}(3^x + 1)$

6 a $f^{-1}(x) = x^2 + 2$ b $f^{-1}(x) = x^2 - 3$

7 a $f^{-1}(x) = \sqrt[3]{x} - 2$ b $f^{-1}(x) = \sqrt[3]{x} + 3$

8 a $f^{-1}(x) = \sqrt[3]{x+2}$ b $f^{-1}(x) = \sqrt[3]{x-5}$

9 a $f^{-1}(x) = \frac{2x+3}{x-1}$ b $f^{-1}(x) = \frac{3x+1}{x-1}$

10 a $f^{-1}(x) = \frac{2x+1}{3x-2}$ b $f^{-1}(x) = \frac{3x-1}{2x-3}$

11 a $x \geqslant 2$ b $x \geqslant -5$

12 a $x \leqslant -1$ b $x \leqslant 3$

13 a $x \geqslant 2$ b $x \geqslant 0$

14 a $-1 \leqslant x \leqslant 1$ b $-2 \leqslant x < +2$

15 a $x \leqslant -1$ b $x \leqslant 1$

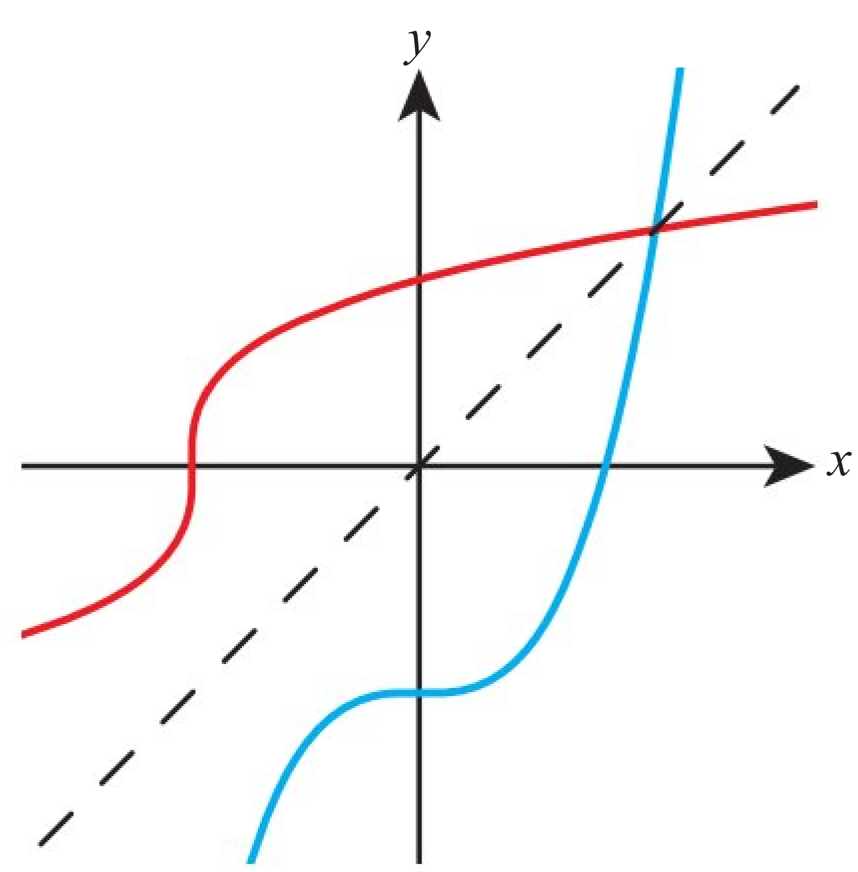
16 a $x \geqslant -2$ b $x \geqslant 3$

17 a $x < -2$ b $x < -1$

18 a $\frac{5}{3}$ b $\frac{3x-4}{x}$

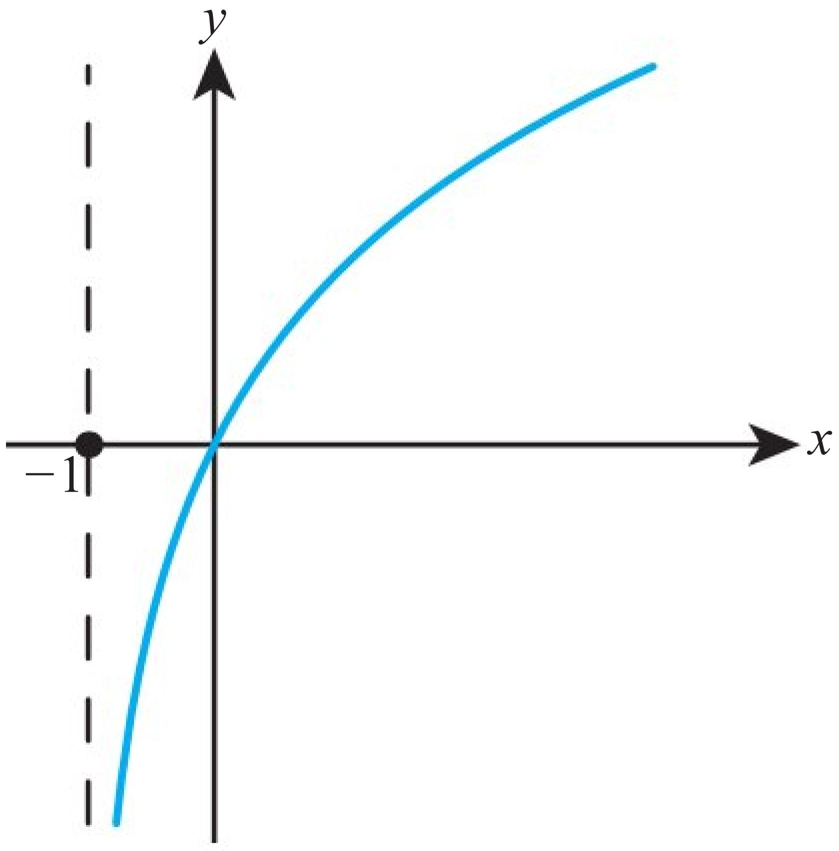
19 $\frac{1}{5}\ln\left(\frac{x}{3}\right)$

20 a



b $f^{-1}(x) = \sqrt[3]{5x + 15}$

21 a

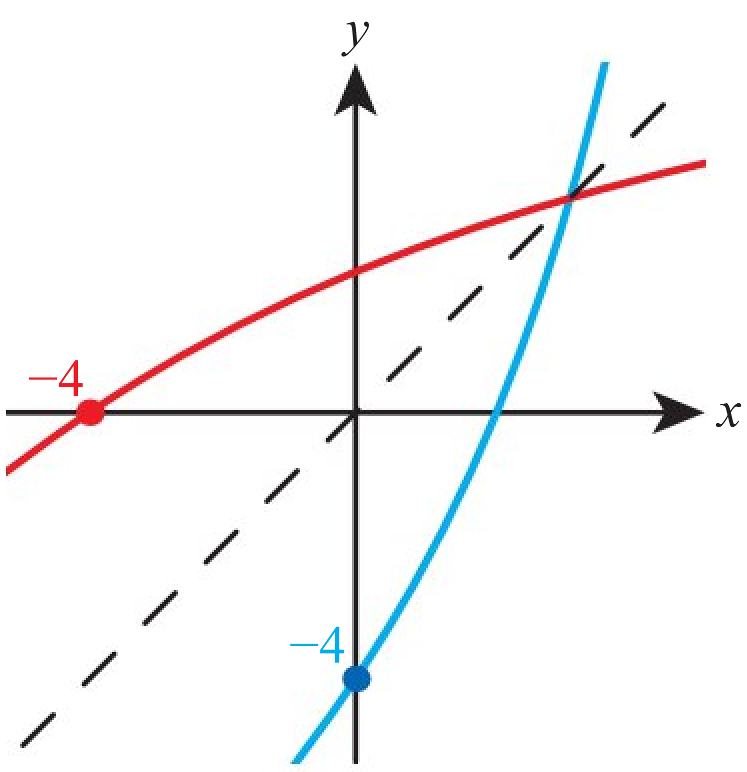


b $f^{-1}(x) = 2\ln(x+1),\ x > -1$

22 a $\sqrt{\dfrac{4x+1}{x-1}}$ b $f^{-1}(x) > 2$

23 $\dfrac{1}{7}$

24 a



b $2\ln 5$

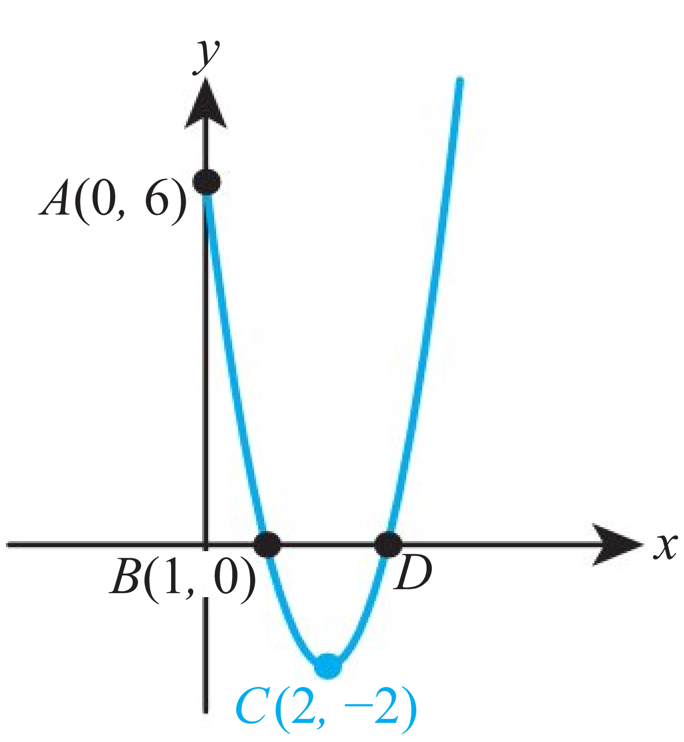
25 a 0 b $-\sqrt{x-3}$

26 a $x \leqslant 5$ b $5 - \dfrac{1}{3}\sqrt{x}$

27 4

Exercise 5C

1 a



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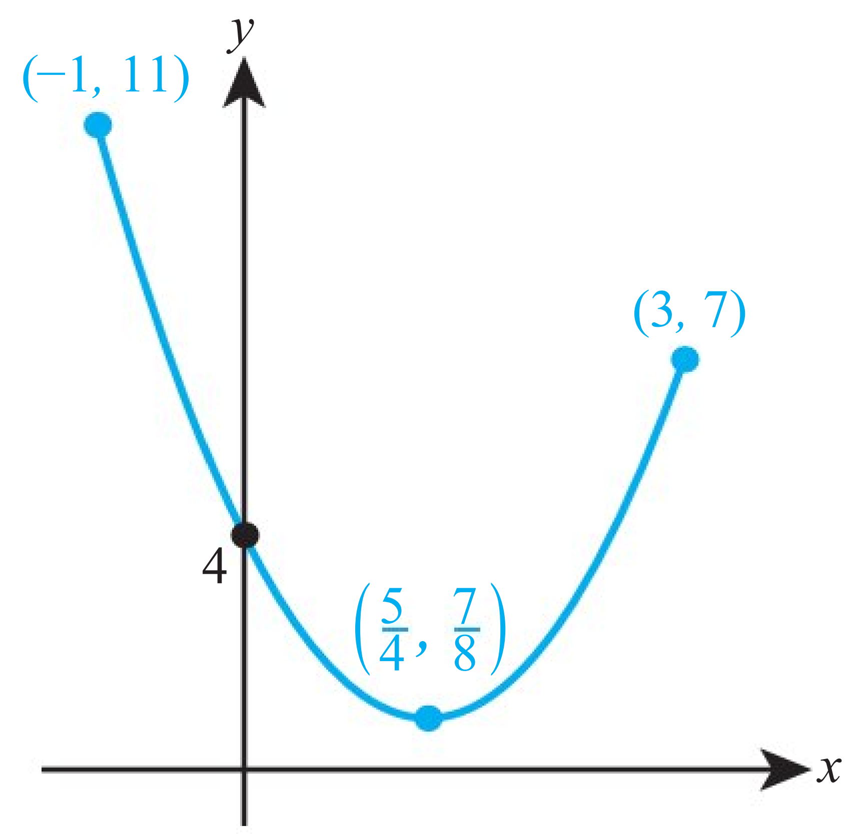
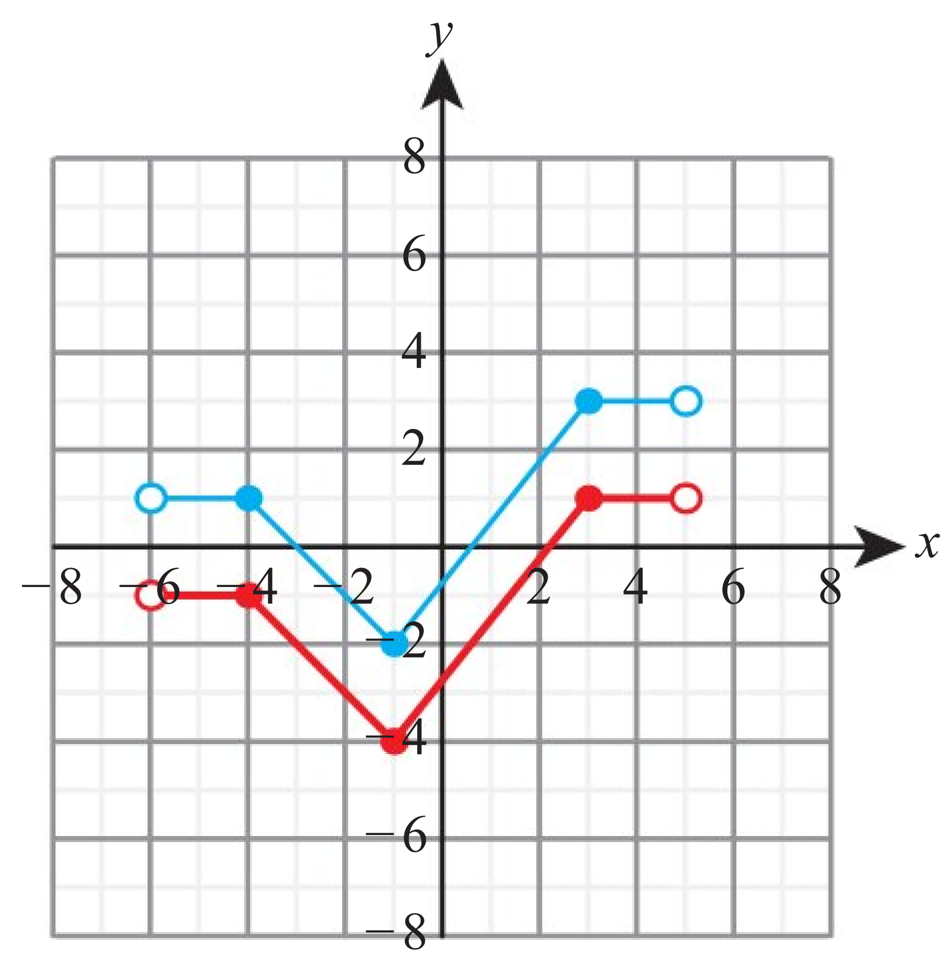


2 a



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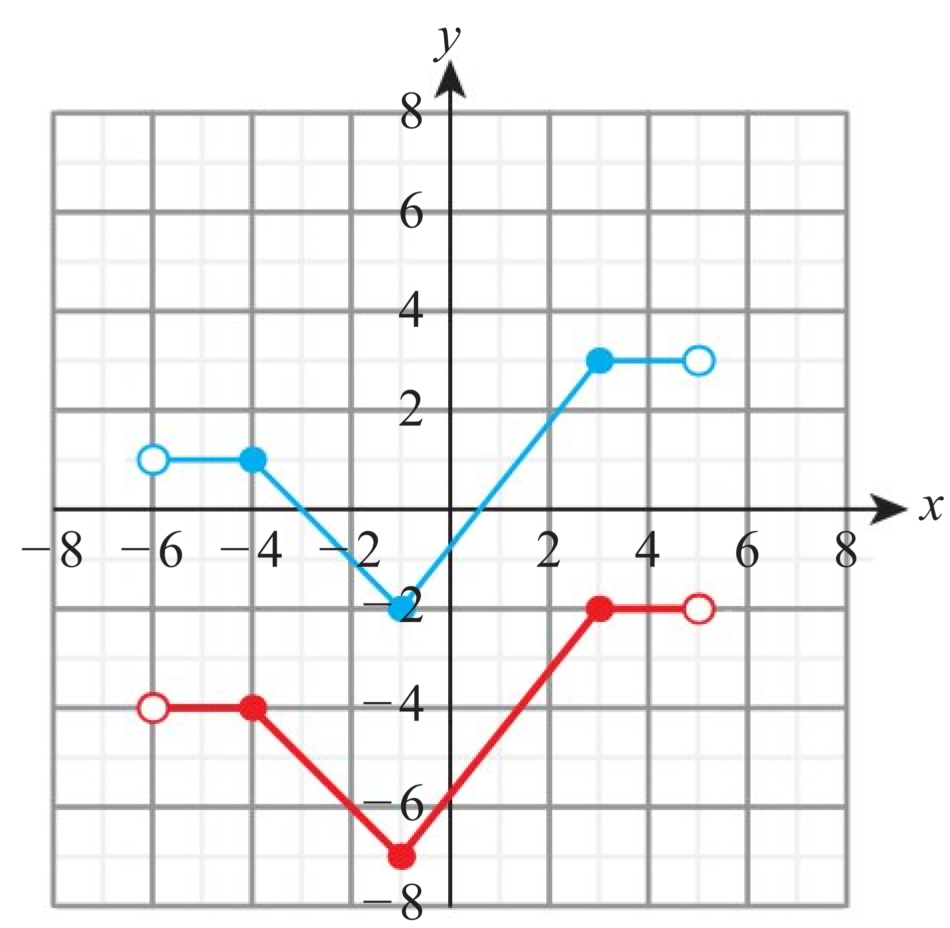
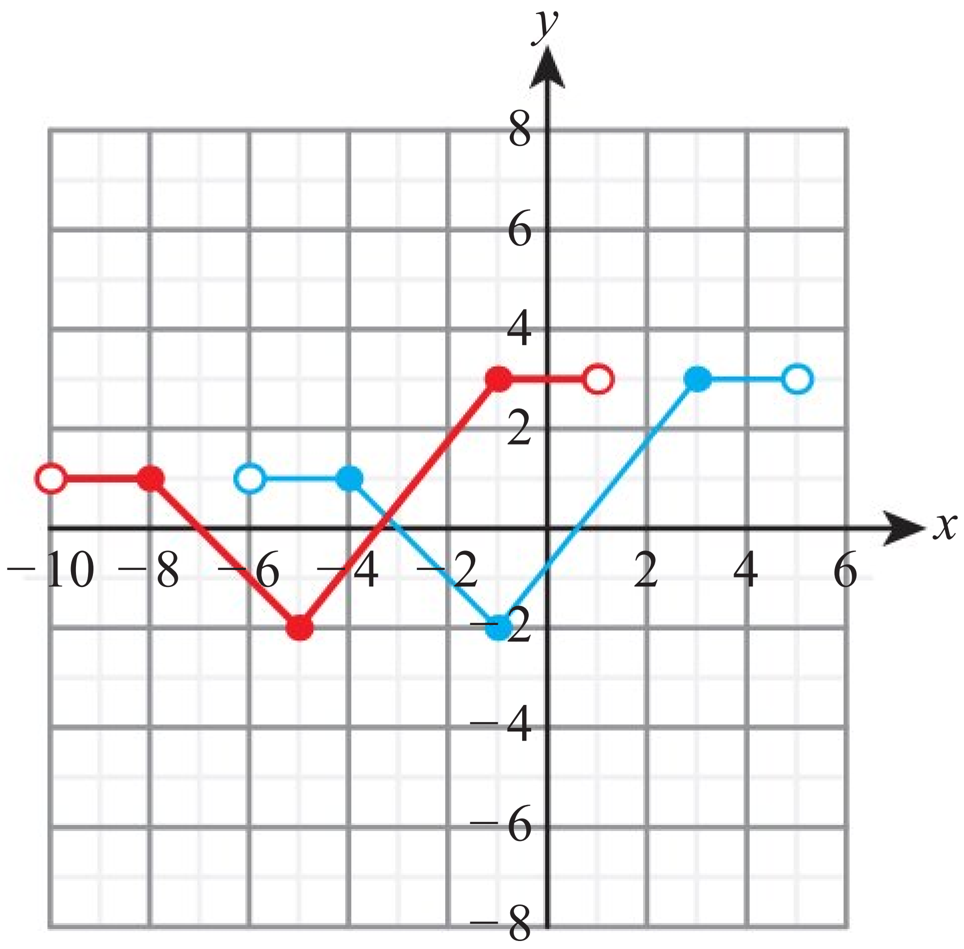


3 a



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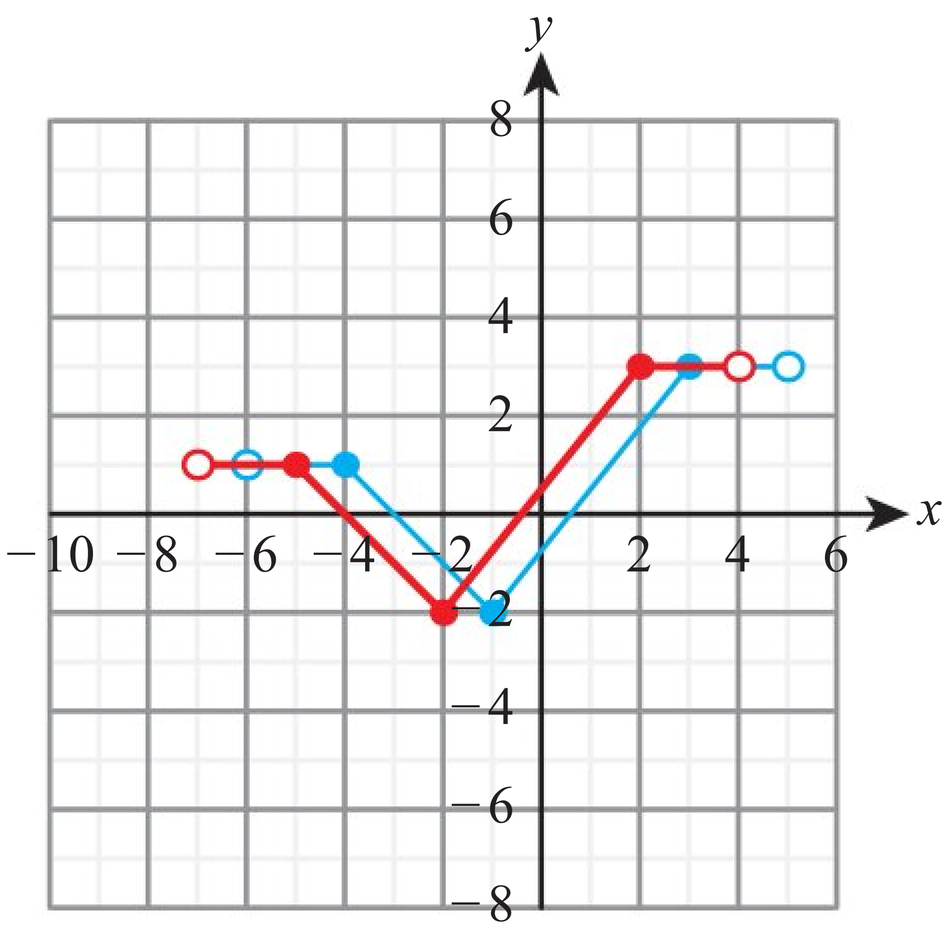
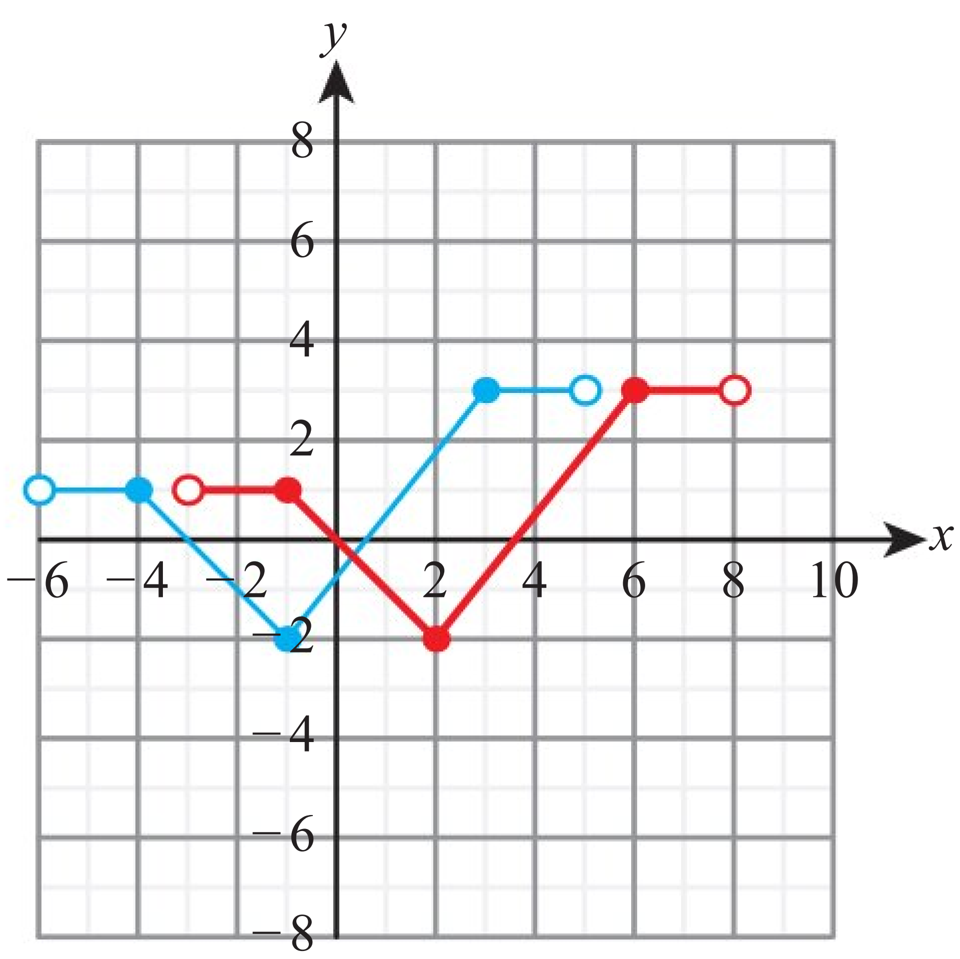
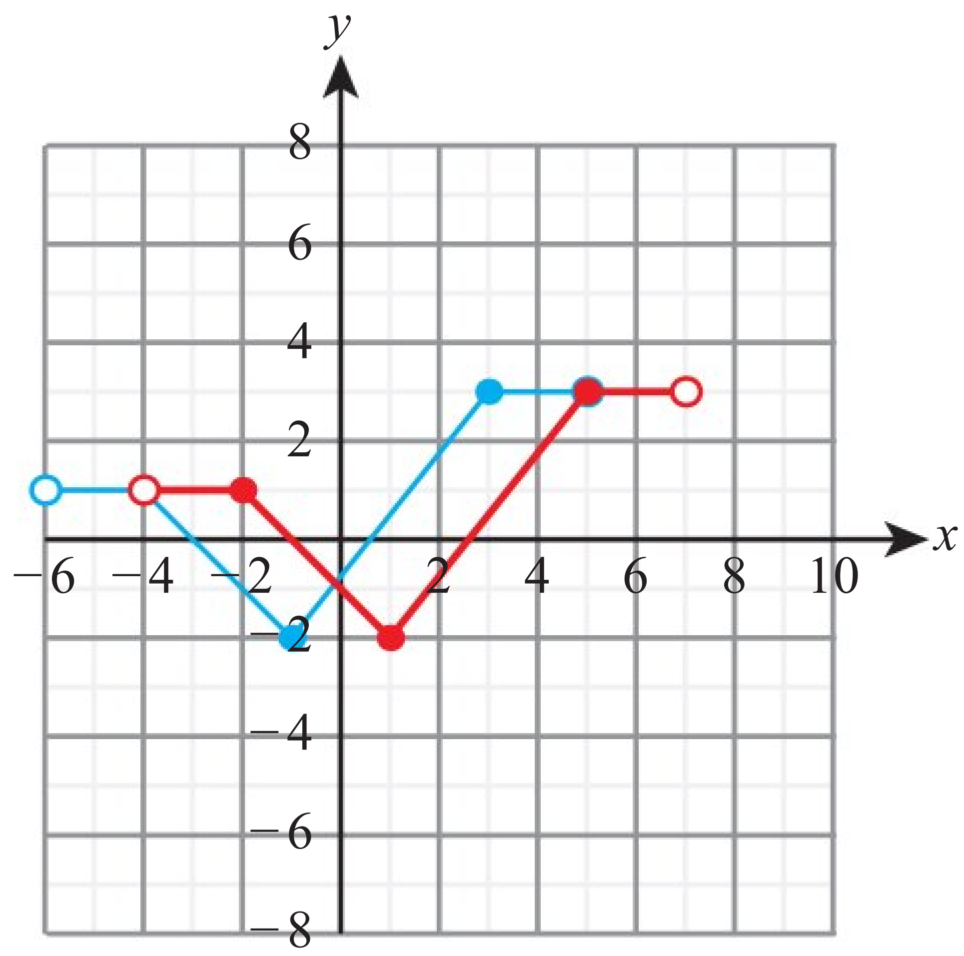


4 a



b



5 a $y = 3x^2 + 3$ **b** $y = 2x^3 + 5$

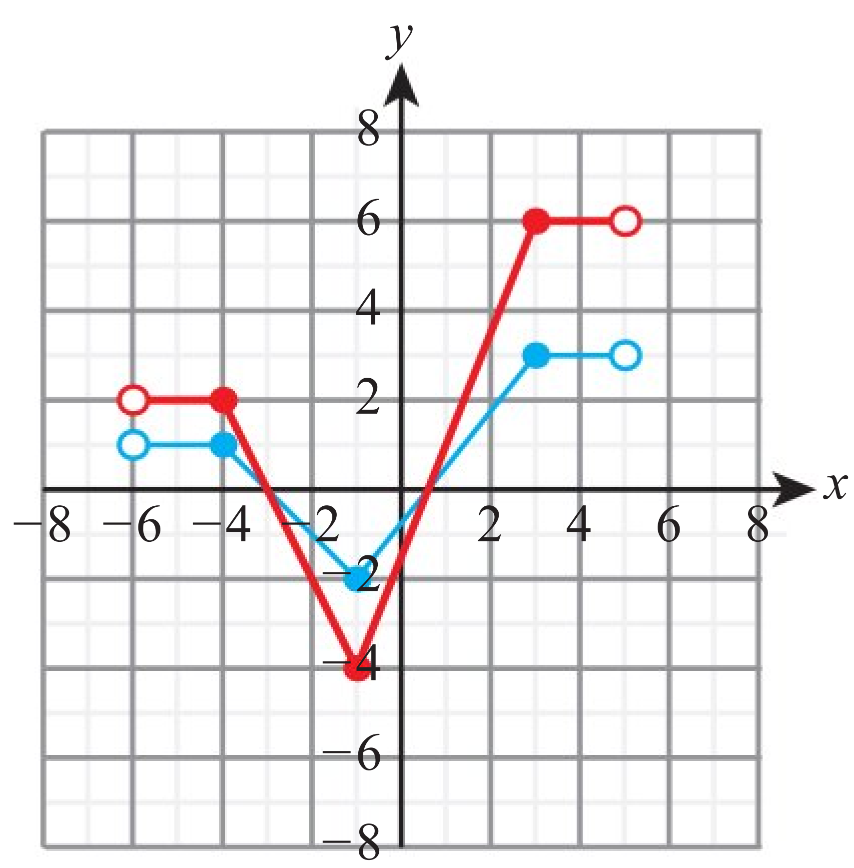
6 a $y = 8x^2 - 7x - 4$ **b** $y = 8x^2 - 7x - 1$

7 a $y = 4(x - 3)^2$ **b** $y = 3(x - 6)^3$

8 a $y = (x + 3)^2 + 6(x + 3) + 2$

b $y = (x + 2)^2 + 5(x + 2) + 4$

9 a



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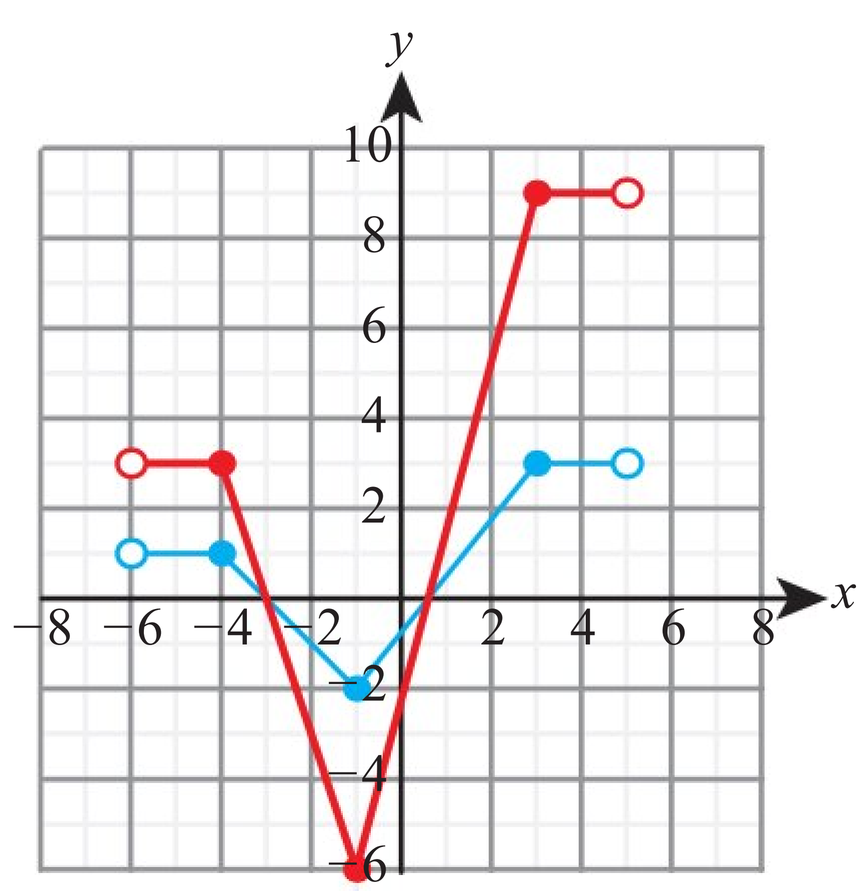
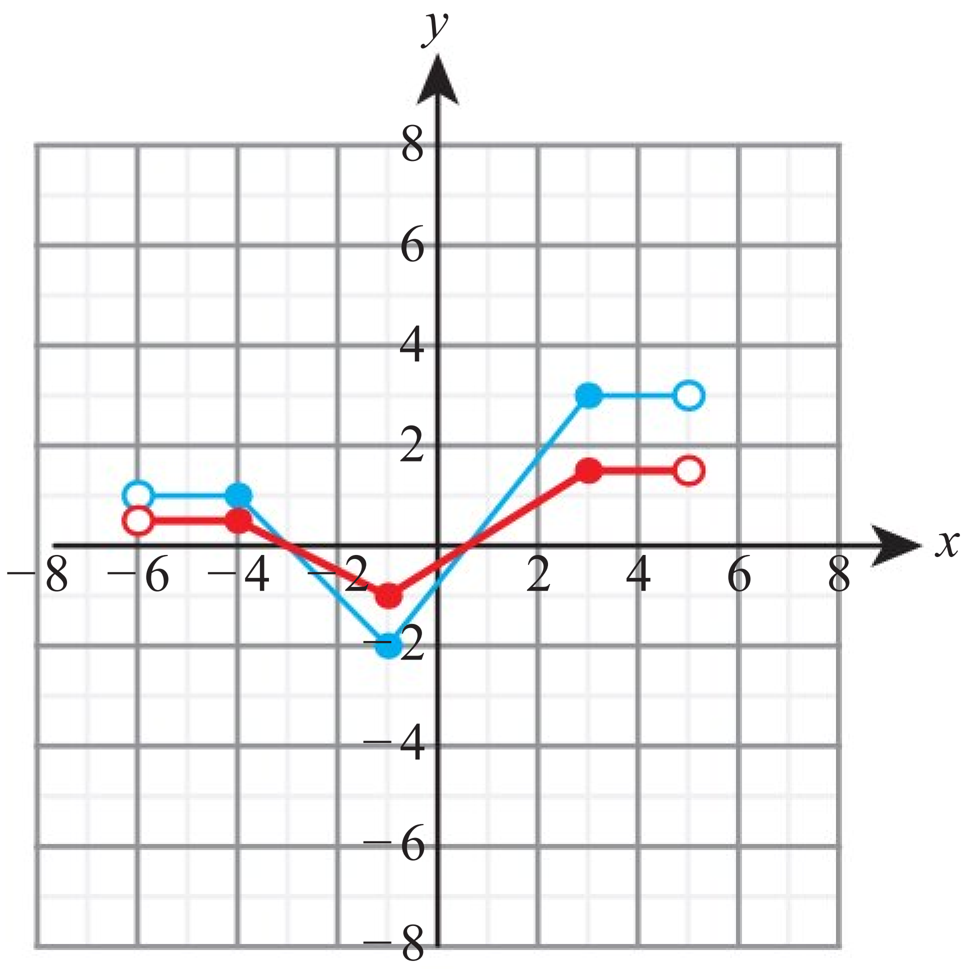


10 a



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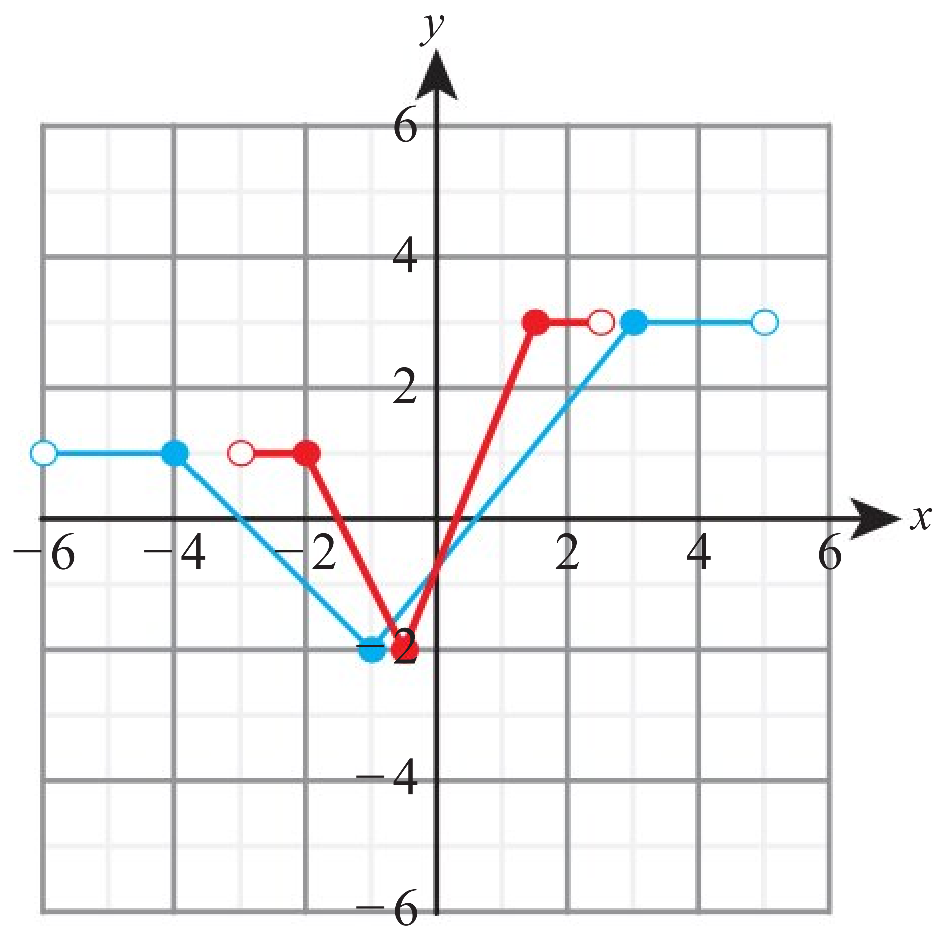

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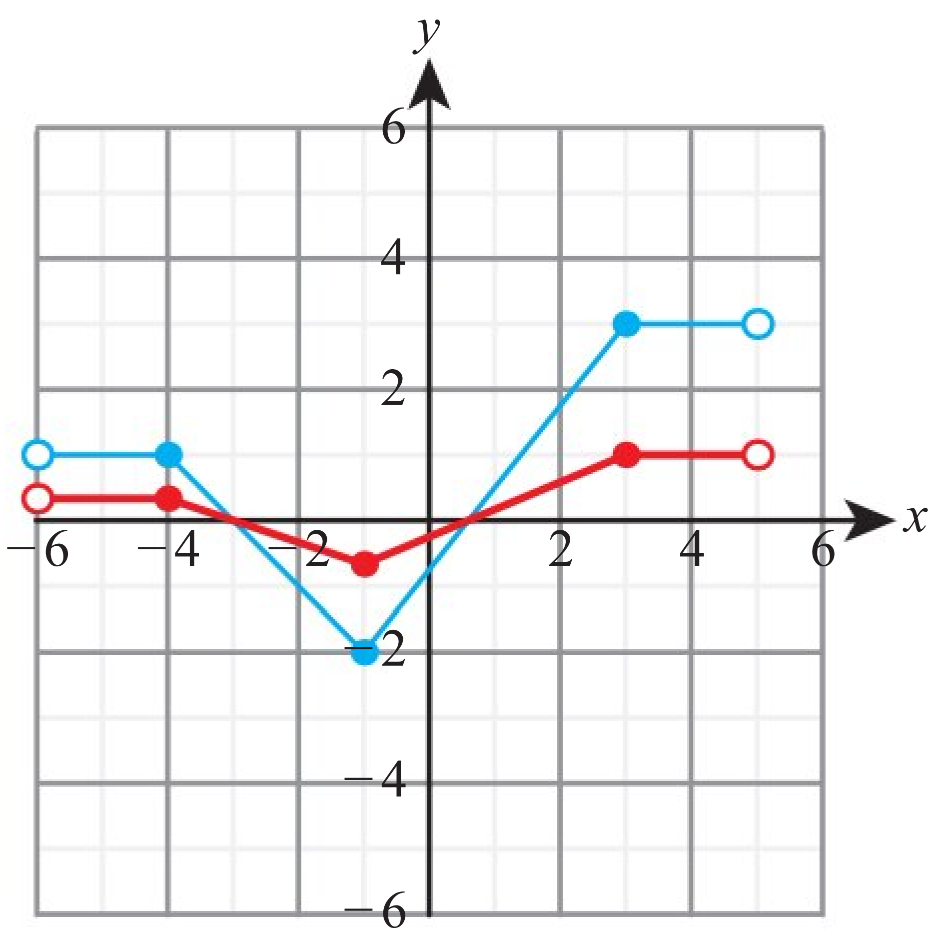
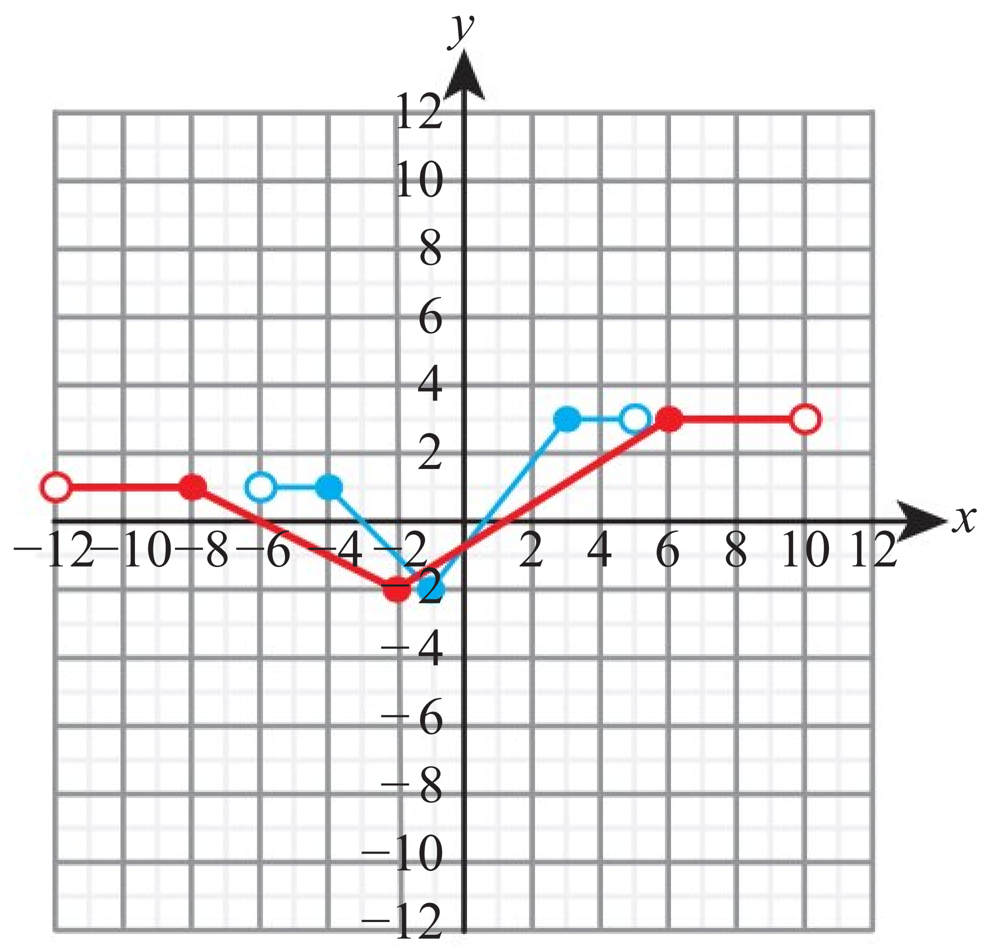
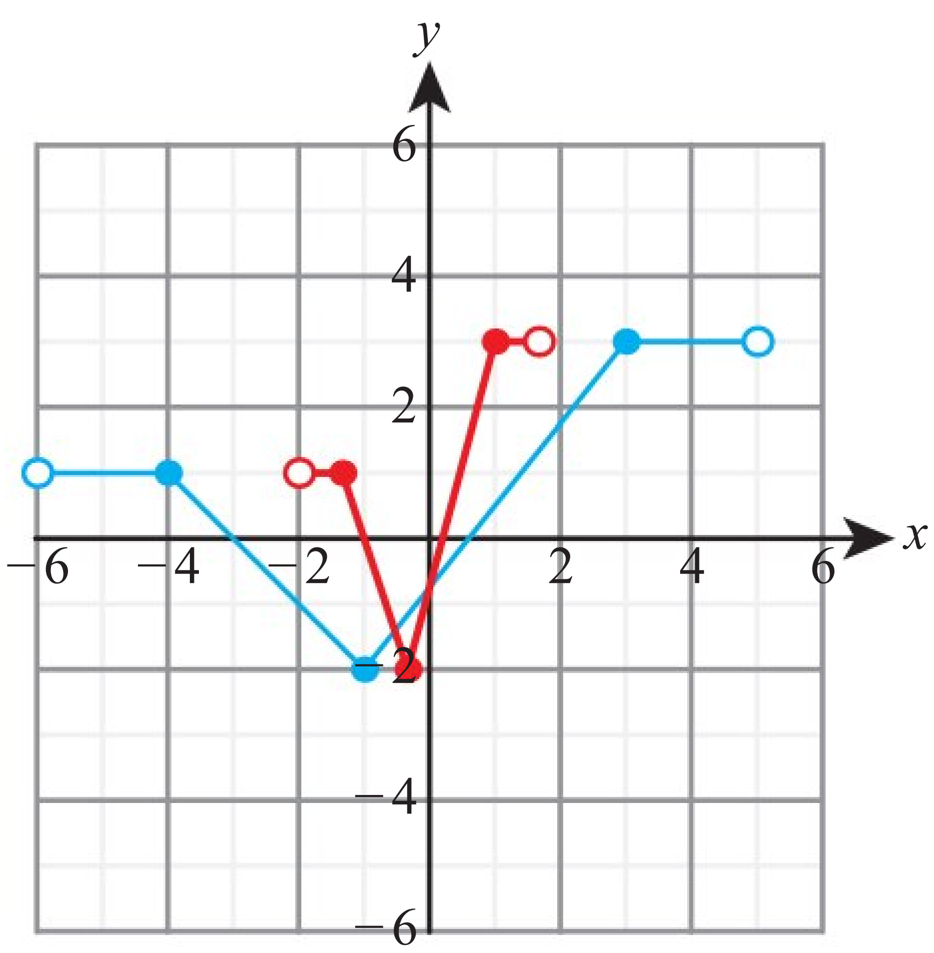


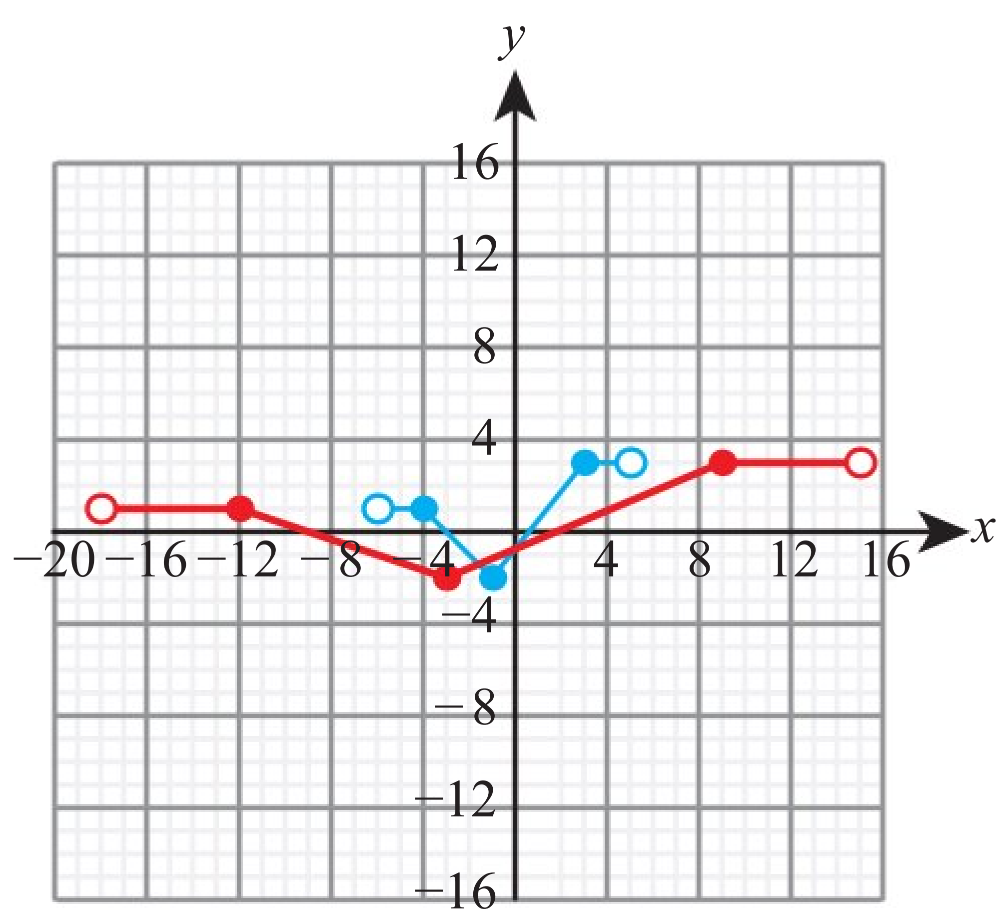
12 a



11 a



b



13 a Vertical stretch, scale factor 3

b Vertical stretch, scale factor 2

14 a Vertical stretch, scale factor $\frac{1}{2}$

b Vertical stretch, scale factor $\frac{1}{3}$

15 a Horizontal stretch, scale factor $\frac{1}{2}$

b Horizontal stretch, scale factor $\frac{1}{3}$

16 a Horizontal stretch, scale factor 2

b Horizontal stretch, scale factor 3

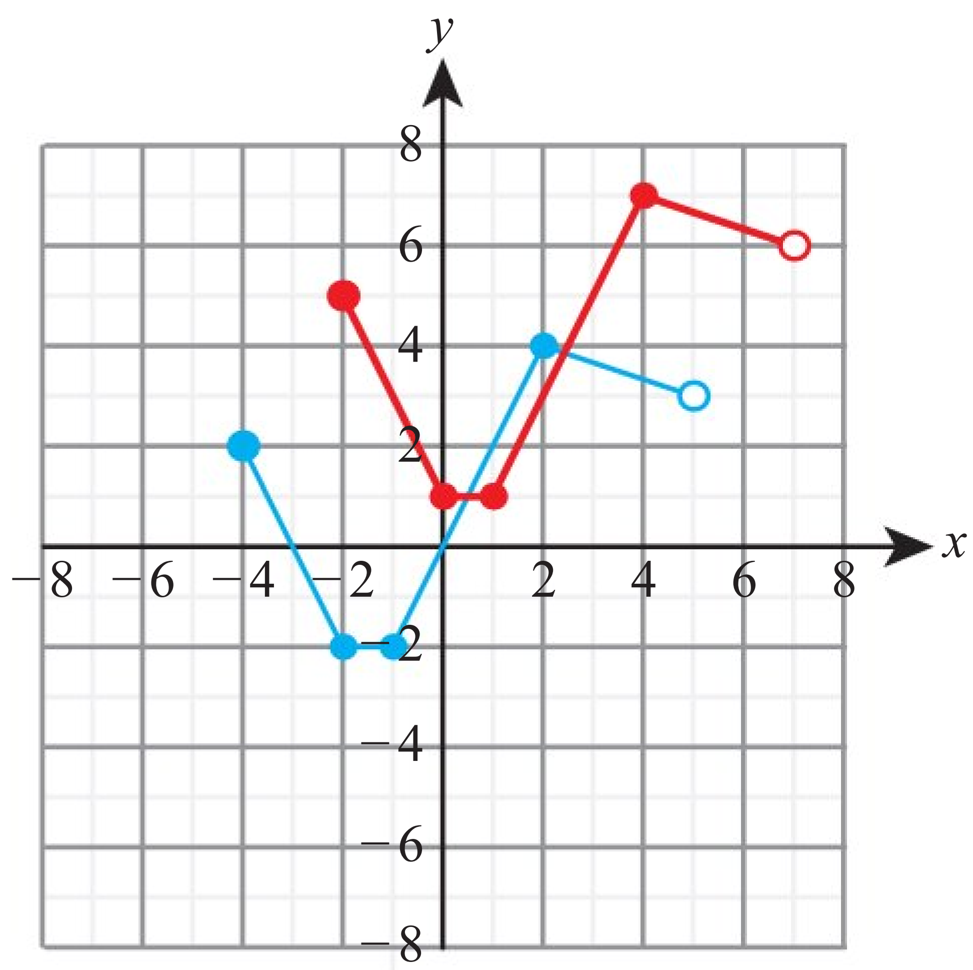
17 a $(2, -3)$ **b** $(5, -1)$

18 a $(-2, 4)$ **b** $(-3, 1)$

19 a $(-2, 3)$ **b** $(-1, 5)$

20 a $(2, -3)$ **b** $(5, -1)$

21 a



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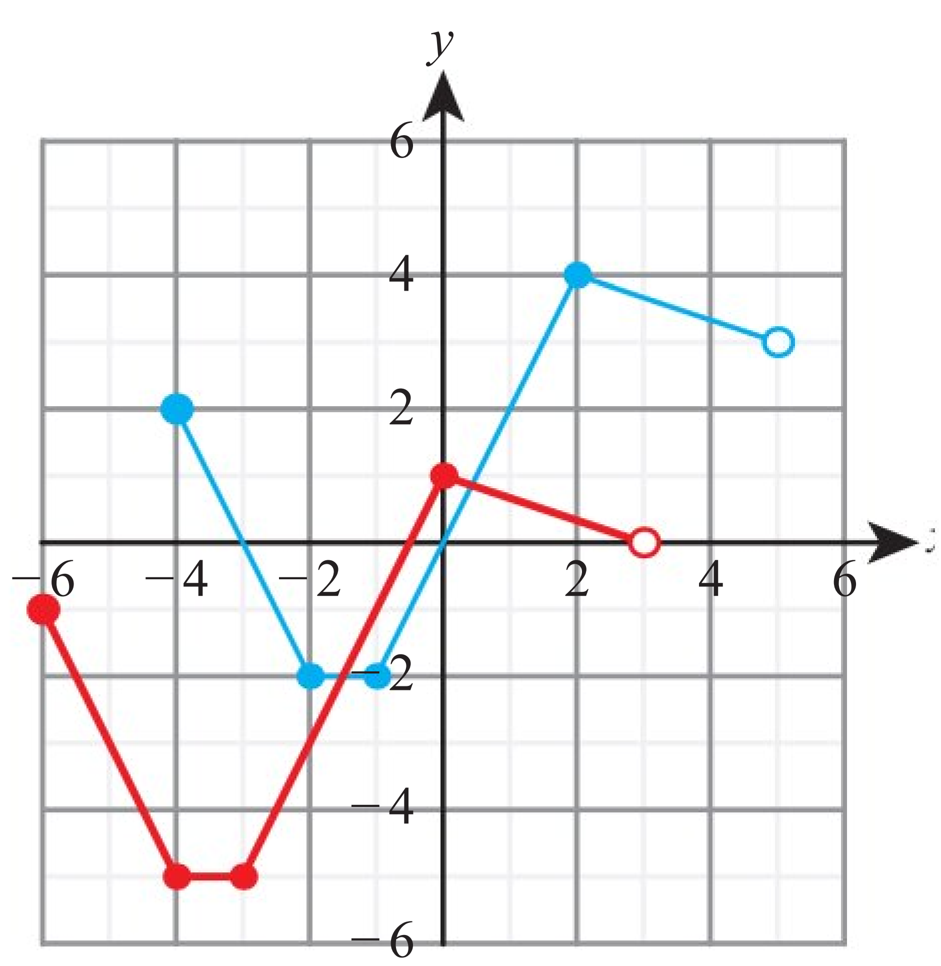
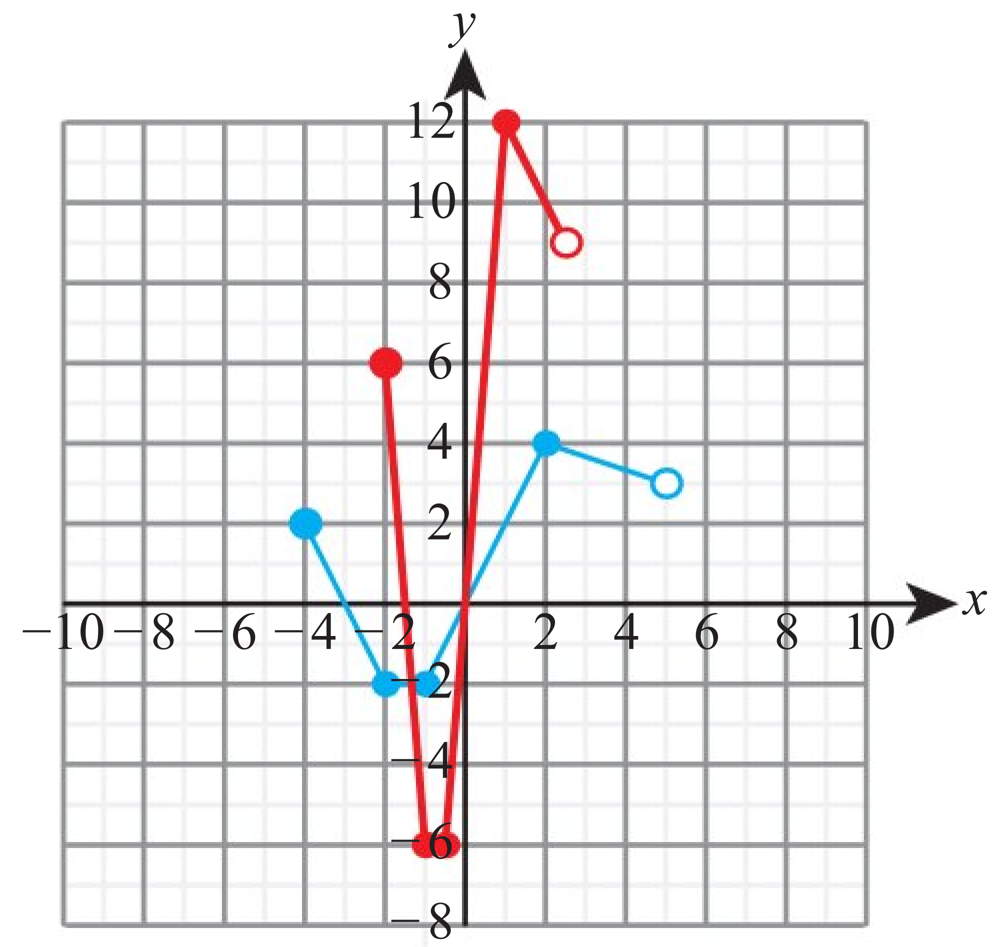


22 a



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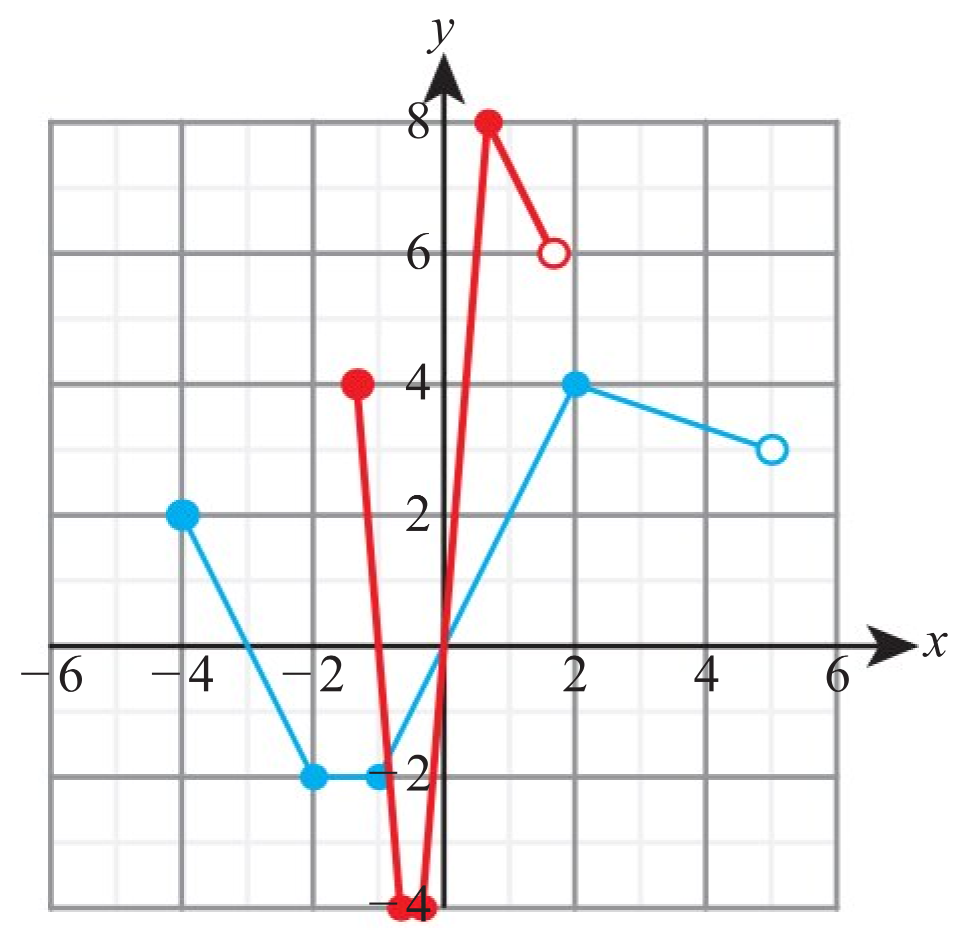
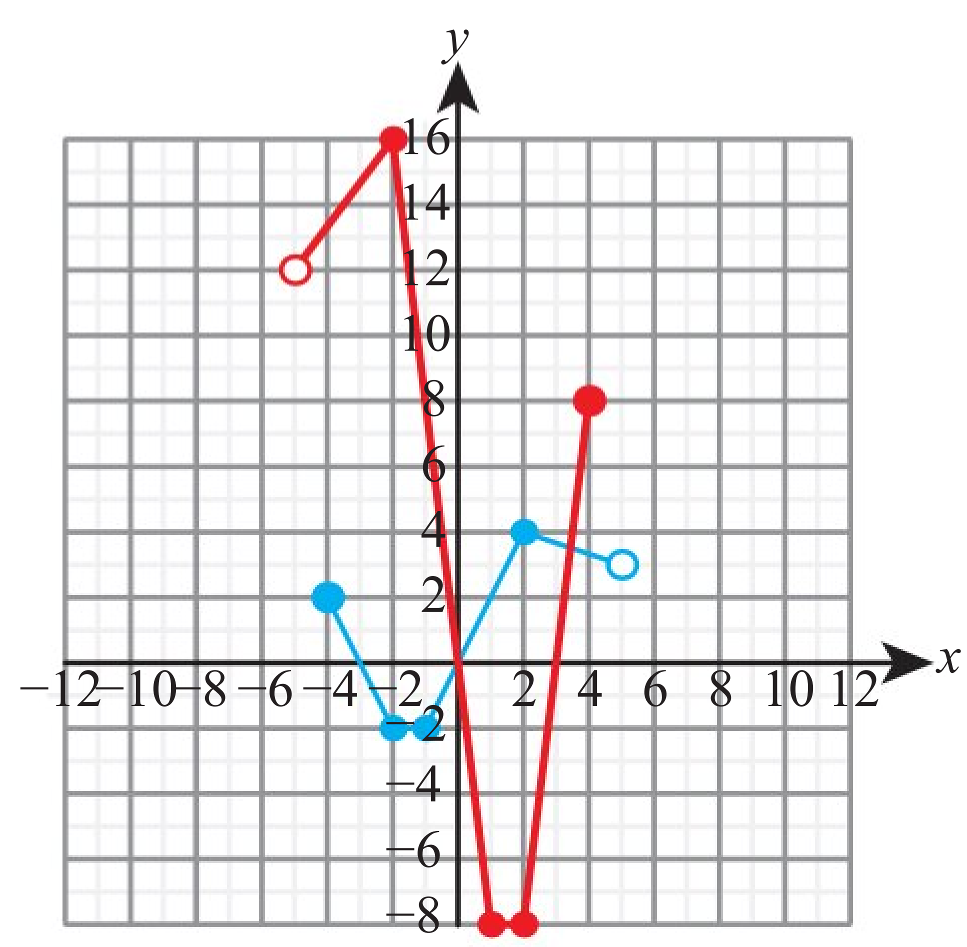


23 a



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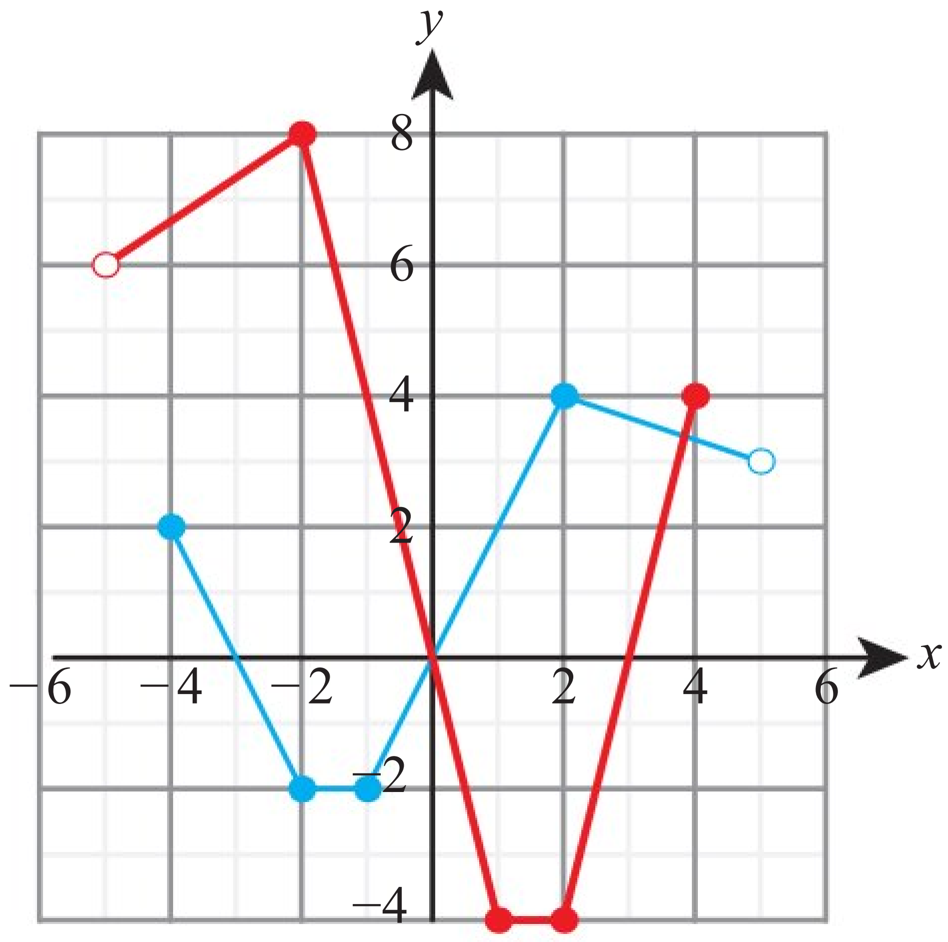
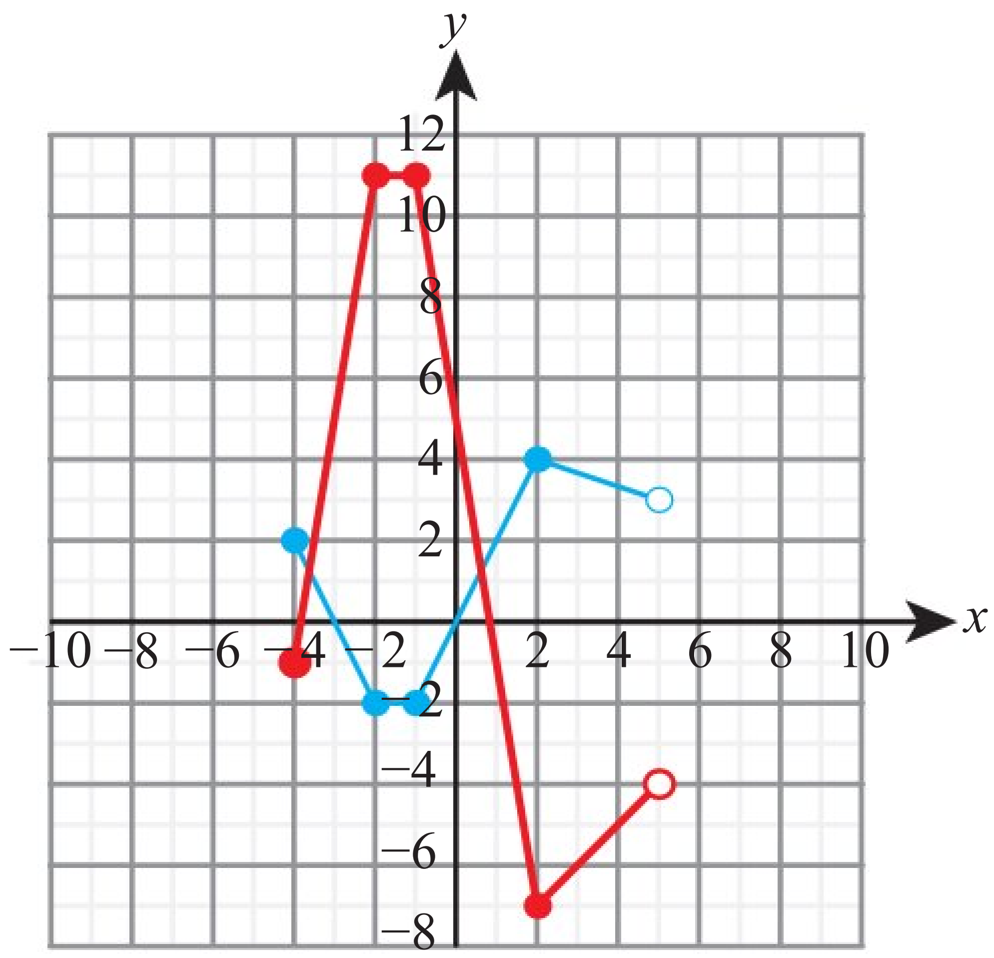


24 a



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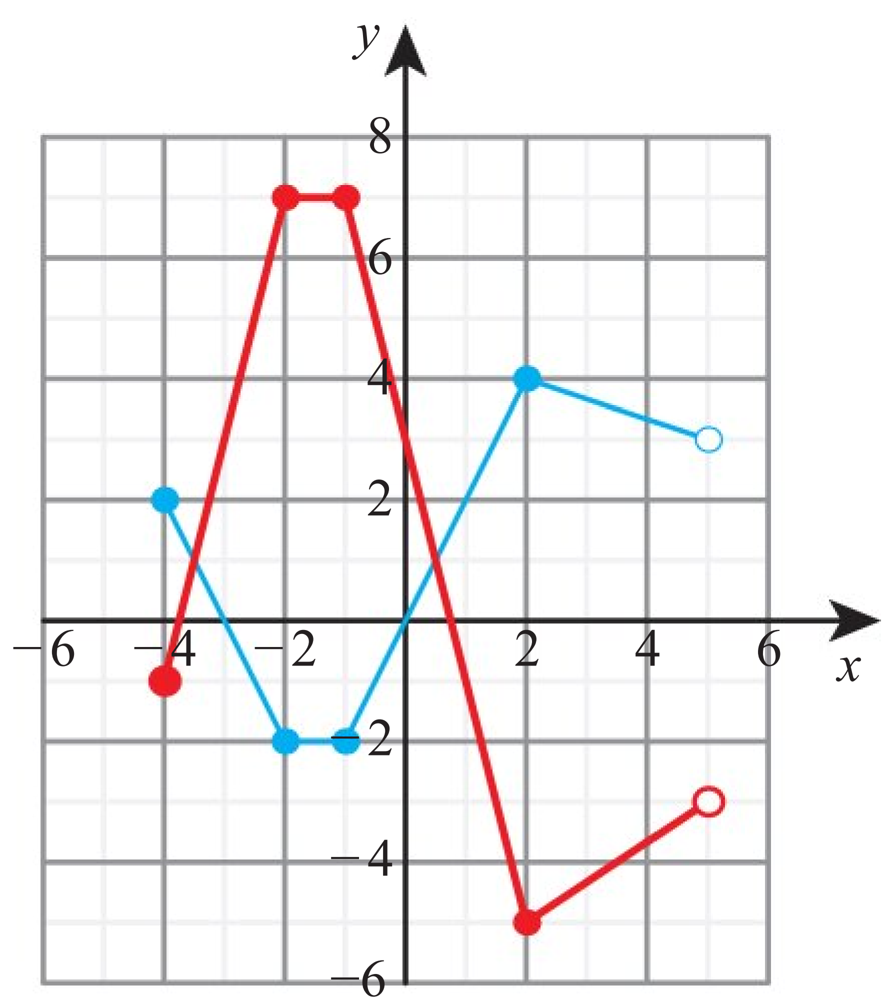
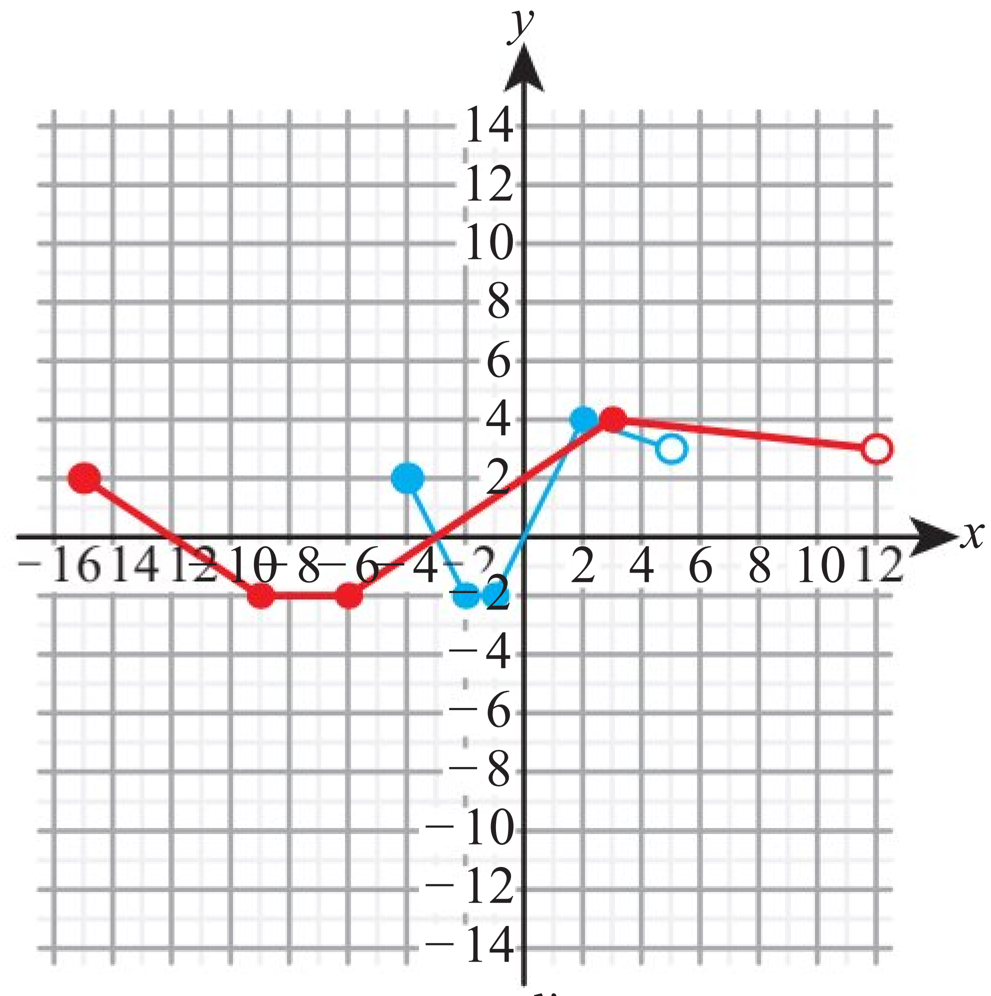


25 a



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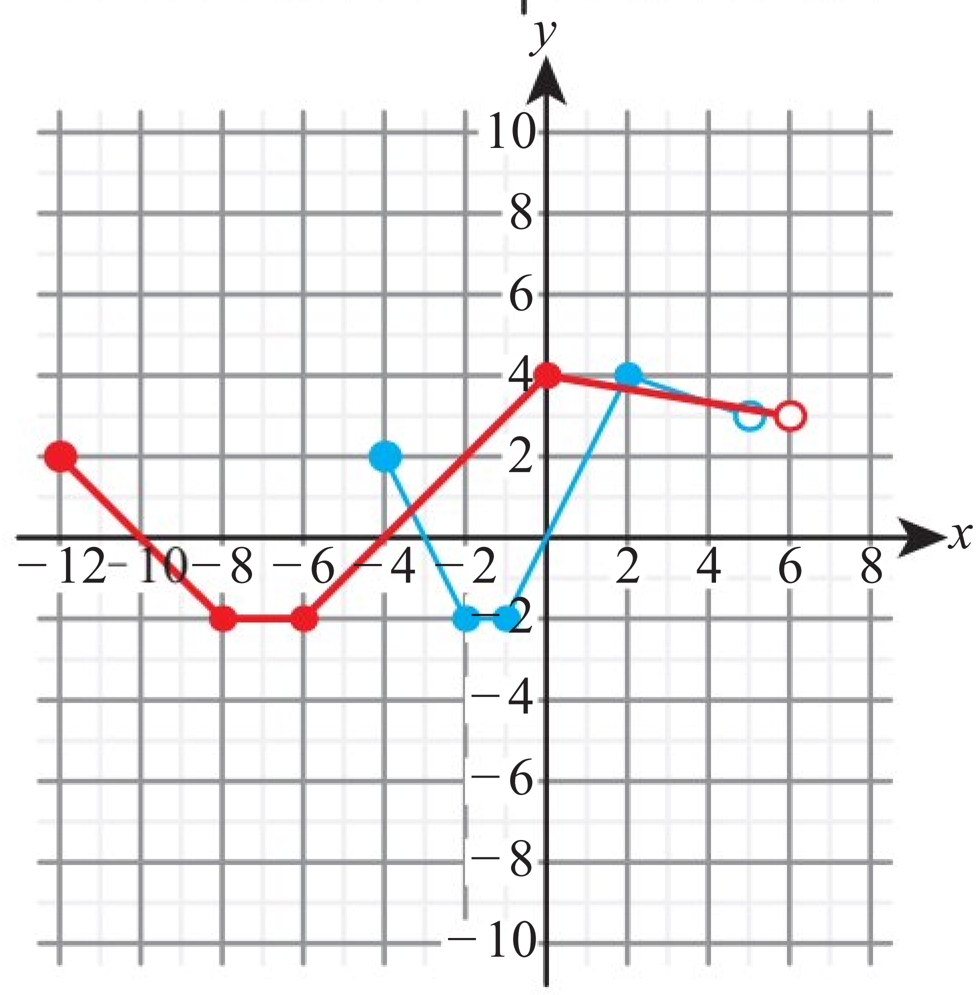
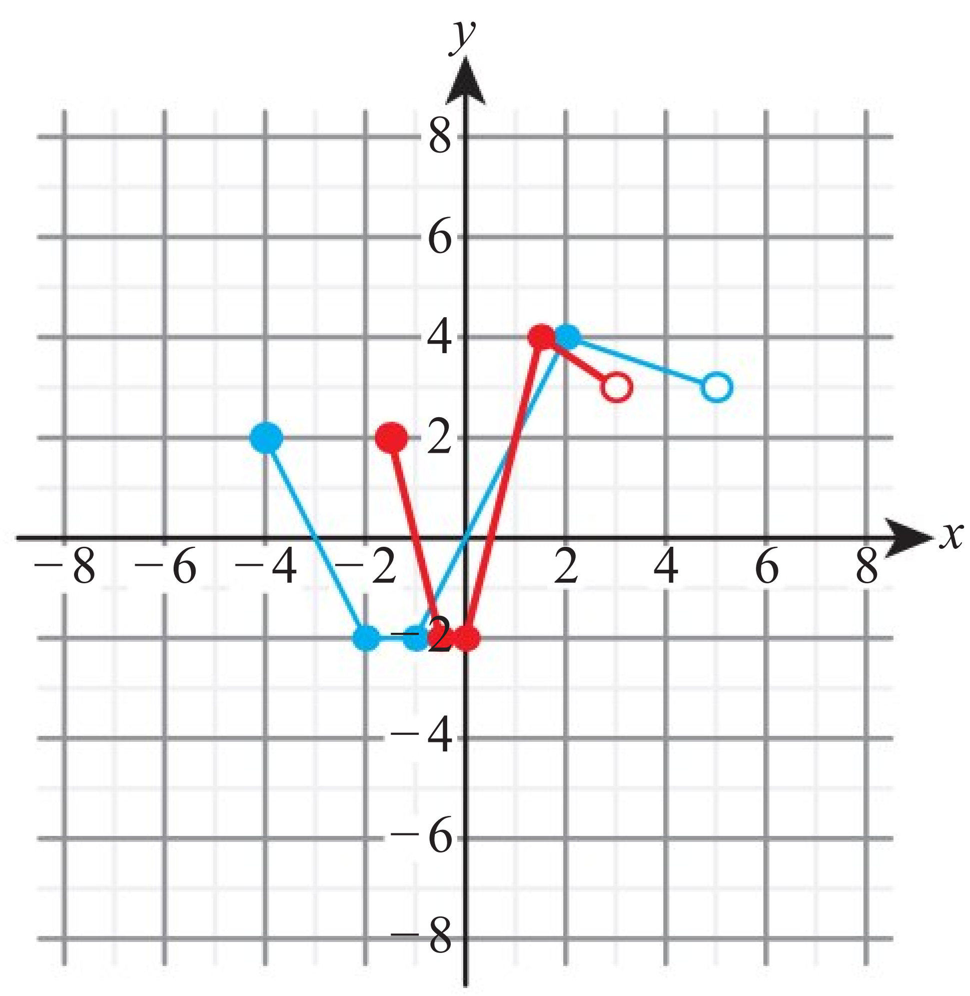
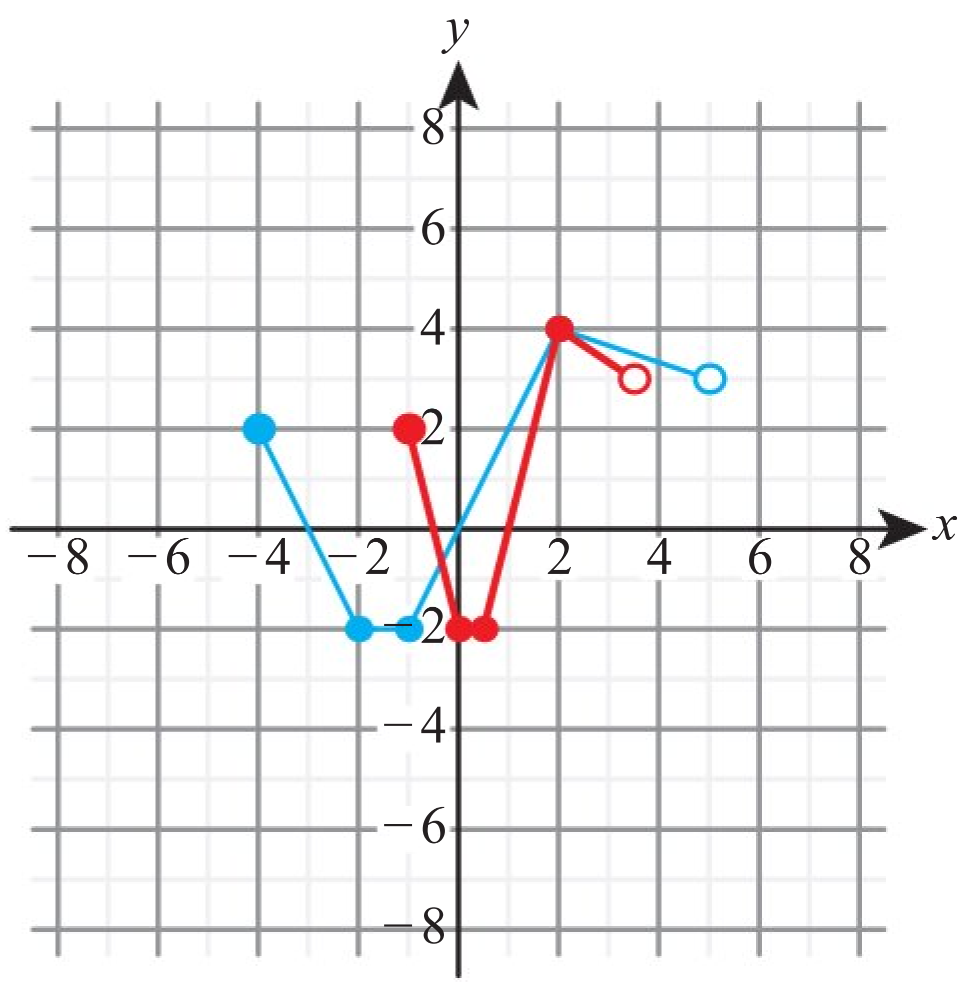


26 a



b



27 a Vertical stretch with scale factor 3, vertical translation −2 units

b Vertical stretch scale factor 5, vertical translation 1 unit

28 a Horizontal stretch scale factor 2, vertical translation −1 unit

b Horizontal stretch scale factor 2, vertical translation 3 units

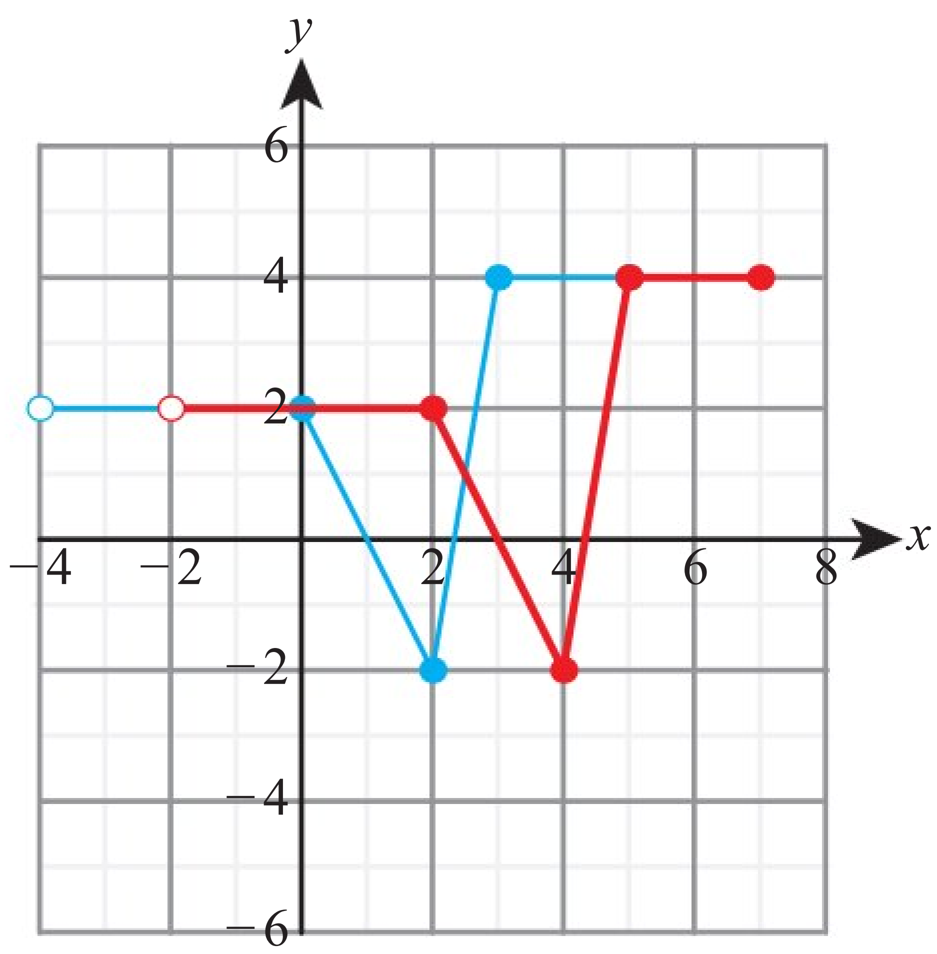
29 a Horizontal translation 2 units, vertical translation 5 units

b Horizontal translation −3 units, vertical translation −4 units

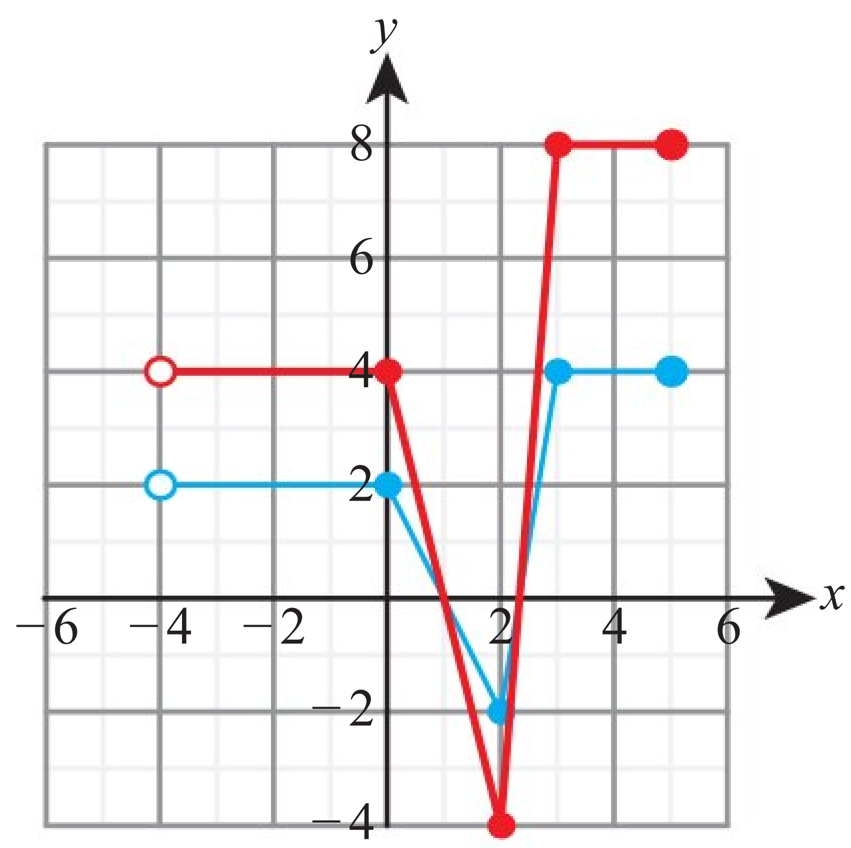
30 a Horizontal stretch scale factor $\frac{1}{3}$, reflection in y-axis

b Horizontal stretch scale factor $\frac{1}{2}$, reflection in y-axis

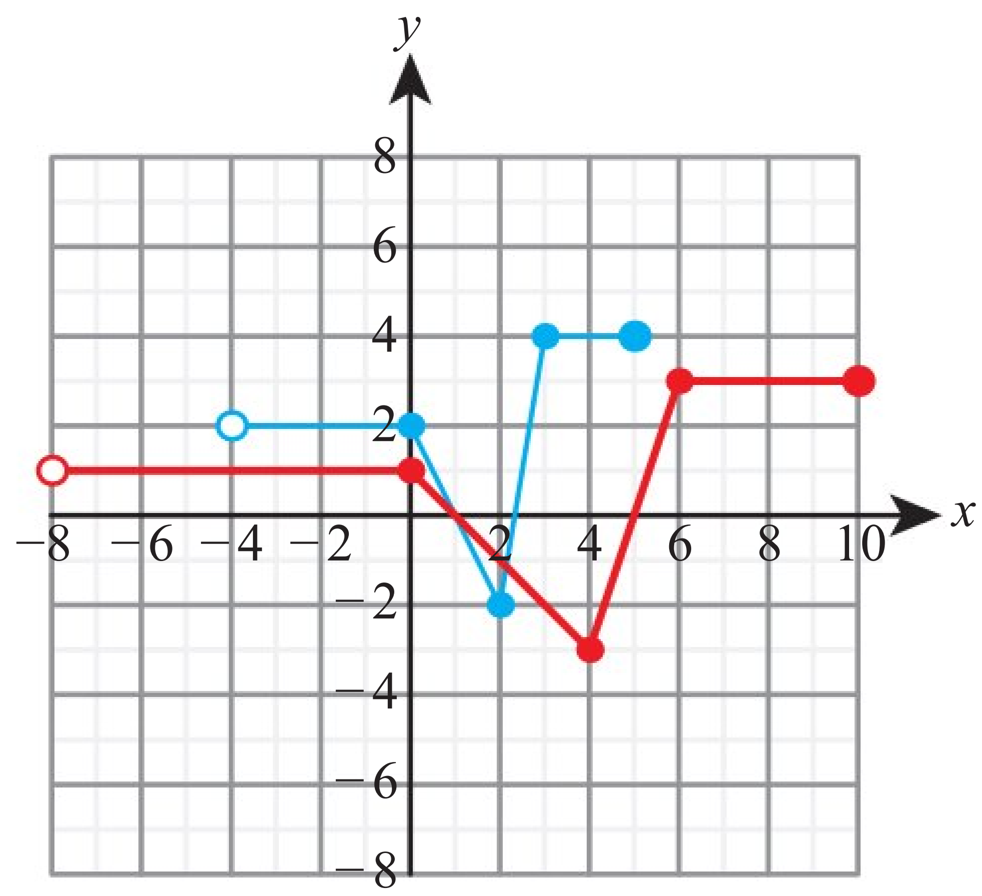
31 a



b



c



32 $y = 12x^2 + 56x + 60$

33 $y = 3e^{x-2}$

34 a $(x-5)^2 - 14$

b Horizontal translation 5 units, vertical translation −14 units

35 a $5(x+3)^2$

b Horizontal translation −3 units, vertical stretch scale factor 5

36 a $y = 9x^2$ **b** 3

37 a $y = 16x^3$ **b** $\frac{1}{2}$

38 $y = 2x^3 + 5x^2$

39 a Translation 1 right followed by a horizontal stretch factor 0.5

b A horizontal stretch factor 0.5 followed by a translation 1 right (or a translation 2 right followed by a horizontal stretch factor 0.5).

40 $a = 9$, $b = 6$, $c = -10$

41 A horizontal stretch factor 0.25 followed by a translation 2 left or a translation 8 left followed by a horizontal stretch factor 0.25.

Exercise 5D

1 a $y = 10e^{-0.347t}$ b $y = 7e^{-0.0693t}$

2 a $y = 10 + 2e^{-0.139t}$ b $y = 15 + 5e^{-0.0990t}$

3 a $y = 1.5 - 0.5e^{-6.93t}$ b $y = 120 - 20e^{-13.9t}$

4 a $y = 3 + 2\ln x$ b $y = 10 + 5\ln x$

5 a $y = -0.419 + 4.93\ln x$ b $y = -2.21 + 4.48\ln x$

6 a $y = 11.5 - 4.98\ln x$ b $y = 6 - 2.89\ln x$

7 a $y = 3 + 5\sin(1.57(x-1))$

b $y = 10\sin(2.09(x-2))$

8 a $y = 2 + 6\sin(0.628(x+2))$

b $y = -4 + 10\sin(1.05(x+1))$

9 a $y = 1.5 + 1.5\sin(0.785(x-4))$

b $y = -1 + 3\sin(3.14(x-0.5))$

10 a $y = \dfrac{500}{1+4e^{-0.980x}}$ b $y = \dfrac{800}{1+19e^{-3.46x}}$

11 a $y = \dfrac{2000}{1+3e^{-0.275x}}$ b $y = \dfrac{70}{1+1.8e^{-0.292x}}$

12 a $y = \dfrac{200}{1+4e^{-1.24x}}$ b $y = \dfrac{200}{1+e^{-0.549x}}$

13 a -2 b 4

14 a 1.5 b 0.5

15 a 6 b -2

16 a 3 b 5

17 a $\frac{1}{2}\ln\frac{1}{2}$ b $\ln 4$

18 a e b $\frac{1}{2}e^4$

19 a 3 b 10

20 a 0.0462 b 22.9 minutes

c e.g. The temperature of the ice remains 0°C.

21 a $p = 0.396,\ q = 2.73$ b 4.55

c $0.16 \leqslant A \leqslant 313$

22 a 25 b 5

c 30 seconds

23 $P = P_0 e^{\frac{\ln 2}{3}t}$

24 a 5 years b 43.2 years

25 a $A = 70,\ B = 20$ b 0.140

c 6.06 minutes

26 a 200, 178 b 178

c 0.173

d 12.6 minutes

27 a 5 years b 0.139

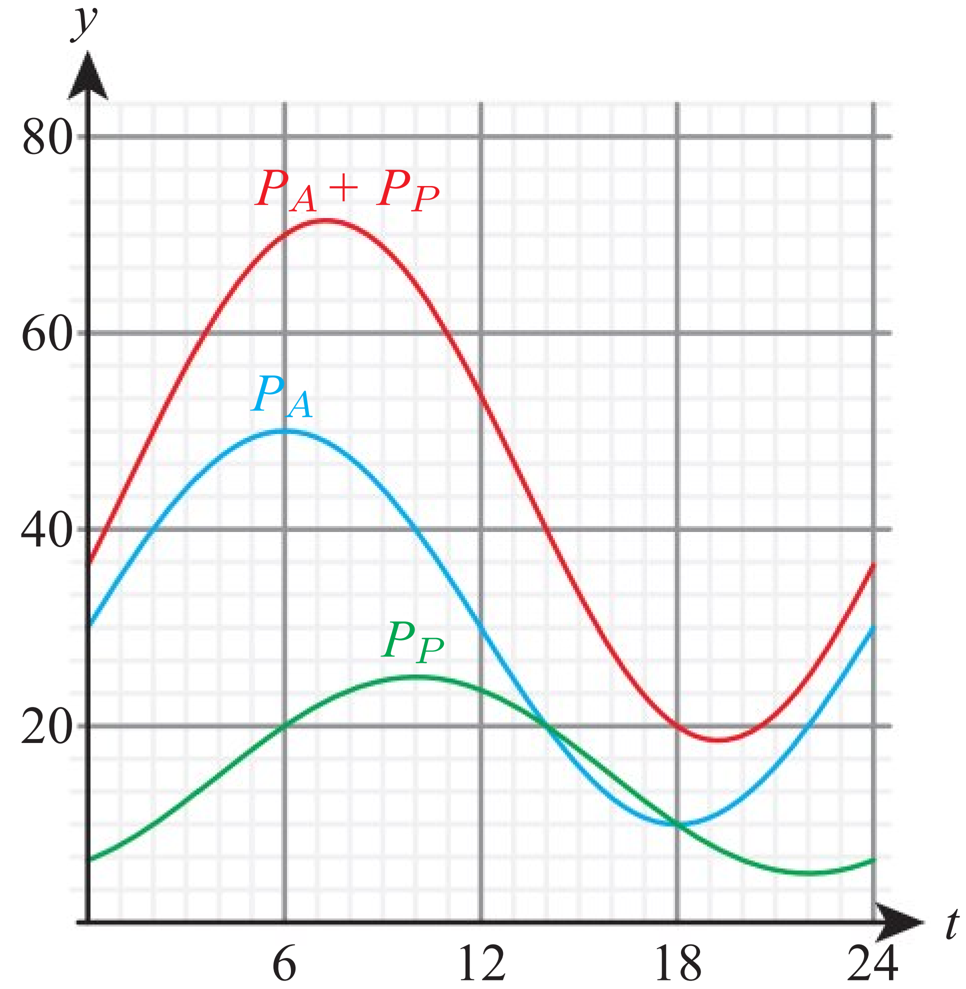
c 9.6 g

28 a $P_A = 20\sin\left(\frac{\pi}{12}t\right) + 30$

b $P_P = 10\sin\left(\frac{\pi}{12}(t-4)\right) + 15$

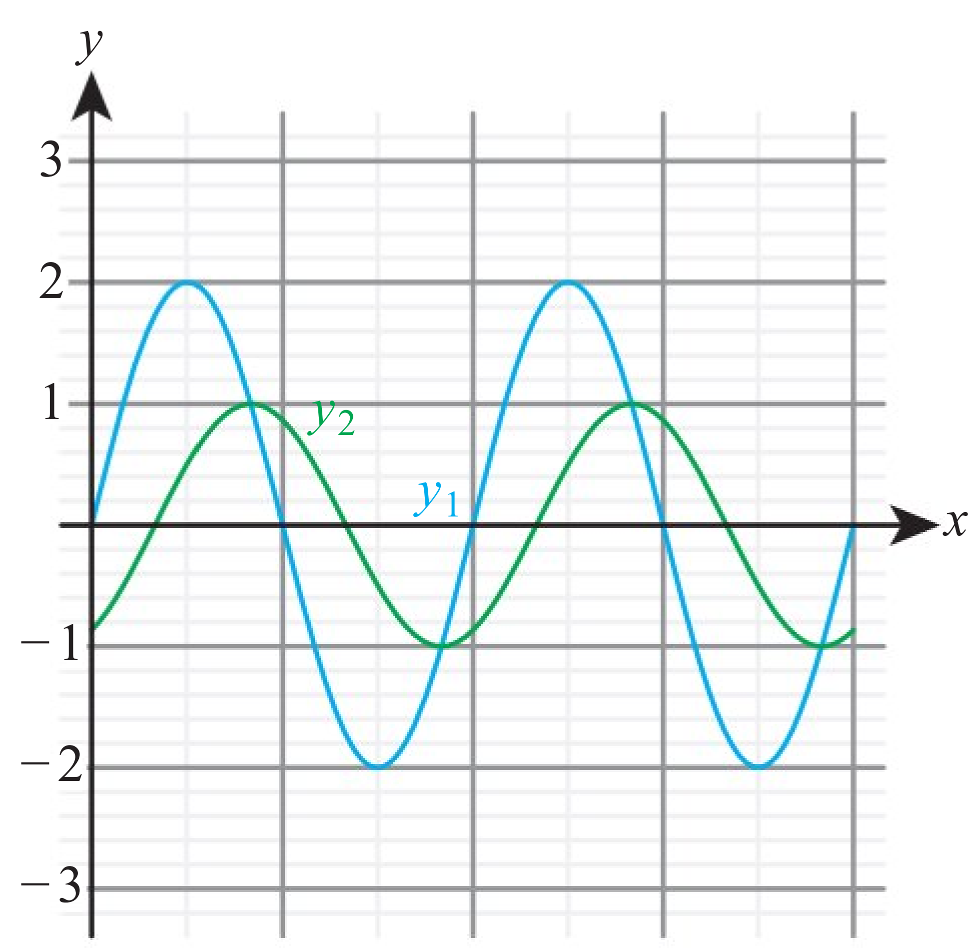
c $t = 14$

d

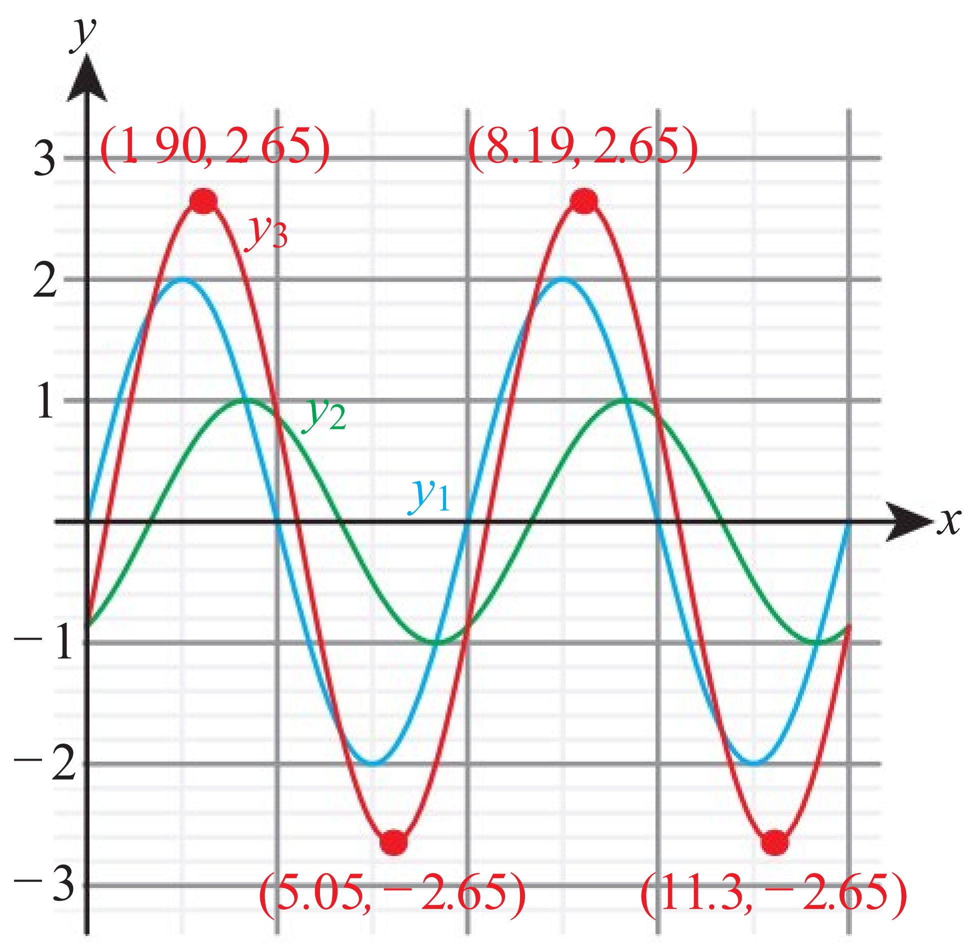


71.5 MW

29 a



b



c $y_3 = 2.65\sin(x - 0.333)$

30 b $C = \frac{1}{4}e^{5k}$

c $k = 0.327$, $C = 1.28$

d 219

e 11

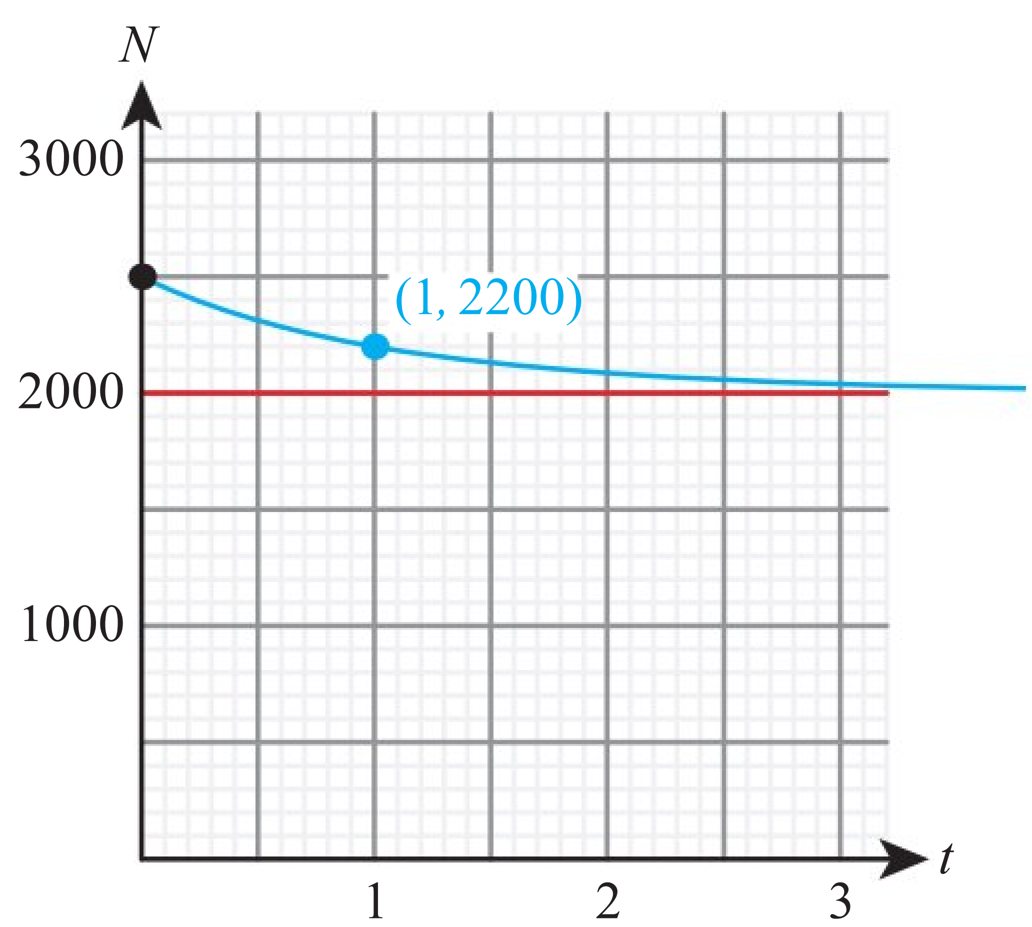
31 a 300; initial population is 0, no rabbit death or breeding

b 610

32 a $L = 2000$, $C = -0.2$

b $k \approx 0.8$

c



33 a $N(t) = \begin{cases} N_0e^{\alpha t} & \text{for } 0 \leqslant t \leqslant 0.5 \\ N_0e^{0.5\alpha}e^{-\beta(t-0.5)} & \text{for } 0.5 < t \leqslant 1 \end{cases}$

b $\alpha > \beta$

34 a $N = 200e^{-\left(\frac{\ln 2}{2}\right)t}$

b $N < 1$

c 15.3 years

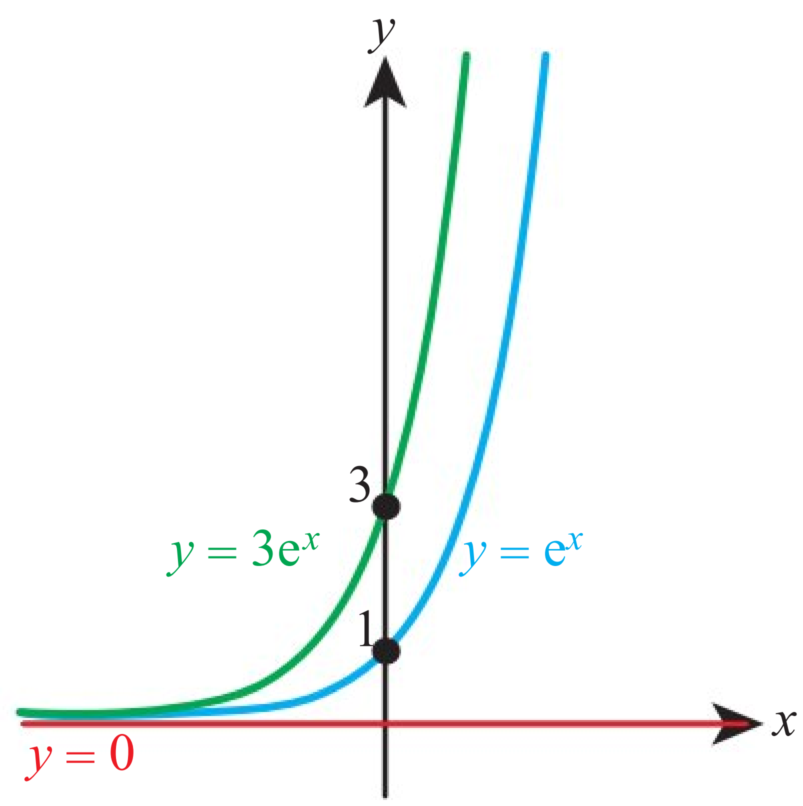
d e.g. The number of tigers is modelled as continuous, which is only a valid approximation when the number is large; for example, the ability to reproduce may be severely impacted when $N < 2$.

35 a $20 + 80e^{-\frac{3\ln 2}{q}t}$

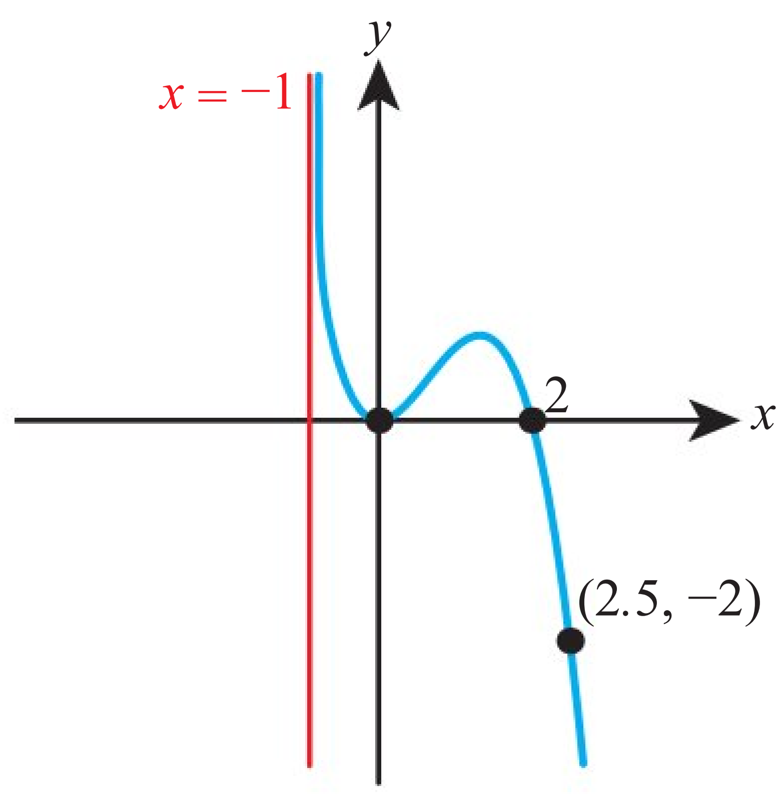
b $-\dfrac{q\ln\frac{3}{8}}{3\ln 2}$

Chapter 5 Mixed Practice

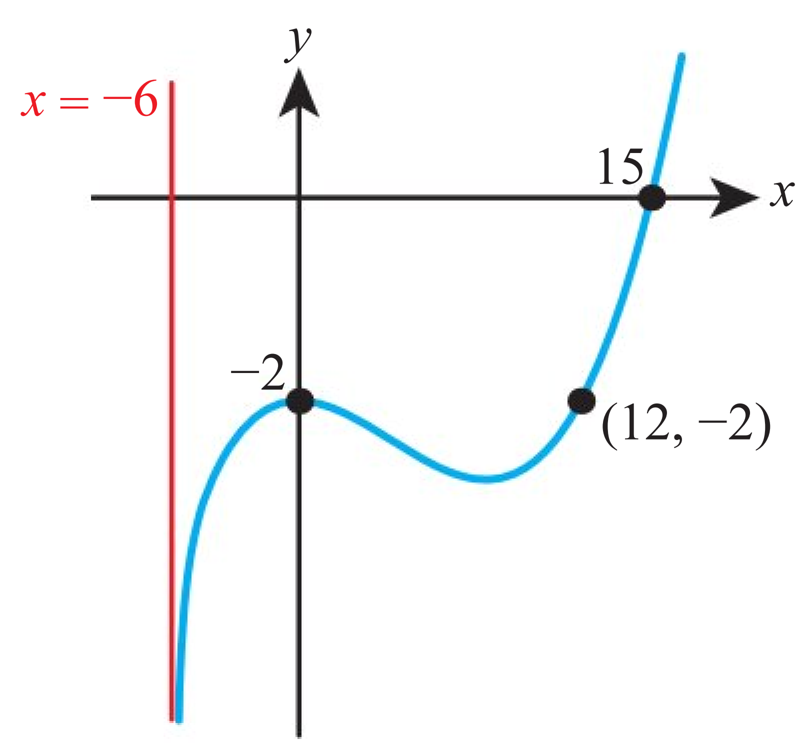
1 a



b



c



2 a $f^{-1}(x) = \dfrac{x+1}{3}$

3 a 5 b -4

4 0.462

5 a 3 b 1

6 -3

7 a $x > 2$ b $2 + e^{\frac{x}{3}}$, $g^{-1}(x) > 2$

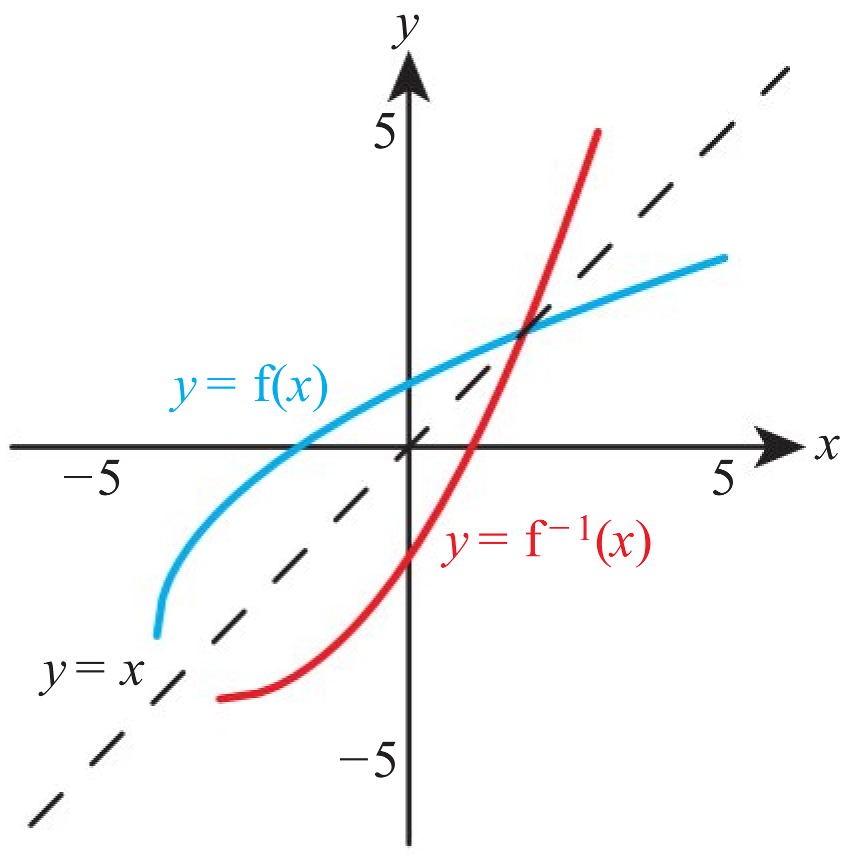
8 $\sqrt[3]{\dfrac{x-1}{3}}$

9 a $2x^3 + 3$ b -1.14

10 a i -1 ii 0

b $-3 \leqslant x \leqslant 3$

c



11 a $\dfrac{1}{x-1} - 2$, $x \neq 1$

12 $\dfrac{4x+1}{3-x}$

13 b 9.39

14 a 3 b $-\dfrac{1}{4}$

15 a $e^x + 2$, $y > 2$ b $x - 2$

16 a -0.5 b $\dfrac{-1-\sqrt{x}}{2}$

17 a 28

b $9 - x^2$

c i Reflection in the line $y = x$

ii $\sqrt{x-3}$ iii $y \geqslant 1$

18 a i 22 ii $\dfrac{4x+11}{x-1}$ iii $9x + 4$

b $f(x)$ can be 1, which is not in the domain of g.

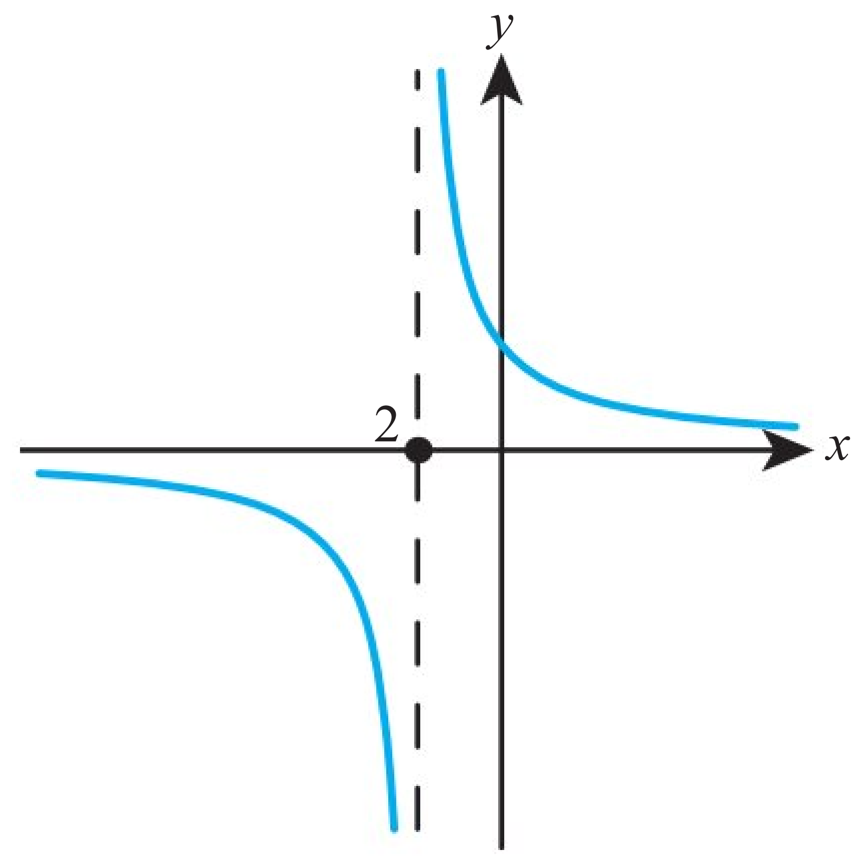
c ii $y \neq 1$

19 a 9 b 5

21 a $\dfrac{x+2}{3}$

c i 2.5

ii



d i 2.5 ii 0

e 1

22 $\dfrac{29}{4}$

23 a 0.577

b Reflection in the line $y = x$

c $\sqrt[3]{2}$

24 a i $\dfrac{1}{2x+3}$, $x \neq -\dfrac{3}{2}$ ii $\dfrac{2}{x} + 3$, $x \neq 0$

b $(-1, 1)$

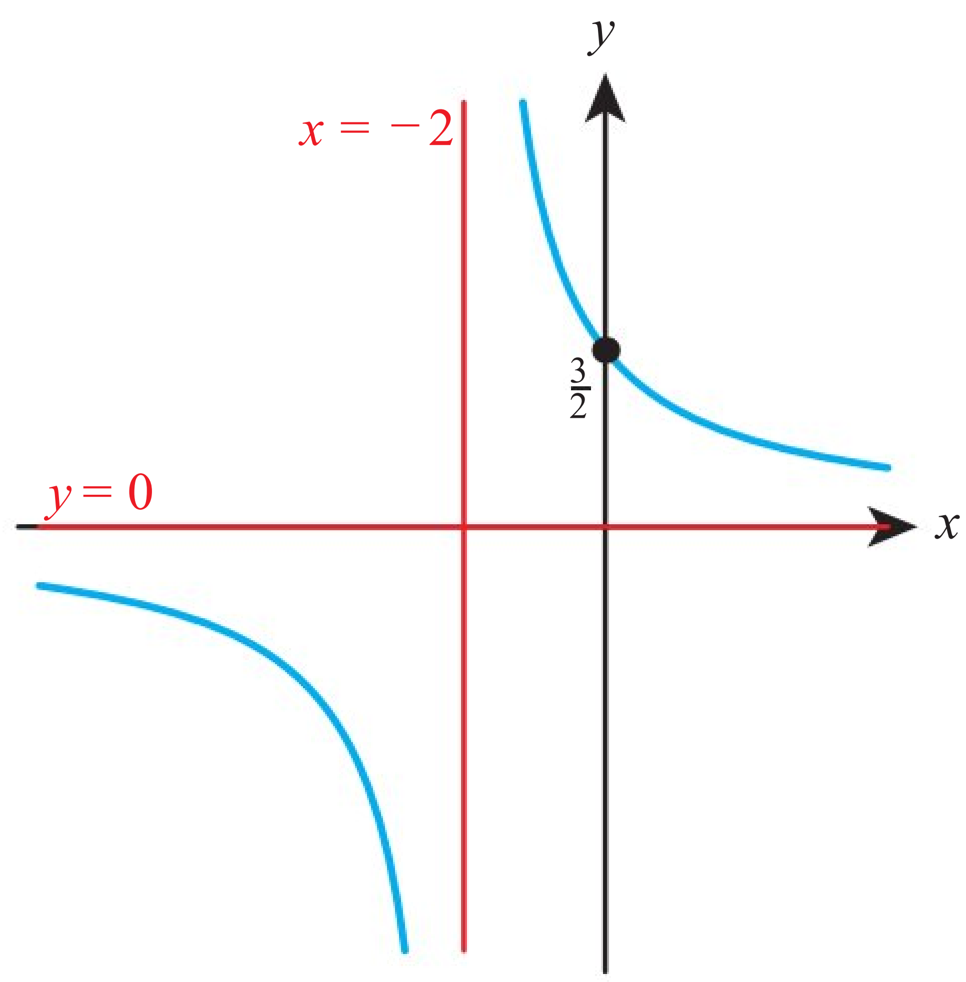
25 $2x^3 - 18x^2 + 50x - 42$

26 a $(x+2)^2 + 5$

b Horizontal translation by -2 units, vertical translation by 5 units

27 a $y = \dfrac{3}{x+2}$

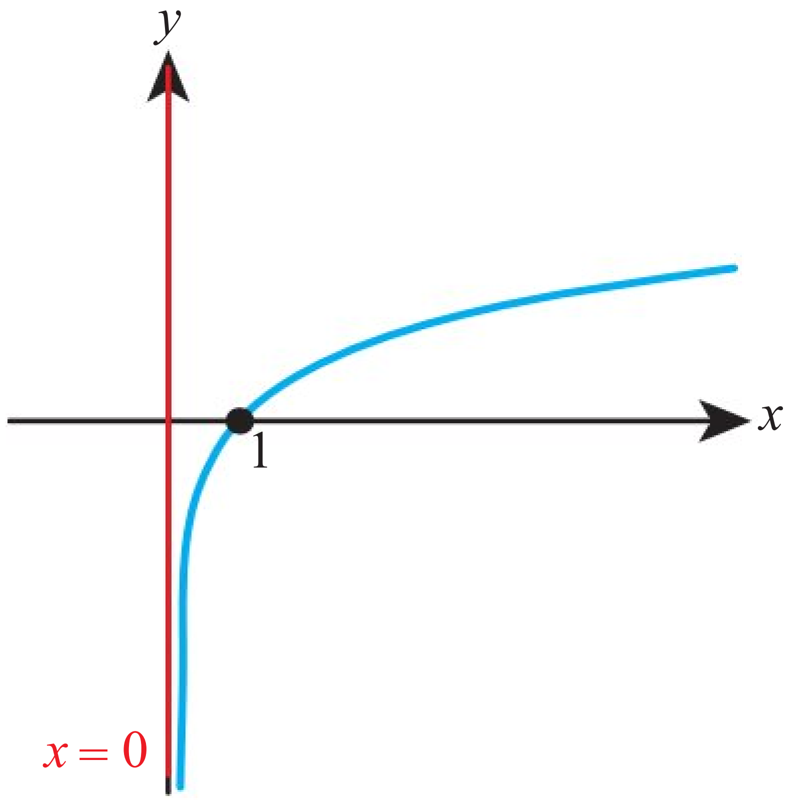
b



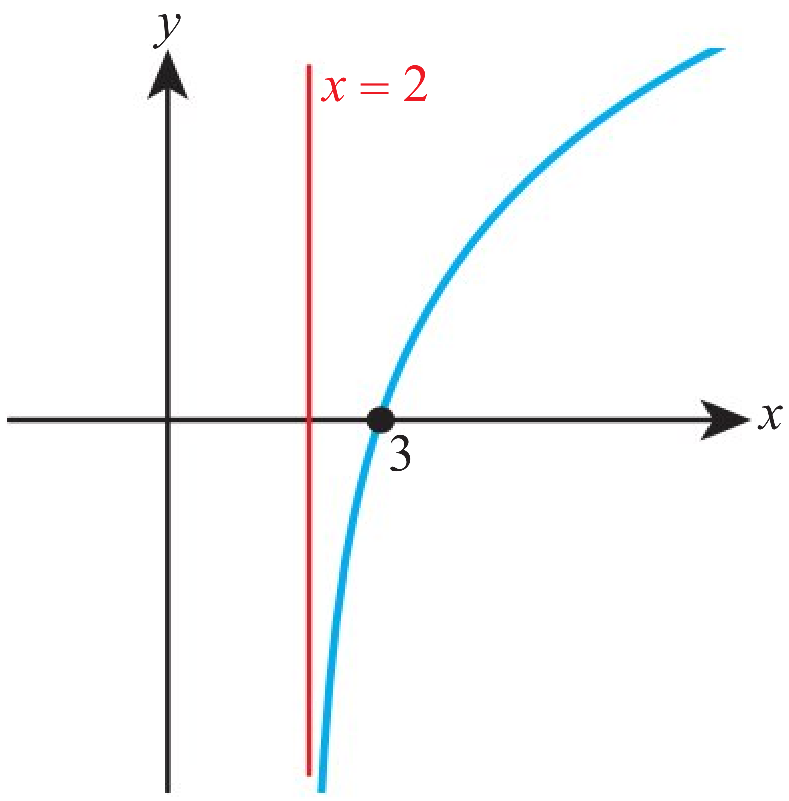
28 b Translation 5 units to the right and 2 units up

c $x = 5,\ y = 2$

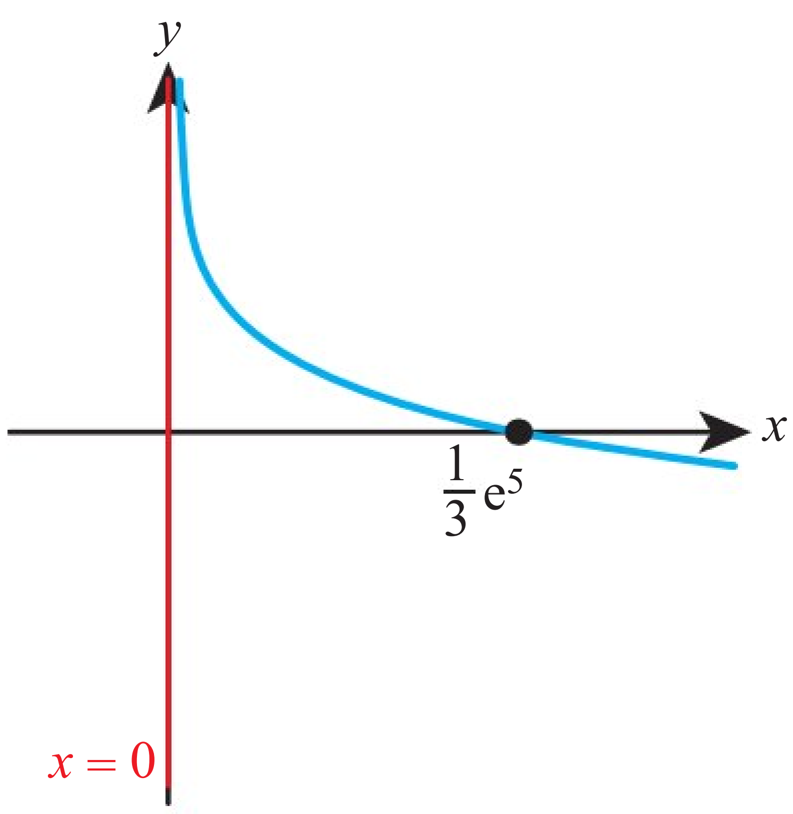
29 a



b



c



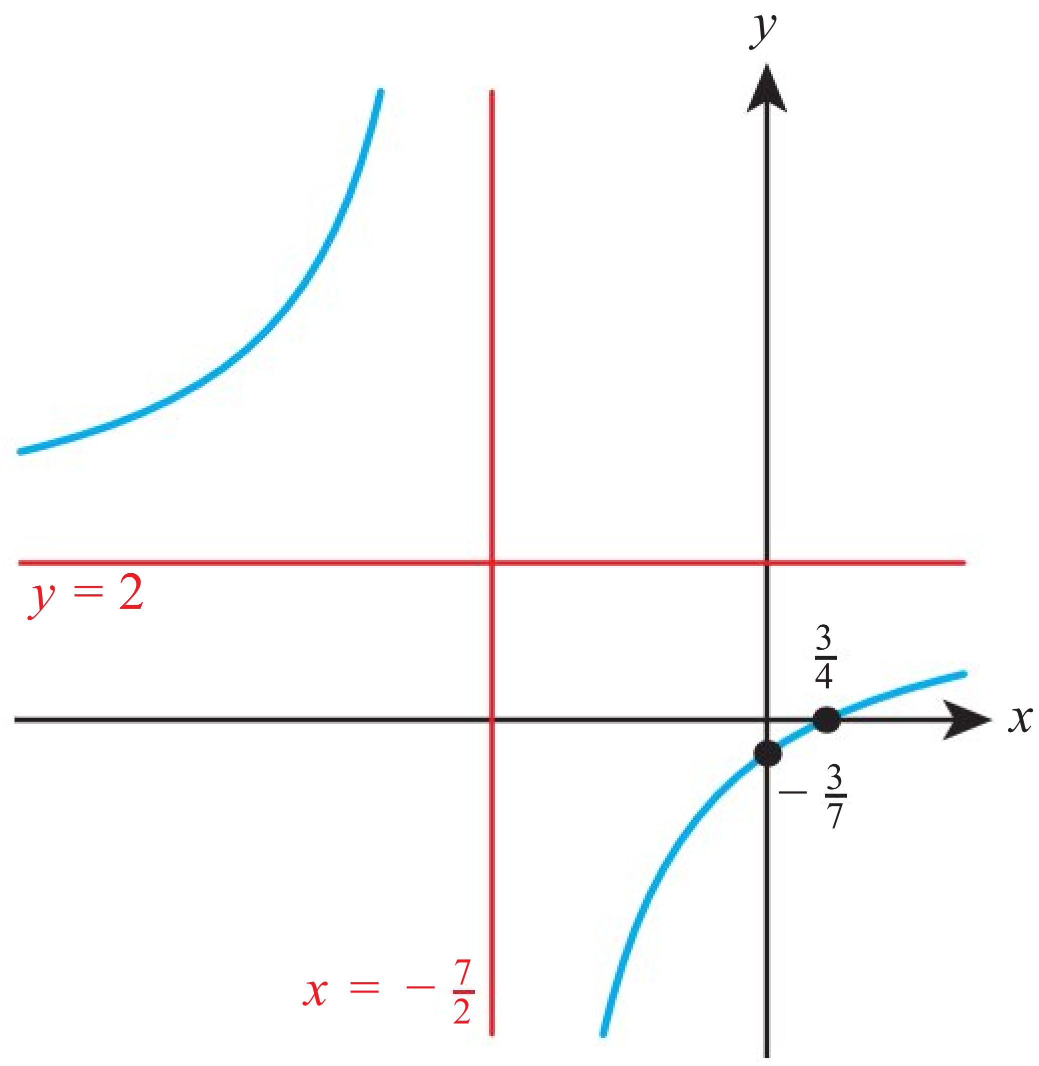
30 $a = 3,\ b = 4$

31 Vertical stretch scale factor $\frac{1}{3}$, translation 5 units to the left

32 $y = \ln\left(e^6 (x+2)^2\right)$

33 a $x = -\frac{7}{2},\ y = 2$ b $\left(\frac{3}{4}, 0\right), \left(0, -\frac{3}{7}\right)$

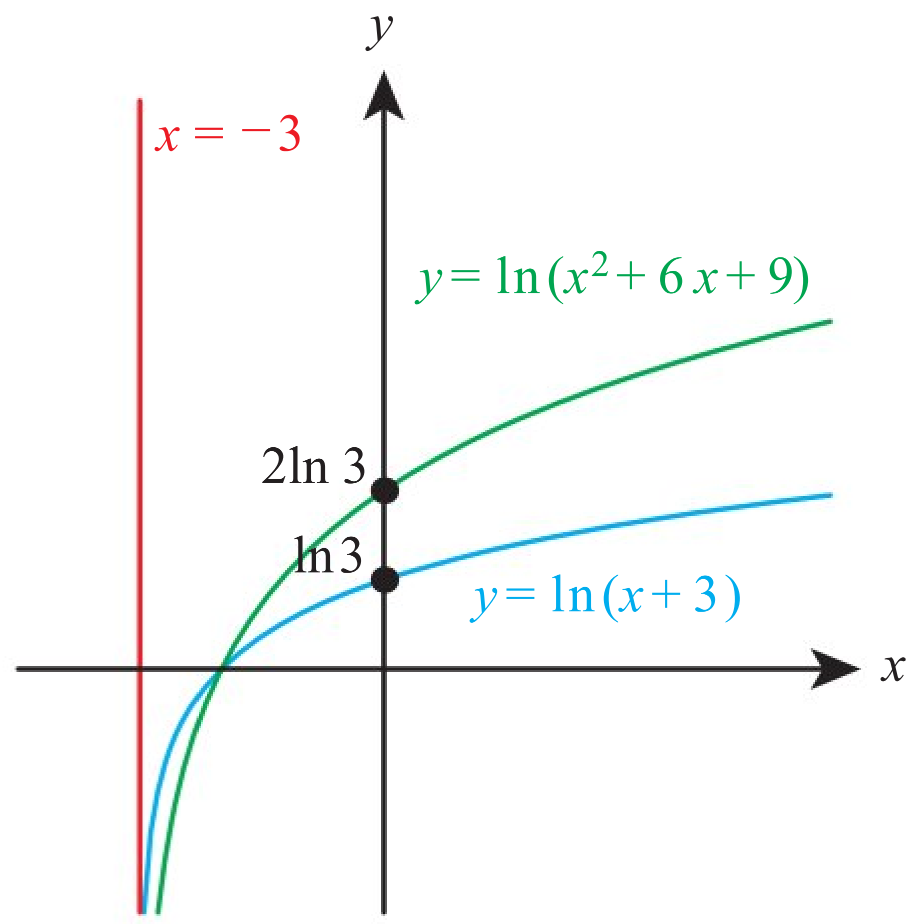
c



34 a $x = -5,\ y = 3$ b $x \neq -5,\ f(x) \neq 3$

35 a Translation 3 units to the left

b



36 a $p = -4,\ q = 6$ b $y = -x^2 + 12x - 31$

37 a 3 b 7 c $y = 7$

38 a $a = 800$ b $b = 2$ c 40

39 $\ln\left(\dfrac{e^2}{x-3}\right)$

40 a $x = 5,\ y = 2$ b $\alpha = 2,\ \beta = 1$

c Horizontal translation 5 to the right, vertical translation 2 up

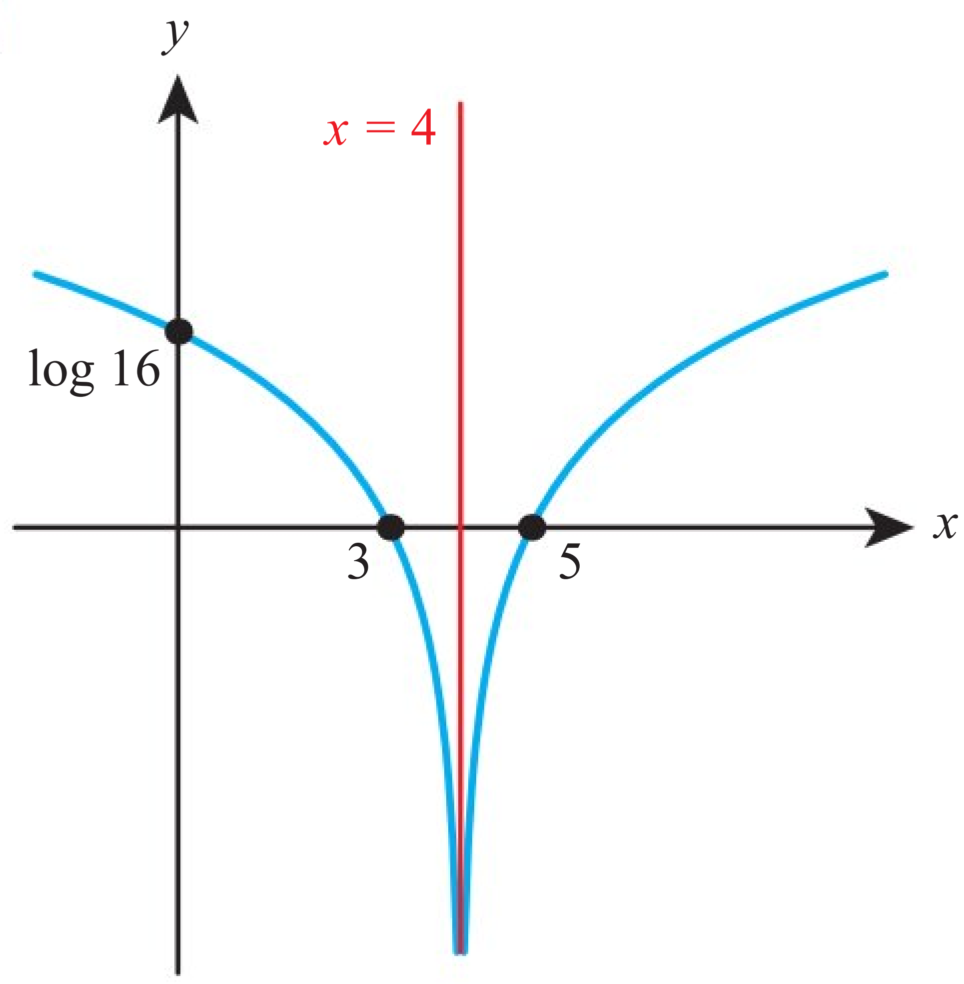
d $\dfrac{5x-9}{x-2},\ x \neq 0$

e Reflection in $y = x$

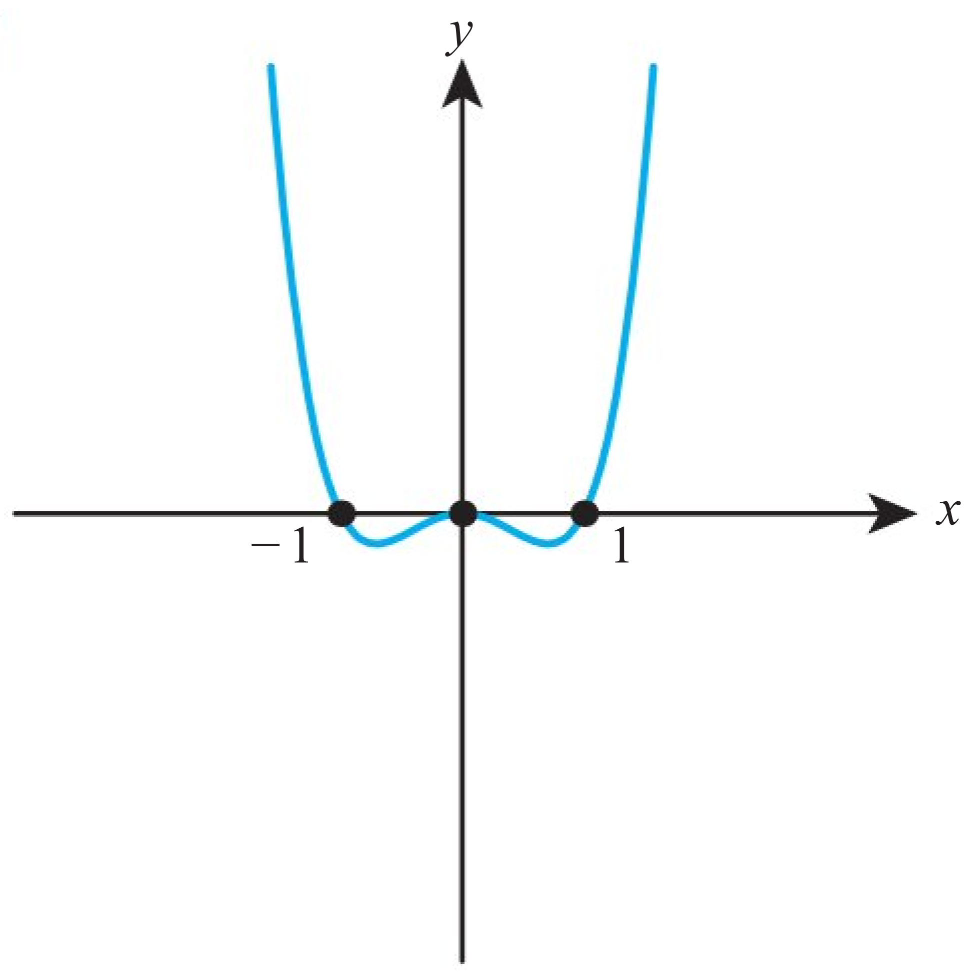
41 Horizontal stretch factor 2

42 Vertical stretch factor $1 / \ln 10$

43



44



45 $2 - \mathrm{f}(x)$

Chapter 6 Prior Knowledge

1 1.58, 0.423

2 **a** $\frac{\pi}{6}$ **b** $\frac{5\pi}{4}$

3 8.06, 0.519

4 **a** $10e^{4x}$ **b** $8e^{14x}$ **c** $32e^{10x}$

Exercise 6A

1 **a** i **b** -1

2 **a** $4\mathrm{i}$ **b** 5

3 **a** -9 **b** $-8\mathrm{i}$

4 **a** -20 **b** 6

5 **a** $11 + 4\mathrm{i}$ **b** $-4 + 2\mathrm{i}$

6 **a** $1 - 2\mathrm{i}$ **b** $-6 + 10\mathrm{i}$

7 **a** $8 - \mathrm{i}$ **b** $16 + 2\mathrm{i}$

8 **a** $8 + 6\mathrm{i}$ **b** $7 - 24\mathrm{i}$

9 **a** $4 - 2\mathrm{i}$ **b** $-3 + 3\mathrm{i}$

10 **a** $4 + 3\mathrm{i}$ **b** $1 - \mathrm{i}$

11 **a** $\frac{1}{2} + \frac{1}{2}\mathrm{i}$ **b** $-\frac{2}{13} - \frac{23}{13}\mathrm{i}$

12 **a** $a = 5,\ b = 7$ **b** $a = -3,\ b = 9$

13 **a** $a = -\frac{3}{2},\ b = \frac{31}{2}$ **b** $a = \frac{1}{2},\ b = -8$

14 **a** $z = -4 - \mathrm{i}$ **b** $z = 1 + 2\mathrm{i}$

15 **a** $z = \frac{2}{3} + 7\mathrm{i}$ **b** $z = -\frac{5}{3} + \frac{1}{3}\mathrm{i}$

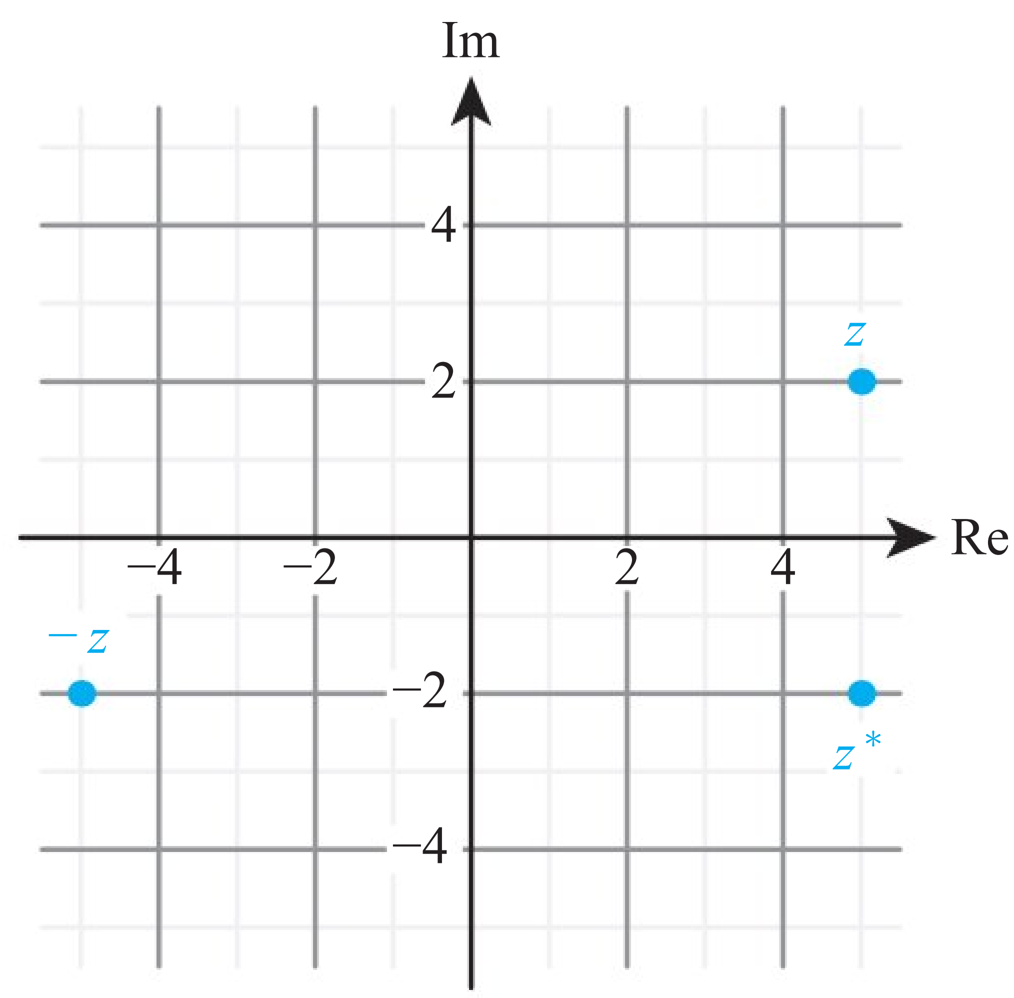
16 **a** $x = \pm 3\mathrm{i}$ **b** $x = \pm 6\mathrm{i}$

17 **a** $x = \pm 2\sqrt{2}\mathrm{i}$ **b** $x = \pm 5\sqrt{3}\mathrm{i}$

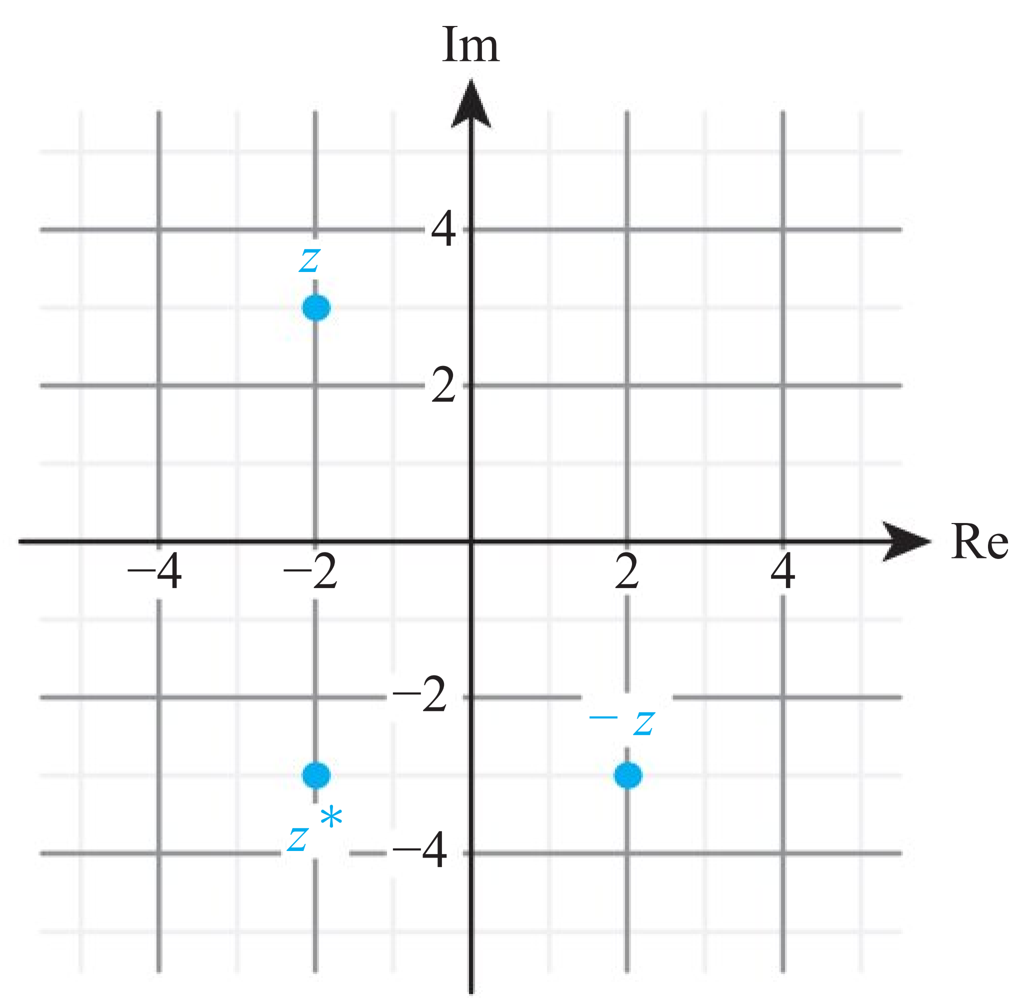
18 **a** $x = 1 \pm 2\mathrm{i}$ **b** $x = 2 \pm 3\mathrm{i}$

19 **a** $x = -1 \pm \frac{\sqrt{2}}{2}\mathrm{i}$ **b** $x = \frac{1}{3} \pm \frac{\sqrt{5}}{3}\mathrm{i}$

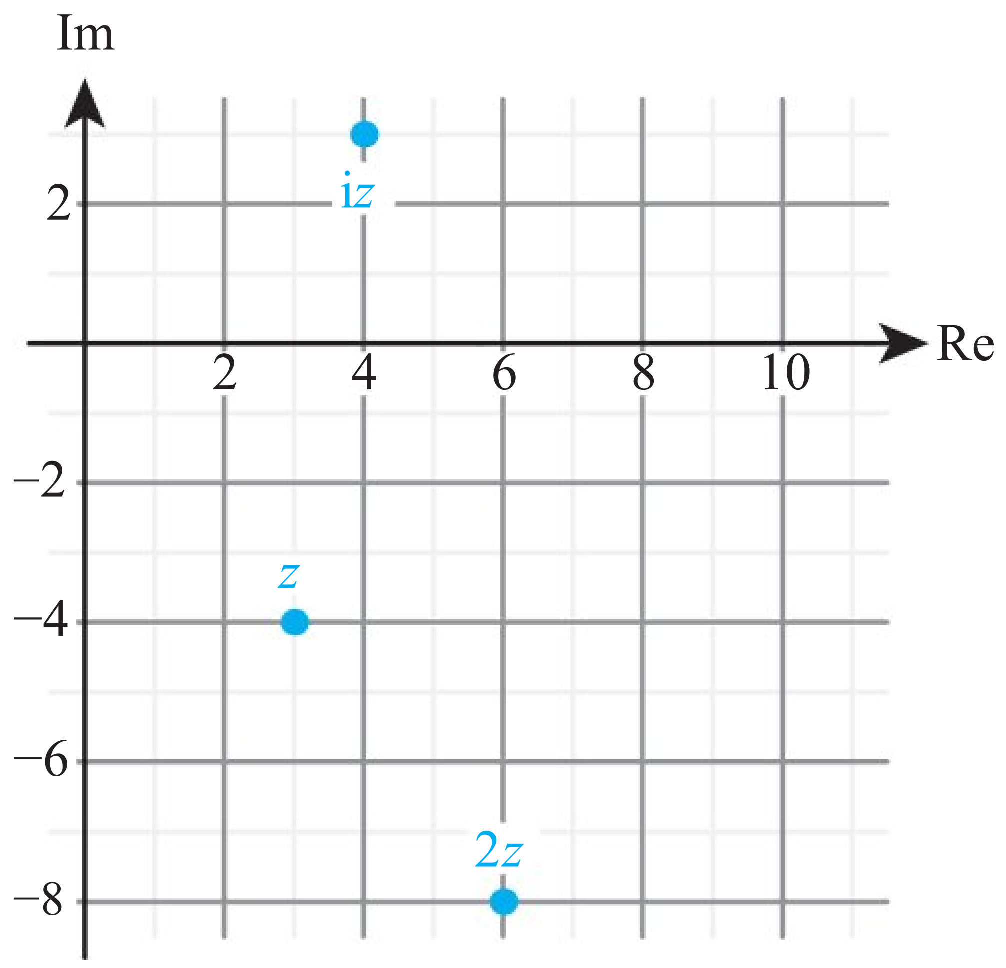
20 **a**



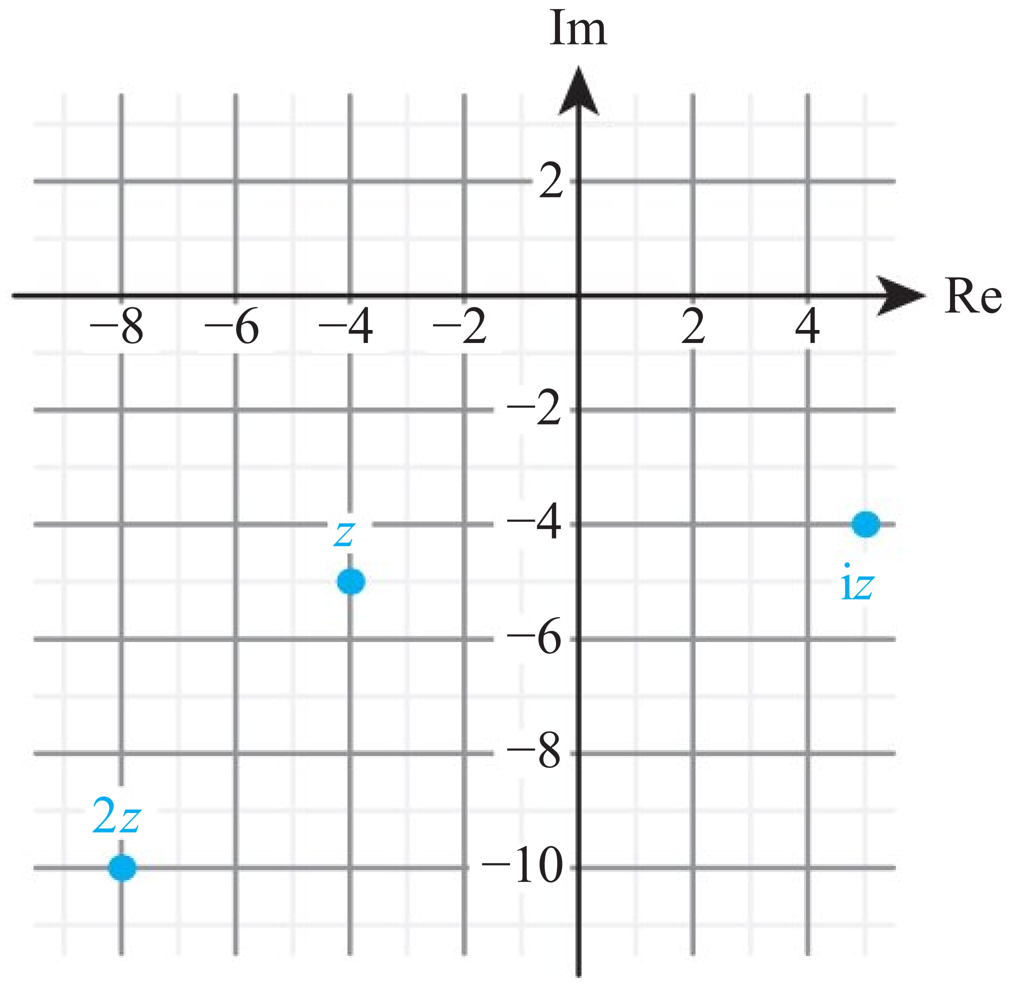
b



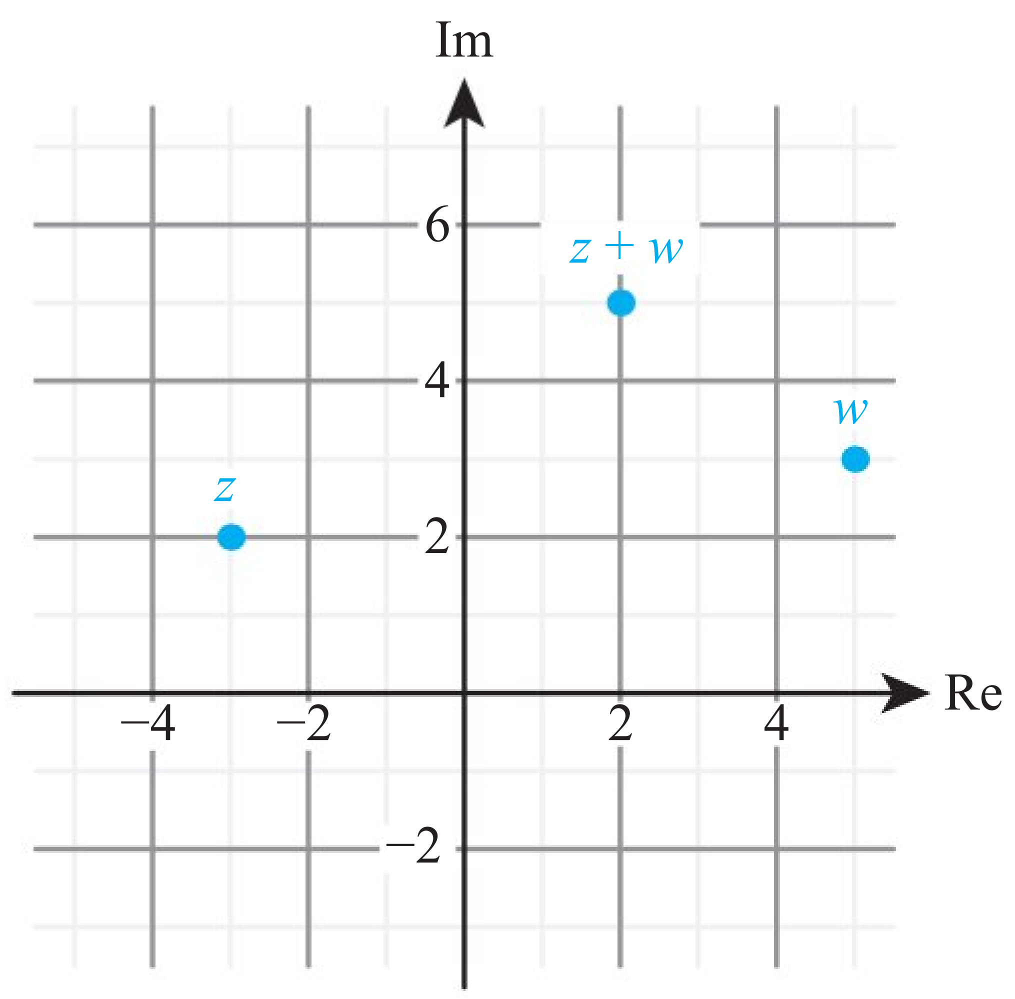
21 a



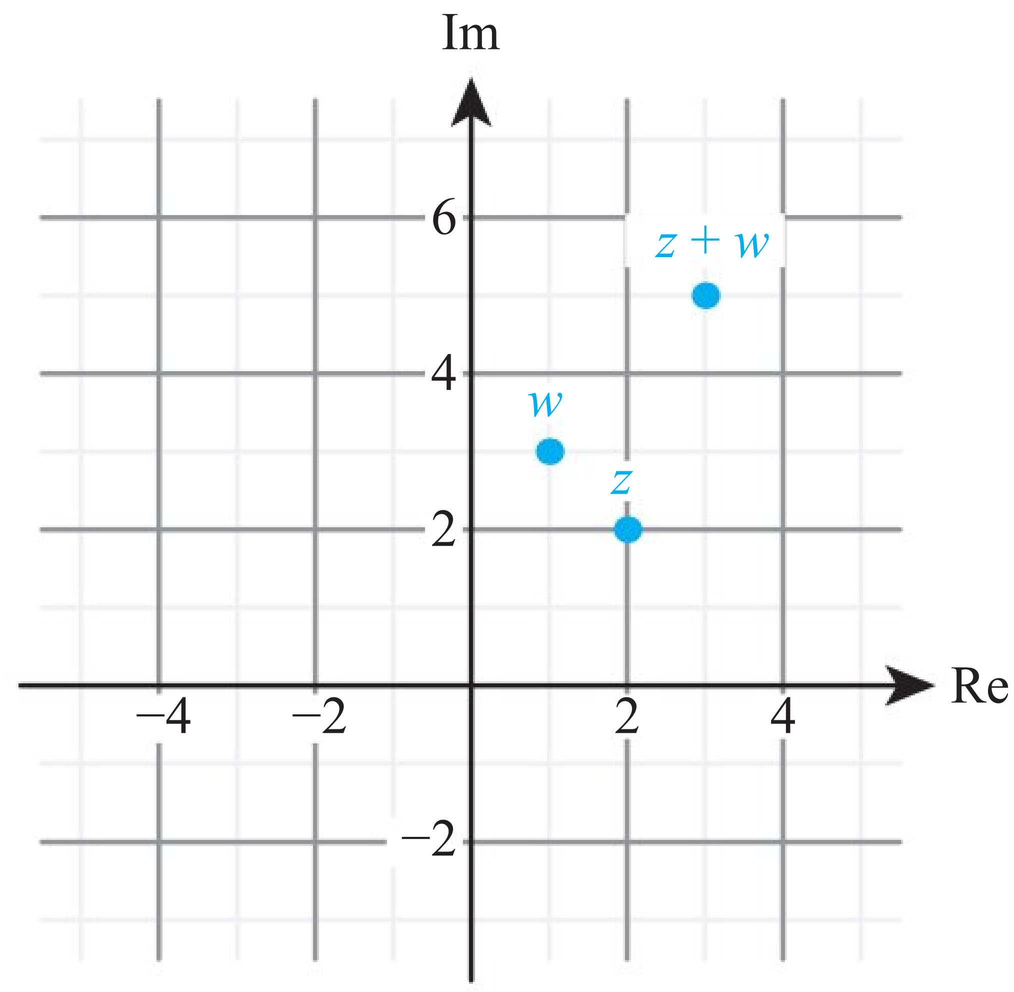
b



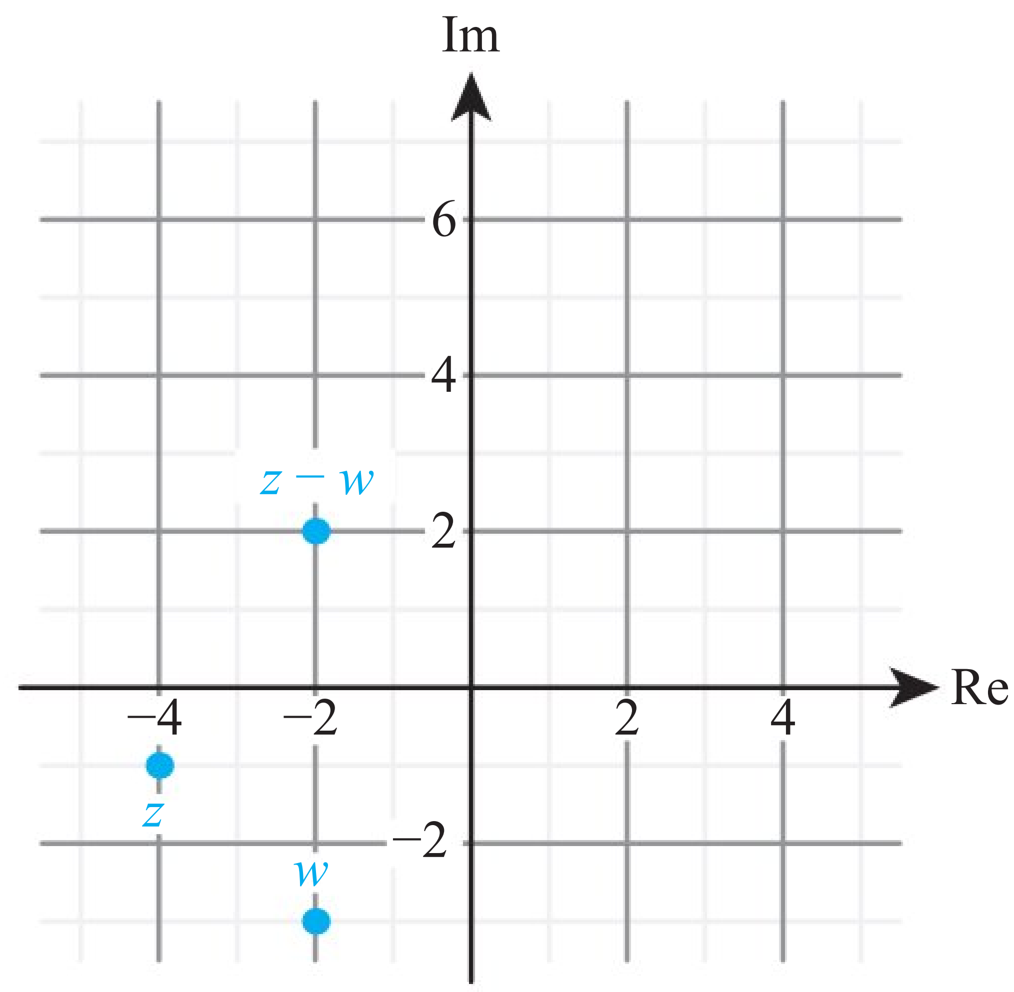
22 a



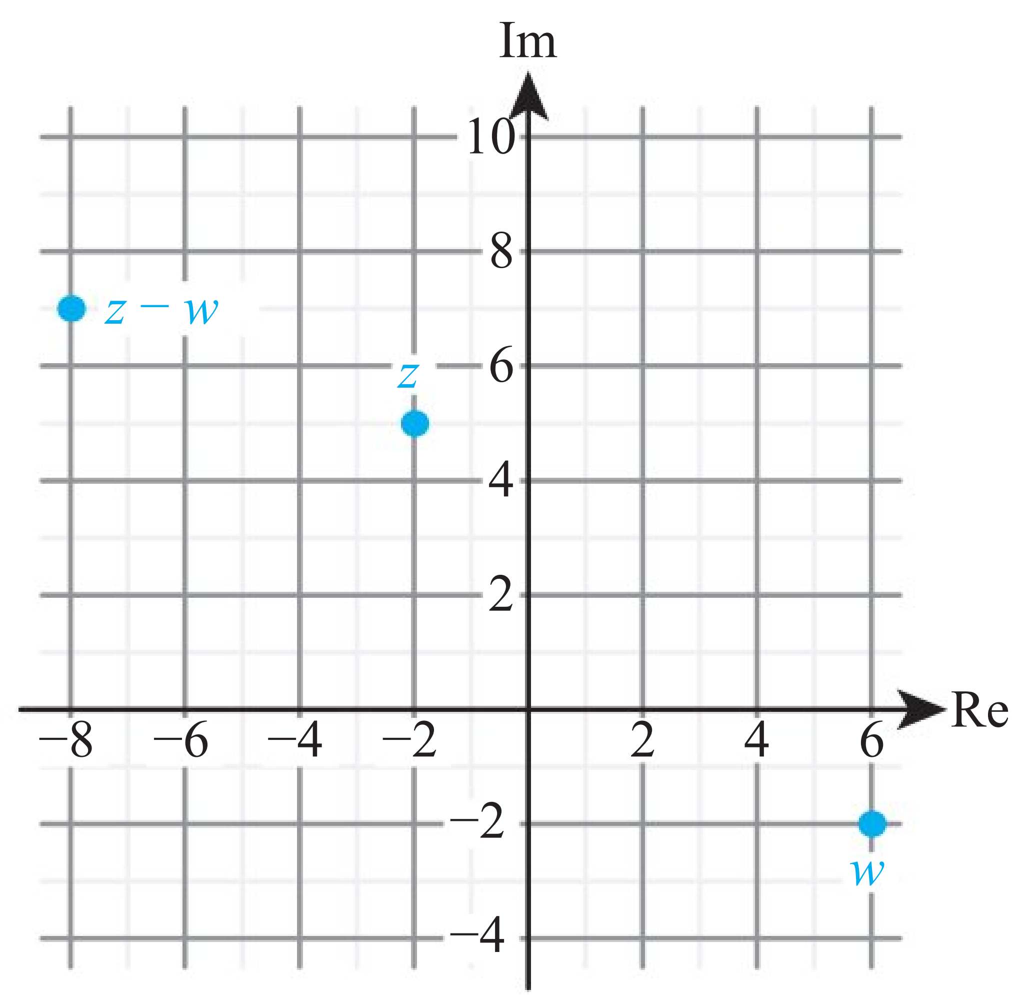
b



23 a

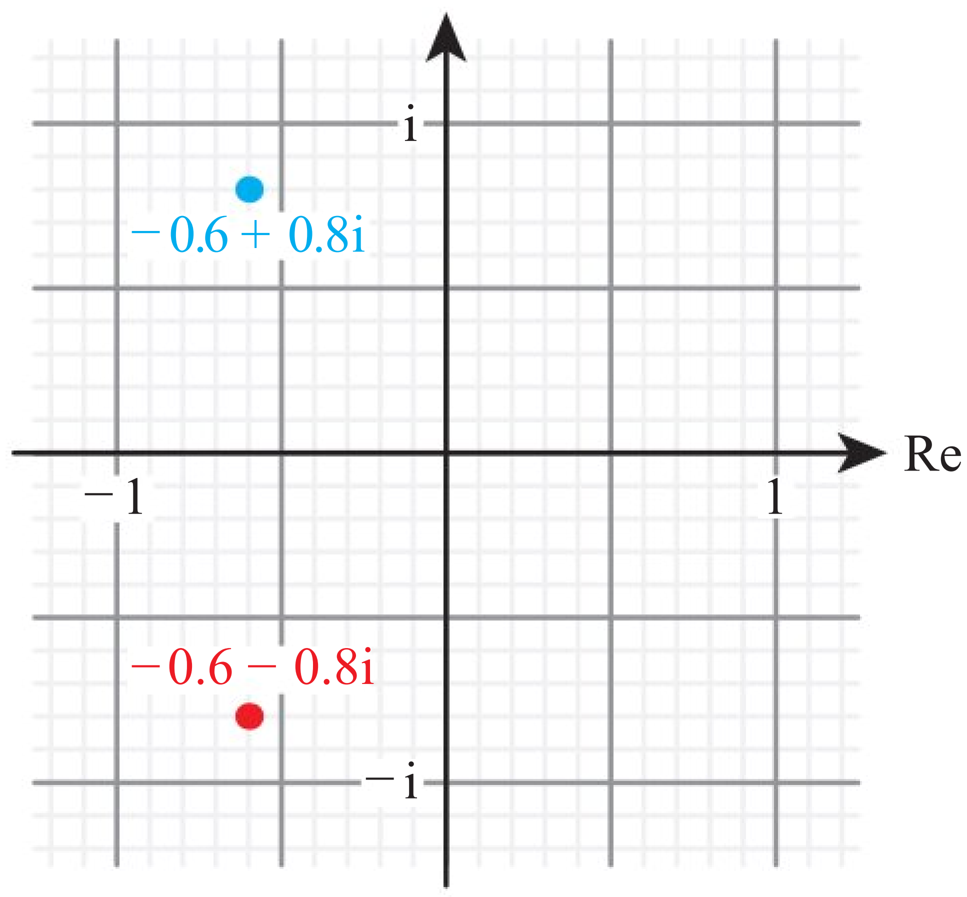


b



24 a $x = -\frac{3}{5} \pm \frac{4}{5}\mathrm{i}$

b

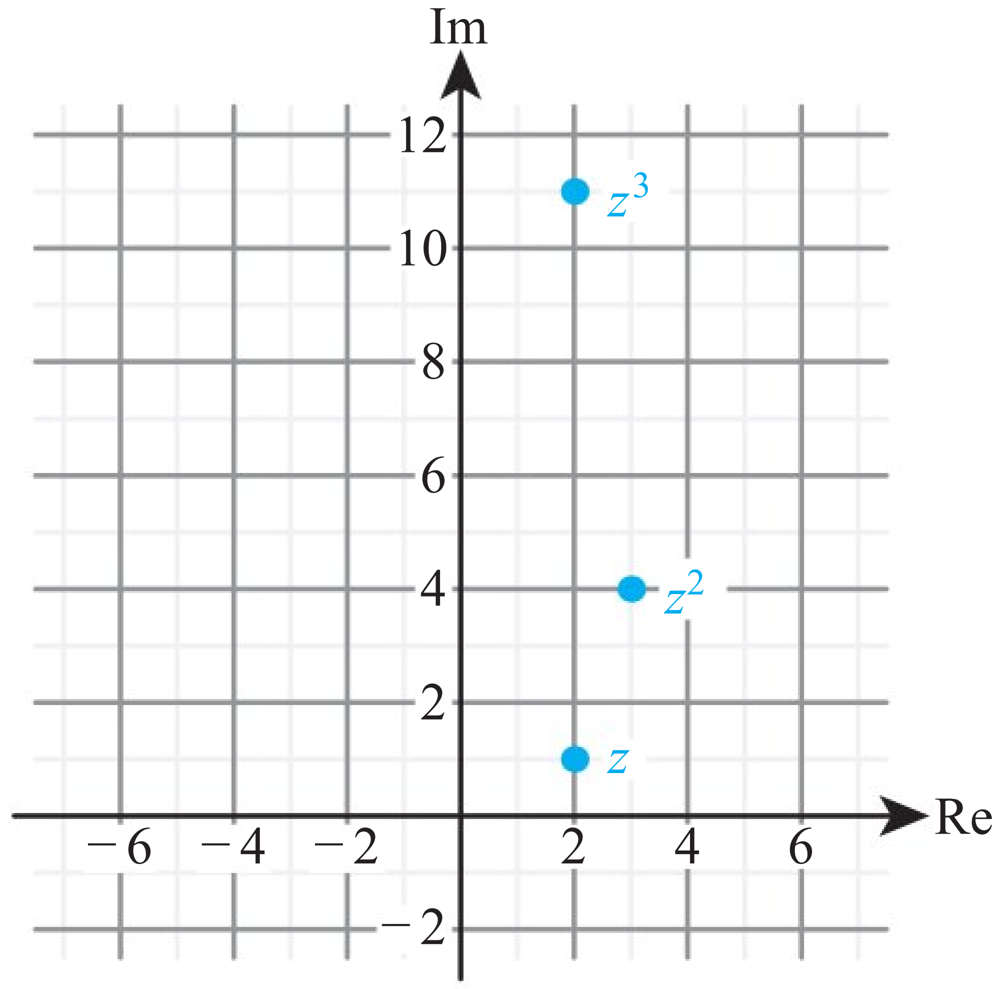


25 $z^* = 5 + \mathrm{i}$

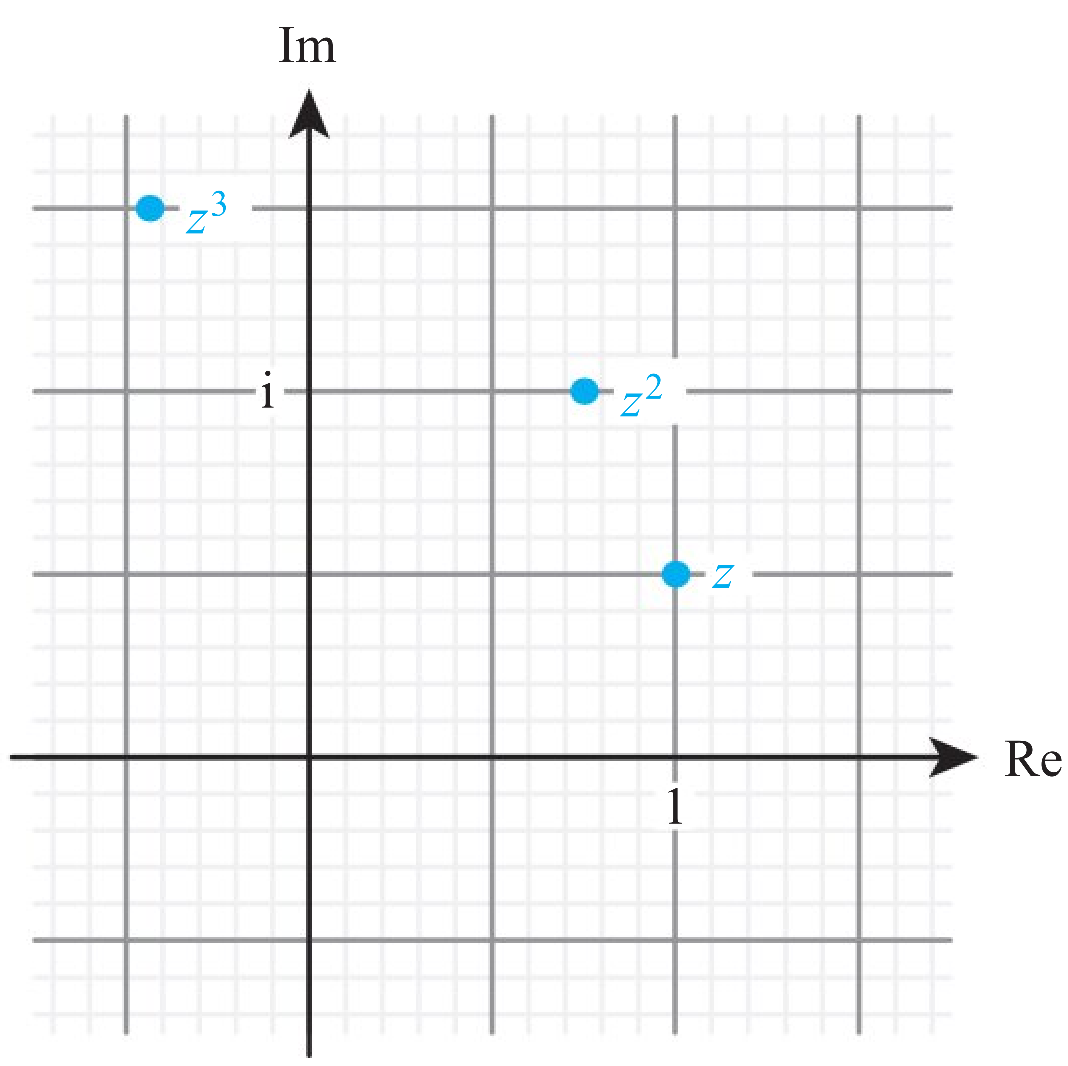
26 $z = 2 + 3\mathrm{i}$

27 **a** $3 + 4\mathrm{i},\ 2 + 11\mathrm{i}$

b



28



29 **a** $a + 2(1 - a)\mathrm{i}$ **b** $2a - a^2\mathrm{i}$

30 $a = 0,\ b = 4$ or $a = 12,\ b = 0$

31 $z = -1 + 2\mathrm{i}$

32 $3 - 5\mathrm{i}$

33 $2 - 3\mathrm{i},\ -2 + 3\mathrm{i}$

34 $\dfrac{k^2 - 1}{k^2 + 1} + \dfrac{2k}{k^2 + 1}\mathrm{i}$

35 $a = \pm 3$

37 $z = 2 - 5\mathrm{i},\ w = 4 + \mathrm{i}$

38 $z = \dfrac{3}{2} + \dfrac{1}{2}\mathrm{i},\ w = 2\mathrm{i}$

39 $a = 8,\ b = 1$

$a = -1,\ b = 10$

40 $a = 4, -\dfrac{3}{2}$

Exercise 6B

1 **a** $5\operatorname{cis}\dfrac{3\pi}{2}$ **b** $7\operatorname{cis}\dfrac{3\pi}{2}$

2 **a** $6\operatorname{cis}\dfrac{5\pi}{3}$ **b** $4\sqrt{2}\operatorname{cis}\dfrac{7\pi}{4}$

3 **a** $2\sqrt{2}\operatorname{cis}\dfrac{5\pi}{4}$ **b** $2\operatorname{cis}\dfrac{4\pi}{3}$

4 **a** $z = -10\mathrm{i}$ **b** $z = 8\mathrm{i}$

5 **a** $z = 2 + 2\sqrt{3}\mathrm{i}$ **b** $z = 1 + \mathrm{i}$

6 **a** $z = -2\sqrt{6} - 2\sqrt{6}\mathrm{i}$ **b** $z = -1 + \sqrt{3}\mathrm{i}$

7 **a** $z = 4\sqrt{3} - 4\mathrm{i}$ **b** $z = \sqrt{2} - \sqrt{2}\mathrm{i}$

8 **a** $12\operatorname{cis}\dfrac{8\pi}{15}$ **b** $5\operatorname{cis}\dfrac{\pi}{8}$

9 **a** $3\operatorname{cis}\dfrac{35\pi}{18}$ **b** $\dfrac{1}{3}\operatorname{cis}\dfrac{4\pi}{7}$

10 **a** $-22.8 - 7.42\mathrm{i}$ **b** $-15.0 - 5.47\mathrm{i}$

11 **a** $0.643 - 0.766\mathrm{i}$ **b** $0.5 + 0.866\mathrm{i}$

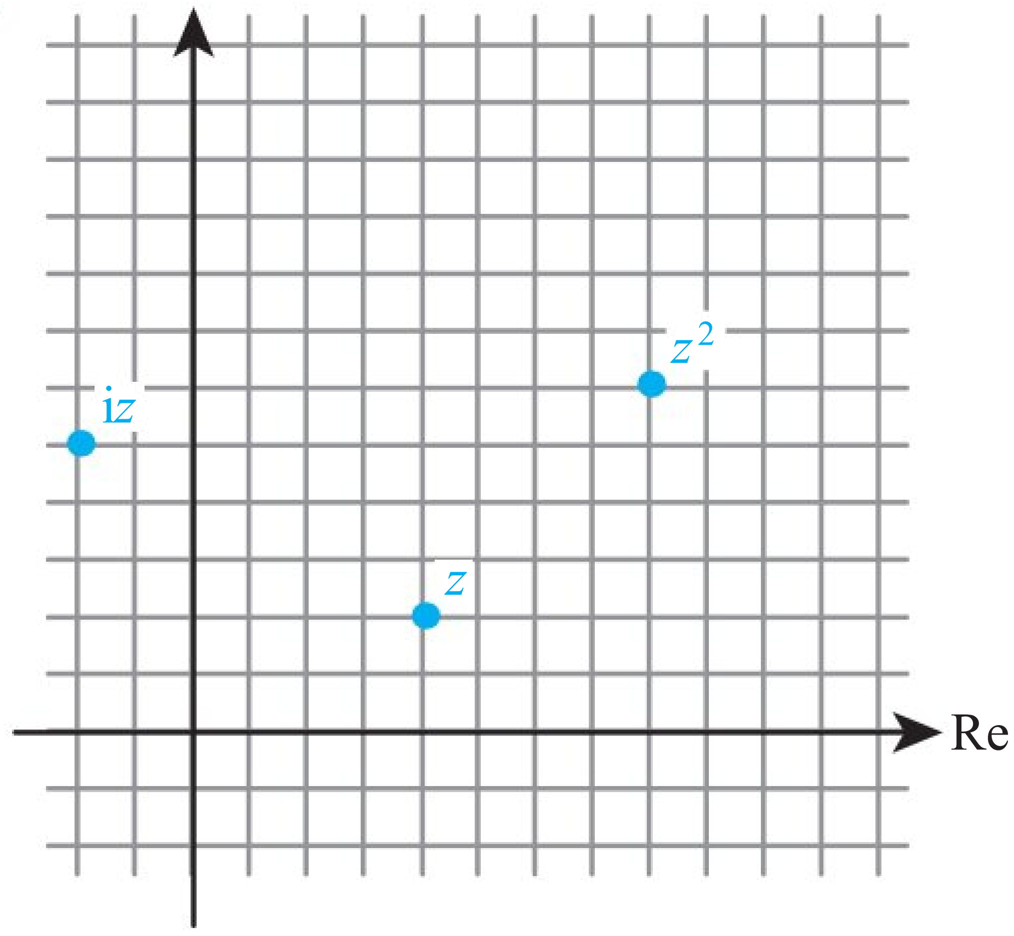
12 **a** $15\mathrm{e}^{-0.1\mathrm{i}}$ **b** $2\mathrm{e}^{2\mathrm{i}}$

13 **a** $2\mathrm{e}^{\frac{3\pi\mathrm{i}}{4}}$ **b** $4\mathrm{e}^{-\frac{\pi\mathrm{i}}{12}}$

14 **a** $\dfrac{1}{2} + \dfrac{\sqrt{3}}{2}\mathrm{i}$ **b** $\dfrac{1}{\sqrt{2}} + \dfrac{1}{\sqrt{2}}\mathrm{i}$

15 **a** i **b** $-\dfrac{1}{2} - \dfrac{\sqrt{3}}{2}$

16



17 a $\sqrt{8}$ b $\frac{3\pi}{4}$

c $8, \frac{3\pi}{2}$ d -8i

18 cis1

19 a $\frac{\pi}{3}$ b $\frac{7\pi}{12}$

20 a i b $\left(\frac{1}{2}+\frac{\sqrt{3}}{2}\right)(1+\text{i})$

21 a $z = 2\sqrt{2}\text{cis}\left(-\frac{\pi}{4}\right)$ b $-128+128\text{i}$

22 a $2\,\text{cis}\left(\frac{\pi}{6}\right)$ b 8i

23 a $w = 2\text{cis}\left(-\frac{3\pi}{4}\right)$ b 64i

24 $-162\sqrt{2}-162\sqrt{2}\text{i}$

25 $n = 24$

26 $n = 9$

27 $|z| = 4, \arg z = \frac{13\pi}{24}$

$|w| = 2, \arg w = \frac{5\pi}{24}$

28 a i 1 ii $\frac{\pi}{4}$

iii 2 iv $\frac{\pi}{3}$

b $2\text{cis}\frac{\pi}{12}$

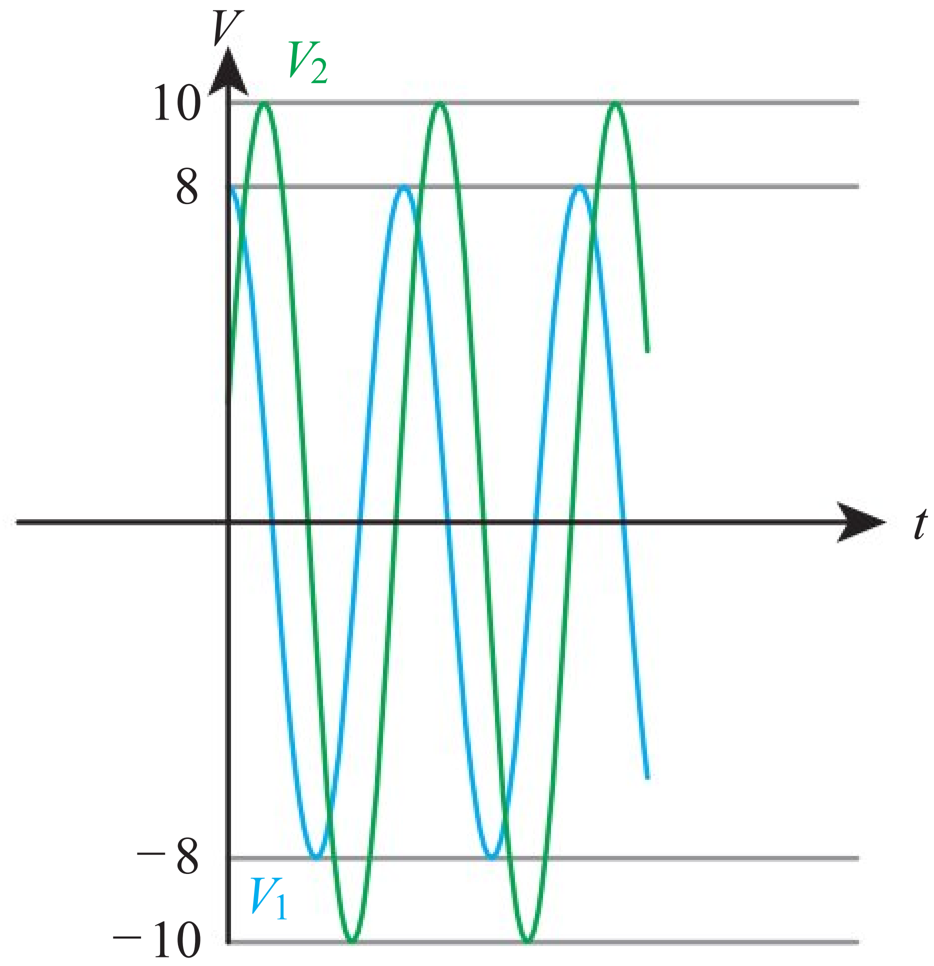
c $\frac{\sqrt{6}+\sqrt{2}}{2}+\frac{\sqrt{6}-\sqrt{2}}{2}\text{i}$

d $\frac{\sqrt{6}+\sqrt{2}}{4}$

29 a 3.03, 0.093 b $3.03\sin(\theta+0.093)$

30 a $3.50\text{e}^{0.317}$ b $350\cos(0.317+t)$

31 a



b $14.5\cos(30t+5.56)$

32 $260+36.5\sin\left(\frac{2\pi}{365}t+0.521\right)$

33 b 0.2

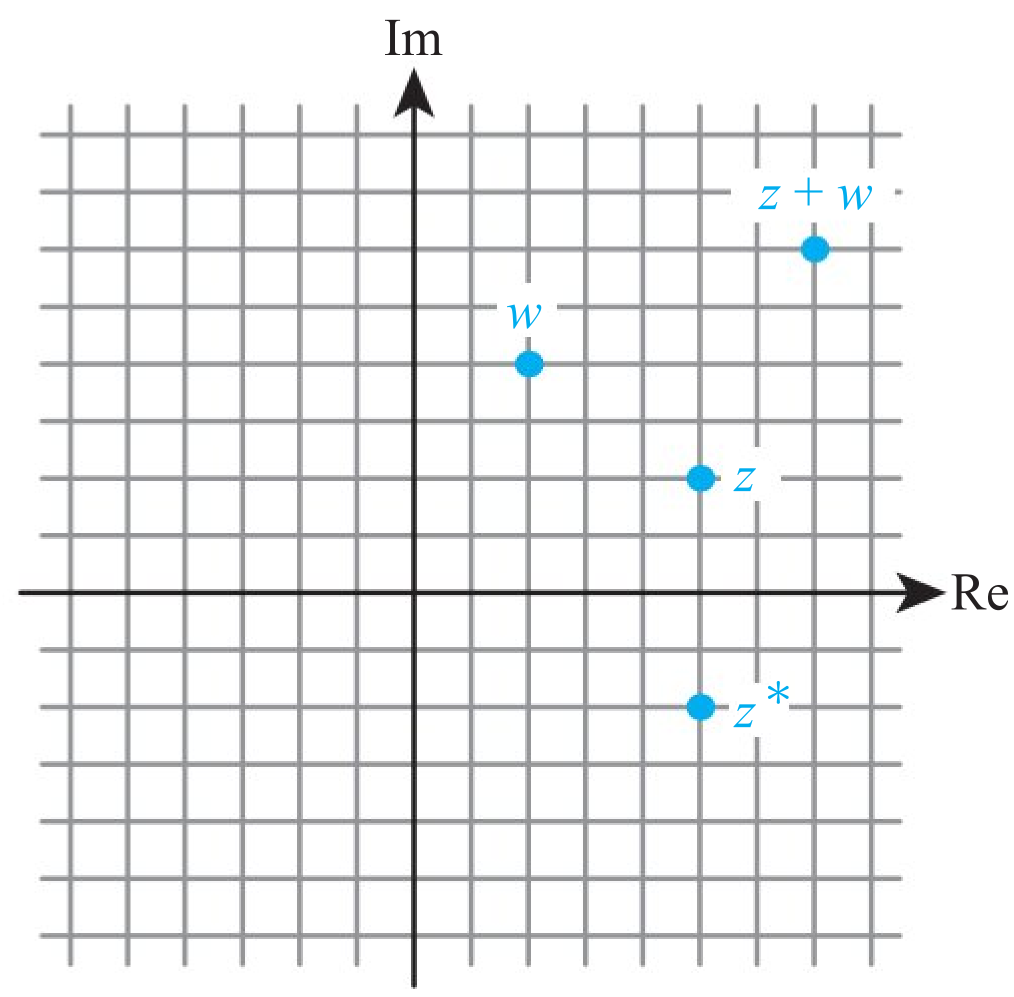
34 a $2\text{e}^{\text{i}\pi}$ b $\ln 2+\text{i}\pi$

35 a $\text{e}^{\frac{\text{i}\pi}{2}}$ b $\frac{\text{i}\pi}{2}$

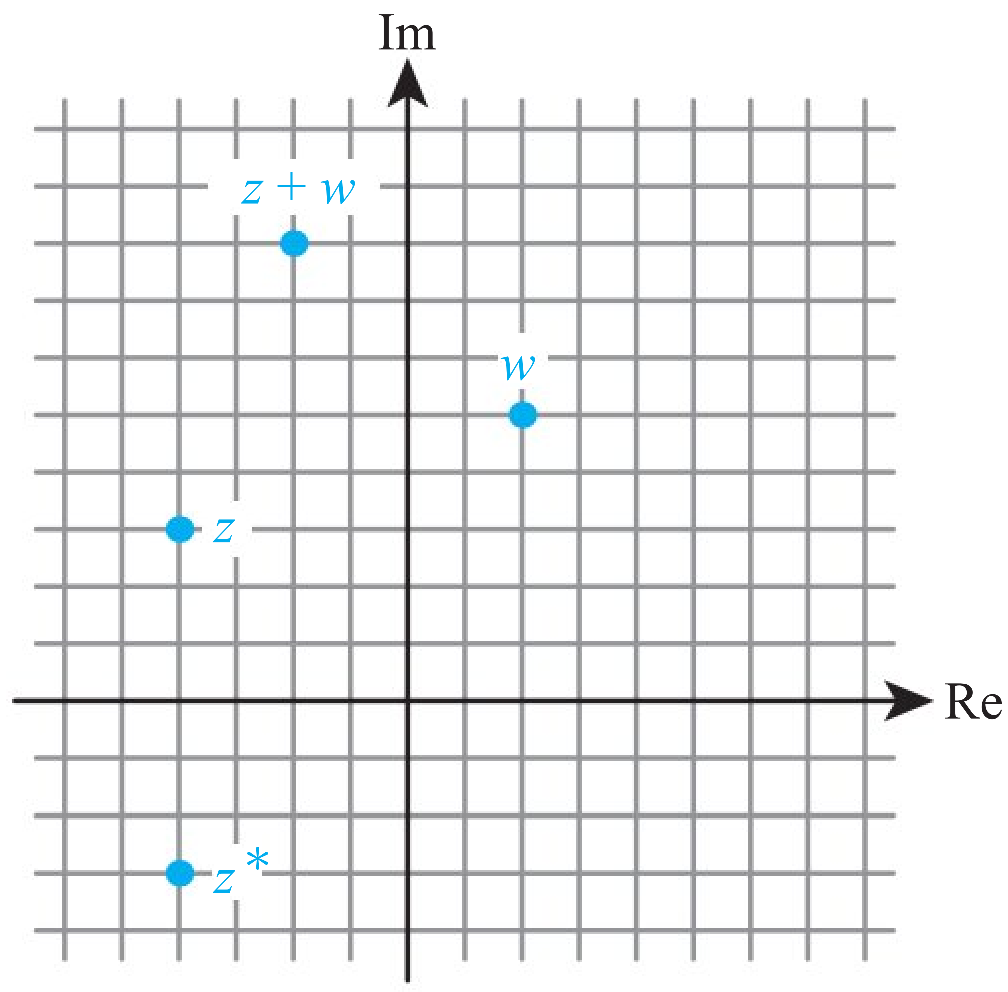
c Could be $\frac{\text{i}\pi}{2}+2k\pi\text{i}$, where $k \in \mathbb{Z}$

Exercise 6C

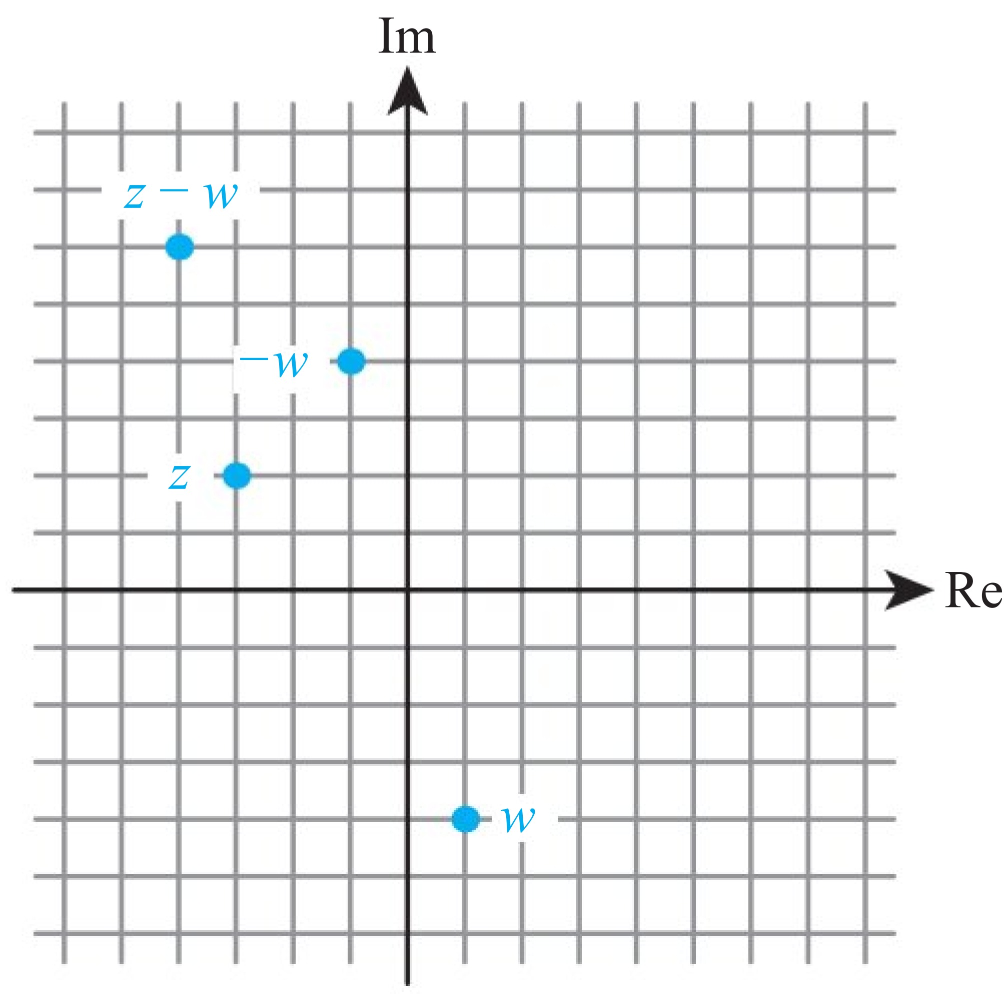
1 a



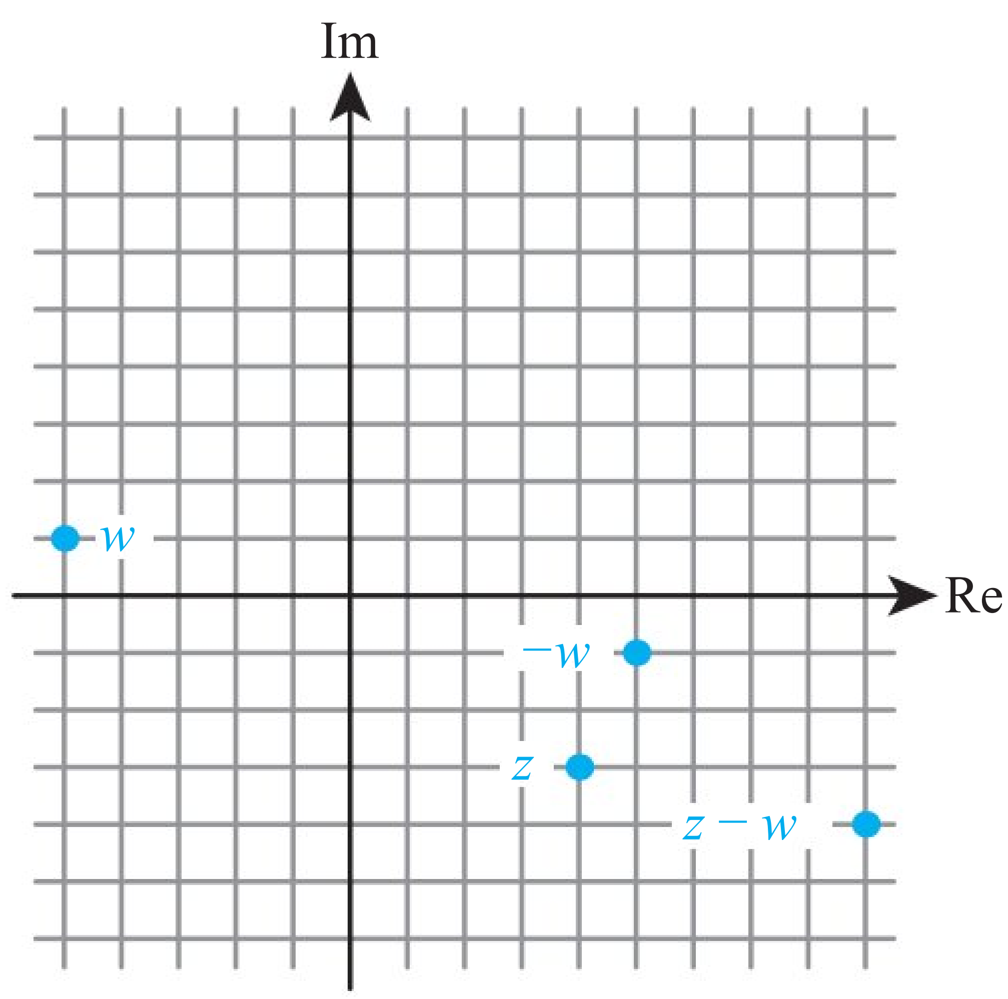
b



2 a



b



3 a $\sqrt{17}$, 0.245; $\sqrt{34}$, 0.540 b $\sqrt{2}$, 0.295 (16.9°)

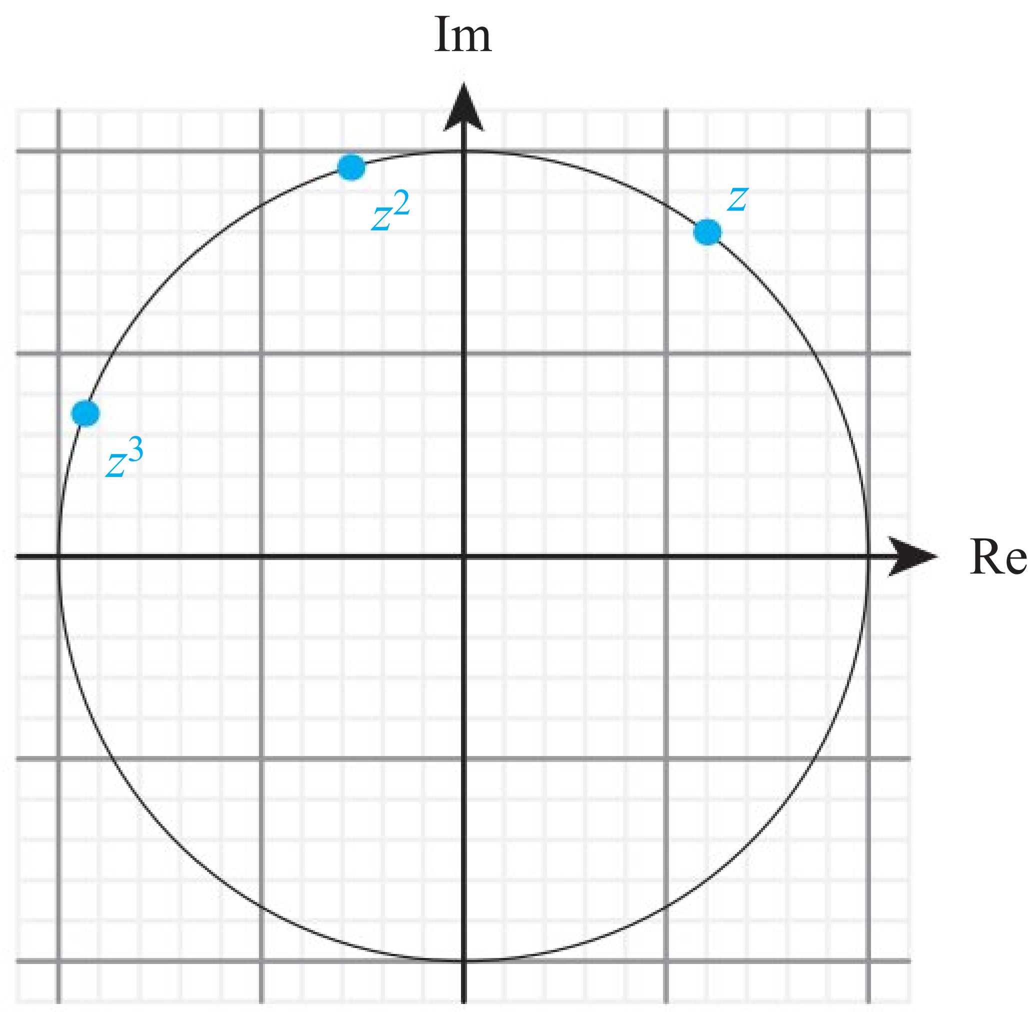
4 b Rotation 1.76 radians (101°) anticlockwise about the origin

5 $\dfrac{3\sqrt{3}-2}{2}+\mathrm{i}\dfrac{3+2\sqrt{3}}{2}$

6 $(7,-3)$, $(2,-5)$

7 a $\sqrt{60}$ b $(6-\sqrt{3},\ 3+2\sqrt{3})$

8 a



b Rotation through 1.85 radians about the origin

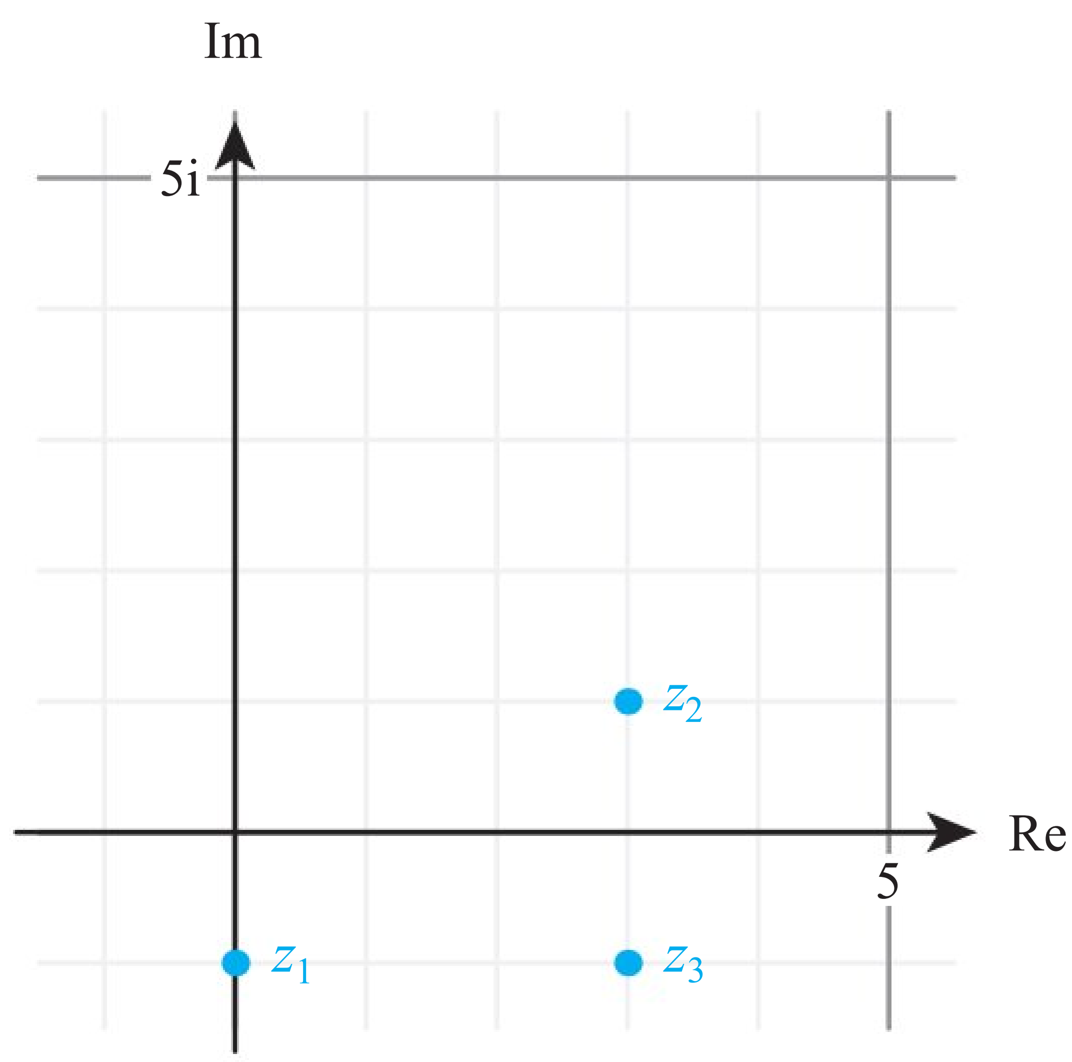
9 a $4-\mathrm{i}$

b $B\left(\dfrac{\sqrt{3}-4}{2}, \dfrac{4\sqrt{3}+1}{2}\right)$, $C\left(-\dfrac{4+\sqrt{3}}{2}, -\dfrac{4\sqrt{3}-1}{2}\right)$

10 b $\left(\dfrac{3-4\sqrt{3}}{2}, \dfrac{2-\sqrt{3}}{2}\right)$

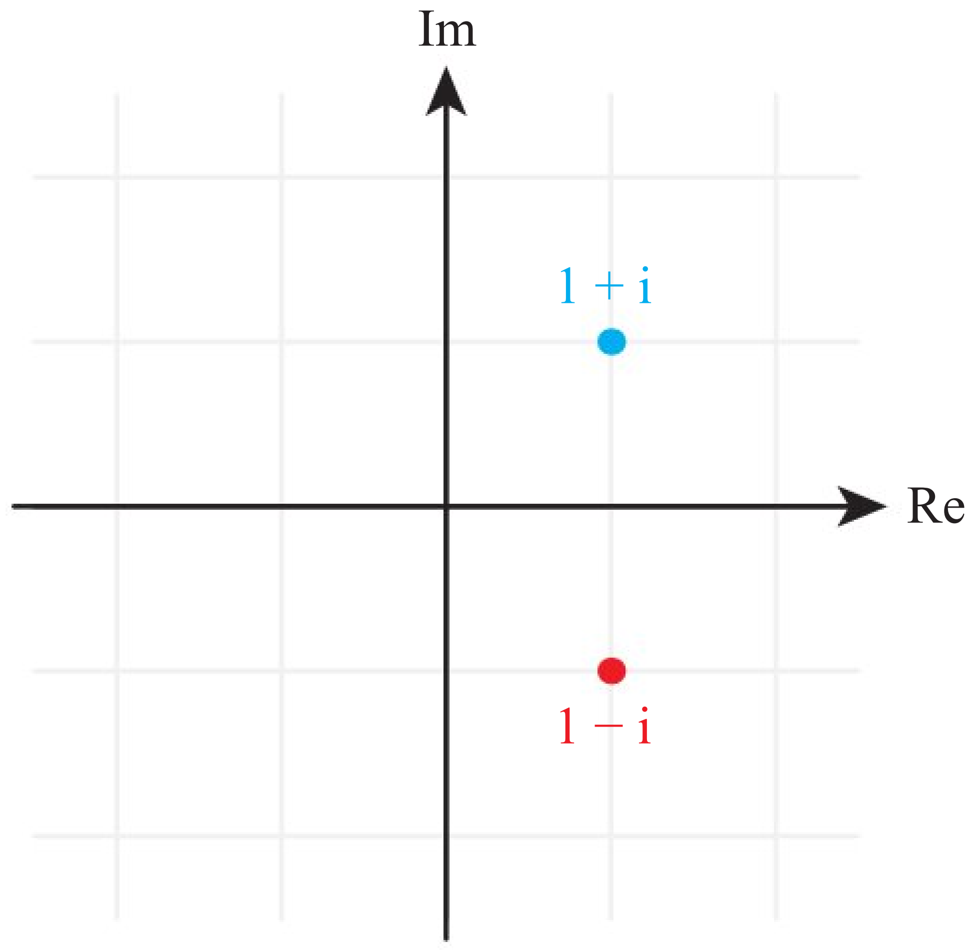
Chapter 6 Mixed Practice

1



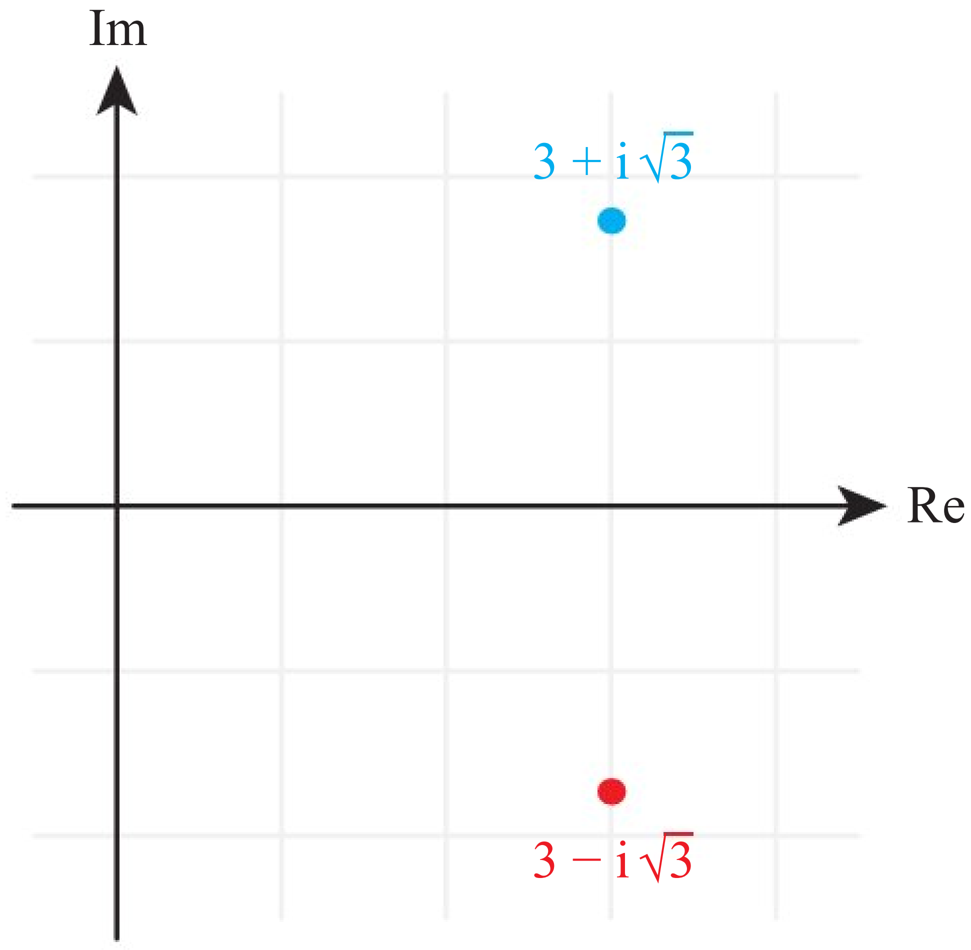
2 a $1 \pm \mathrm{i}$

b



3 **a** $3 \pm \sqrt{3}\mathrm{i}$

b



4 $\frac{3}{5} + \frac{1}{5}\mathrm{i}$

5 $-0.5 - \mathrm{i}$

6 $-\frac{\mathrm{i}}{3}$

7 $z = -2 - 2\mathrm{i}$

8 **a** $|z| = \sqrt{2}$, $\arg z = \frac{\pi}{4}$ **b** $8\mathrm{i}$

9 $16\sqrt{2}\,\mathrm{cis}\frac{3\pi}{4}$

10 **a** $p = 3$, $q = 0.5$ **b** $p = \frac{1}{3} - \frac{2}{3}\mathrm{i}$, $q = \frac{1}{3} + \frac{2}{3}\mathrm{i}$

11 **a** $\frac{1}{2} + \frac{\sqrt{3}}{2}\mathrm{i}$ **b** $-\frac{\sqrt{3}}{2} + \frac{1}{2}\mathrm{i}$

12 $\frac{a}{a^2+1} - \frac{1}{a^2+1}\mathrm{i}$

13 $x = -\frac{\mathrm{i}}{3}$

14 $z = 4 + 3\mathrm{i}$

15 $1, \frac{\pi}{6}$

16 $\pm\sqrt{\frac{3}{2}}$

17 $2 \pm \mathrm{i}$

18 $5 + 12\mathrm{i}$

19 $b = -2$, $c = 5$

20 $3 + 4\mathrm{i}$

21 **a** $0.387, 1.22$ **b** $0.387\cos(3t + 1.22)$

22 **a** $\sqrt{17}, 2.90$ **b** $e^{\mathrm{i}\frac{\pi}{6}}$
c $(-3.96, -1.13)$

23 $2 \pm \mathrm{i}\sqrt{5}$

24 $x^2 + y^2 - 2x = 3$

25 **a** $(1 - \sqrt{3}) + (1 + \sqrt{3})\mathrm{i}$

b $z = \sqrt{2}\left(\cos\frac{\pi}{4} + \mathrm{i}\sin\frac{\pi}{4}\right)$, $w = 2\left(\cos\frac{\pi}{3} + \mathrm{i}\sin\frac{\pi}{3}\right)$,

$|zw| = 2\sqrt{2}$, $\arg(zw) = \frac{7\pi}{12}$

c $\frac{\sqrt{6} + \sqrt{2}}{4}$

26 **a** $2, \frac{\pi}{6}$ **b** $-128\sqrt{3}$

27 $3 + 4\mathrm{i}$

28 1.30 radians, 2.28

29 **a** $2\sqrt{2} - \sqrt{3}$ **b** $\frac{\pi}{12}$

Chapter 7 Prior Knowledge

1 $\begin{pmatrix} 106 & 91.8 & 67.2 & 69.4 \\ 245 & 220 & 163 & 153 \\ 222 & 203 & 149 & 136 \\ 130 & 112 & 81.9 & 83.2 \end{pmatrix}$

Exercise 7A

1 **a** Connected
b Complete, connected, simple

2 **a** Connected, simple, tree
b None

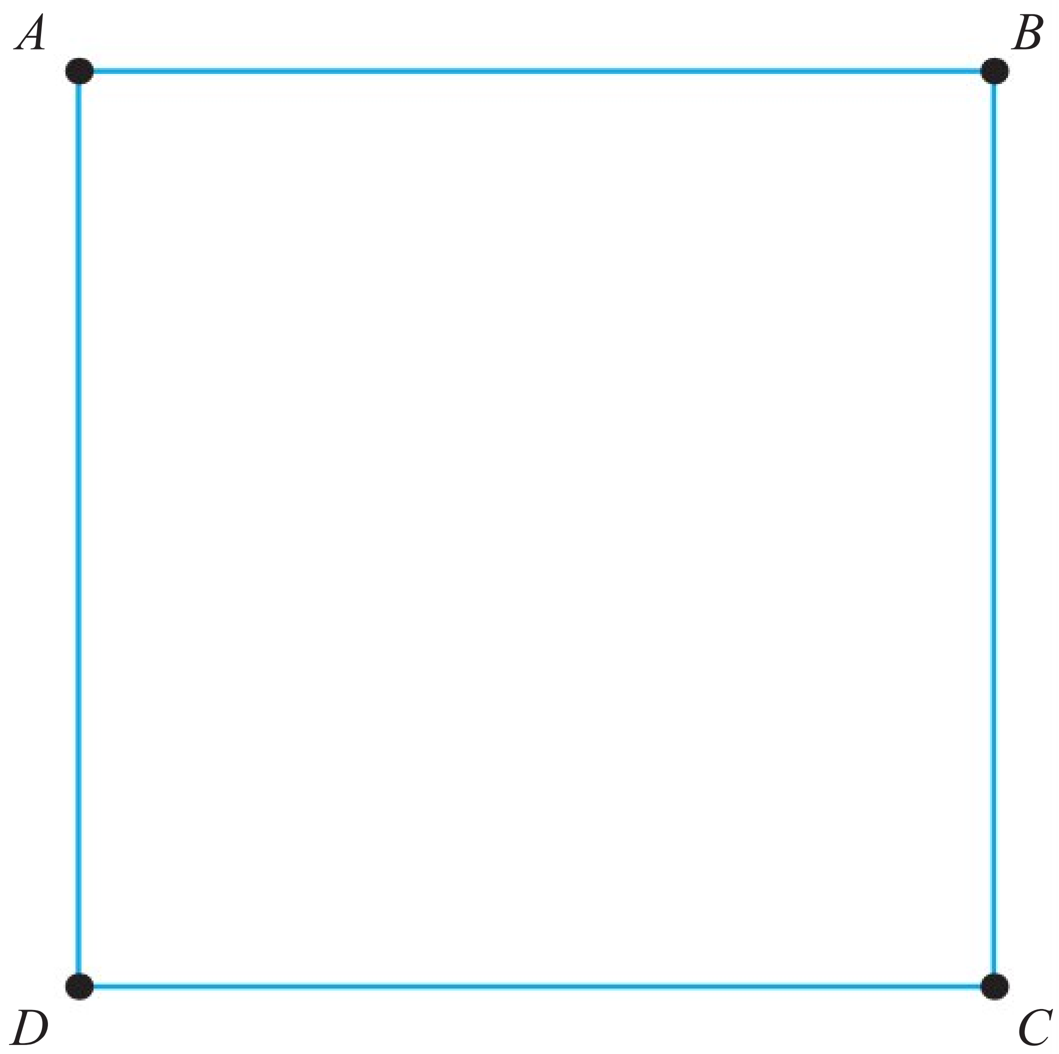
3 **a** No **b** Yes

4 **a** Yes **b** No

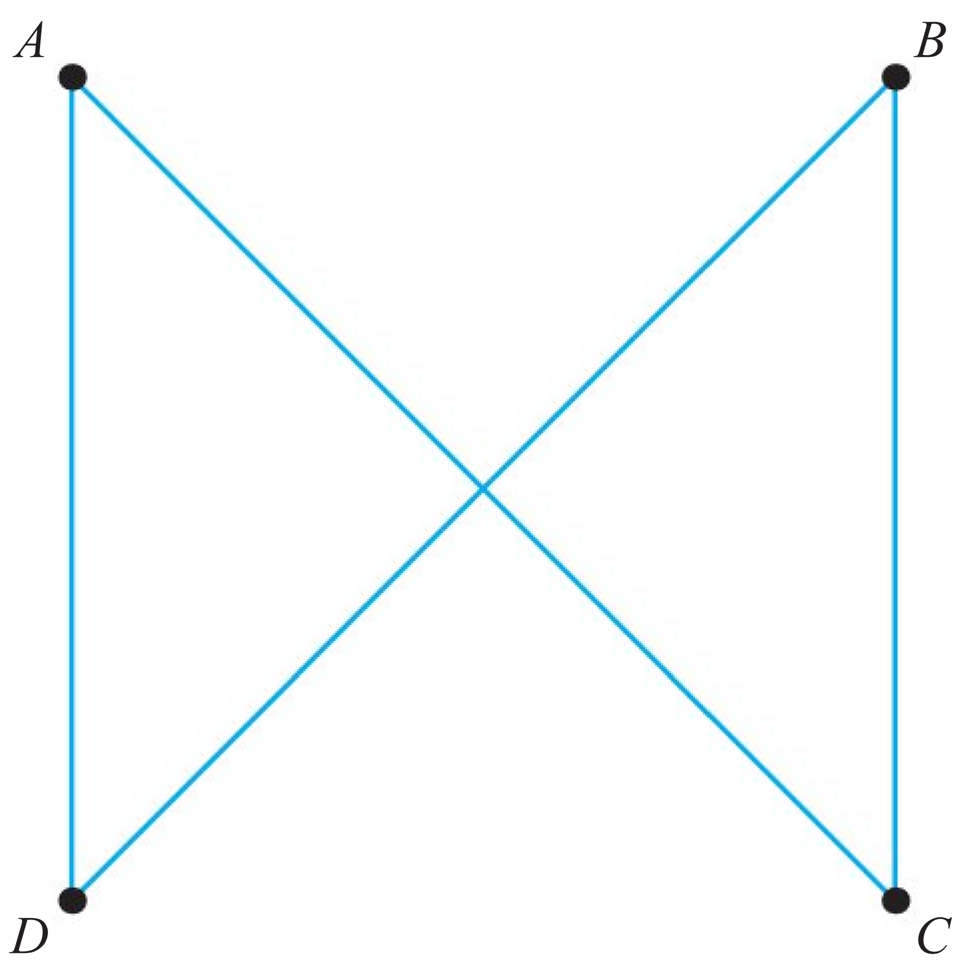
5 a $\begin{array}{c} A \\ B \\ C \\ D \end{array}\begin{pmatrix} 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 0 \\ 1 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \end{pmatrix}$ **b** $\begin{array}{c} A \\ B \\ C \\ D \end{array}\begin{pmatrix} 0 & 1 & 1 & 1 \\ 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 \\ 1 & 0 & 1 & 0 \end{pmatrix}$

6 a $\begin{array}{c} A \\ B \\ C \\ D \end{array}\begin{pmatrix} 2 & 2 & 0 & 1 \\ 2 & 0 & 1 & 1 \\ 0 & 1 & 2 & 2 \\ 1 & 1 & 2 & 0 \end{pmatrix}$ **b** $\begin{array}{c} A \\ B \\ C \\ D \end{array}\begin{pmatrix} 2 & 0 & 0 & 2 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 2 & 1 & 0 & 2 \end{pmatrix}$

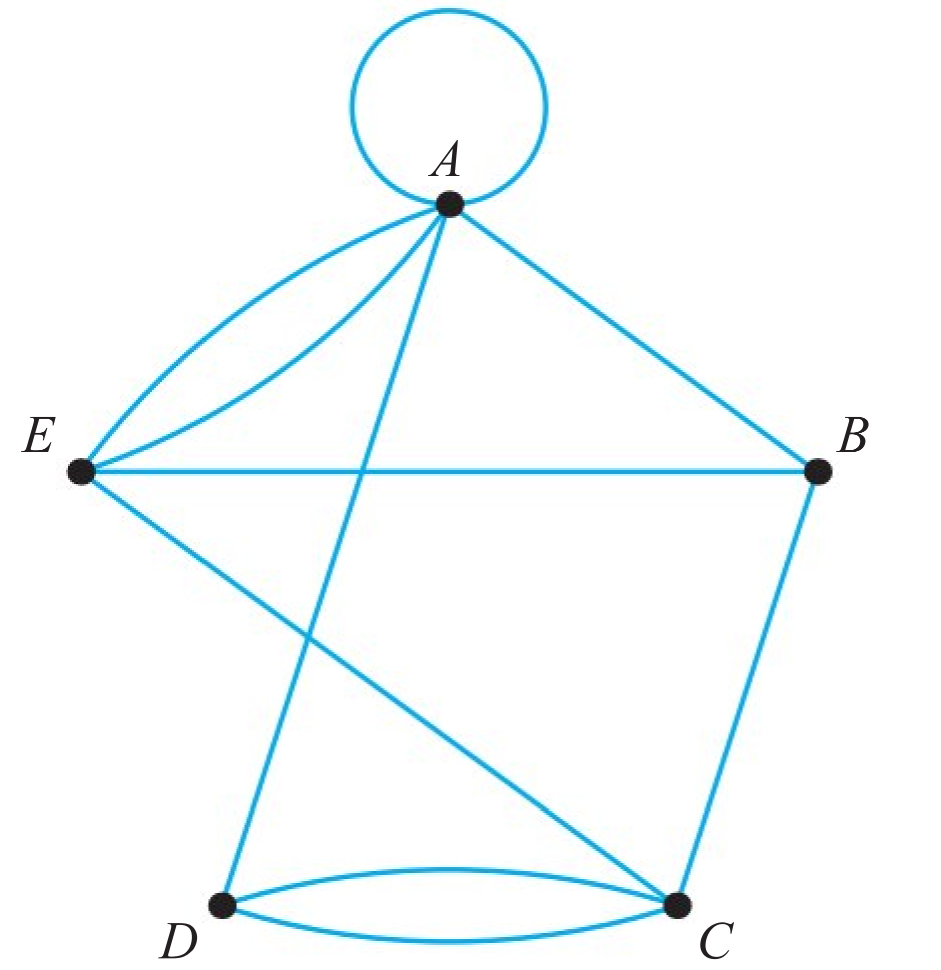
7 a



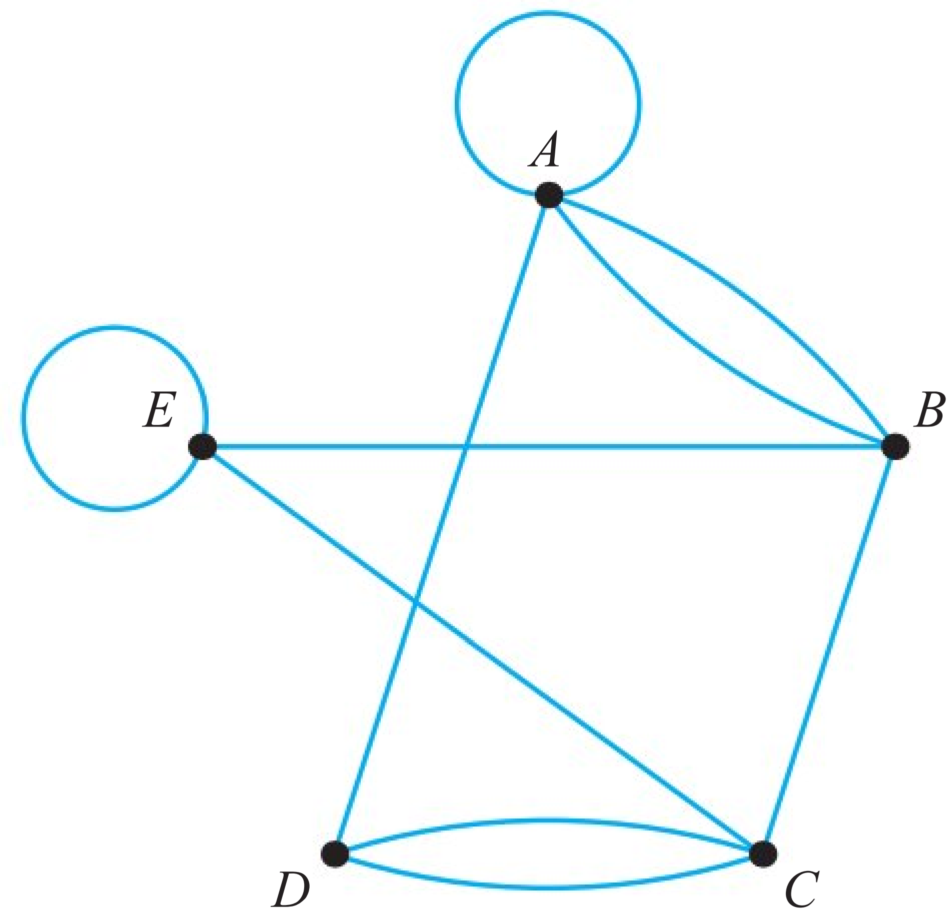
b



8 a



b



9 a i B, D **ii** B, E
iii 3 **iv** 2
b i A, C **ii** B, D, E
iii 2 **iv** 3

10 a i B, D **ii** B, D, E
iii 3 **iv** 5
b i C, D, E **ii** A, B, C
iii 8 **iv** 3

11 a i 2 **ii** 3
b i 2 **ii** 2

12 a i 4 **ii** 6
b i 2 **ii** 3

13 a
$$\begin{array}{c} A \\ B \\ C \\ D \\ E \\ F \\ G \end{array} \begin{pmatrix} 0 & 1 & 1 & 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 & 1 & 1 \\ 1 & 0 & 0 & 0 & 1 & 0 & 1 \\ 1 & 0 & 0 & 0 & 1 & 1 & 0 \end{pmatrix}$$

b 4; C is friends with 4 out of the other 6 people.

c Everyone is friends with the other six people.

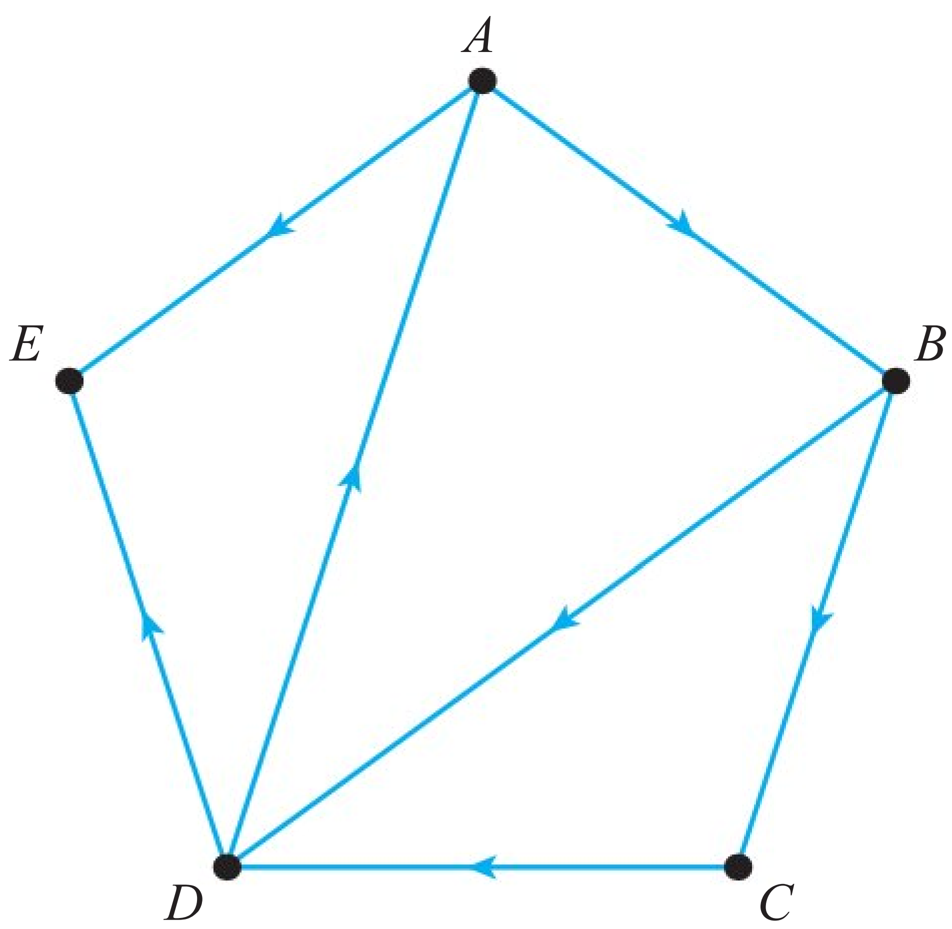
d 8

14 a There is a loop $ABDEA$.

b So that power can be supplied to each device.

c Any one of AB, BD, DE or EA.

15 a



b It is not possible to get from E to any other vertex.

c EA or ED

Exercise 7B

1 a $AEDB$, $ADCB$ b $CBAE$, $CDAE$

2 a $ABCDA$, $ABDEA$ b $CDABC$

3 a 6 b 7

4 a 14 b 19

5 a 0 b 1

6 a 1 b 1

7 a Eulerian; e.g. $ADBECABCDEA$

b Semi-Eulerian; e.g. $DCBAECA$

8 a Semi-Eulerian; e.g. $FEDCABCFD$

b Neither

9 a $ACBDEA$ (there are alternative solutions)

b $ABDCEA$

10 a $ABFECDA$

b $ADEBCFA$ (there are alternative solutions)

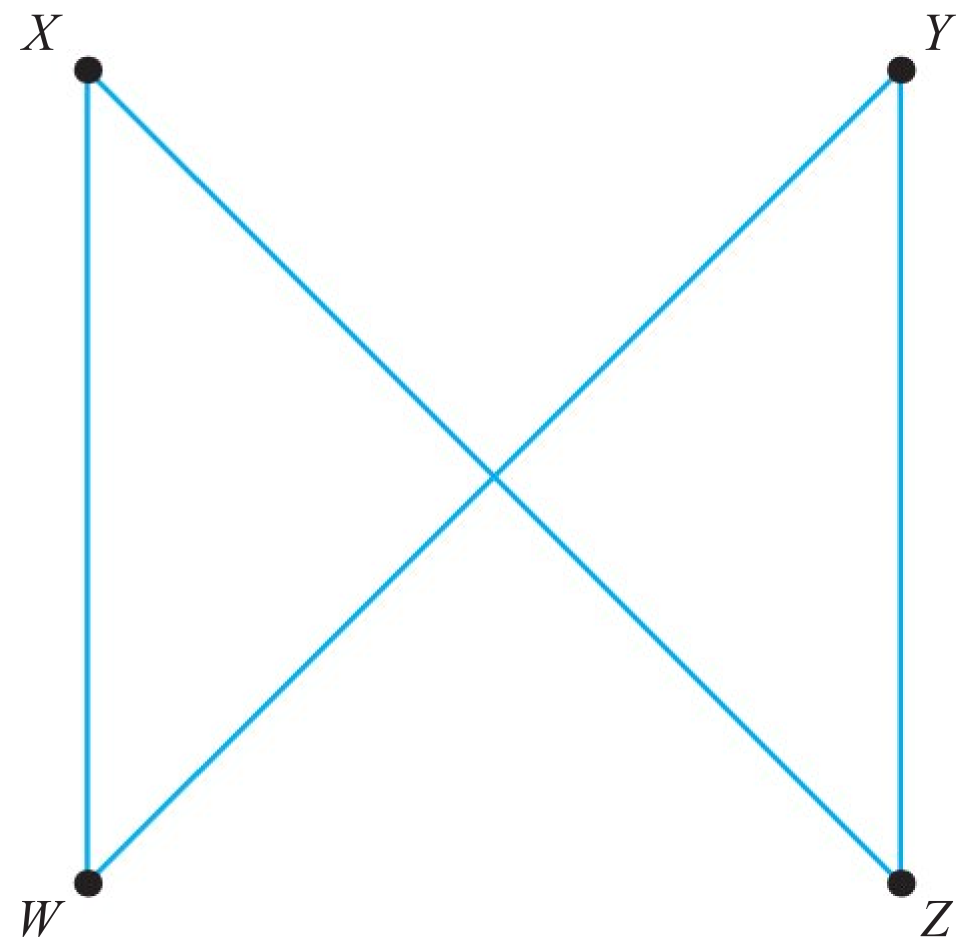
11 a Four vertices have odd degree

b $SMNPQNRQMRS$ (there are alternative solutions)

12 There are two vertices of odd degree: $UWXVYZWUVZ$ (there are alternative solutions)

13 a B and C

b



c 16

14 a e.g. Hamiltonian cycle $ABEDCA$

b 80

Exercise 7C

1 a

	A	B	C	D	E
A	–	34	20	18	–
B	34	–	20	–	25
C	20	20	–	12	–
D	18	–	12	–	15
E	–	25	–	15	–

b

	A	B	C	D	E
A	–	9	–	–	9
B	9	–	6	12	–
C	–	6	–	–	14
D	–	12	–	–	8
E	9	–	14	8	–

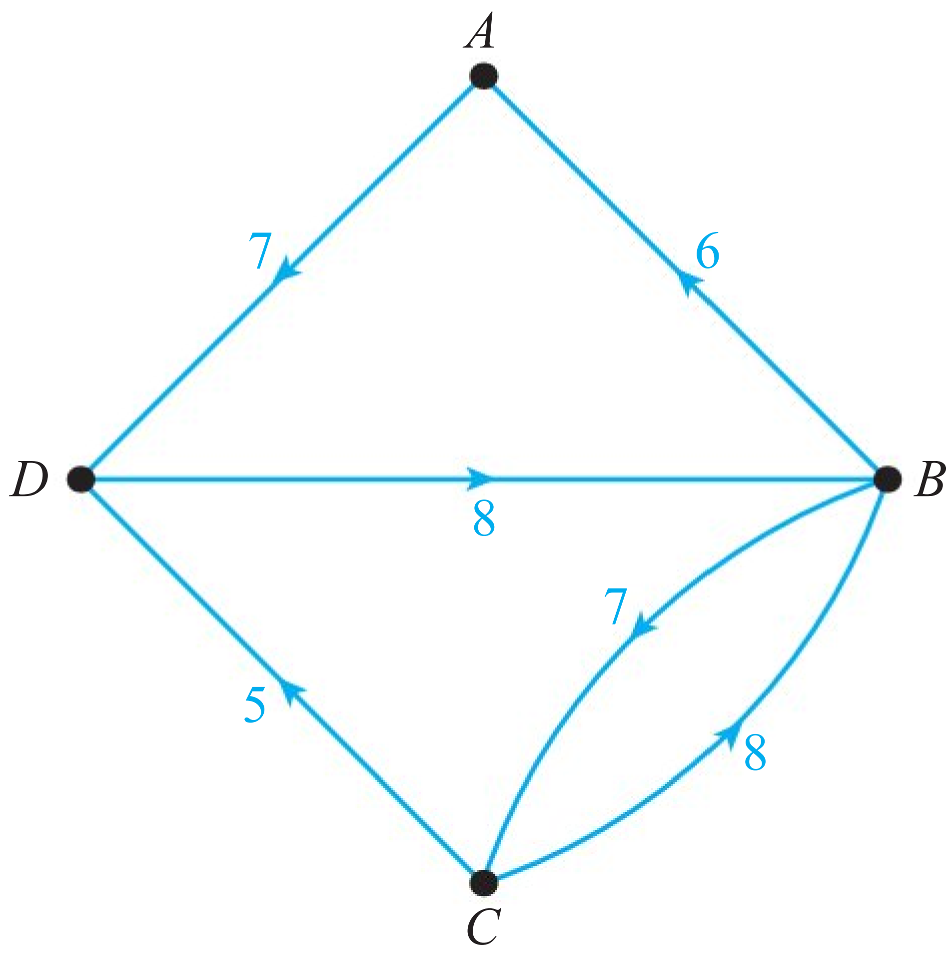
2 a

	A	B	C	D
A	–	6	5	7
B	8	–	10	–
C	–	–	–	–
D	7	–	6	–

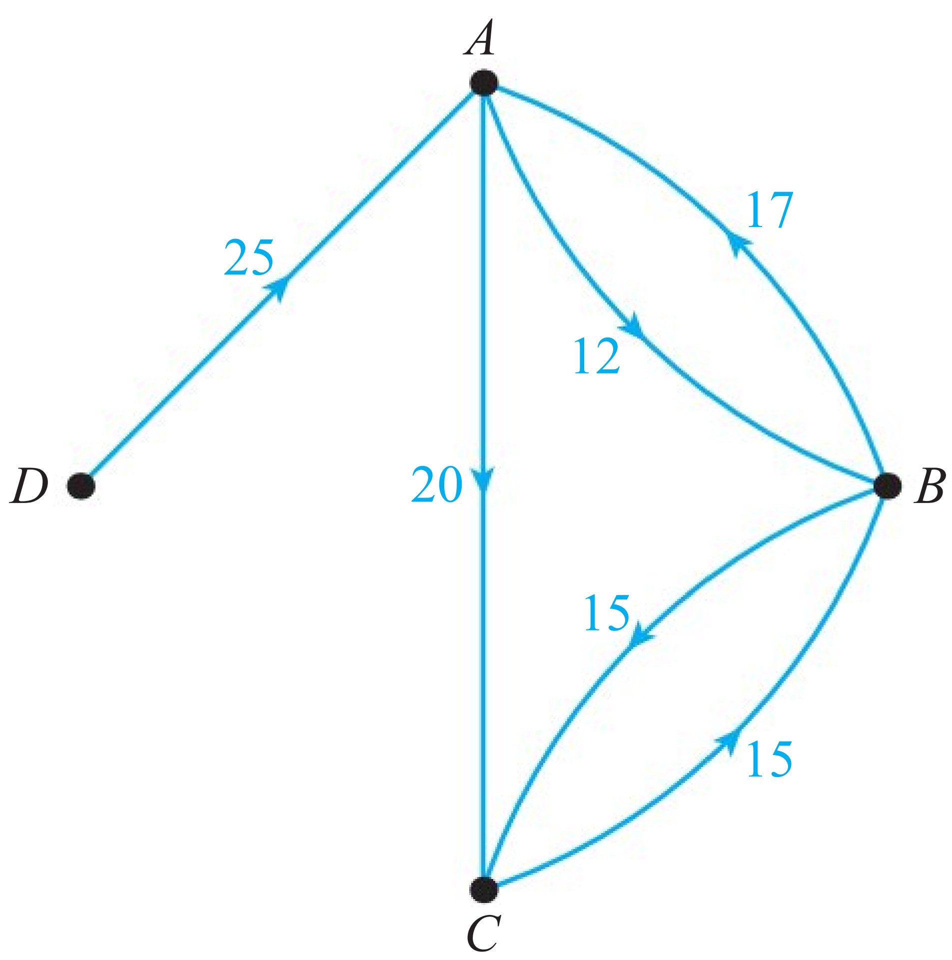
b

	A	B	C	D
A	–	–	8	8
B	5	–	6	–
C	10	–	–	6
D	–	–	7	–

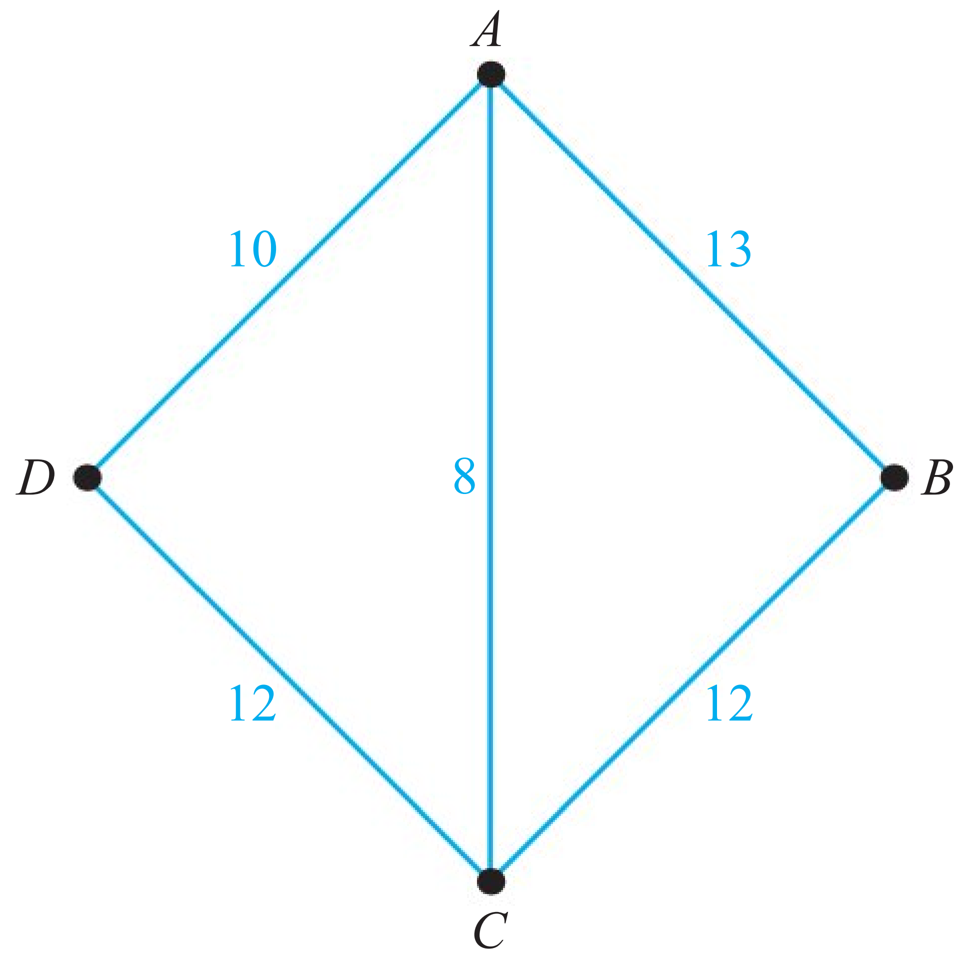
3 a



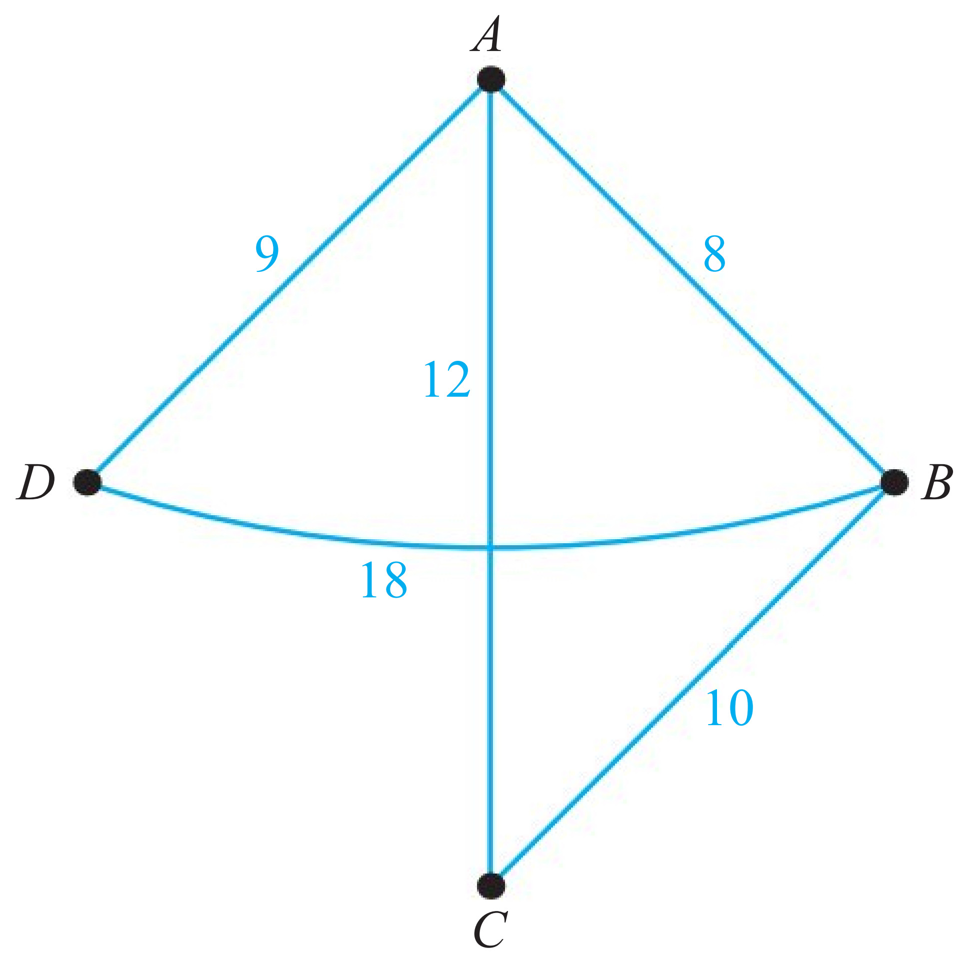
b



4 a



b



5 a

$$\begin{pmatrix} 0 & 1/2 & 0 & 1/4 & 1/3 \\ 1/3 & 0 & 0 & 1/4 & 0 \\ 0 & 0 & 0 & 1/4 & 1/3 \\ 1/3 & 1/2 & 1/2 & 0 & 1/3 \\ 1/3 & 0 & 1/2 & 1/4 & 0 \end{pmatrix}$$

b

$$\begin{pmatrix} 0 & 1/4 & 0 & 1/4 & 1/3 \\ 1/3 & 0 & 1/2 & 1/4 & 1/3 \\ 0 & 1/4 & 0 & 1/4 & 0 \\ 1/3 & 1/4 & 1/2 & 0 & 1/3 \\ 1/3 & 1/4 & 0 & 1/4 & 0 \end{pmatrix}$$

6 a $\begin{pmatrix} 0 & 1/2 & 0 & 0 \\ 1/2 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1/2 & 1/2 & 0 & 0 \end{pmatrix}$

b $\begin{pmatrix} 0 & 0 & 0 & 1/2 \\ 1/3 & 0 & 0 & 0 \\ 1/3 & 1 & 0 & 1/2 \\ 1/3 & 0 & 1 & 0 \end{pmatrix}$

7 a 0.143 b 0.125

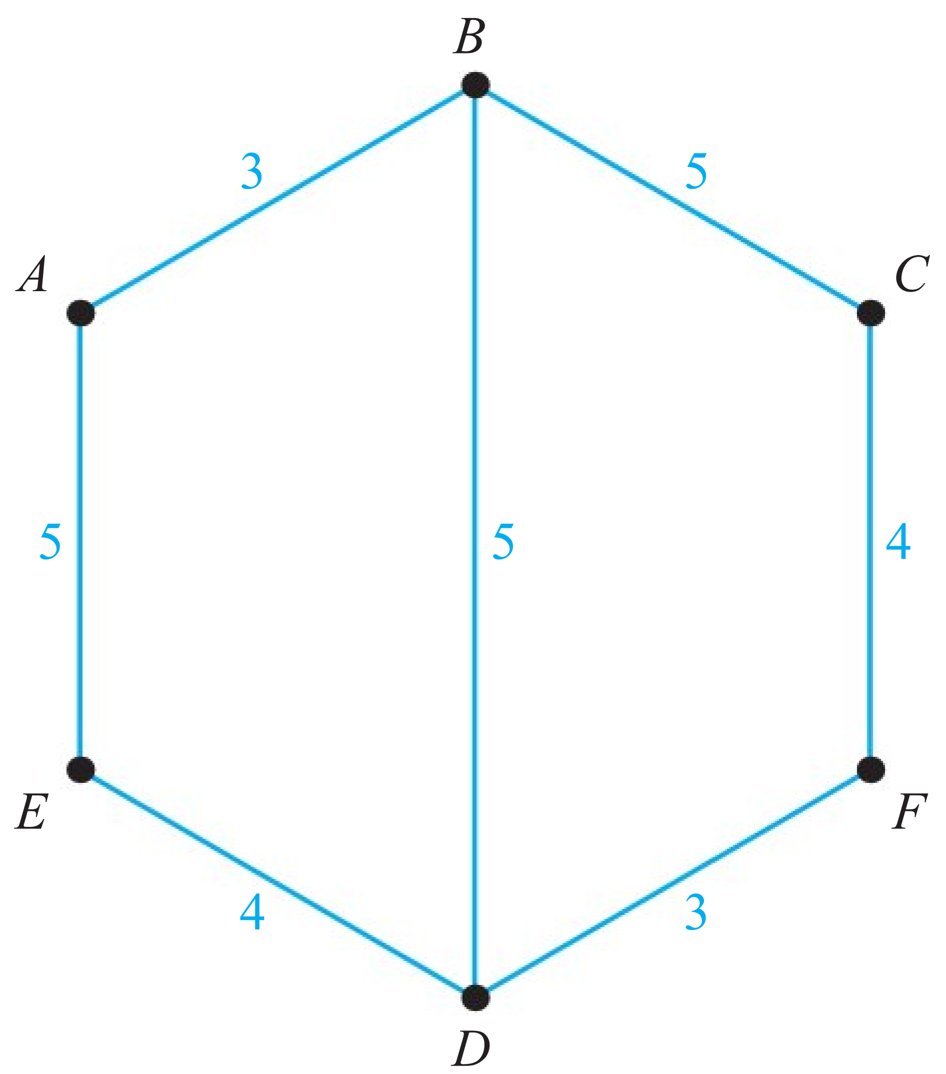
8 a 0.167 b 0.400

9 a B (0.352) b C (0.421)

10 a A and B (0.286) b B and D (0.333)

11 a 1500 km (via A) b \$350 (via E)

12 a



b ABD, 8 minutes

13 a 12 km ($SCBDT$)

b $SCBT$ or $SCDT$ (13 km)

14 a $\begin{pmatrix} 0 & 1/4 & 0 & 0 & 0 \\ 1 & 0 & 1/2 & 1/3 & 1/2 \\ 0 & 1/4 & 0 & 1/3 & 0 \\ 0 & 1/4 & 1/2 & 0 & 1/2 \\ 0 & 1/4 & 0 & 1/3 & 0 \end{pmatrix}$

b 0.145 c B (33.3%)

15 A (0.293), E (0.220), C (0.195), B and D (0.146)

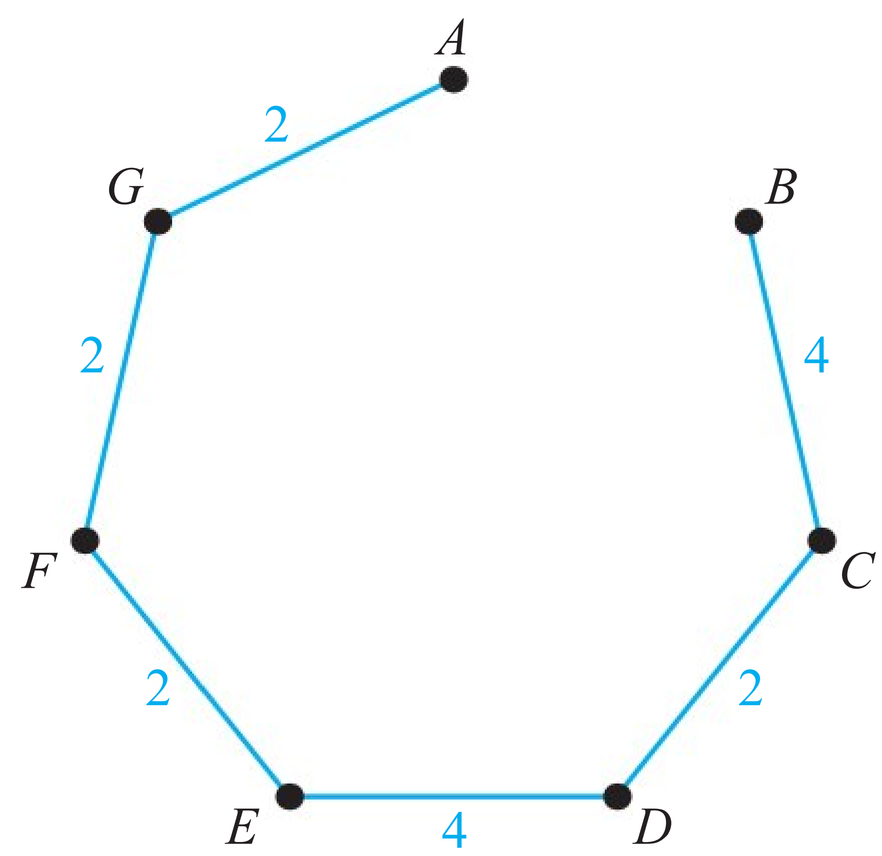
16 a D (0.387)

b $\begin{pmatrix} 0 & 0 & 0 & 0.3 & 0 \\ 0.45 & 0 & 0 & 0.3 & 0 \\ 0.45 & 0.45 & 0 & 0.3 & 0 \\ 0 & 0.45 & 0.9 & 0 & 0 \\ 0.1 & 0.1 & 0.1 & 0.1 & 1 \end{pmatrix}$

c No, relative probabilities stay the same.

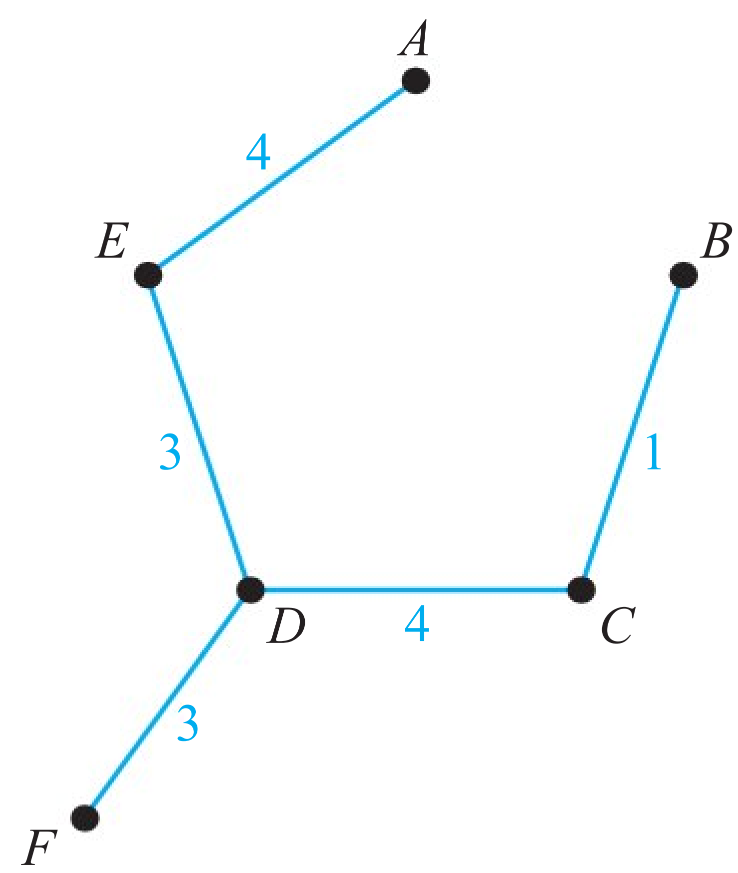
Exercise 7D

1 a



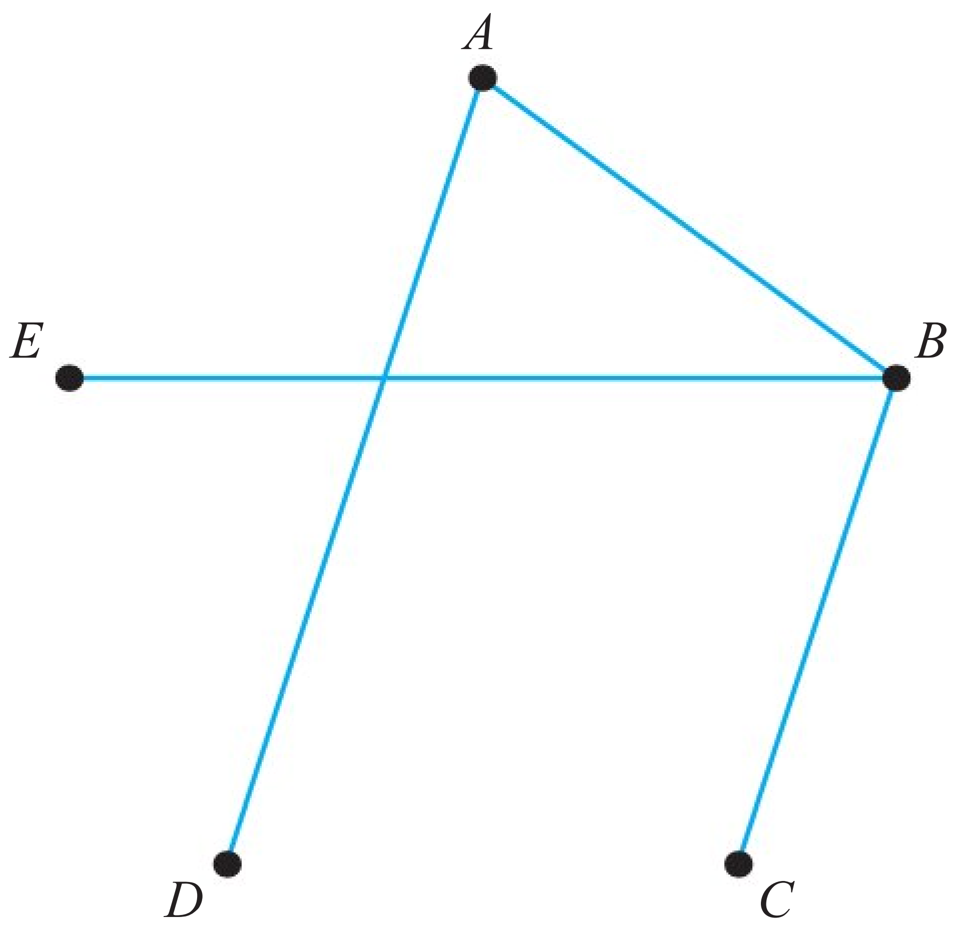
Weight = 16

b



Weight = 15

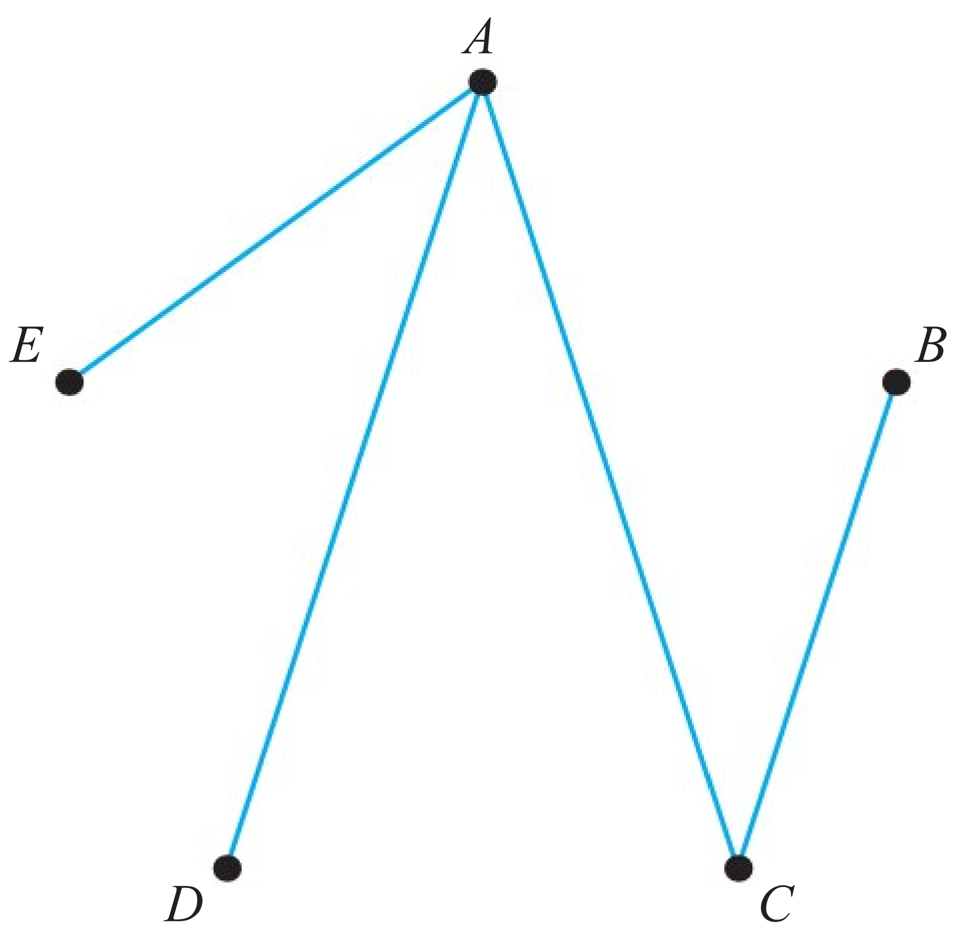
2 a



Weight = 41

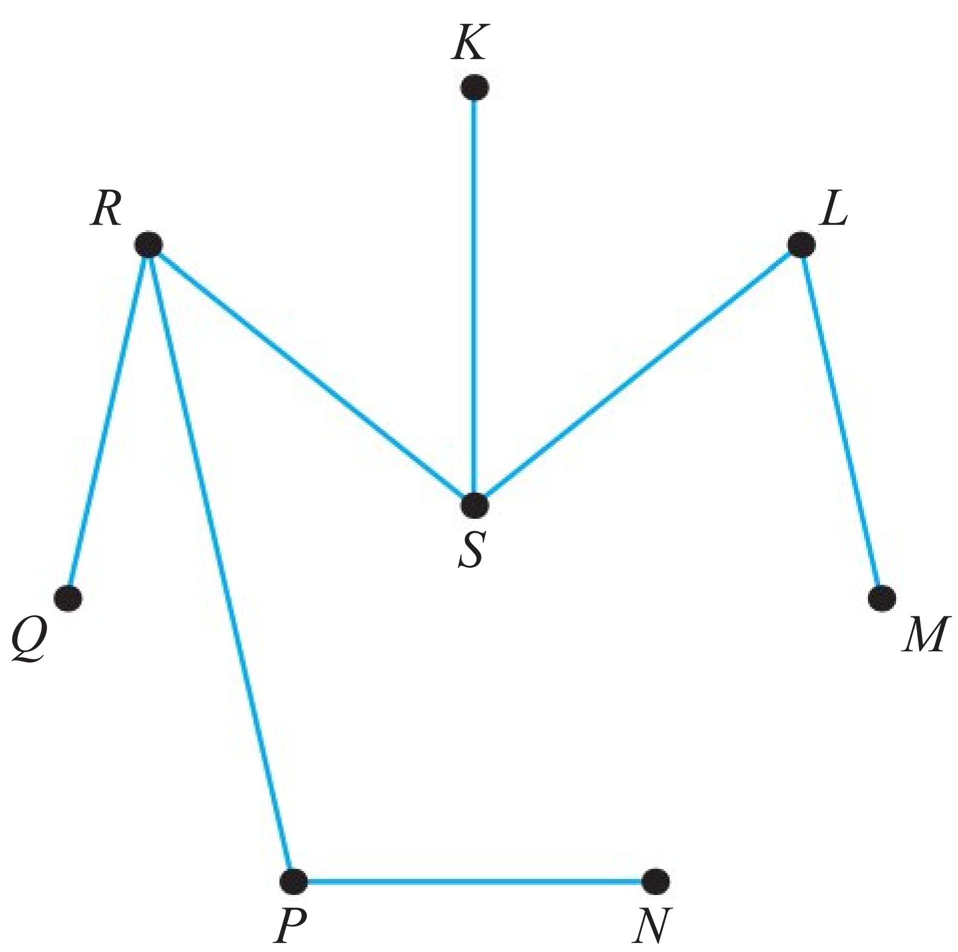
(can replace *BE* by *CE*)

b



Weight = 28

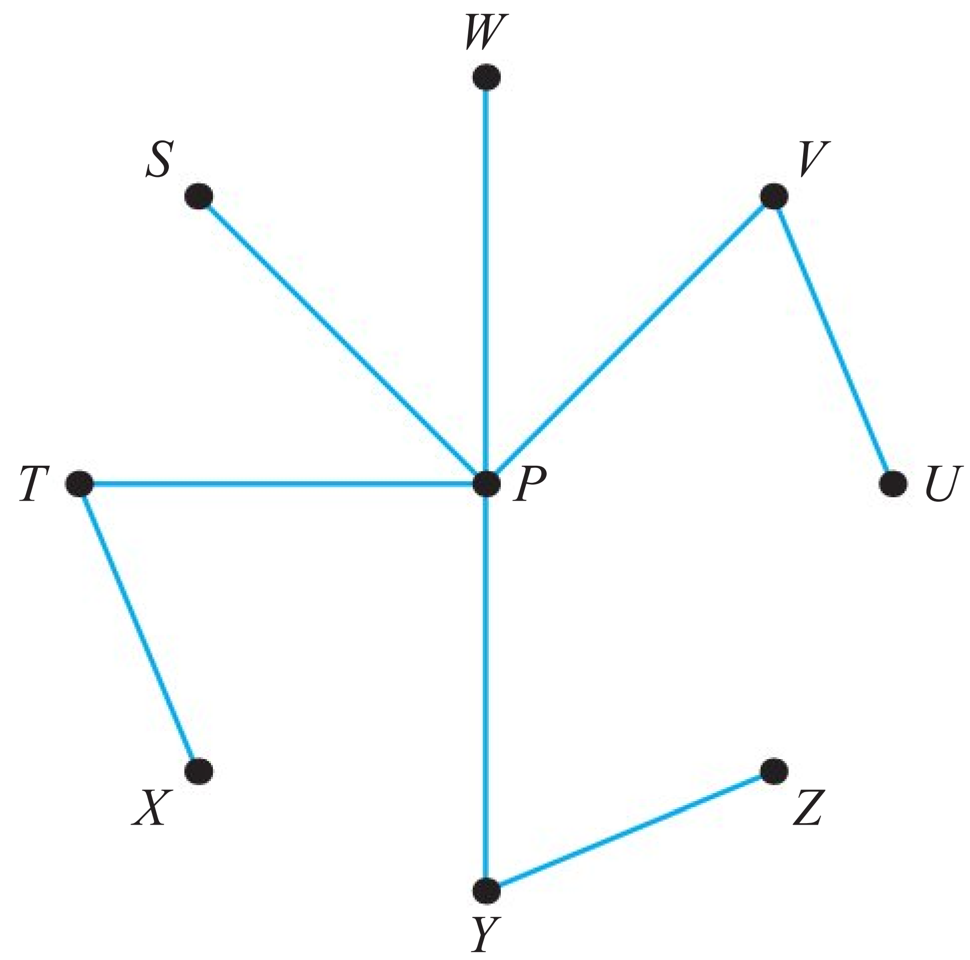
3 a



Weight = 47

(can replace *RP* by *RS*)

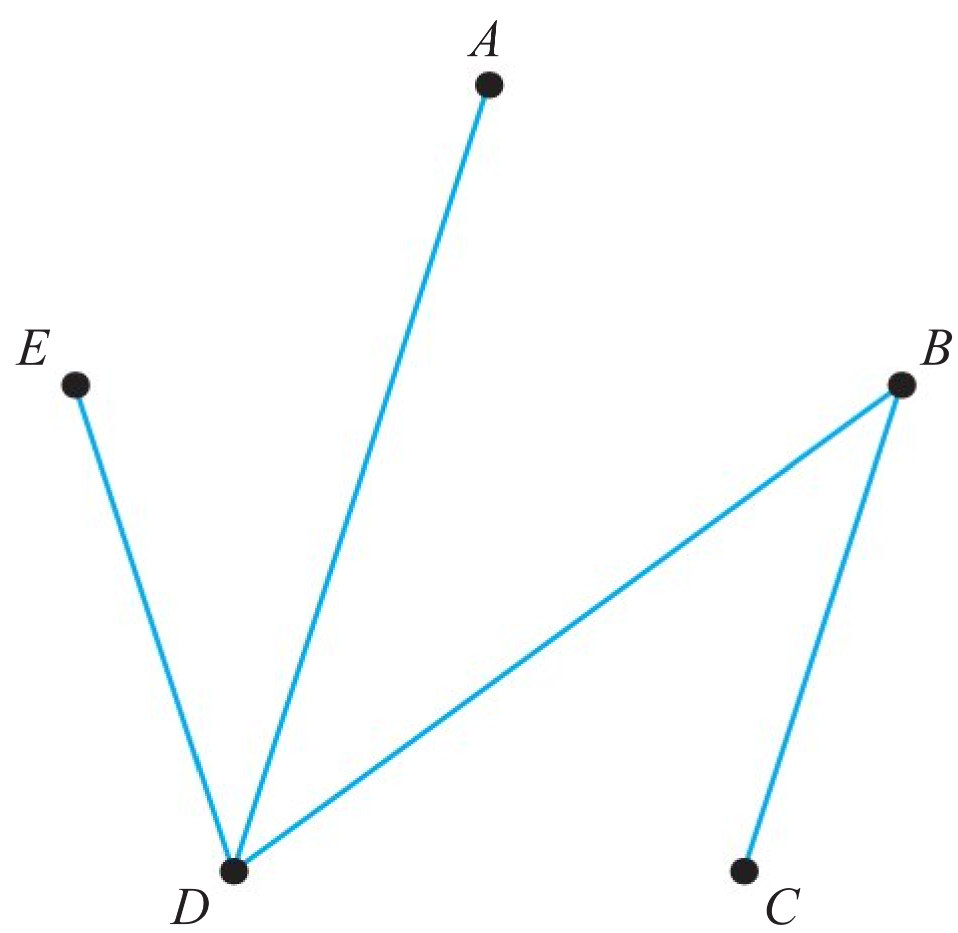
b



Weight = 92

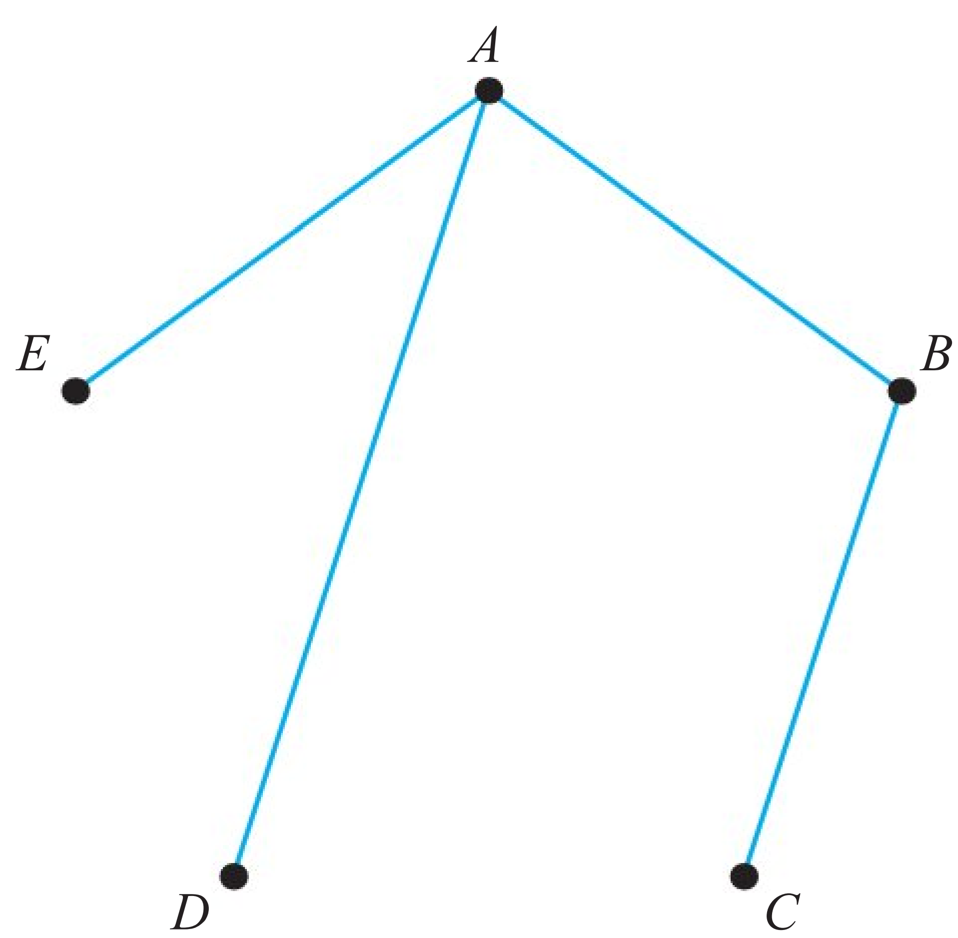
(can replace *PS* by *WS*)

4 a



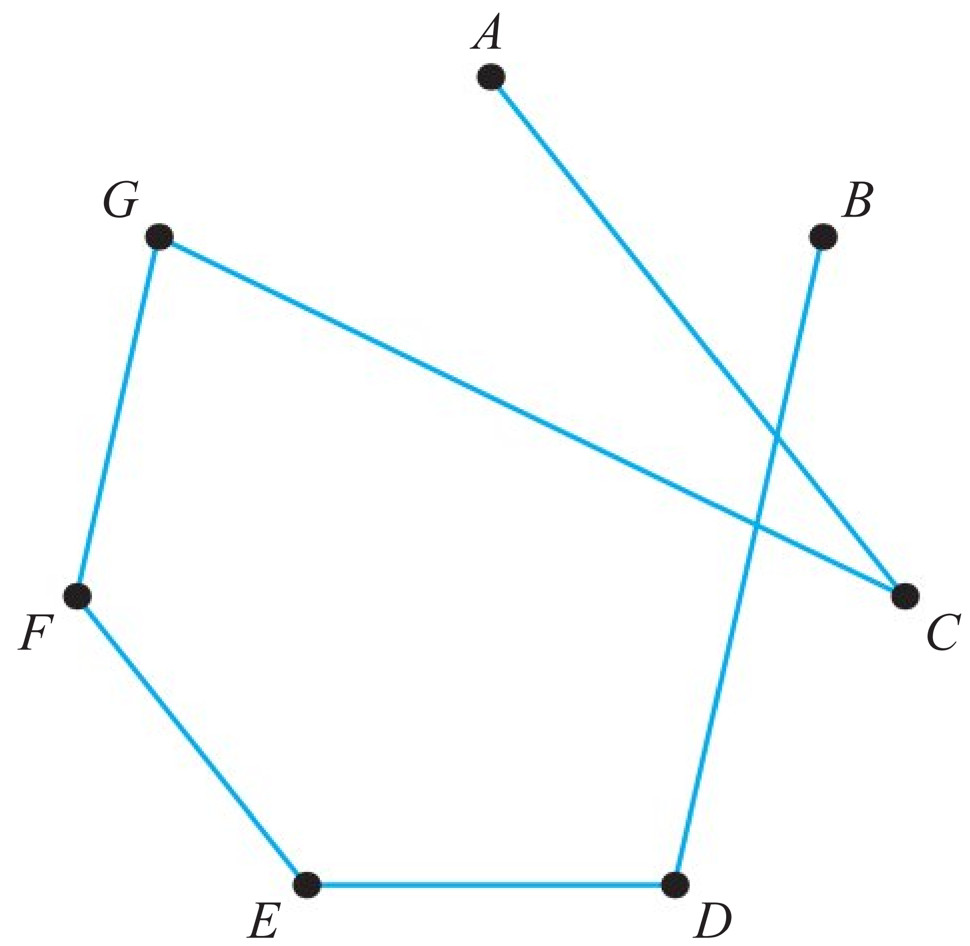
Weight = 24

b



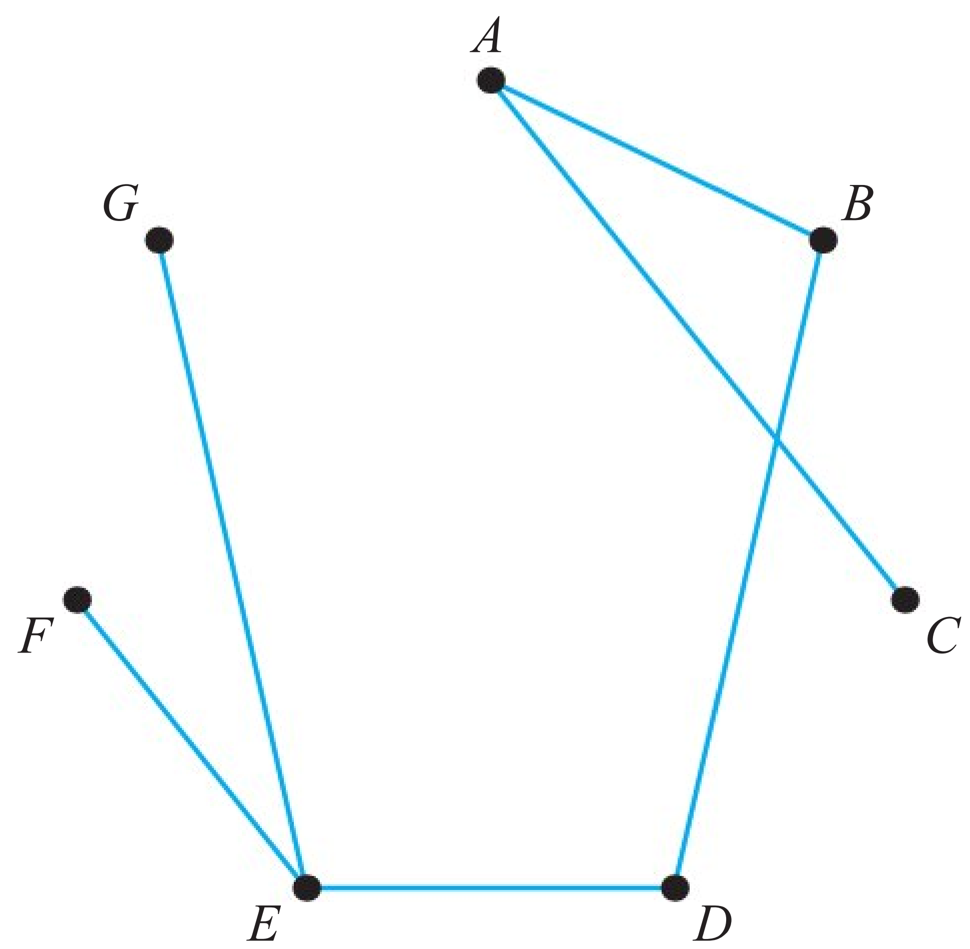
Weight = 52

5 a



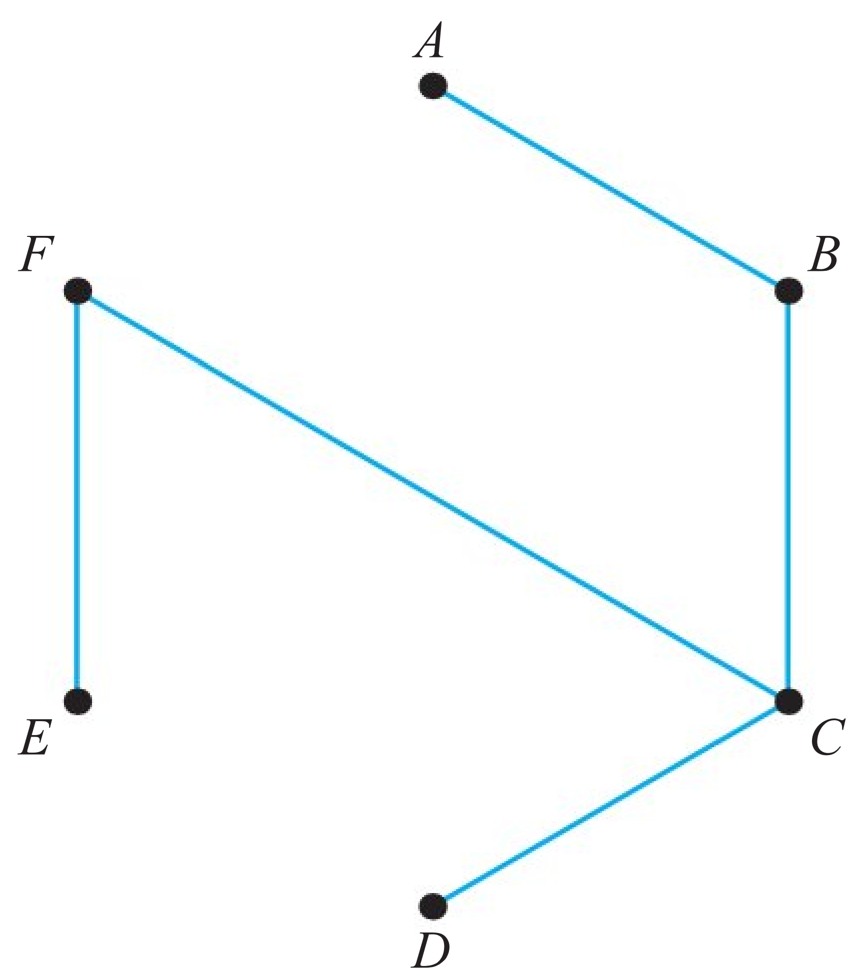
Weight = 120

b



Weight = 155

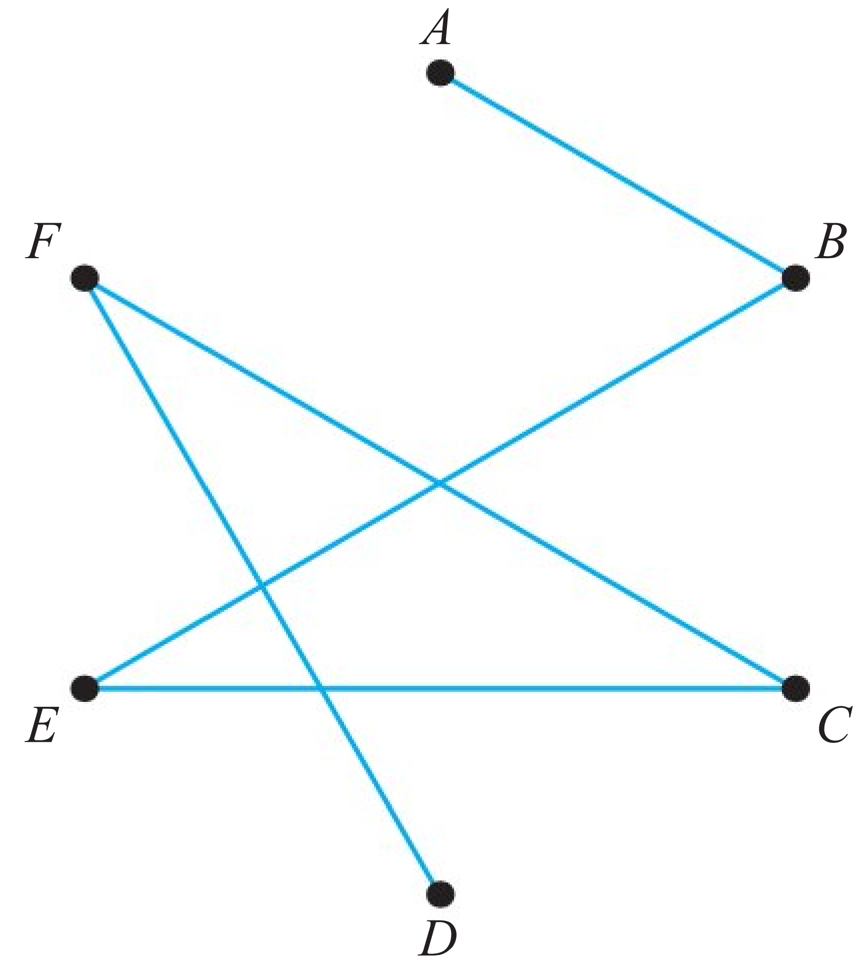
6 a



(or *DA* instead of *DC*)

Weight = 21

b



Weight = 19

7 a *AB*, *BD*, *DF*, *AC*, *CE*

b *AB*, *AD*, *DF*, *FC*, *CE*

8 a *AB*, *AE*, *ED*, *EC* (or *AC*)
or *AE*, *ED*, *AB*, *AC* (or *EC*)

b *AB*, *BC*, *CD*, *DE*
or *AE*, *AB*, *ED*, *BC*
or other combinations

9 a *AB*, *BG*, *BH*, *HE*, *HD*, *DC*, *GF*

b *AG*, *GB*, *BC*, *GD*, *DE*, *AF* (or *EF*)
or *AG*, *GD*, *DE*, *GB*, *BC*, *AF* (or *EF*)

10 a *AD*, *DB*, *BC*, *DE*

b *AD*, *AB*, *BC*, *EA*

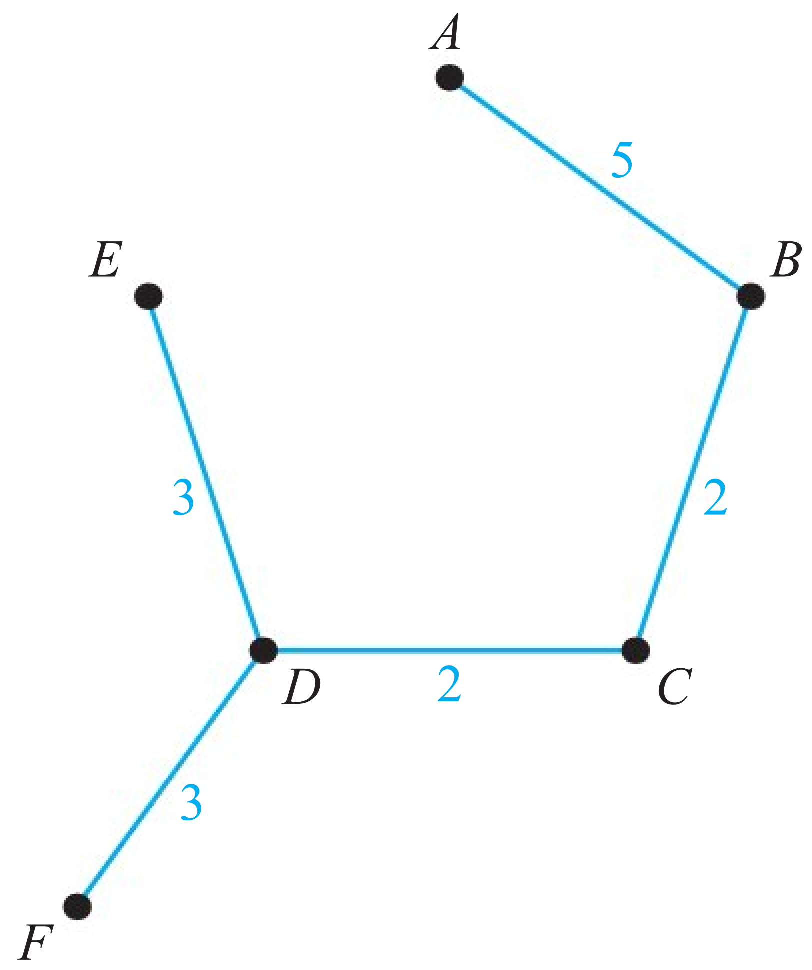
11 a *ED*, *EF*, *FG*, *GC*, *CA*, *DB*
or *ED*, *DG*, *EF*, *GC*, *CA*, *DB*

b *EF*, *ED*, *EG*, *DB*, *BA*, *AC*

12 a *CF*, *FE*, *EB*, *BA*, *DA* (or *DC*)
or *CF*, *FE*, *CB*, *BA*, *DC* (or *DA*)

b *CF*, *FD*, *EC*, *EB*, *BA*

13 a



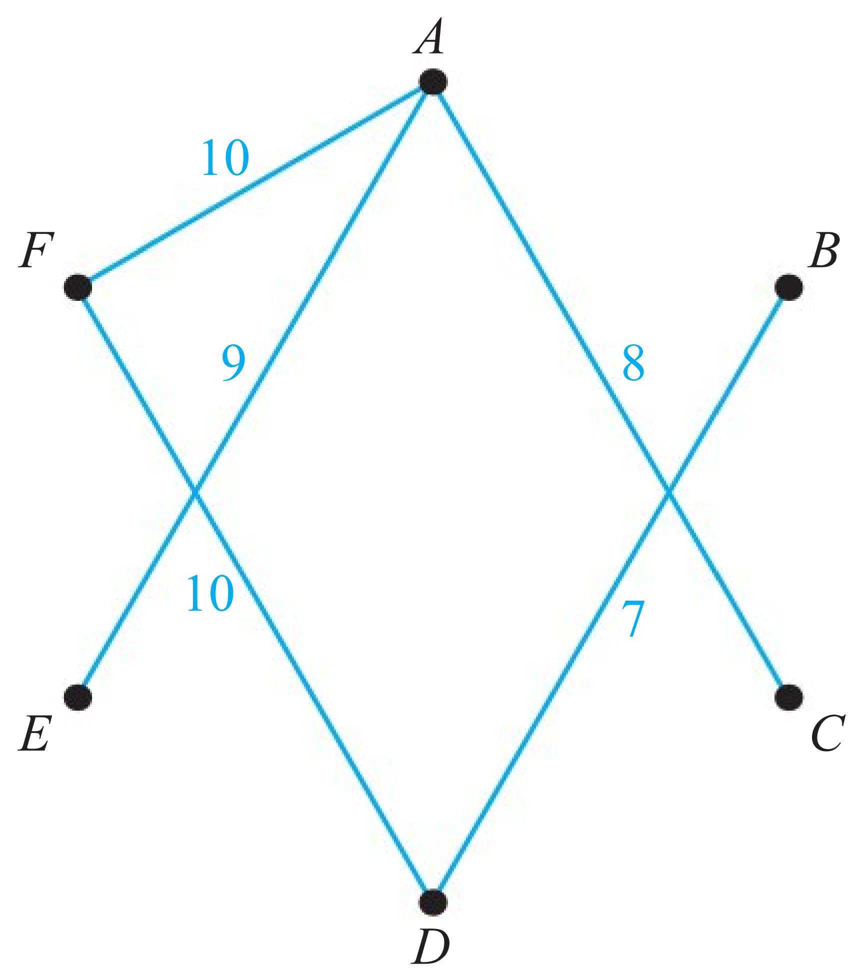
b 15 km

14 a BD, CD, CE, AB, AF (or EF)

b Add CF first, then apply the usual algorithm.

15 a Kruskal adds one edge at a time, Prim adds one vertex at a time.

b



c 44 m

d i Each workstation is connected to the power supply.

ii There are no unnecessary connections.

16 a 49 b 58

17 $0 < x < 7$ or $21 < x < 28$

Exercise 7E

1 a DE, EF (70) b DF, FG (45)

2 a AF, FE (52) b AF, FE (34)

3 a AB, EG (78) b BC, CD, AF, FE (93)

4 a AB, CF, FH (137) b BE, EF, DG (107)

5 a O, C b 9

c 81 d OB, BC

6 a F and D ($FABFEDCBD$); 6800 m

b Vertices D and F have odd degrees.

c 8050 m

d FE and ED

7 e.g. $CBAJIHGFEDBDFHJCFDC$, 8.7 km

8 a 99 minutes (NB, I_1, SB, I_1, I_2, SB, I_1, NB)

b 130 minutes

c e.g. The time to walk from one bridge to another

9 a AD and EH

b 109 minutes

c 100 minutes

d e.g. $ABCDACFEDEHFGH$

10 a 208 m b FG, DH

c 208 m d 230 m

11 a C and F have odd degrees.

b CEF (15)

c $DEFDBACBECFECD$ (180)

d i 165 ii C and F

Exercise 7F

1 a 36 ($ADCBA$) b 85 ($ACBDA$)

2 a 63 ($ACDEBA$) b 98 ($ADBECA$)

3 a 98 ($AECBDA$) b 110 ($ADCBEA$)

4 a 32 b 77

5 a 46 b 81

6 a 95 b 105

7 a

	A	B	C	D	E
A	–	4	10	9	5
B	4	–	6	5	9
C	10	6	–	11	15
D	9	5	11	–	4
E	5	9	15	4	–

b

	A	B	C	D	E
A	–	10	9	20	8
B	10	–	12	18	6
C	9	12	–	12	11
D	20	18	12	–	12
E	8	6	11	12	–

8 a

	A	B	C	D
A	–	8	10	7
B	8	–	7	9
C	10	7	–	6
D	7	9	6	–

b

	A	B	C	D
A	–	30	28	12
B	30	–	12	18
C	28	12	–	17
D	12	18	17	–

9 a

	A	B	C	D
A	–	4	7	6
B	4	–	3	8
C	7	3	–	5
D	6	8	5	–

b

	A	B	C	D
A	–	20	38	35
B	20	–	18	15
C	38	18	–	35
D	35	15	35	–

10 a *ABCDA*, *ABDCA*, *ACBDA*

b *ABDCA* (45)

11 a 32

b 31

c It is at most 31 km long.

12 a The cycle *ACBEDA* has length 29.

b 28

c 24

d It is 24, 25, 26, 27 or 28 minutes.

13 a *PUSR* is 40 km.

b *PUQ*, 25 km

c

	P	Q	R	S	T	U
P	–	**25**	40	**24**	**26**	14
Q	**25**	–	20	21	23	11
R	40	20	–	**16**	**28**	26
S	**24**	21	**16**	–	**12**	10
T	**26**	23	**28**	**12**	–	12
U	14	11	26	10	12	–

d 119 km

e 79 km

14 a 40

b 40

c The lower and upper bounds are equal.

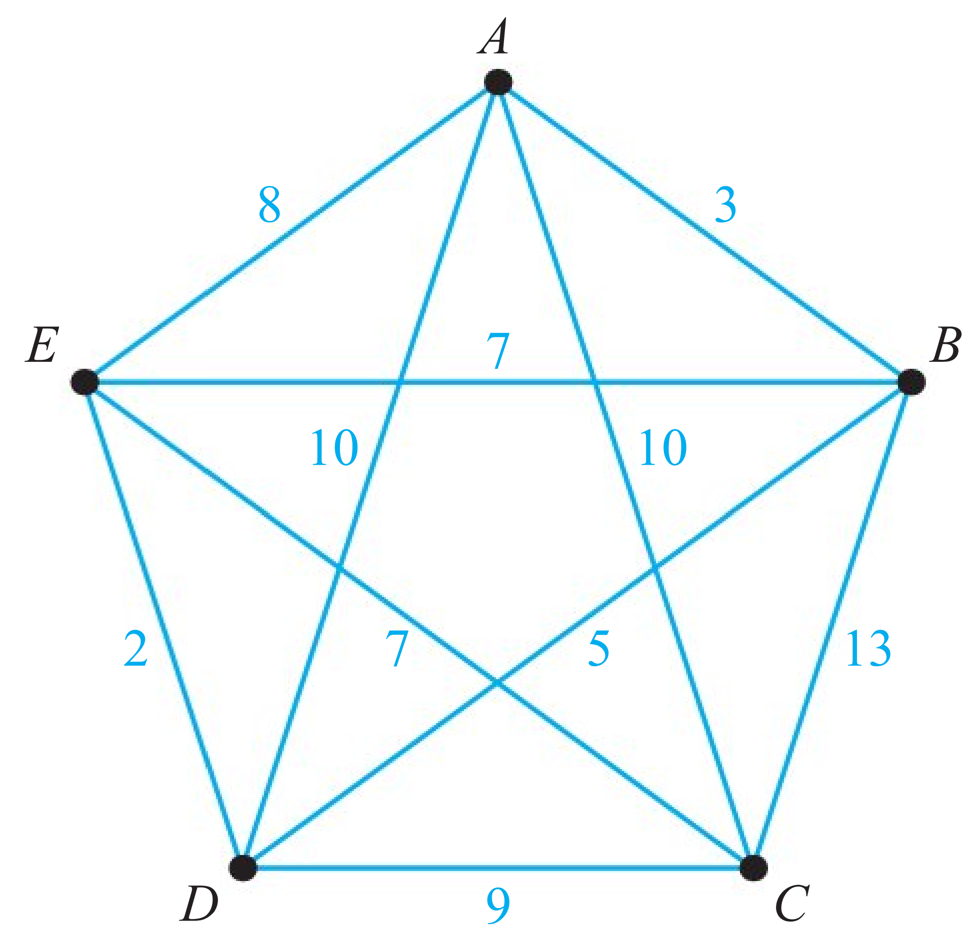
d *ABCDEFA*

15 a i *BDE* (7)

ii *BAC* (13)

iii *CED* (9)

b



c 27 (*CEDBAC*)

d 26

e The lower bound is unattainable; the paths *CD* and *CE* cannot be linked to either end of a route, nor can *CE* twice. *CD* and *CA* is the next best option, and can form a route.

f 27 minutes: *CEDBAC*

Chapter 7 Mixed Practice

1 a The second one, all vertices have even degree.

b e.g. *ABCDEBDA*

c *C* or *E*

2 a *RS*

b e.g. *PQRSUV*

c e.g. *PQRSVUP*

3 a

	A	B	C	D	E
A	0	0	0	1	1
B	0	0	1	1	1
C	0	1	0	1	0
D	1	1	1	0	0
E	1	1	0	0	0

b 20

c B and D

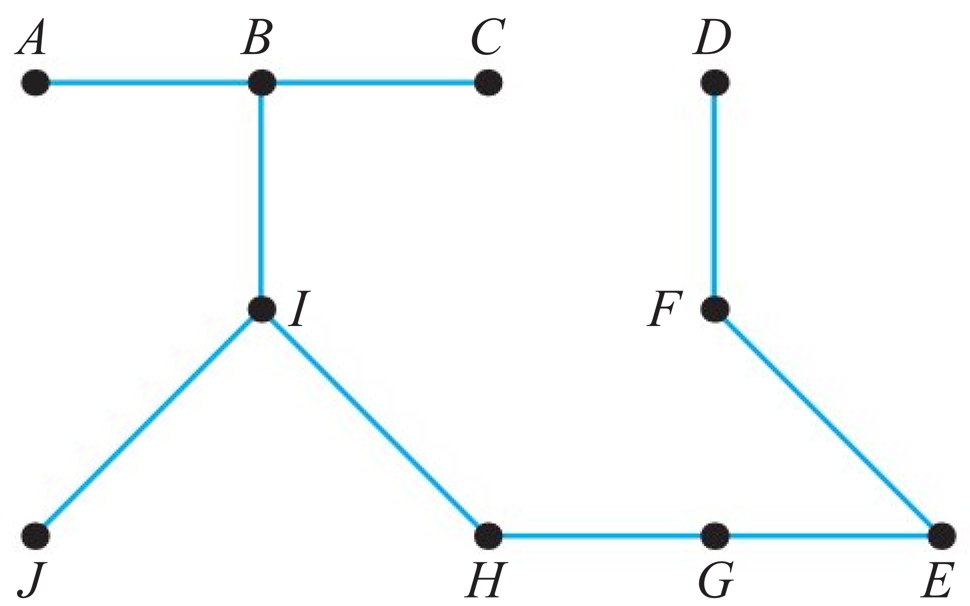
4 a A tree of minimum weight which includes every vertex of the graph.

b BH, NF, HN, HA, BE, CN; weight = 48

c BH, NF, HN, HA, BE, CN

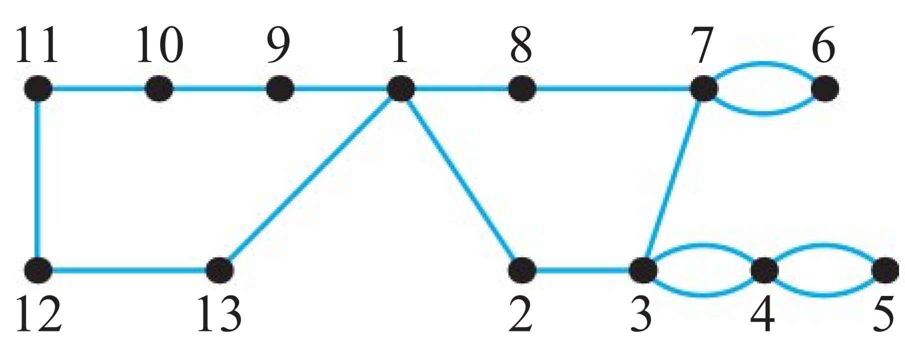
5 a Kruskal's adds one edges at a time, Prim's adds one vertex at a time.

b



Length = 117

6 a i



ii deg 2: 2, 5, 6, 8, 9, 10, 11, 12, 13;
deg 4: 1, 3, 4, 7

iii All vertices have even degree.

b i G: 15, H : 22

ii G: exactly two vertices have odd degrees (E and F).

iii Both graphs have vertices of odd degree.

7 a Hamiltonian cycle

b 111 ($BAFCDEB$)

c 106

d $106 \leqslant T \leqslant 111$

8 a $n-1$

b i AB, BC, AD, DE

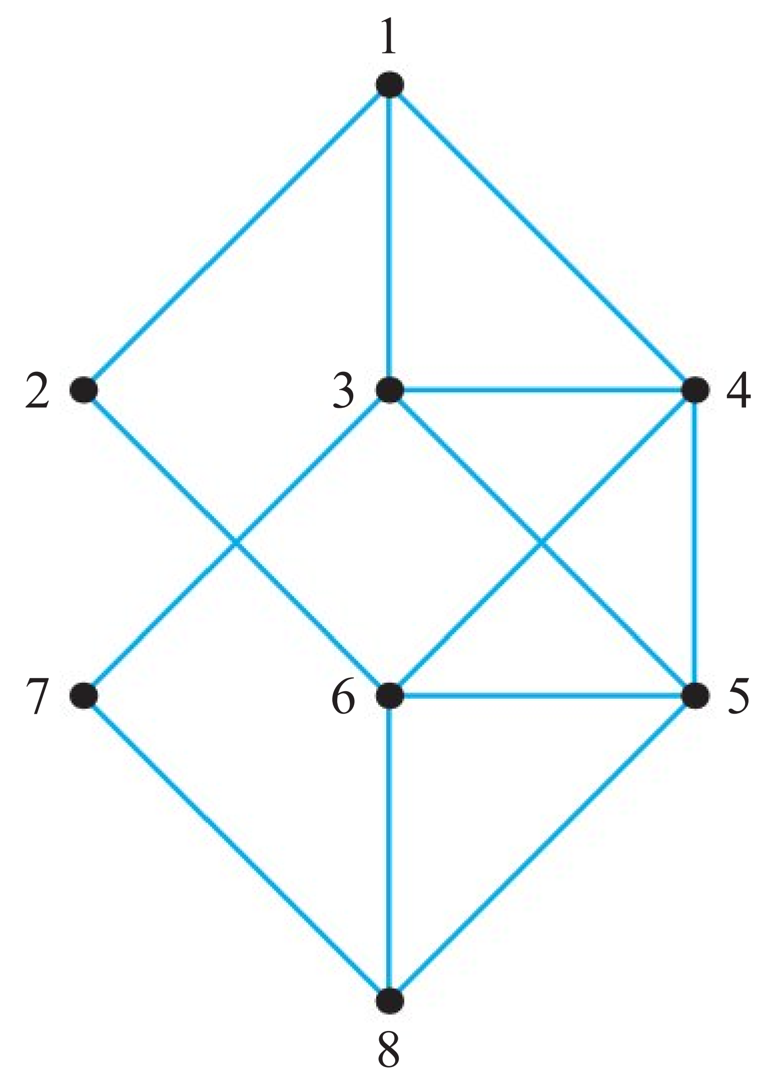
c i All vertices have even degree.

ii e.g. $ABCDEACEBDA$, 46

9 $C(0.293)$, $E(0.244)$, $D(0.195)$, $B(0.171)$, $A(0.098)$

10 276

11 a



b Vertices 1 and 8 have odd degrees.

c

0	1/2	1/4	1/4	0	0	0	0
1/3	0	0	0	0	1/4	0	0
1/3	0	0	1/4	1/4	0	1/2	0
1/3	0	1/4	0	1/4	1/4	0	0
0	0	1/4	1/4	0	1/4	0	1/3
0	1/2	0	1/4	1/4	0	0	1/3
0	0	1/4	0	0	0	0	1/3
0	0	0	0	1/4	1/4	1/2	0

d 0.215 e 11.5%

12 a A and E have odd degrees.

b 250 m c A and E

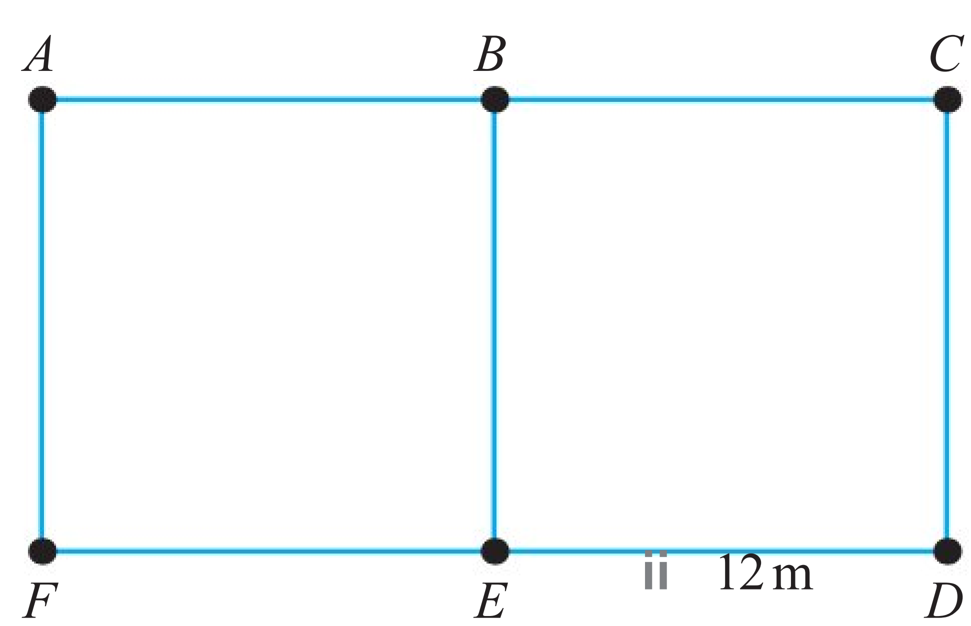
13 b i e.g $PQRSTSRTQP$

ii 34

c i To find a Hamiltonian cycle of least weight.

ii There is no cycle containing vertex P.

14 a



ii 12 m

b The number of walks of length 2 from a vertex to itself

c 8

d $ABEDC$, $AFEBC$, $AFEDC$

15 a 63

b 63

c $ADCBEA$ is a solution to the travelling salesman problem.

16 a i There are vertices of odd degree.

ii No, there are more than two vertices of odd degree.

b i All degrees have doubled, so all are even

ii 306 km

17 \$92, repeat AE, EB, DG, GC

18 a

	A	B	C	D	E	F
A	–	8	8	16	18	18
B	8	–	7	15	17	17
C	8	7	–	8	10	10
D	16	15	8	–	6	8
E	18	17	10	6	–	9
F	18	17	10	8	9	–

b 52 c 56

d $52 \leqslant T \leqslant 56$ e $CBACDEFC$

19 a CD, AB, BC, BF, CE

b i 231

ii 239

iii 239

c $239 \leqslant L \leqslant 253$

20 a 33

b There is a cycle of length 33 ($ABCDEA$).

Chapter 8 Prior Knowledge

1 a 0.4 b 2.5

2 0.345

3 0.174

4 a 0.12 b 0.833

5 $\begin{pmatrix} 1319 \\ 1082 \\ 2541 \end{pmatrix}$

6 $x = -10.25$, $y = 7.25$, $z = 10.75$

7 a $\lambda = 1, 0.1$, $v = \begin{pmatrix} 5 \\ 4 \end{pmatrix}, \begin{pmatrix} -1 \\ 1 \end{pmatrix}$

b $\begin{pmatrix} 5 & -1 \\ 4 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & 0.1 \end{pmatrix}\begin{pmatrix} 5 & -1 \\ 4 & 1 \end{pmatrix}^{-1}$

Exercise 8A

1 a $\mathrm{E}(Y) = 21$, $\mathrm{Var}(Y) = 20$

b $\mathrm{E}(Y) = 31$, $\mathrm{Var}(Y) = 80$

2 a $\mathrm{E}(Y) = -4$, $\mathrm{Var}(Y) = \frac{5}{9}$

b $\mathrm{E}(Y) = 4.9$, $\mathrm{Var}(Y) = 0.05$

3 a $\mathrm{E}(Y) = -25$, $\mathrm{Var}(Y) = 45$

b $\mathrm{E}(Y) = -3$, $\mathrm{Var}(Y) = 5$

4 a $\mathrm{E}(Y) = 24$, $\mathrm{Var}(Y) = 20$

b $\mathrm{E}(Y) = 21$, $\mathrm{Var}(Y) = 45$

5 a $\mathrm{E}(Y) = 1$, $\mathrm{Var}(Y) = 0.2$

b $\mathrm{E}(Y) = 5$, $\mathrm{Var}(Y) = 1.25$

6 a $\mathrm{E}(Z) = 30$, $\mathrm{Var}(Z) = 35$

b $\mathrm{E}(Z) = -37$, $\mathrm{Var}(Z) = 107$

7 a $\mathrm{E}(Z) = -37.75$, $\mathrm{Var}(Z) = 48.5$

b $\mathrm{E}(Z) = \frac{7}{6}$, $\mathrm{Var}(Z) = \frac{7}{3}$

8 a $\mathrm{E}(Z) = 28.5$, $\mathrm{Var}(Z) = 45$

b $\mathrm{E}(Z) = -74$, $\mathrm{Var}(Z) = 224$

9 a $\mathrm{E}(Z) = 4.7$, $\mathrm{Var}(Z) = 1.16$

b $\mathrm{E}(Z) = 3$, $\mathrm{Var}(Z) = \frac{11}{16}$

10 a $\mathrm{E}(Z) = 1.5$, $\mathrm{Var}(Z) = 6$

b $\mathrm{E}(Z) = 0$, $\mathrm{Var}(Z) = 6$

11 a $\mathrm{E}(Z) = 17$, $\mathrm{Var}(Z) = 14$

b $\mathrm{E}(Z) = 17$, $\mathrm{Var}(Z) = 14$

12 a $\mathrm{E}(\bar{X}) = 5$, $\mathrm{Var}(\bar{X}) = 0.171$

b $\mathrm{E}(\bar{X}) = 6$, $\mathrm{Var}(\bar{X}) = 0.208$

13 a $\mathrm{E}(\bar{X}) = -4.7$, $\mathrm{Var}(\bar{X}) = 0.04$

b $\mathrm{E}(\bar{X}) = -15.1$, $\mathrm{Var}(\bar{X}) = 0.0467$

14 a $\mathrm{E}(\bar{X}) = 12$, $\mathrm{Var}(\bar{X}) = 0.9$

b $\mathrm{E}(\bar{X}) = 8$, $\mathrm{Var}(\bar{X}) = 0.0208$

15 a $\mathrm{E}(\bar{X}) = 3$, $\mathrm{Var}(\bar{X}) = 0.15$

b $\mathrm{E}(\bar{X}) = 3.6$, $\mathrm{Var}(\bar{X}) = 0.315$

16 a 4.2 b 13.6
c 44.64

17 a $\frac{8}{15}$ b 13.8

18 a $\frac{25}{8}$ b 34.25
c 85.9

19 a 15 b 17

20 6.6 cm, 0.6 cm

21 198.8 g, 4.16 g

22 102 g, 1.92 g

23 $\mathrm{E}(B) = 3.1$, $\mathrm{Var}(B) = 0.3$

24 $\mathrm{E}(V) = -6$, $\mathrm{Var}(V) = 2.56$

25 1076 kg, 30.2 kg

26 0, 1.41

27 0, 35.4

28 68.3 g

29 65 minutes, 8.06 minutes

30 a 19 b 10

Exercise 8B

1 a $W \sim \mathrm{N}(21, 16)$ b $W \sim \mathrm{N}(42, 67)$

2 a $W \sim \mathrm{N}(7, 111)$ b $W \sim \mathrm{N}(-1, 7)$

3 a $W \sim \mathrm{N}(7, 1.48)$ b $W \sim \mathrm{N}(-5.5, 6.91)$

4 a $W \sim \mathrm{N}(0, 8)$ b $W \sim \mathrm{N}(15, 9)$

5 a $W \sim \mathrm{N}(10, 0.5)$ b $W \sim \mathrm{N}(5, 0.3)$

6 a $\bar{X} \sim \mathrm{N}(30, 1.25)$ b $\bar{X} \sim \mathrm{N}(30, 0.5)$

7 a Cannot say b Cannot say

8 a $\sum_{i=1}^{50} X_i \sim \mathrm{N}(1500, 5000)$
b $\sum_{i=1}^{40} X_i \sim \mathrm{N}(1200, 4000)$

9 a 0.543 b 0.996

10 a 0.331 b 0.137

11 a $\mathrm{N}(91.3, 16.3)$ b 0.0156

12 a 0.3 s, 0.721 s b 0.339
c 0.166

13 a 65 cm, 0.005 cm^2 b 0.00235

14 a 0.252 b 0.0175

15 0.0352

16 0.0336

17 0.0820

18 a 0.4 kg, 0.223 kg^2
b 0.198
c 0.0209

19 a 0.196
b 0.0211

20 0.272 m

21 0.0281

22 a 2500 g, 1.79 g
b 0.995
c CLT shows that the distribution of the mass of a ream is approximately normal.

23 a 0.321
b The distribution of the times for each company were unknown.

24 a 0.208
b 0.196

25 $\mu = 7.33$ mm, $\sigma = 0.525$ mm

26 a 0.0228
b i 0.868
ii 0.315
iii 0.868
c 0.691
d 0.645

27 a 0.0173
b The mean of normal variables is normal.

28 44

Exercise 8C

1 a No, $X \sim \mathrm{B}(60, 0.801)$
b No, $X \sim \mathrm{B}(5, 0.195)$

2 a Yes, $X \sim \mathrm{Po}(3)$
Conditions well met.
b Yes, $X \sim \mathrm{Po}(4)$
But occurrences of fish unlikely to be independent of each other as live in shoals.

3 a No – this is a continuous distribution.
b No – this is a continuous distribution.

4 a No – not Poisson as not a number of events in a given period; not binomial as n not fixed.

b No – not Poisson as not a number of events in a given period; not binomial as n not fixed.

5 a 0.132 b 0.108

6 a 0.0789 b 0.666

7 a 0.774 b 0.399

8 a 0.843 b 0.990

9 a 0.085 b 0.601

10 a 0.507 b 0.458

11 a 0.138 b 0.590

12 a 0.249 b 0.213

13 a 3.1 b 3.1

14 a 2.30 b 5.3

15 a $X \sim \text{Po}(36)$ b $X \sim \text{Po}(24)$

16 a $X \sim \text{Po}(2.4)$ b $X \sim \text{Po}(3)$

17 a $X \sim \text{Po}(54)$ b $X \sim \text{Po}(9.6)$

18 a $X \sim \text{Po}(6.5)$ b $X \sim \text{Po}(8.9)$

19 a $X \sim \text{Po}(14)$ b $X \sim \text{Po}(7)$

20 0.0116

21 a 0.0595 b 0.0548

22 a 0.298 b 0.973

23 a 0.868

b The rate of arrival of messages is unlikely to be constant – there will probably be more at some times of the day than others. Within each distribution messages are not likely to be independent as they may occur as part of a conversation. The two distributions are also probably not independent of each other, as times when more emails arrive might be similar to times when more texts might arrive.

24 a 0.916 b 0.0656

25 a 0.430

b No – likely to be more outbreaks at certain times of the year so not constant rate, and outbreaks not independent since contagious.

26 a i 0.161 ii 0.0514

b 0.0132

27 a 0.0537 b 0.321

28 a 0.475 b 1 c 0.00413

29 a 0.273 b 0.143 c 201

30 a 0.747 b 78

Exercise 8D

1 a $\begin{pmatrix} 0.25 & 0.7 \\ 0.75 & 0.3 \end{pmatrix}$ b $\begin{pmatrix} 0.45 & 0.9 \\ 0.55 & 0.1 \end{pmatrix}$

2 a $\begin{pmatrix} 0.82 & 0.08 & 0.22 \\ 0.06 & 0.73 & 0.11 \\ 0.12 & 0.19 & 0.67 \end{pmatrix}$

b $\begin{pmatrix} 0.64 & 0 & 0.45 \\ 0.02 & 0.57 & 0 \\ 0.34 & 0.43 & 0.55 \end{pmatrix}$

3 a $\begin{pmatrix} 0.71 & 0 & 0.24 & 0.45 \\ 0.29 & 0.88 & 0 & 0 \\ 0 & 0.05 & 0.57 & 0.07 \\ 0 & 0.07 & 0.19 & 0.48 \end{pmatrix}$

b $\begin{pmatrix} 0.35 & 0.15 & 0.1 & 0.2 \\ 0.1 & 0.4 & 0.05 & 0.2 \\ 0.25 & 0.2 & 0.5 & 0.05 \\ 0.3 & 0.25 & 0.35 & 0.55 \end{pmatrix}$

4 a $\begin{pmatrix} 0.14 & 0.23 \\ 0.86 & 0.77 \end{pmatrix}$ b $\begin{pmatrix} 0.5 & 1 \\ 0.5 & 0 \end{pmatrix}$

5 a $\begin{pmatrix} 0.3 & 0 & 0.4 \\ 0.1 & 1 & 0.4 \\ 0.6 & 0 & 0.2 \end{pmatrix}$

b $\begin{pmatrix} 0.22 & 0.5 & 0.12 \\ 0.33 & 0.29 & 0.8 \\ 0.45 & 0.21 & 0.08 \end{pmatrix}$

6 a $\begin{pmatrix} 0 & 0.55 & 0.05 & 0.3 \\ 0.3 & 0 & 0.7 & 0.1 \\ 0.35 & 0.25 & 0 & 0.6 \\ 0.35 & 0.2 & 0.25 & 0 \end{pmatrix}$

b $\begin{pmatrix} 0.84 & 0 & 0.31 & 0.34 \\ 0 & 0 & 0.13 & 0.22 \\ 0 & 0.5 & 0.56 & 0.44 \\ 0.16 & 0.5 & 0 & 0 \end{pmatrix}$

7 a 0.666 b 0.590

8 a 0.316 b 0.378

9 a 0.112 b 0.0426

10 a $\begin{pmatrix} 0.610 \\ 0.390 \end{pmatrix}$ b $\begin{pmatrix} 0.503 \\ 0.497 \end{pmatrix}$

11 a $\begin{pmatrix} 0.277 \\ 0.325 \\ 0.398 \end{pmatrix}$ b $\begin{pmatrix} 0.264 \\ 0.365 \\ 0.371 \end{pmatrix}$

12 a $\begin{pmatrix} 0.260 \\ 0.254 \\ 0.267 \\ 0.220 \end{pmatrix}$ b $\begin{pmatrix} 0.153 \\ 0.448 \\ 0.237 \\ 0.162 \end{pmatrix}$

13 a $\begin{pmatrix} 0.591 \\ 0.409 \end{pmatrix}$ b $\begin{pmatrix} 0.517 \\ 0.483 \end{pmatrix}$

14 a $\begin{pmatrix} 0.209 \\ 0.579 \\ 0.213 \end{pmatrix}$ b $\begin{pmatrix} 0.333 \\ 0.333 \\ 0.333 \end{pmatrix}$

15 a $\begin{pmatrix} 0.253 \\ 0.259 \\ 0.261 \\ 0.227 \end{pmatrix}$ b $\begin{pmatrix} 0.153 \\ 0.444 \\ 0.239 \\ 0.164 \end{pmatrix}$

16 a $\begin{pmatrix} 13/22 \\ 9/22 \end{pmatrix}$ b $\begin{pmatrix} 61/118 \\ 57/118 \end{pmatrix}$

17 a $\begin{pmatrix} 49/235 \\ 136/235 \\ 10/47 \end{pmatrix}$ b $\begin{pmatrix} 1/3 \\ 1/3 \\ 1/3 \end{pmatrix}$

18 a $\begin{pmatrix} 249/985 \\ 51/197 \\ 257/985 \\ 224/985 \end{pmatrix}$ b $\begin{pmatrix} 9/59 \\ 4/9 \\ 127/531 \\ 29/177 \end{pmatrix}$

19 a 0.35 b $\begin{pmatrix} 0.875 & 0.875 \\ 0.125 & 0.125 \end{pmatrix}$

c 12.5%

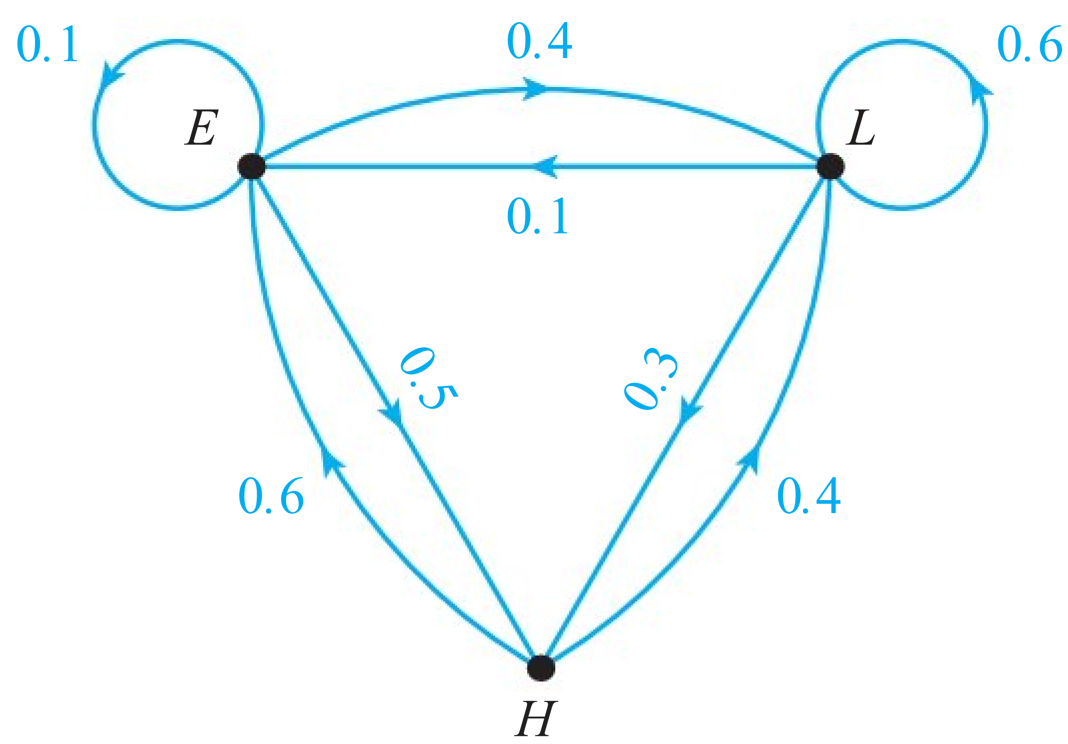
20 a $\begin{pmatrix} 0.95 & 0.8 \\ 0.05 & 0.2 \end{pmatrix}$ b 0.938

c $\begin{pmatrix} 0.941 \\ 0.059 \end{pmatrix}$

21 a $\begin{pmatrix} 0.8 & 0.2 & 0.1 \\ 0.1 & 0.7 & 0.3 \\ 0.1 & 0.1 & 0.6 \end{pmatrix}$ b $\begin{pmatrix} 433 \\ 357 \\ 210 \end{pmatrix}$

c $\begin{pmatrix} 450 \\ 350 \\ 200 \end{pmatrix}$

22 a



b 0.319 c 0.5

23 a $\begin{pmatrix} 0.85 & 0.1 & 0.25 \\ 0.05 & 0.75 & 0.25 \\ 0.1 & 0.15 & 0.5 \end{pmatrix}$

b 0.278

c Bull 51.5%, Bear 29.4%, Stagnant 19.1%

24 a $\begin{pmatrix} 1 & 0.5 & 0 & 0 \\ 0 & 0 & 0.5 & 0 \\ 0 & 0.5 & 0 & 0 \\ 0 & 0 & 0.5 & 1 \end{pmatrix}$ b $\frac{2}{3}$

25 a $\begin{pmatrix} 0.7 & 0.4 \\ 0.3 & 0.6 \end{pmatrix}$ b 0.561

c $\begin{cases} 0.7a + 0.4b = a \\ 0.3a + 0.6b = b \\ a + b = 1 \end{cases}$ d $a = \frac{4}{7},\ b = \frac{3}{7}$

26 a 0.125

b $\begin{cases} 0.5d + 0.25h = d \\ 0.5d + 0.5h + 0.5r = h \\ 0.25h + 0.5r = r \\ d + h + r = 1 \end{cases}$

c $d = 0.25,\ h = 0.5,\ r = 0.25$

27 a $\begin{pmatrix} 0.9 & 0.15 \\ 0.1 & 0.85 \end{pmatrix}$

b $\lambda = 1, \frac{3}{4}$

$v_1 = \begin{pmatrix} 3 \\ 2 \end{pmatrix}, v_2 = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$

c $\mathbf{P} = \begin{pmatrix} 3 & 1 \\ 2 & -1 \end{pmatrix}, \mathbf{D} = \begin{pmatrix} 1 & 0 \\ 0 & \frac{3}{4} \end{pmatrix}$

d $0.6 + 0.05(0.75^n)$

e 0.6

28 a $\begin{pmatrix} 0.8 & 0.6 \\ 0.2 & 0.4 \end{pmatrix}$

b $\lambda = 1, \frac{1}{5}$

$v_1 = \begin{pmatrix} 3 \\ 1 \end{pmatrix}, v_2 = \begin{pmatrix} -1 \\ 1 \end{pmatrix}$

c $\mathbf{P} = \begin{pmatrix} 3 & -1 \\ 1 & 1 \end{pmatrix}, \mathbf{D} = \begin{pmatrix} 1 & 0 \\ 0 & \frac{1}{5} \end{pmatrix}$

d $\frac{1}{4} - \frac{1}{4}\left(\frac{1}{5}\right)^n$

e $\frac{1}{4}$

Chapter 8 Mixed Practice

1 a 3.2 b −7.6 c 14.0

2 a 0.1 b 4.2 c 0.6 d 4.28

3 a

x	1	2	3	4
$\mathrm{P}(X = x)$	$\frac{1}{13}$	$\frac{5}{26}$	$\frac{4}{13}$	$\frac{11}{26}$

b $\frac{40}{13}$ c 22.9

4 a i 6.4 kg ii $0.52\,\text{kg}^2$

b Masses of individual components are independent.

5 a 20 m, $1.57\,\text{m}^2$ b 0.0553

6 0.123

7 a 45, 6.71

b Number of tweets on any given day independent from other days.

8 a Independent events; constant rate of success

b 0.731

9 a The average rate must be constant. However, we might expect it to vary over different times of the day and with different weather conditions. Birds must arrive independently, but we know they may come in flocks.

b 0.156

10 a 0.04 b $\begin{pmatrix} 24.6\% \\ 42.6\% \\ 32.8\% \end{pmatrix}$

c $\begin{pmatrix} 16.2\% \\ 33.7\% \\ 50.1\% \end{pmatrix}$

11 a i $2\mu, 2\sigma^2$ ii $3\mu, 9\sigma^2$

iii $\mu, 3\sigma^2$ iv $\mu, \frac{\sigma^2}{n}$

12 0.00537

13 a 0.110 b 0, 168 c 0.350

14 0.882

15 a $X \sim \mathrm{Po}(45)$, calls arrive independently and at a constant rate.

b 0.204 c 0.0207

16 a 0.547 b 3

17 a 0.362 b 0.279

c 0.659 d 0.0464

e 0.247 f 0.641

18 a 0.189 b 0.372 c 0.208

19 a

$$\begin{array}{c c} \begin{array}{ccc} R & F & S \end{array} & \\ \begin{pmatrix} 0.6 & 0.4 & 0.5 \\ 0.1 & 0.4 & 0.2 \\ 0.3 & 0.2 & 0.3 \end{pmatrix} & \begin{array}{c} R \\ F \\ S \end{array} \end{array}$$

b 0.526

c i $\begin{cases} 0.6r + 0.4f + 0.5s = r \\ 0.1r + 0.4f + 0.2s = f \\ 0.3r + 0.2f + 0.3s = s \\ r + f + s = 1 \end{cases}$

ii $\begin{pmatrix} 38/71 \\ 13/71 \\ 20/71 \end{pmatrix}$

20 a 0.785 b 0.122 c 0.00976

21 i 0.549 ii 0.0231
iii 0.964 iv 0.169

22 a i 0.122 ii 0.248
b i $\mathrm{E}(\bar{X}) = 8, \mathrm{Var}(\bar{X}) = \frac{8}{n}$
ii $\mathrm{E}(\bar{X}) \neq \mathrm{Var}(\bar{X})$ for $n > 1$
c i 0.846
ii $k = 0.736$

23 85; assume independent lengths

24 8.20 g

25 a 0.431 b 0.274

26 a 0.261 b 0.131 c 1.72

27 a 0.112 b 42 c 58

28 a $\begin{pmatrix} 0.1 & 0.6 \\ 0.9 & 0.4 \end{pmatrix}$
b $\lambda = 1, -\frac{1}{2}$
$v_1 = \begin{pmatrix} 2 \\ 3 \end{pmatrix}, v_2 = \begin{pmatrix} -1 \\ 1 \end{pmatrix}$
c $\mathbf{P} = \begin{pmatrix} 2 & -1 \\ 3 & 1 \end{pmatrix}, \mathbf{D} = \begin{pmatrix} 1 & 0 \\ 0 & -\frac{1}{2} \end{pmatrix}$
d $\frac{2}{5} + \frac{3}{5}\left(-\frac{1}{2}\right)^n$
e $\frac{2}{5}$

Chapter 9 Prior Knowledge

1 $\bar{x} = 5.8, \sigma = 2.32$
2 $103 > 12.8$ so reject H_0. Data do not come from B(5, 0.7).
3 $p = 0.0248 < 0.1$ so reject H_0. Data are not independent.
4 a $r = 0.920$: strong positive correlation
b $y = 1.20x + 0.862$
5 $p = 0.192 > 0.1$ so do not reject H_0. Insufficient evidence that $\mu_1 < \mu_2$.
6 0.227
7 0.834
8 0.268

Exercise 9A

1 a Not valid, reliable b Not valid, reliable
2 a Valid, reliable b Valid, reliable
3 a Not valid, not reliable
b Not valid, not reliable
4 a 20.8, 17.4 b 24.5, 171.5
5 a 0.833, 67.4 b 1.83, 221
6 a 0.1, 0.072 b 1.3, 4.46

7 13.1

8 5

9 a 4.5 b 3.5
c The sampling is not representative of the whole snake population.
d She could repeat the experiment and use a t-test to see if the population means were very different.

10 a It is not a feasible result – someone is likely not taking the questionnaire seriously.
b 2, 1.5
c e.g. Criminal activities is not clearly defined. Affecting could be very indirect. In the last year might mean last calendar year or most recent 12 months.
d The sample was self selecting, and based on those attending a police station, so biased and therefore not valid.
e Overestimate, since people attending a police station are more likely to be victims of crime.
f Use a larger sample size.

11 a Reliability is when similar conclusions are be reached on each occasion the test is conducted in similar circumstances.
b Parallel forms
c Give the same test again some time later (test-retest) and check that the results are similar.

12 a Validity is the extent to which you are measuring what you really want to measure.
b e.g. Success in business might not be just measured by salary. People's own opinions are not the same as actual success.
c e.g. Politeness might not then be a confounding variable.
d Criterion validity

Exercise 9B

1 a $\chi^2 = 3.10,\ \nu = 2,\ p = 0.212$, plausibly independent

b $\chi^2 = 0.287,\ \nu = 2,\ p = 0.866$, plausibly independent

2 a $\chi^2 = 6.10,\ \nu = 2,\ p = 0.0473$, not independent

b $\chi^2 = 1.86,\ \nu = 2,\ p = 0.394$, plausibly independent

3 a $\chi^2 = 10,\ \nu = 3,\ p = 0.0186$, not independent

b $\chi^2 = 4.02,\ \nu = 6,\ p = 0.674$, plausibly independent

4 a $p = 0.0258$; not Poisson

b $p = 0.200$, plausibly Poisson

5 a $p = 0.263$, plausibly Poisson

b $p = 1.16 \times 10^{-8}$, not Poisson

6 a $p = 0.426$, plausibly Poisson

b $p = 0.191$, plausibly Poisson

7 a $p = 0.477$, plausibly normal

b $p = 7.38 \times 10^{-6}$, not normal

8 a $p = 0.219$, plausibly normal

b $p = 0.805$, plausibly normal

9 a $p = 0.000214$, not normal

b $p = 0.0167$, not normal

10 $p = 0.436$, accept Mendel's suggestion.

11 a H_0: $N(12, 2.5^2)$ is a good model,

H_1: $N(12, 2.5^2)$ is not a good model.

b 4.60, 9.18, 12.4, 9.18, 4.60

c Some expected frequencies are smaller than 5.

d 2

e $p = 0.578$; no evidence that $N(12, 2.5^2)$ is not a good model.

12 a H_0: Diet choices are independent of age; H_1: Diet choices are dependent on age.

b The expected frequency of the 17–18 vegetarian group is less than 5.

c Sufficient evidence that diet choices depend on age ($p = 0.0485$).

d i e.g. Ask for the response which is the best description of their usual behaviour or have an 'other' category.

ii Having too many categories with a fixed sample size would make the chi-squared test invalid as some expected frequencies would drop below 5.

13 a H_0: Po(3.5) is a good model, H_1: Po(3.5) is not a good model.

b 0 and 1

c 5

d 0.737; Po(3.5) is a good model.

14 a 4.46, 5.20, 5.57, 5.57, 5.2

b Sufficient evidence that the model is not appropriate ($p = 0.0811$).

15 Insufficient evidence of bias ($p = 0.459$)

16 $\nu = 6,\ p \approx 0$; sufficient evidence that city and more of transport are dependent.

17 a i $p = 0.0235$, not plausible

ii $p = 0.0575$, plausible

iii $p = 0.00404$, not plausible

b If there were theoretical reasons to believe that the mean is 25 before the data is observed.

18 a H_0: $B(3, 0.6)$ is a good model.

H_1: $B(3, 0.6)$ is not a good model.

b 12.8, 57.6, 86.4, 43.2, $\nu = 3$

c $p = 1.04 \times 10^{-7} < 0.05$ so reject H_0. There is evidence, at the 5% significance level, that $B(3, 0.6)$ is not a good model.

d Mean = 2.14, $p \approx 0.713$

e Binomial is a good model ($p = 0.249 > 0.05$).

19 a 2.01

b H_0: a Poisson model is appropriate.

H_1: a Poisson model is not appropriate.

c 13.40, 26.93, 27.07, 18.13, 9.11, 5.36, $\nu = 4$

d No reason to conclude a Poisson model is inappropriate ($p = 0.060 > 0.05$).

20 a 1.22

b 5.44, 11.59, 15.95, 11.59, 5.44. Expected value for $t < 20$ needed combining with $20 < t < 21.5$ and likewise for $t > 26$ with $24.5 < t < 26$.

c 3

d No reason to conclude a Normal distribution with mean 23 mins is inappropriate ($p = 0.103 > 0.05$).

21 a 28.410, 40.265, 5.264

b 2

c H_0: a normal model is appropriate.

H_1: a normal model is not appropriate.

d No reason to conclude that a normal model is inappropriate ($p = 0.194 > 0.1$).

22 a 2.67

b 1.57

c i Mean and variance are very different, suggesting not a Poisson distribution.

ii $\chi^2 = 6.67,\ p = 0.0830 > 0.05$; possibly Poisson

iii $\chi^2 = 60.7,\ p = 2.08 \times 10^{-12} < 0.05$; not Poisson

d The first method does not have any measure of uncertainty. It is subjective. The third method takes into account the fact that no observations of 5 or more were made, so it provides a more authentic view of the observed data so is more valid.

Exercise 9C

1 a $y = 1.70x^2 - 4.97x + 4.26$

b $y = -3.72x^2 + 12.6x + 6.14$

2 a $y = -5.89x^3 + 17.7x^2 - 6.86x + 4.53$

b $y = 0.497x^3 - 3.15x^2 + 7.42x + 0.868$

3 a $y = 1.68e^{0.217x}$ b $y = 72.7e^{-1.07x}$

4 a $y = 55.1x^{-0.706}$ b $y = 1.37x^{4.39}$

5 a $y = -3.80\sin(2.18x + 0.785) + 3.56$

b $y = 2.83\sin(0.754x - 0.824) + 2.14$

6 a $R^2 = 0.938$ b $R^2 = 0.819$

7 a $R^2 = 0.851$ b $R^2 = 0.712$

8 a $R^2 = 0.650$ b $R^2 = 0.899$

9 0.773

10 a $T = -0.00160x^3 + 0.179x^2 + 1.57x - 0.671$

b i 242 years

ii Need to extrapolate so treat with caution.

11 a $h = 1.9\sin(0.52t - 0.15) + 6.2$

b i 8.1 m

ii 4.3 m

12 a 0.861, 0.911

b Model B a better fit

c $D = 89.5p^{-0.785}$

d $10.7 \approx 11$

e Demand tends to zero as price increases so potentially suitable, but extrapolation means any predictions should be treated cautiously.

13 a A: $SS_{res} = 6.75$

B: $SS_{res} = 8.35$

b Model A

14 a $s_{girls} = -0.171t^2 + 13.2t + 25.1$

$s_{boys} = -1.26t^2 + 19.1t + 13.1$

b Different number of data points in each model

c $R^2_{girls} = 0.915$

$R^2_{boys} = 0.844$

So, girls' model is better fit.

15 a $T = 0.2t^2 - 7.13t + 99.8$

b 0.994

c No – this quadratic model predicts that after $t \approx 17.8$ minutes the temperature will increase again (without bound). Clearly the temperature model needs to predict the temperature tending to room temperature.

16 The extra parameter in a cubic model compared to a quadratic model makes the comparison invalid.

Exercise 9D

Hint: The answers in this section are all given using interval notation, but you can use inequality notation or words if you prefer.

1 a (6.06, 9.94) b (2.76, 5.35)

2 a (135, 205) b (377,567)

3 a (5.90,8.20) b (5.63, 8.47)

4 a (11.4, 13.2) b (12.2, 16.2)

5 a (165, 207) b (209, 245)

6 a (8.04, 8.36) b (7.48, 7.92)

7 a (16.6, 21.2) b (15.5, 22.3)

8 a (11.5, 26.3) b (29.1, 43.9)

9 a (156, 190) b (325, 385)

10 a (7.07, 8.93) b (3.55, 4.56)

11 a (175, 193) b (455, 489)

12 a (6.87, 7.23) b (6.52, 7.58)

13 (3.66, 4.62)

14 (62.1, 65.5)

15 (780, 832)

16 a $(23.8, 29.8)$

b z-interval because the population standard deviation is known.

17 a $(8.33, 8.87)$

b Population distribution is normal.

18 a $15.5\ (°C)^2$ b $(16.8, 20.4)$

19 a Population of concentrations is normally distributed.

b $(20.1, 23.9)$

c No; the confidence interval does not include 20.

20 a $(2.85, 3.55)$; t-interval as the population variance unknown. Population distribution is normal.

b There is no evidence against his belief, as the confidence interval extends below 3 minutes.

21 a $12\ \text{hours}^2$

b $(25.2, 27.8)$; assuming the population distribution is normal.

c The claim is not supported, as the confidence interval does not include 28.

22 a 4.72

b $(147, 151)$

c No; the confidence interval is entirely below 153.

d No; 150 is contained within the confidence interval.

23 a $(166, 172)$

b Used midpoints instead of actual values.

24 a $(16.7, 18.9)$

b Yes – the entire interval is above 16.5.

c Yes – the new interval is $(16.4, 19.2)$ and contains 16.5.

25 a $(3.13, 4.27)$

b Yes, because the sample mean is approximately normal because of the central limit theorem.

26 a 14, 2, 17, 23, −9, 0, 15, 11, −13, 20

b $(0.507, 12.8)$

c Yes – the confidence interval suggests that the average increase is greater than zero.

Exercise 9E

1 a $p = 0.0852$, insufficient evidence to reject H_0

b $p = 0.0196$, sufficient evidence to reject H_0

2 a $p = 0.0138$, sufficient evidence to reject H_0

b $p = 0.0227$, insufficient evidence to reject H_0

3 a $p = 0.438$, insufficient evidence to reject H_0

b $p = 0.207$, insufficient evidence to reject H_0

4 a $p = 0.00338$, sufficient evidence to reject H_0

b $p = 0.0125$, sufficient evidence to reject H_0

5 a $\bar{x} < 78.1$ b $\bar{x} < 91.4$

6 a $\bar{X} > 85.5$ b $\bar{X} > 753$

7 a $\bar{X} < 55.1$, $\bar{X} > 64.9$ b $\bar{X} < 117$, $\bar{X} > 123$

8 a 0.0288 b 0.0190

9 a 0.303 b 0.0707

10 a 0.00455 b 0.238

11 a 0.0702 b 0.0271

12 a Sufficient evidence to reject H_0 ($p = 0.000919$)

b Insufficient evidence to reject H_0 ($p = 0.221$)

13 a Sufficient evidence to reject H_0 ($p = 0.00691$)

b Sufficient evidence to reject H_0 ($p = 0.00106$)

14 a Insufficient evidence to reject H_0 ($p = 0.155$)

b Insufficient evidence to reject H_0 ($p = 0.199$)

15 The suspicion is justified ($p = 0.00329$).

16 a 13.3

b Population variance has been estimated from the sample.

c Insufficient evidence for belief ($p = 0.179$)

17 a H_0: $\mu = 7.8$, H_1: $\mu < 7.8$

b $N\left(7.8, \frac{0.8}{\sqrt{10}}\right)$ c $\bar{X} < 7.38$

18 a The population variance is known.

b Insufficient evidence ($p = 0.366$)

19 a We have a pair of values for each tree.

b $p = 0.0352$, evidence that claim is justified.

20 a 0.2, 0.3, 0, 0.1, 0.4, 0.1, 0.1, 0.3, 0.2, 0

b 0.127

c H_0: $\mu_D = 0$, H_1: $\mu_D > 0$, belief supported ($p = 0.00151$).

21 Yes ($p = 0.00921$)

22 $\bar{X} < 59.1$ or $\bar{X} > 60.9$

23 a 144, 122.4

b Sufficient evidence to support Celine's belief ($p = 0.00949 < 0.1$)

c Homework times distributed normally, with equal variances.

24 a Evidence to support claim ($p = 0.0188$)

b No; large sample, so CLT means that the distribution of sample mean is approximately normal.

25 a Weights before and after follow normal distributions with equal variances.

b 0.359

c 6, 16, 0, 3, 6, 1

d Neither; the change in weight needs to be normally distributed.

e Fewer assumptions; eliminates differences between individual mice.

f 0.0368

g Paired test; there is evidence that the weight is increased after taking the drug.

26 a 89 b $N(0, 89)$

c Insufficient evidence that the mean is positive ($p = 0.174$).

d There is insufficient evidence to support Johannes's belief.

27 a H_0: $\mu_D = 0$, H_1: $\mu_D < 0$

b $D \sim N(0, 1.4792)$, $\bar{D} \sim N(0, 0.1233)$

c $\bar{D} < -0.702$

Exercise 9F

1 a 0.110, do not reject H_0

b 0.169, do not reject H_0

2 a 0.0321, reject H_0

b 0.0478, reject H_0

3 a 0.0221, do not reject H_0

b 0.00635, reject H_0

4 a 0.0393, reject H_0

b 0.0603, do not reject H_0

5 a $X \leqslant 23$ b $X \leqslant 22$

6 a $X \leqslant 10$ b $X \leqslant 10$

7 a $X \geqslant 37$ b $X \geqslant 34$

8 a $X \geqslant 22$ b $X \geqslant 21$

9 a 0.155, do not reject H_0

b 0.287, do not reject H_0

10 a 0.0986, reject H_0 b 0.0430, reject H_0

11 a 0.156, do not reject H_0

b 0.201, do not reject H_0

12 a 0.0185, reject H_0 b 0.00987, reject H_0

13 a $X \leqslant 6$ b $X \leqslant 12$

14 a $X \leqslant 4$ b $X \leqslant 5$

15 a $X \geqslant 19$ b $X \geqslant 29$

16 a $X \geqslant 3$ b $X \geqslant 4$

17 a Reject H_0 ($p = 0.0169$)

b Do not reject H_0 ($p = 0.0649$)

18 a Reject H_0 ($p = 0.0631$)

b Reject H_0 ($p = 0.0189$)

19 a Reject H_0 ($p = 0.0482$)

b Reject H_0 ($p = 0.00163$)

20 a Reject H_0 ($p = 0.00251$)

b Do not reject H_0 ($p = 0.0362$)

21 Not supported ($p = 0.0659$)

22 Justified ($p = 0.0761$)

23 Insufficient evidence ($p = 0.0707$)

24 a Po(47)

b Sufficient evidence ($p = 0.0420$)

25 a Po(2.4)

b Insufficient evidence of increase ($p = 0.0959$)

26 a 0.900

b H_0: $\rho = 0$, H_1: $\rho > 0$

c 0.0185

d Sufficient evidence of positive correlation

27 Sufficient evidence of positive correlation ($p = 0.0447$)

28 a H_0: $\rho = 0$, H_1: $\rho < 0$

b Sufficient evidence of negative correlation ($p = 0.0415$)

29 a H_0: $\rho = 0.38$, H_1: $\rho > 0.38$

b $X \geqslant 88$

c Insufficient evidence that the support has increased.

30 Insufficient evidence of correlation ($p = 0.0884$)

31 a 0.820

b Insufficient evidence of correlation ($p = 0.0239$)

c Sufficient evidence of positive correlation ($p = 0.0120$)

32 a H_0: $\lambda = 1$, H_1: $\lambda < 1$, where λ is the mean number of busses in a ten-minute interval.

b $X \leqslant 2$ c $X \leqslant 22$

33 a H_0: $\lambda = 6.3$, H_1: $\lambda > 6.3$ b $X \geqslant 42$

34 a At least 5% b At least 6.5%

35 a H_0: $\lambda = 1.7$, H_1: $\lambda < 1.7$

b $P(X = 0) = 0.183 > 0.05$; H_0 will never be rejected.

c 19 d $X \leqslant 10$ e 3

36 a H_0: $p = 0.01$, H_1: $p > 0.01$, where $X \sim B(10, p)$; sufficient evidence of increase ($p = 0.00427$)

b Sufficient evidence of increase ($p = 0.00468$)

c e.g. 0.45%

Exercise 9G

1 a 0.05 b 0.1

2 a 0.1 b 0.01

3 a 0.0119 b 0.0331

4 a 0.136 b 0.0477

5 a 0.0181 b 0.0592

6 a 0.0548 b 0.0781

7 a 0.238 b 0.202

8 a 0.0909 b 0.0527

9 a 0.829 b 0.581

10 a 0.538 b 0.589

11 a 0.304 b 0.485

12 a 0.618 b 0.633

13 a 0.731 b 0.735

14 a 0.450 b 0.285

15 a H_0: defendant is innocent.
H_1: defendant is guilty.

b Finding an innocent person guilty.

c Finding a guilty person innocent.

16 a Type II – they are judging the dice to be fair (not rejecting H_0) when in fact they are biased (H_0 is false).

b H_0: $p = \frac{1}{6}$
H_1: $p < \frac{1}{6}$

c 0.0649

17 a 0.05 b 0.114

18 a H_0: $\lambda = 20$
H_1: $\lambda > 20$

b 0.0343

19 a 5% b 0.756

c Increase the sample size.

20 a i Claiming the proportion that germinate is lower than 80% when it isn't.

ii Claiming the proportion that germinate isn't lower than 80% when it is.

b 0.0867

c 0.786

d Test more seeds, increase the limit of less than 14 to reject the distributor's claim.

21 a H_0: $\lambda = 2$, H_1: $\lambda > 2$

b 4.87% c 56.8%

22 a No change in chance of Type I error.

b Reduces the chance of Type II error.

23 a H_0: $p = \frac{2}{3}$
H_1: $p > \frac{2}{3}$

b 0.619

24 a H_0: $\lambda = 18.4$
H_1: $\lambda < 18.4$

b 0.183

25 a $\frac{17}{56}$ b $\frac{3}{14}$

Chapter 9 Mixed Practice

1 a $25.3\,\text{cm}^2$ b $121 < \mu < 126$

2 a 7 and 8

b Insufficient evidence of dependence ($p = 0.173$)

c Not valid; the test is looking for evidence of dependence, not independence.

3 a Qingqing's model: $0.983 > 0.870$

b 7.36

4 a $p = 0.0918$; evidence of a decrease
b It is still 3.7 minutes. The training may have decreased differences between individual students.
c Repeat the test with a different sample.

5 a 73.2 g, 134 g^2 b (66.5, 79.9)

6 (85.8, 90.6)

7 a H_0: $\mu_1 = \mu_2$, H_1: $\mu_1 \neq \mu_2$
b 0.197
c Insufficient evidence that the means are different

8 a H_0: $\rho = 0$, H_1: $\rho < 0$
b Both are normal.
c 0.0312
d Significant evidence of negative correlation
e e.g. The season has a greater effect on temperature.

9 a 0.119
b H_0: $\lambda = 17$, H_1: $\lambda > 17$
c Insufficient evidence of increase ($p = 0.119$)

10 a e.g. Students may not answer honestly.
b The expected frequency in the bottom left cell is less than 5.
c They are not adjacent values.
d $p = 0.0740$

11 H_0: $p = 0.5$, H_1: $p > 0.5$
$P(X \geqslant 19) = 0.100 > 0.05$
Insufficient evidence

12 Insufficient evidence of linear correlation ($p = 0.0578$)

13 a 9.761 to 9.825
b If a large number of intervals are formed in this way, 99% of them will contain μ.

14 a H_0: $\rho = 0$, H_1: $\rho > 0$
b 0.853
c 0.00173; sufficient evidence of positive correlation (e.g. at 1% SL)
d $y = 1.78x + 40.5$
e 74.3

15 a H_0: $\rho = 0$, H_1: $\rho > 0$
$p = 0.0574$
b Only shows insufficient evidence of a linear relationship.
c $P = -0.202n^2 + 1.72n + 1.79$
d $R^2 = 0.979$, which suggests the quadratic model is a good fit.

16 a Model A: $SS_{\text{res}} = 6.01$
Model B: $SS_{\text{res}} = 5.17$
b $5.17 < 6.01$, which suggests Model B is a better fit.

17 a (21 200, 27 800)
b No, cannot reject H_0.
c The population of wages is normal.

18 a $(31.3, 53.5)$ b $(25.4, 31.4)$
c Confidence intervals barely overlap, so not very reliable.
d Increase sample size.

19 a H_0: $\mu = 2.7$; H_1: $\mu \neq 2.7$
b $\bar{x} < 2.53$ or $\bar{x} > 2.87$
c Evidence that the average height is different.
d Yes; $\bar{X}$ is approximately normal by CLT.

20 a 7.09
b H_0: Poisson is a suitable model, H_1: Poisson is not a suitable model.
c 6.95, 7.90, 11.2, 13.2, 13.4, 11.9, 9.35, 6.63, 9.45
d 7
e Sufficient evidence to reject H_0 $\left(\chi^2 = 27.3, p = 2.89 \times 10^{-4}\right)$. Suggests Poisson is not a suitable model.

21 a H_0: $p = 0.72$, H_1: $p > 0.72$
b $X \geqslant 26$ c 0.0495

22 a 24, 32, 26, 23 b 21.5 g to 31.0 g
c The differences in weights are normally distributed.

23 a $X \leqslant 2$ b 0.0824 c 0.857

24 a H_0: a normal distribution is a suitable model.
H_1: a normal distribution is not a suitable model.
b $\bar{x} = 15.3$, $s_{n-1} = 4.99$
c 0.61, 3.69, 8.65, 30.11, 20.96, 11.08, 13.51, 2.24, 0.15
d Some expected frequencies are < 5; $\nu = 2$.
e Evidence that normal is not a good model ($\chi^2 = 11.2$, $p = 0.00370$).

25 a e.g. Students may not answer honestly.
b H_0: $p = 0.6$, H_1: $p < 0.6$
c $X \leqslant 23$
d 0.283 e 0.989

26 a H_0: $p = 0.1$, H_1: $p > 0.1$
p-value $= 0.0684 > 0.05$
Do not reject H_0; insufficient evidence that probability trains are late is greater than 0.1.
b 0.0170 c 0.854 d 0.146

27 b H_0: $X \sim B(6, 0.4)$, H_1: X does not follow $B(6, 0.4)$
$\chi^2 = 6.06$, $p = 0.194$; claim consistent with data

28 a H_0: $\mu = 2.5$, H_1: $\mu \neq 2.5$
b $\bar{x} < 2.45$ or $\bar{x} > 2.55$
c 0.228

29 a i $\bar{x} < 2.863$
ii Rejecting H_0 when it is true.
iii Accepting H_0 when it is false.
iv 0.05
v 0.0877
b i t-test
ii Do not reject H_0, $p = 0.0509$.
iii $2.719 < \mu < 3.001$

30 a $p = -0.000964y^3 + 0.111y^2 - 3.43y + 151$
b $p = -0.000140m^3 + 0.0628m^2 - 6.72m + 332$
c Using R^2, Nathan's model accounts for 76.7% of the variability in p whereas Marc's only accounts for 68.0%.
d Form a model that depends on both age and weight.

31 a $T = 4.21Q^2 - 44.9Q + 656$
b $R^2 = 0.973$
c i \$630 ii \$5600
d Interpolation for $Q = 10$ with good fit of model so reliable. Extrapolation for $Q = 40$ so treat with caution.
e $T = 0.257Q^2 + 31.7Q + 192$
f Different number of data points in each model
g $R^2_{\text{Irina}} = 0.973$
$R^2_{\text{Julian}} = 0.935$
So, Irina's model is better fit.
h R^2 is better for the cubic model (0.993 cf. 0.973) but the extra parameter makes the comparison invalid.

32 a H_0: $\mu = 6.4$ b 10%
H_1: $\mu \neq 6.4$

33 a Test B; Test A also measures the difference between the two groups.
b 0.494
c Insufficient evidence of improvement ($p = 0.312$)

34 a $X \geqslant 243$
b Reto (0.0461 vs 0.0437)

35 a H_0: $\mu = 1.2$, H_1: $\mu < 1.2$
b i 3.45% (accept between 3.45% and 5.13%)
ii 0.257

36 a 0.0835 b 0.105

37 a $\bar{X} \leqslant 4.71$ b 0.361 c 0.109

Chapter 10 Prior Knowledge

1 $f''(x) = 2$
2 $y - 1 = 3(x - 1)$
3 $x = \ln 5 \approx 1.61$
4 $(2, -16), (-2, 16)$

Exercise 10A

1 a $f'(x) = \frac{2}{3}x^{-\frac{1}{3}}$ b $f'(x) = \frac{3}{4}x^{-\frac{1}{4}}$
2 a $f'(x) = -\frac{1}{2}x^{-\frac{3}{2}}$ b $f'(x) = -\frac{4}{3}x^{-\frac{7}{3}}$
3 a $f'(x) = 9x^{\frac{1}{2}}$ b $f'(x) = 2x^{-\frac{3}{4}}$
4 a $f'(x) = -6x^{-\frac{5}{3}}$ b $f'(x) = -4x^{-\frac{7}{5}}$
5 a $f'(x) = x^{-\frac{1}{2}}$ b $f'(x) = \frac{3}{2}x^{-\frac{3}{4}}$
6 a $f'(x) = 4x^{-\frac{4}{3}}$ b $f'(x) = x^{-\frac{6}{5}}$
7 a $-3x^{-\frac{4}{3}} - \frac{4}{3}x^{-\frac{7}{3}}$ b $-x^{-\frac{3}{2}} + \frac{3}{2}x^{-\frac{5}{2}}$
8 a $3x^{\frac{1}{4}} - \frac{1}{20}x^{-\frac{3}{4}}$ b $2x^{\frac{4}{3}} + \frac{1}{21}x^{-\frac{2}{3}}$
9 a $-\frac{2}{3}x^{-\frac{4}{3}} - 2x^{-\frac{1}{3}}$ b $-\frac{5}{4}x^{-\frac{5}{4}} + 6x^{-\frac{1}{4}}$
10 a $f'(x) = 3\cos x$ b $f'(x) = -4\sin x$
11 a $f'(x) = -\frac{1}{2}\sin x - 5\cos x$
b $f'(x) = \frac{3}{4}\cos x + 2\sin x$
12 a $f'(x) = \frac{2}{\cos^2 x}$ b $f'(x) = 1 - \frac{1}{\cos^2 x}$

13 a $\frac{dy}{dx} = \frac{3}{x}$ b $\frac{dy}{dx} = -\frac{4}{x}$

14 a $\frac{dy}{dx} = \frac{2}{x}$ b $\frac{dy}{dx} = -\frac{1}{x}$

15 a $y' = 5e^x$ b $y' = -6e^x$

16 a $y' = -\frac{e^x}{2}$ b $y' = \frac{3e^x}{4}$

17 0.5

18 $-5\,\text{m}\,\text{s}^{-1}$; falcon is descending.

19 (0, 1)

20 5

21 $y = -\frac{3x}{2} + \frac{\pi}{4} + \frac{3\sqrt{3}}{2}$

22 $y = x + 3 - 3\ln 3$

23 (9, 3)

24 a $y = 2x - \frac{5}{2}$

25 $y - \tan k = \frac{1}{\cos^2 k}(x - k)$

26 a 1 million b 9 hours
c 1.5 million per hour d No upper limit

27 $x > 0.5$

28 0.5

29 a $\frac{\pi^2}{4}$ b $\frac{\pi^2}{4}$

30 (4, 2)

31 $\left(\frac{13\pi}{6}, 0\right)$

32 0.5

Exercise 10B

1 a $\frac{dy}{dx} = 12(3x+2)^3$ b $\frac{dy}{dx} = 10(2x-7)^4$

2 a $\frac{dy}{dx} = \frac{3}{2}x(x^2+3)^{-\frac{1}{4}}$ b $\frac{dy}{dx} = -\frac{4}{3}x(4-x^2)^{-\frac{1}{3}}$

3 a $\frac{dy}{dx} = \frac{1}{2}(2x^3+x)^{-\frac{1}{2}}(6x+1)$
b $\frac{dy}{dx} = \frac{1}{3}(5x-x^3)^{-\frac{2}{3}}(5-3x^2)$

4 a $f'(x) = 2\cos 2x$ b $f'(x) = \frac{1}{3}\cos\frac{1}{3}x$

5 a $f'(x) = -\pi\sin\pi x$ b $f'(x) = -5\sin 5x$

6 a $f'(x) = 3\cos(3x+1)$
b $f'(x) = -4\cos(1-4x)$

7 a $f'(x) = 3\sin(2-3x)$
b $f'(x) = -\frac{1}{2}\sin\left(\frac{1}{2}x+4\right)$

8 a $3e^{3x}$ b $\frac{1}{2}e^{\frac{x}{2}}$

9 a $-3x^2e^{-x^3}$ b $8xe^{4x^2}$

10 a $\frac{1}{x}$ b $\frac{1}{x}$

11 a $\frac{2x}{x^2+1}$ b $-\frac{8x}{3-4x^2}$

12 a $y' = 6\sin 3x\cos 3x$
b $y' = -8\cos 4x\sin 4x$

13 a $y' = -2\sin 2x e^{\cos 2x}$
b $y' = 5\cos 5x e^{\sin 5x}$

14 a $y' = \frac{1}{2x}(\ln 3x)^{-\frac{1}{2}}$ b $y' = \frac{1}{3x}(\ln 2x)^{-\frac{2}{3}}$

15 $-e^{-x}$

16 23

17 $y = \frac{4x}{5} + \frac{9}{5}$

18 (3, 0)

19 $y = \frac{\pi^2 x}{4} - \frac{\pi}{2}$

20 $y = \frac{x}{3} - \frac{5}{3} + \ln 3$

21 a $2\sin x\cos x$ b $\frac{\pi}{2}, \pi, \frac{3\pi}{2}$

25 $x = 3$

26 b No, also true for e.g. $y = 3e^{2x}$

Exercise 10C

1 a $\frac{dy}{dx} = x\cos x + \sin x$
b $\frac{dy}{dx} = x^{\frac{1}{2}}\cos x + \frac{1}{2}x^{-\frac{1}{2}}\sin x$

2 a $\frac{dy}{dx} = -x^2\sin x + 2x\cos x$
b $\frac{dy}{dx} = -x^{-1}\sin x - x^{-2}\cos x$

3 a $\frac{dy}{dx} = x^{-\frac{1}{2}}e^x - \frac{1}{2}x^{-\frac{3}{2}}e^x$
b $\frac{dy}{dx} = x^3e^x + 3x^2e^x$

4 a $\frac{dy}{dx} = x^{-\frac{1}{3}} + \frac{2}{3}x^{-\frac{1}{3}}\ln x$

b $\frac{dy}{dx} = x^3 + 4x^3 \ln x$

5 a $f'(x) = x(2x+1)^{-\frac{1}{2}} + (2x+1)^{\frac{1}{2}}$

b $f'(x) = \frac{9}{2}x^2(3x-4)^{-\frac{1}{2}} + 2x(3x-4)^{\frac{3}{2}}$

6 a $f'(x) = 3x^2\cos 3x + 2x\sin 3x$

b $f'(x) = 2x^{\frac{3}{4}}\cos 2x + \frac{3}{4}x^{-\frac{1}{4}}\sin 2x$

7 a $f'(x) = -4x\sin(4x-1) + \cos(4x-1)$

b $f'(x) = -5x^{-2}\sin 5x - 2x^{-3}\cos 5x$

8 a $f'(x) = 4x^{\frac{1}{2}}e^{4x+5} + \frac{1}{2}x^{-\frac{1}{2}}e^{4x+5}$

b $f'(x) = -x^{-2}e^{1-x} - 2x^{-3}e^{1-x}$

9 a $f'(x) = \frac{2x^{-1}}{2x-3} - x^{-2}\ln(2x-3)$

b $f'(x) = -\frac{x^3}{5-x} + 3x^2\ln(5-x)$

10 a $y' = \frac{x^2+4x}{(x+2)^2}$ b $y' = \frac{-12}{(x-4)^2}$

11 a $y' = \frac{\frac{1}{2}x+2}{(x+1)^{\frac{3}{2}}}$ b $y' = \frac{x-5}{(x-3)^3}$

12 a $y' = \frac{e^x(3x-5)}{(3x+1)^3}$ b $y' = \frac{e^x(2x-7)}{(2x-1)^4}$

13 $y = \frac{4}{\pi} - \frac{4x}{\pi^2}$

14 $y = \frac{2}{\pi}x - \frac{1}{2}$

15 $7e^6$

16 $e^x((x+1)\ln x + 1)$

17 $e^{x\sin x}(\sin x + x\cos x)$

18 $(\ln 2, 8 - 11\ln 2)$

19 $\left(1, \frac{1}{e}\right)$

20 (e, e)

21 $\frac{k}{(k+x^2)^{1.5}}$

22 $a = 3$, $b = 4$

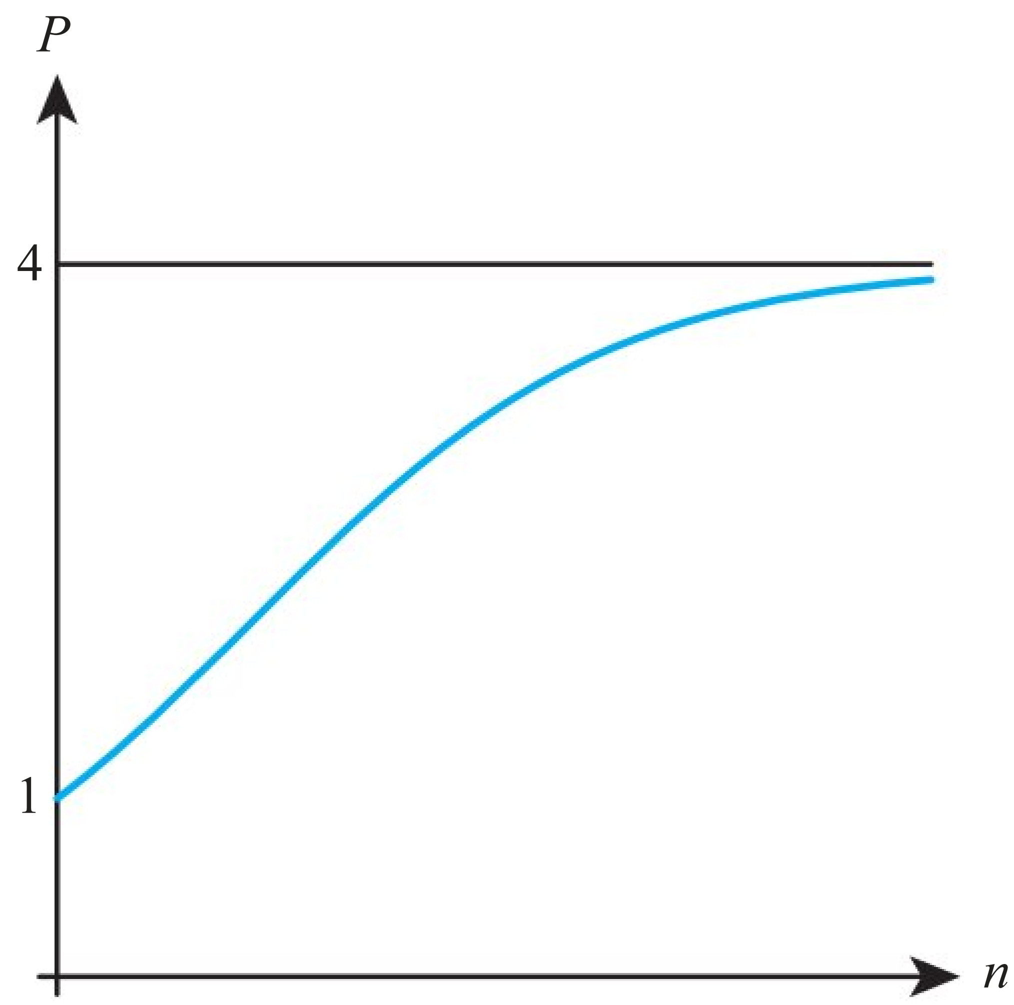
23 $a = 0$, $b = -2$

24 a $\frac{12e^n}{(e^n+3)^2}$; derivative > 0 for all n

b i 1000 rabbits

ii 750 rabbits per year

c 4

d


25 b $y = x$

Exercise 10D

1 a -3 b 4

2 a 12 b $7e$

3 a $-\frac{3}{4}$ per second b -4 per second

4 a -2 per hour b $-\frac{3}{2}$ per hour

5 -2

6 -21

7 $\frac{15}{16}$

8 $20\,\text{cm}^2\,\text{s}^{-1}$

9 $22.6\,\text{mm}^2$ per day

10 $1.92\,\text{cm}\,\text{s}^{-1}$

11 a $10\,\text{cm}^2\,\text{s}^{-1}$ b $0.8\,\text{cm}\,\text{s}^{-1}$

12 $\frac{5}{9\pi}\,\text{cm}\,\text{s}^{-1}$

13 7.65 cm

14 $2.68\,\text{m}\,\text{s}^{-1}$

15 Increasing

16 $\frac{0.2}{\sqrt{5}} \approx 0.0894\,\text{m}\,\text{s}^{-1}$

Exercise 10E

1 a 10 b 12

2 a 62 b 8

3 a $4e^{-4}$ b $18e^{6}$

4 a $-\frac{1}{4}$ b 1

5 a i Increasing, concave-down
ii Concave-up
iii Concave-down
b i Decreasing, concave-up
ii Increasing
iii Decreasing, concave-down

6 a $x < \frac{4}{3}$ b $x > -3$

7 a All x b No values

8 a Max: $(1, 11)$; min: $(2, 10)$
b Max: $(-2, 20)$; min: $(2, -12)$

9 a Min: $(-3, -84)$; max: $(0, -3)$; min: $(3, -84)$
b Min: $\left(\frac{1}{2}, \frac{5}{8}\right)$

10 a Max: $\left(\frac{1}{9}, \frac{4}{3}\right)$ b Min: $(0.25, 1)$

11 a Min: $(\ln 5, 5 - 5\ln 5)$
b Max: $\left(-\ln 2, \frac{1}{2}(-1 - \ln 2)\right)$

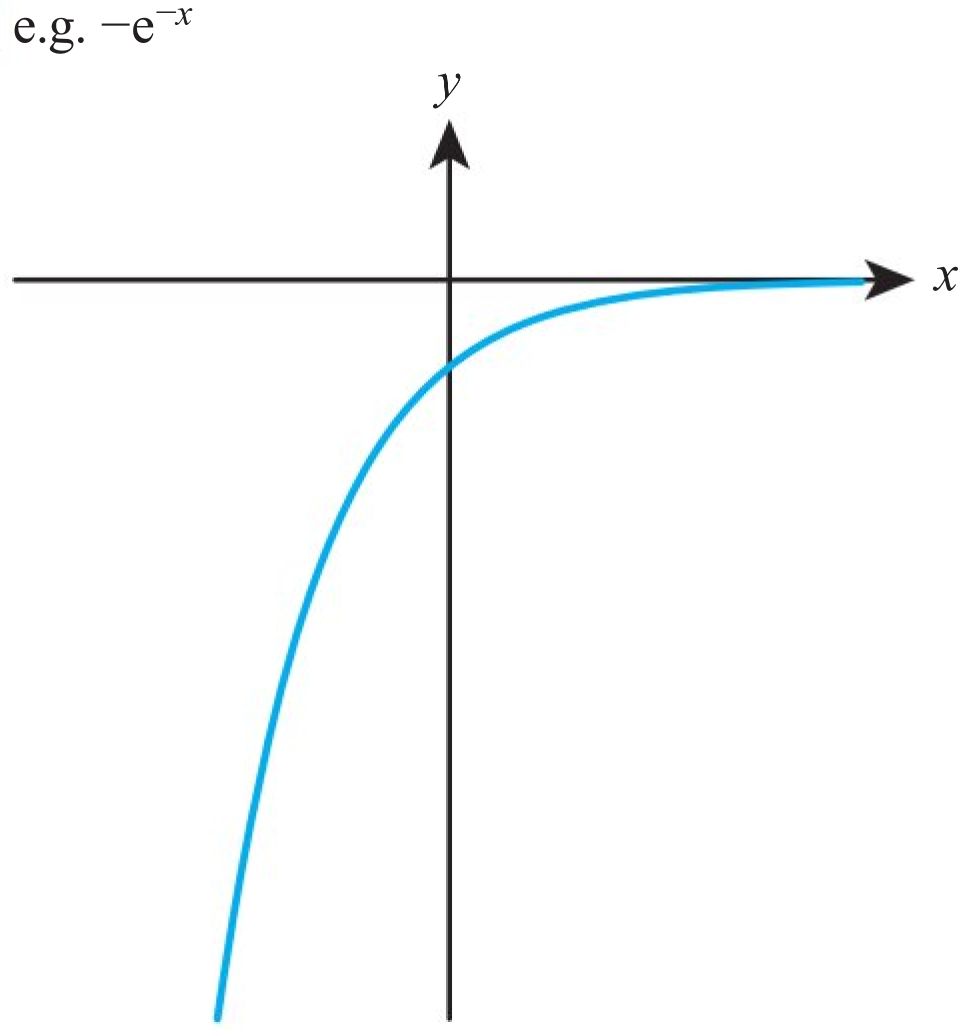
12 a Max: $\left(\frac{3}{2}, \ln\left(\frac{27}{8}\right) - 3\right)$
b Min: $(4, 2 - \ln 4)$

13 2

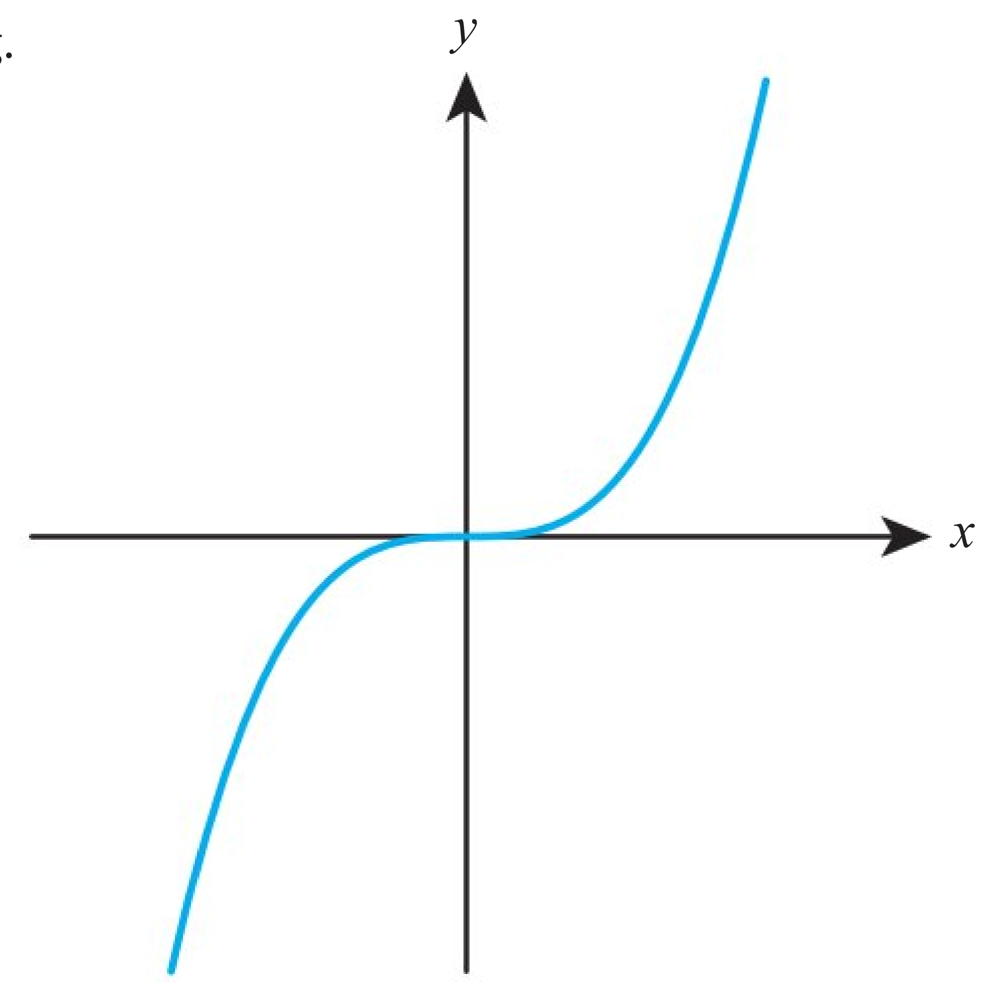
14 $a = 1, b = -1$

15 $(x + 2)e^{x}$

16 e.g. $-e^{-x}$



17 e.g.



18 a $k = 3$
b It is a point of inflection.

19 $4, -3$

20 $b = -2, c = 3$

21 $b = -4, c = 1$

22 $(1, -1)$, local min

25 $-\frac{1}{(a + x)^2}$

26 a $\left(\frac{1}{e}, -\frac{1}{e^2}\right)$ b $e^{-\frac{3}{2}}$

27 a $x < \frac{5}{3}$ b $\left(\frac{5}{3}, -\frac{124}{27}\right)$

28 a $\frac{1 - \ln x}{x^2}$ b $\left(e, \frac{1}{e}\right)$
d Local max

29 Max: $\left(-\frac{1}{p}, -2p\right)$ Min: $\left(\frac{1}{p}, 2p\right)$

30 $\left(\frac{\pi}{2}, e\right)$, local max $\left(\frac{3\pi}{2}, \frac{1}{e}\right)$, local min

31 Max: $\left(\frac{\pi}{3}, \frac{\sqrt{3}}{2} - \frac{\pi}{6}\right)$ Min: $\left(\frac{5\pi}{3}, -\frac{\sqrt{3}}{2} - \frac{5\pi}{6}\right)$

32 Min: $(0, 0)$ and max: $(-2, -4)$

33 a 4000 b \$39 000

34 $\frac{4a^3}{27}$ cm^3

35 $\left(-\frac{1}{p}, -\frac{1}{pe}\right)$; max if $p<0$, min if $p>0$

36 $t=1, \frac{dV}{dt}=2$

37 250

38 $\begin{cases} f(x) > a(1-a\ln a) & (\text{if } a>0) \\ f(x)>0 & (\text{if } a=0) \\ f(x)\in\mathbb{R} & (\text{if } a<0)\end{cases}$

39 (0, 3) local max

$(\ln 2, \ln 16)$ local min

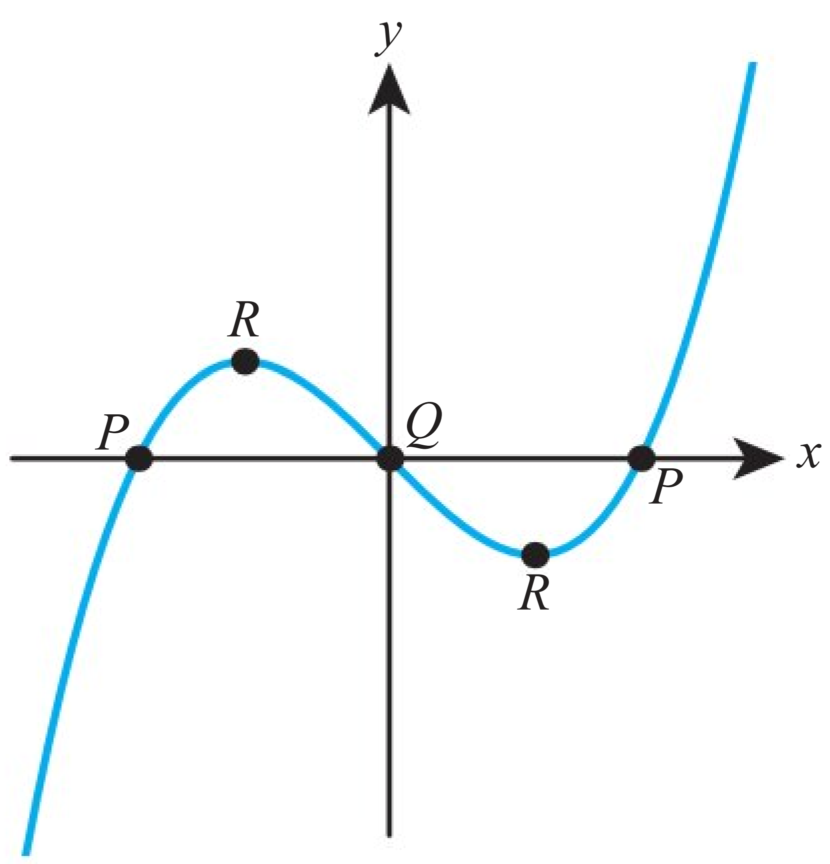
40 $1.5\sqrt[3]{2}$ m

41 a -6 b $b>12$

42 a 6

43 $5x^2+10x+2$

44 a



d No; $f'(x)\neq 0$ at these points

Chapter 10 Mixed Practice

1 $b=-4, c=7$

2 $2xe^x$

3 $\frac{2}{(x+2)^3}$

4 $\frac{x\cos x-\sin x}{x^2}$

5 $1+\ln x$

6 0.25

7 $y=x-1$

8 $y=1-\frac{x}{2}$

9 $259.2\,\text{cm}^3\,\text{s}^{-1}$

10 -16.3

11 b 10 c $600\,\text{cm}^3$

12 $y=x$

13 $2e^{2x}(2x^2+4x+1)$

14 a $4.5\,\text{cm}^2$ b $(6+3\sqrt{2})$cm

15 $6e^5$

16 a 12 b $(4, -94)$

17 $e^x((x+1)\sin x+x\cos x)$

18 $a=1, b=8, c=1.5$

19 a $x<2$ b $(2, 2e^{-2})$

20 $\left(\frac{\pi}{4}, \frac{e^{-\frac{\pi}{4}}}{\sqrt{2}}\right)$

21 0

22 a 5 million m^3 b $\left(5+\frac{2}{e}\right)$ million m^3

c 2 hours after the storm

23 $0.577\,\text{mg}\,\text{l}^{-1}$

24 $\frac{ds}{dt}=25$ (constant)

25 b $\left(-1, \frac{1}{9}\right)$ c $\frac{1}{9}<k<1$

27 a $\frac{100}{51}$ b 34.3

c $0<f(x)<100$ e 5

28 b No

29 a $-2\leqslant x\leqslant 2$

30 a 0.5 b $\pm\sqrt{2}$

31 $0.243\,\text{m}\,\text{s}^{-1}$

Chapter 11 Prior Knowledge

1 $y=\frac{1}{3}x^3+2x+4$

2 48.4

3 $\frac{dy}{dx}=2e^{2x}+\cos x$

4 $f'(x)=x(x^2+3)^{-\frac{1}{2}}$

Exercise 11A

1 a $\frac{3}{5}x^{\frac{5}{3}}+c$ b $\frac{4}{7}x^{\frac{7}{4}}+c$

2 a $2x^{\frac{1}{2}}+c$ b $-3x^{-\frac{1}{3}}+c$

3 a $4x^{\frac{5}{2}}+c$ b $4x^{\frac{5}{4}}+c$

4 a $12x^{\frac{1}{3}}+c$ b $5x^{\frac{3}{5}}+c$

5 a $\frac{3}{8}x^{\frac{4}{3}}+c$ b $\frac{5}{18}x^{\frac{6}{5}}+c$

6 a $8x^{\frac{3}{4}}+c$ b $14x^{\frac{1}{2}}+c$

7 a $y=3\sin x+c$ b $y=-\sin x+c$

8 a $y=2\cos x+c$ b $y=-\frac{1}{2}\cos x+c$

9 a $y=5e^{x}+c$ b $y=-\frac{4}{3}e^{x}+c$

10 a $y=2\ln x+c$ b $y=\frac{1}{2}\ln x+c$

11 a $\frac{1}{3}(2x+1)^{\frac{3}{2}}+c$ b $-\frac{3}{8}(1-2x)^{\frac{4}{3}}+c$

12 a $-\frac{2}{5}(3-5x)^{\frac{1}{2}}+c$ b $\frac{2}{3}(2x-7)^{\frac{3}{4}}+c$

13 a $-\frac{1}{3}\sin(2-3x)+c$ b $\frac{1}{4}\sin(4x+3)+c$

14 a $-2\cos\left(\frac{1}{2}x-5\right)+c$ b $\frac{1}{2}\cos(5-2x)+c$

15 a $\frac{1}{5}e^{5x+2}+c$ b $-\frac{1}{3}e^{1-3x}+c$

16 a $\frac{1}{4}\ln|4x-5|+c$ b $-\frac{1}{2}\ln|3-2x|+c$

17 a $\frac{1}{4}(x^2+4)^4+c$ b $\frac{2}{5}(x^4-2)^{\frac{5}{2}}+c$

18 a $\frac{1}{4}\sin^4 x+c$ b $\frac{1}{3}\cos^3 x+x$

19 a $\ln|x^3+2x|+c$ b $\ln|x^4-5x|+c$

20 a $\frac{2}{9}(x^3+4)^{\frac{3}{2}}+c$ b $-\frac{3}{8}(1-x^2)^{\frac{4}{3}}+c$

21 a $-2e^{-x^2}+c$ b $3e^{x^3}+c$

22 a $\frac{1}{3}\ln|x^3+5|+c$ b $\frac{1}{4}\ln|x^4-3|+c$

23 $y=2x^{1.5}-42$

24 $2\sin x+3\cos x+2$

25 $y=2(e^{x}-e)-5\ln|x|$

26 $x^2+\frac{3}{2}\ln|x|+c$

27 $V=\frac{1}{2}t^2+\frac{1}{2}\cos t+\frac{3}{2}$

28 $x=5t-2e^{t}+7$

29 $-\frac{1}{14}(5-2x)^7+c$

30 $-\frac{3}{2}\cos(2x)-\frac{2}{3}\sin(3x)+c$

31 $\frac{1}{6}(x^2+1)^6+c$

32 $-\frac{1}{6}\cos(3x^2)+c$

33 $3\sqrt{x^2+2}+c$

34 $-\frac{1}{2(2x+3)}+\frac{3}{2}$

35 $f(x)=1+\frac{(\ln x)^2}{3}$

36 $2\ln 3+5$

37 $-\cos^4 x+\frac{kx^3}{3}+x+c$

38 $e^{x}+\frac{1}{3}e^{-3x}+c$

39 $\frac{1}{3}(x^2+1)^{1.5}+c$

40 a $1-2\sin^2 x$ b $\frac{1}{2}x-\frac{1}{4}\sin 2x+c$

41 $-\ln|\cos x|+c$

42 $-\cos x+\frac{1}{3}\cos^3 x+c$

43 $\ln|\ln x|+c$

Exercise 11B

1 a $\frac{14}{3}$ b 12

2 a $\frac{33}{5}$ b $-\frac{1}{6}$

3 a $\frac{26}{3}$ b –60

4 a $\frac{1}{2}$ b 1

5 a 2 b 5

6 a $\ln 5$ b $\ln\frac{3}{2}$

7 a 0.774 b –0.152

8 a 1.12 b 3.02

9 a $\frac{4}{3}$ b $\frac{4}{3}$

10 a $\frac{27}{2}$ b $\frac{5}{2}$

11 a 1.63 b 9.77

12 a 2.83 b 1.83

13 $\frac{112}{9}$

14 3

15 3.29

16 25

17 a $(-3, 0), (3, 0)$ b 36

18 a $\frac{8}{3}, -\frac{5}{12}$ b $\frac{37}{12}$

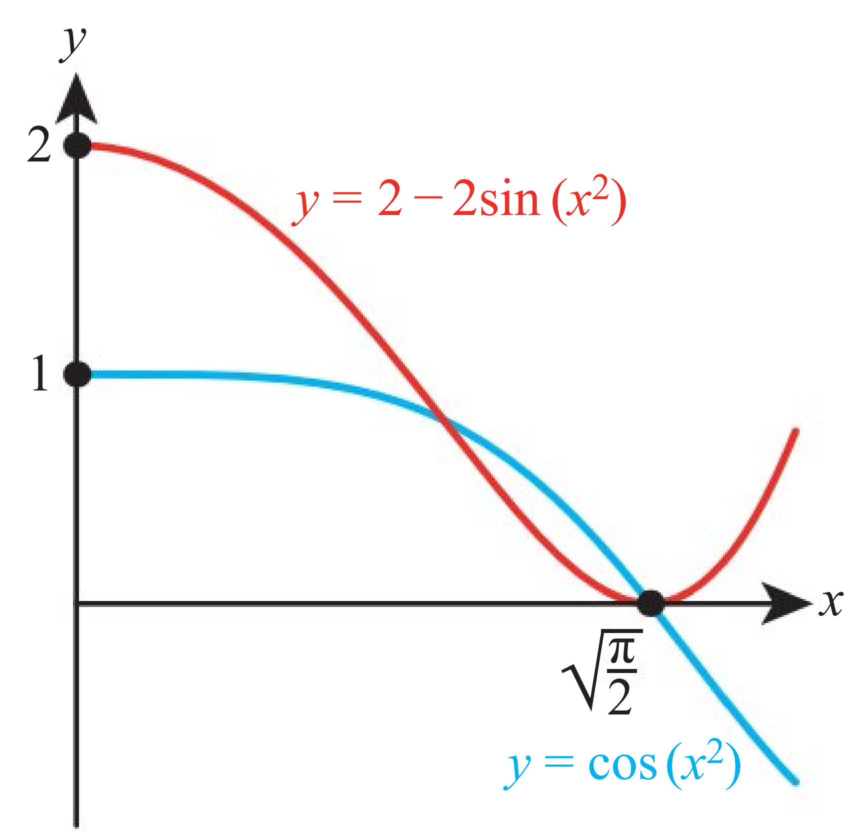
19 11.0

20 $\frac{52}{81}$

21 $-5\ln 3$

23 a $\left(\frac{\pi}{4}, \frac{\sqrt{2}}{2}\right)$ b $2-\sqrt{2}$

24 a



b 0.0701

26 $\frac{7}{3}$

27 $\frac{3}{2}\left(1-e^{-2}\right)$

28 $\frac{1}{2}\ln\left(\frac{11}{5}\right)$

29 $\ln\left(\frac{2}{3}\right)$

30 $a = 6$

31 $a = 2$ or $-\frac{8}{3}$

32 a $A(0, 1), B\left(1, \frac{1}{e}\right)$ b 0.115

33 27

34 17.5

35 a $\ln x + 1$ b 1

36 -2

Exercise 11C

1 a $\frac{26}{3}$ b $\frac{45}{4}$

2 a 1.83 b 0.848

3 a 5.10 b 8.77

4 a 2.48 b 0.527

5 a 1370 b 230

6 a 91.7 b 512

7 a 11.8 b 33.0

8 a 3.14 b 7.07

9 a 101 b 133

10 a 12.6 b 45.7

11 a 3.59 b 0.771

12 a 93.2 b 48.7

13 a $\ln 5$ b $\frac{4}{5}\pi$

14 3

15 $a = \pm 2$

16 a $(-3, 0)$ b 18π

17 a $\frac{32}{3}$ b $\frac{512\pi}{15}$

18 a 2.43 b 7.75

19 b 5.25

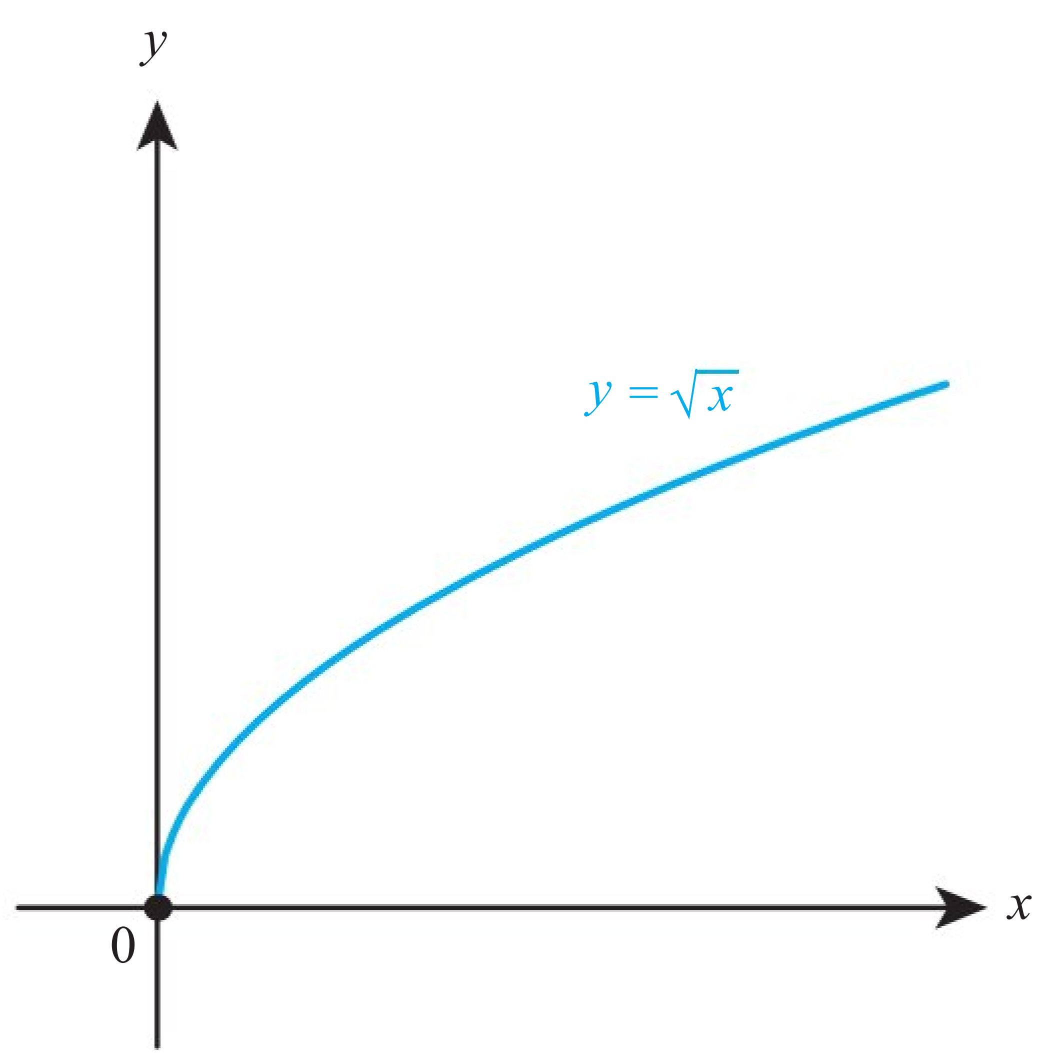
c i 44.1 ii 30.1

20 2π

21 3646

22 $\frac{\pi^2}{2}$

23 a



b 127 c 153

24 a $A(0, 2)$, $B(2, 0)$ b 4.39 c 32.7

25 $\frac{20}{3}\pi$

26 $\frac{32}{5}\pi$

27 a $a - 1$

28 a $hx + ry = rh$

29 a $x^2 + y^2 = r^2$

30 a $(0, 0)$ and $(1, 1)$ b $\frac{3}{10}\pi$

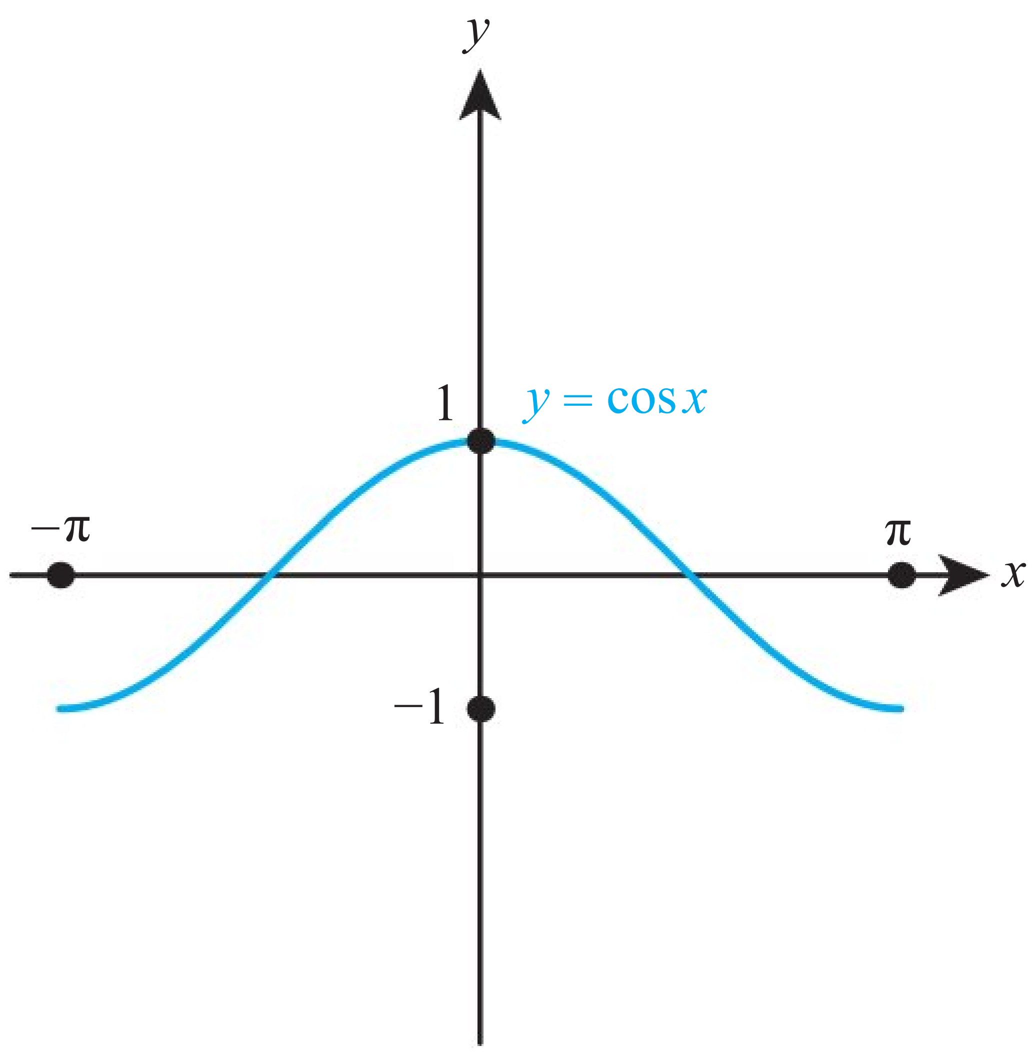
31 $\frac{\pi^2 - 2\pi}{4}$

32 a $(0, 0)$ and $(2, 4)$ b $\frac{8\pi}{3}$

33 $1:3$

34 a $y = \ln(x - 2)$ b $\frac{\pi}{2}(e^2 + 8e - 1)$

35 a



c $3\pi^2$

Exercise 11D

1 a $\frac{1}{2}e^{2x} + c$ b $\frac{1}{5}\sin(5x) + c$

2 a $-\frac{1}{5}\ln|2 - 5x$ b $\frac{1}{4}e^{1+4x} + c$

3 a $\ln|1 + e^x| + c$ b $\frac{1}{3}\ln|1 + x^3| + c$

4 a $\frac{1}{3}(x^2 + 2)^{\frac{3}{2}} + c$ b $\frac{2}{3}(\sin x)^{\frac{3}{2}} + c$

5 a $-2\cos\sqrt{x} + c$ b $\sin(\ln x) + c$

6 a $-\frac{1}{4}\cos(4x) + c$ b $\frac{1}{3}e^{3x} + c$

7 a $\frac{1}{4}\ln|3 + 4x| + c$ b $-\ln|2 - x| + c$

8 a $\frac{1}{3}e^{(x^3)} + c$ b $\frac{1}{5}\sin(x^5) + c$

9 a $\frac{1}{6}\ln|1 + 3x^2| + c$ b $\ln|1 + \sin x| + c$

10 a $\frac{\sin^2 x}{2} + c$ b $-\ln|\cos x| + c$

11 $\frac{2}{5}(x + 1)^{\frac{5}{2}} - \frac{2}{3}(x + 1)^{\frac{2}{3}} + c$

12 $\arcsin(x) + c$

Chapter 11 Mixed Practice

1 $a^3 - 4a$

2 $y = \frac{1}{2}\sqrt{x} + 2$

3 $\frac{\sqrt{3} - 1}{2}$

4 $\frac{3}{4}\ln x + \frac{1}{2x} + c$

5 19.0

6 $\frac{\pi^2}{4} \approx 2.47$

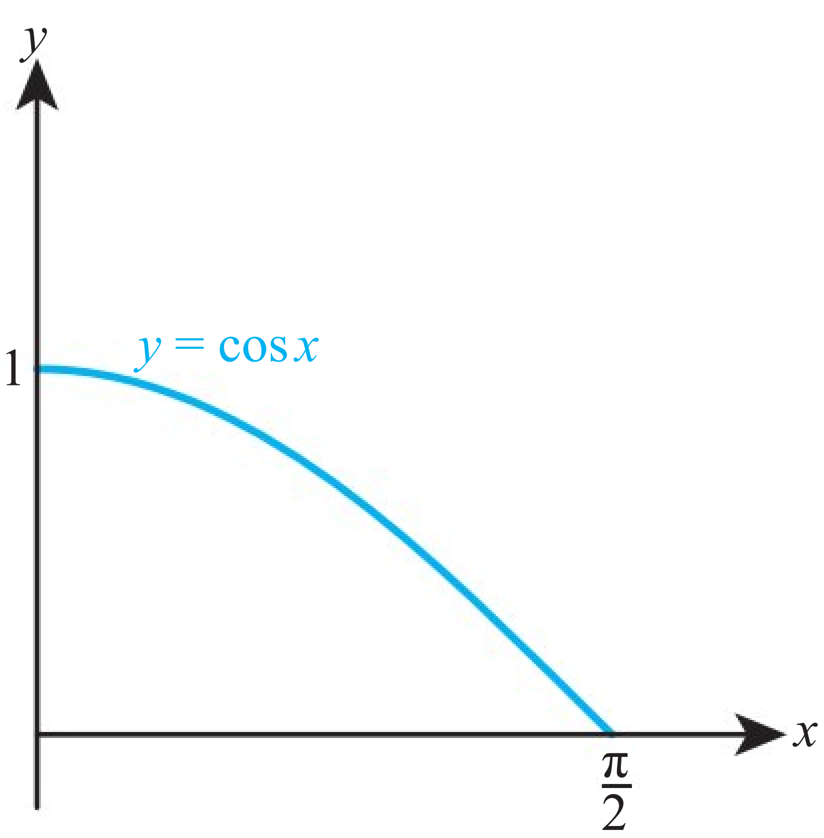
7 $\frac{7}{3}$

8 $6\ln\left(\frac{2x - 5}{3}\right)$

9 $2x^{\frac{3}{2}} + 4x^{-\frac{1}{2}} + c$

10 $y = \sin^3 x + 2$

11 a



b $A(0, 1)$, $B\left(\frac{\pi}{2}, 0\right)$ c $y = -\frac{2}{\pi}x + 1$

d $1 - \frac{\pi}{4}$

12 $\frac{4}{3}$

13 a $A(0, 1)$, $B(2, 5)$, $C(7, 0)$ b 17.2

14 a $\ln 11$ b 1.52 c 5.33

15 $\pi(e^4 - 1)$

16 π

18 a $\pi \tan a$ b $\frac{\pi}{4} \approx 0.786$

19 $\frac{1}{2}\ln 17$

20 $\frac{5\pi}{3}$

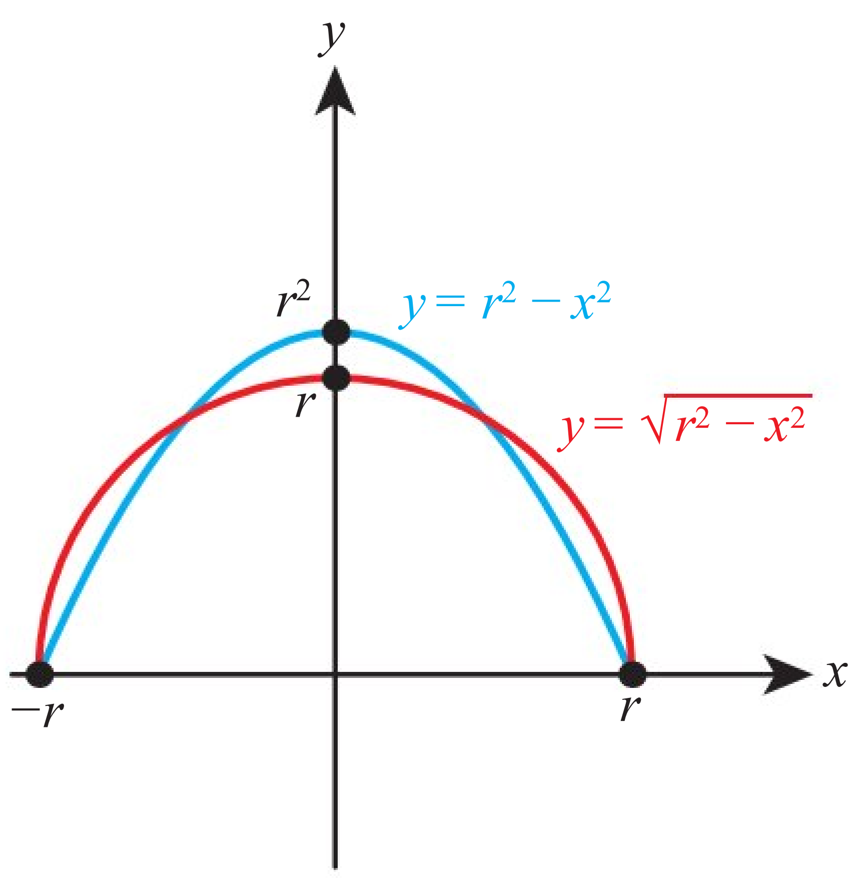
21 a $y = 0.000545x^3 - 0.0582x^2 + 1.69x + 10$
b $74.4\ l$

22 6

23 a $0 < x < d$ b $a < x < b$ and $x > c$
c 15

24 a $hx + (a - b)y = ah$

25 a



b $r = \frac{4}{3}$

26 a $a - 1$ b $a \ln a - a + 1$

Chapter 12
Prior Knowledge

1 a $-3e^{-3t}$ b $\frac{1}{2}\cos\left(2t - \frac{\pi}{6}\right)$

2 $-\frac{1}{t^2}$

3 $1.5t^2 + \frac{1}{3}\cos(3t) + c$

4 5

5 60.3°

Exercise 12A

1 a $v = 3t^2 + 6t$, $a = 6t + 6$ b $v = 4t^3 - 5$, $a = 12t^2$

2 a $v = -2\sin 2t$, $a = -4\cos 2t$ b $v = -\frac{1}{4}\sin\frac{t}{2}$, $a = -\frac{1}{4}\sin\frac{t}{2}$

3 a $v = 3e^{3t}$, $a = 9e^{3t}$ b $v = -4e^{-4t}$, $a = 16e^{-4t}$

4 a $v = \frac{1}{t}$, $a = -\frac{1}{t^2}$ b $v = \frac{1}{t}$, $a = -\frac{1}{t^2}$

5 a $3s^5 - 3s^2$ b $2s^3 + 3s^2 + s$

6 a $-\frac{1}{s^3}$ b $-e^{-2s}$

7 a 45 b 124

8 a 4 b $\frac{4}{5}$

9 a $3\sqrt{3}$ b 3

10 a $\ln\frac{5}{3}$ b $\ln\frac{16}{3}$

11 a 1.21 b 0.935

12 a 1.27 b 0.984

13 a 0.672 b 2.53

14 a 1.54 b 0.795

15 a 0.657 b 1.435

16 a 16 m b $4\ \mathrm{m\,s^{-1}}$ c $-2\ \mathrm{m\,s^{-2}}$

17 $2 - 2e^{-1.5} \approx 1.55$ m

18 a -48 m b $\frac{176}{3} \approx 58.7$ m

19 $23\ \mathrm{m\,s^{-1}}$

20 a $256\ \mathrm{m\,s^{-1}}$ b $-32\ \mathrm{m\,s^{-2}}$
c 4 s d 819.2 m

21 $e^2 + \ln 16 - 7 \approx 3.16$

22 a 0 s, 6 s b 4 s c 4 s

23 a $\sin t + t\cos t$ b $0\ \mathrm{m\,s^{-1}}$
c $\frac{4 + \pi}{4\sqrt{2}}\ \mathrm{m\,s^{-1}}$ d $2\ \mathrm{m\,s^{-2}}$
e 0.556 s, 1.57 s, 5.10 s f 13.3 m

26 a $-10\ \mathrm{m\,s^{-2}}$ b 0.5 s, 61.25 m
c 4 s d 62.5 m
e e.g. No obstructions to path of ball

27 a $18\,\mathrm{m\,s^{-1}}$ b $3\,\mathrm{s}$
c $-8\,\mathrm{m\,s^{-2}}$ d $108\,\mathrm{m}$
e $-36\,\mathrm{m}$ f $36\,\mathrm{m}$

28 $10\,\mathrm{m\,s^{-1}}$

29 $8\,\mathrm{m\,s^{-1}}$

30 $0.5\,\mathrm{m}$

31 a $10\,\mathrm{m\,s^{-1}}$ b $44.6\,\mathrm{m\,s^{-1}}$ c $11.3\,\mathrm{m\,s^{-2}}$

32 a $0\,\mathrm{m}$ b $50\,\mathrm{m}$

33 a $6.06\,\mathrm{m\,s^{-1}}$ b 0
c $\dfrac{71}{24} \approx 2.96\,\mathrm{m\,s^{-1}}$

34 a i $v = \dfrac{At^2}{2} + u$ ii $s = \dfrac{At^3}{6} + ut$

35 $6\,\mathrm{s}$

36 $0.938\,\mathrm{m}$

37 $v = \sqrt{\dfrac{1}{2}s^4 + s^2 + k}$

Exercise 12B

1 a $\mathbf{a} = \begin{pmatrix} 4 \\ -6t \end{pmatrix}$, $\mathbf{r} = \begin{pmatrix} 2t^2 + t \\ t - t^3 \end{pmatrix}$

b $\mathbf{a} = \begin{pmatrix} 12t \\ 6 \end{pmatrix}$, $\mathbf{r} = \begin{pmatrix} 2t^3 - t \\ 3t^3 + 3t \end{pmatrix}$

2 a $\mathbf{a} = \begin{pmatrix} 4e^t \\ 3e^t \end{pmatrix}$, $\mathbf{r} = \begin{pmatrix} 4e^t \\ 3e^t \end{pmatrix}$

b $\mathbf{a} = \begin{pmatrix} 3e^t \\ -5e^t \end{pmatrix}$, $\mathbf{r} = \begin{pmatrix} 3e^t \\ -5e^t \end{pmatrix}$

3 a $\mathbf{a} = \begin{pmatrix} 48e^{4t} \\ -18e^{-3t} \end{pmatrix}$, $\mathbf{r} = \begin{pmatrix} 3e^{4t} \\ -2e^{-3t} \end{pmatrix}$

b $\mathbf{a} = \begin{pmatrix} -5e^{-t} \\ 50e^{5t} \end{pmatrix}$, $\mathbf{r} = \begin{pmatrix} -5e^{-t} \\ 2e^{5t} \end{pmatrix}$

4 a $\mathbf{a} = \begin{pmatrix} -6\sin 2t \\ 8\cos 2t \end{pmatrix}$, $\mathbf{r} = \begin{pmatrix} 1.5\sin 2t \\ -2\cos 2t \end{pmatrix}$

b $\mathbf{a} = \begin{pmatrix} -25\cos 5t \\ -50\sin 5t \end{pmatrix}$, $\mathbf{r} = \begin{pmatrix} \cos 5t \\ 2\sin 5t \end{pmatrix}$

5 a $14.2\,\mathrm{m\,s^{-1}}$ b $57.0\,\mathrm{m\,s^{-1}}$

6 a $8.24\,\mathrm{m\,s^{-1}}$ b $19.4\,\mathrm{m\,s^{-1}}$

7 a $59.5\,\mathrm{m\,s^{-1}}$ b $18\,100\,\mathrm{m\,s^{-1}}$

8 a $3.98\,\mathrm{m\,s^{-1}}$ b $8.26\,\mathrm{m\,s^{-1}}$

9 a $\begin{pmatrix} 1.5\cos 3t \\ 4 \end{pmatrix}$ b $4.20\,\mathrm{m\,s^{-1}}$

10 a $60.3°$ b $3.61\,\mathrm{m\,s^{-2}}$

11 a $0.249\,\mathrm{m\,s^{-1}}$ b $\begin{pmatrix} -3e^{-t} \\ -4e^{-t} \end{pmatrix}$ c $0.249\,\mathrm{m}$

12 a $53.9°$ b $11.9\,\mathrm{m\,s^{-1}}$ c $3.03\,\mathrm{m\,s^{-2}}$

13 a $16.8\,\mathrm{m\,s^{-1}}$ b $3.61\,\mathrm{m\,s^{-2}}$

14 a $9.8\,\mathrm{m\,s^{-2}}$ downwards b $20\,\mathrm{m\,s^{-1}}$
c $\sqrt{256 + (12 - 9.8t)^2}$; $16\,\mathrm{m\,s^{-1}}$

15 a $\begin{pmatrix} 9t \\ 7t - 4.9t^2 \end{pmatrix}$ b $12.9\,\mathrm{m}$

16 a i $26.6°$ above the horizontal
ii $25.6°$ below the horizontal
b 1.02 seconds
c $\begin{pmatrix} 20t \\ 10t - 4.9t^2 \end{pmatrix}$; $5.10\,\mathrm{m}$

17 a $\begin{pmatrix} 12t \\ 9t - 4.9t^2 \end{pmatrix}$ b $22.0\,\mathrm{m}$

18 a $5\,\mathrm{m}$

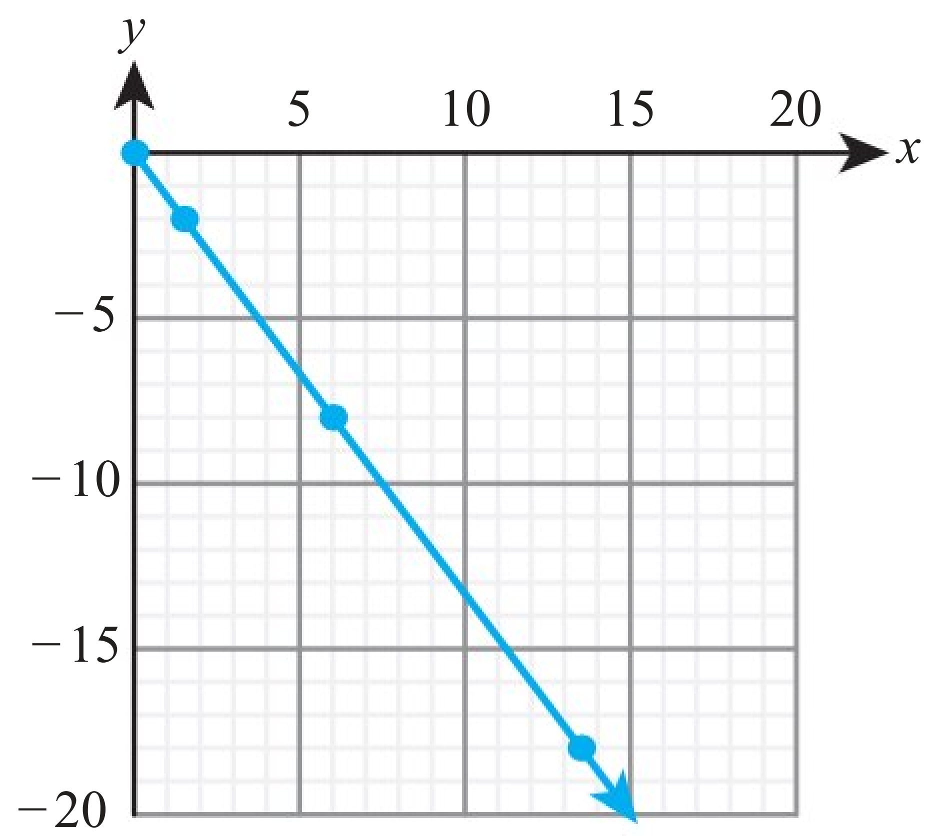
19 a $\mathbf{v} = \begin{pmatrix} -6\sin\left(\dfrac{t}{2}\right) \\ 6\cos\left(\dfrac{t}{2}\right) \end{pmatrix}$ b $6\,\mathrm{m\,s^{-1}}$
d $3\,\mathrm{m\,s^{-2}}$

20 a $0.9\,\mathrm{m\,s^{-1}}$
b $\begin{pmatrix} 0.3\sin 3t \\ -0.3\cos 3t \end{pmatrix}$; $r = 0.3\,\mathrm{m}$
c $\begin{pmatrix} -2.7\sin 3t \\ 2.7\cos 3t \end{pmatrix}$; $k = -9$
d Towards the origin

21 a $0.5\,\mathrm{m}$ b $\begin{pmatrix} 0.5 \\ 0 \end{pmatrix}$
c 1.57 seconds d $\mathbf{v} = \begin{pmatrix} -2\sin 4t \\ 2\cos 4t \end{pmatrix}$

22 a $\begin{pmatrix} 4t - 0.75t^2 \\ 0.1t^3 + 5t \end{pmatrix}$ b $20.2\,\mathrm{m}$

23 a $\mathbf{v} = \begin{pmatrix} 3t \\ -4t \end{pmatrix}$, $\mathbf{r} = \begin{pmatrix} 1.5t^2 \\ -2t^2 \end{pmatrix}$
b 4 seconds
c Points (0, 0), (1.5, −2), (6, −8), (13.5, −18); straight line

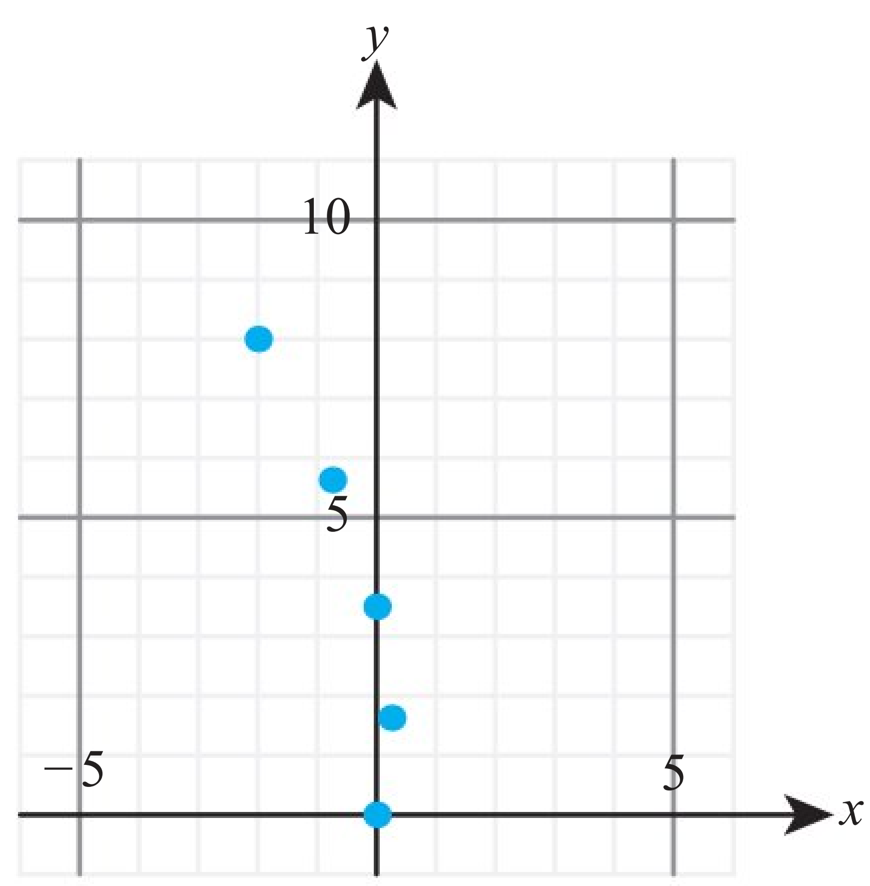


Hint: You can find the equation of the line expressing y in terms of x, where $x = 1.5t^2$ and $y = -2t^2$.

24 a $\mathbf{v} = \begin{pmatrix} 1 - 2t \\ 3 + t \end{pmatrix}$, $\mathbf{r} = \begin{pmatrix} t - t^2 \\ 3t + 0.5t^2 \end{pmatrix}$

b 49.4°

c Points (0,0), (0.25, 1.63), (0, 3.5), (−0.75, 5.63), (−2, 8); no



25 $4.7\,\mathrm{m\,s^{-2}}$

27 18.2

28 a $\begin{pmatrix} 12t \\ 30t - 4.9t^2 \end{pmatrix}$ b $\begin{pmatrix} 20(t-1) \\ b(t-1) - 4.9(t-1)^2 \end{pmatrix}$

c 36.9 d 53.6 m

29 a $\sqrt{1200^2 + (1390 - 98t)^2}$

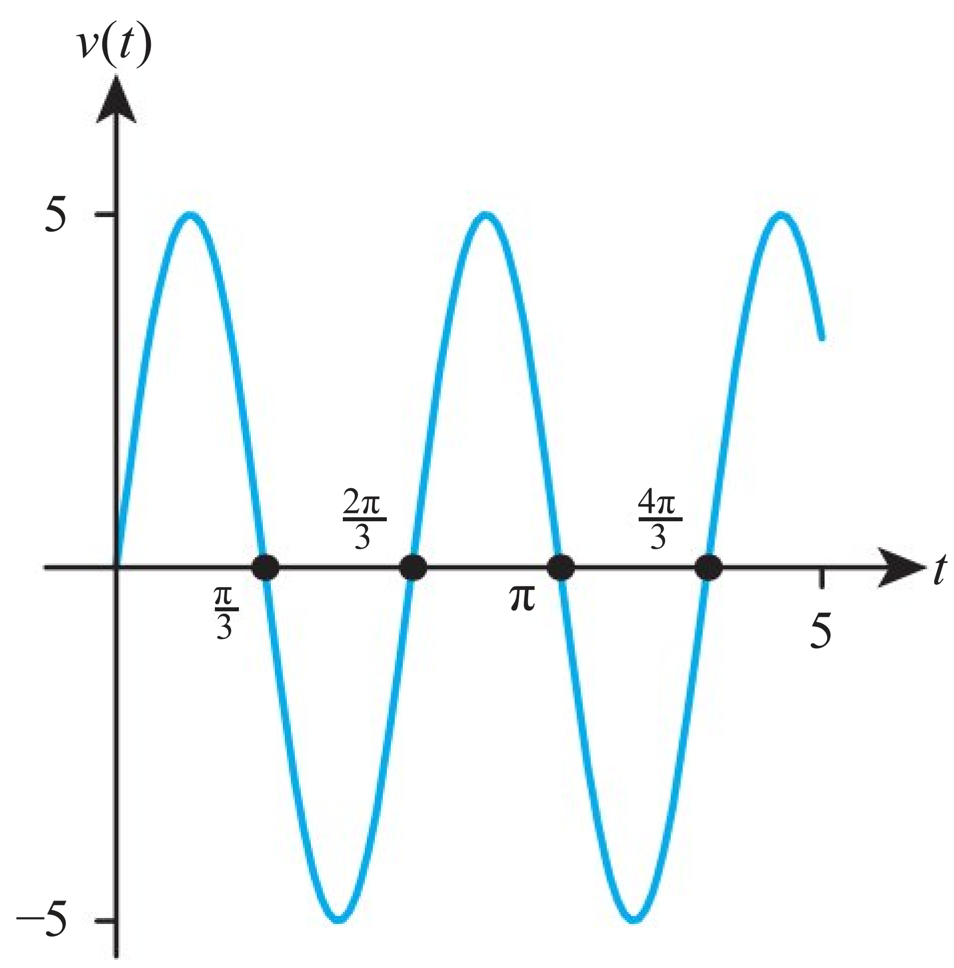
b 14.2 seconds

Chapter 12 Mixed Practice

1 $v = -10.3\,\mathrm{m\,s^{-1}}$; $a = -3.10\,\mathrm{m\,s^{-2}}$

2 a $-0.147\,\mathrm{m\,s^{-2}}$ b 1.96 m

3 a



b 1.05 s, 2.09 s, 3.14 s, 4.19 s

c 16.3 m

4 a $\begin{pmatrix} 7 \\ 0 \end{pmatrix}$ b 7 cm c $21\,\mathrm{cm\,s^{-1}}$

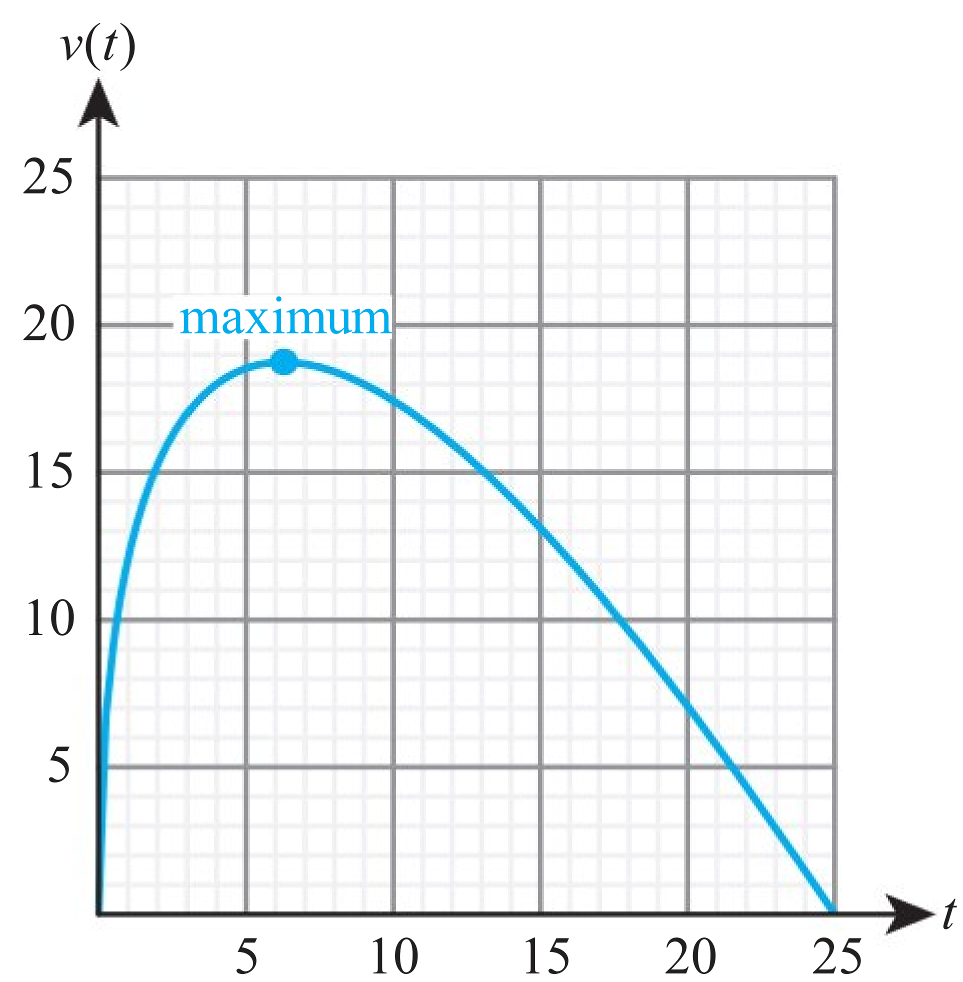
5 a $21.3\,\mathrm{m\,s^{-1}}$ b $\begin{pmatrix} 16 \\ -5.6 \end{pmatrix}$; down

c $\begin{pmatrix} 16t \\ 14t - 4.9t^2 \end{pmatrix}$ d 33.1 m

6 a $2\,\mathrm{cm\,s^{-1}}$ b $\begin{pmatrix} 3 \\ -1 \end{pmatrix}$

7 a $-31.7\,\mathrm{cm\,s^{-2}}$ b 7.41 cm

8 a



b i $d = \int_0^9 (15\sqrt{t} - 3t)\,\mathrm{d}t$

ii 149 m

9 a $-9\,\mathrm{m\,s^{-2}}$ b 18.5 m

10 a $\begin{pmatrix} 1.5t^2 \\ 4 - 4e^{-0.5t} \end{pmatrix}$ b 6.51 m

11 a $\mathbf{r} = \begin{pmatrix} -3\cos t \\ 4\sin t \end{pmatrix}$ c 4 m

d 73.7°

12 a $\begin{pmatrix} 2 \\ 17 - 9.8t \end{pmatrix}$

b $\begin{pmatrix} 2t \\ 17t - 4.9t^2 \end{pmatrix}$; 3.47 seconds

c $\sqrt{(2t)^2 + (17t - 4.9t^2)^2}$

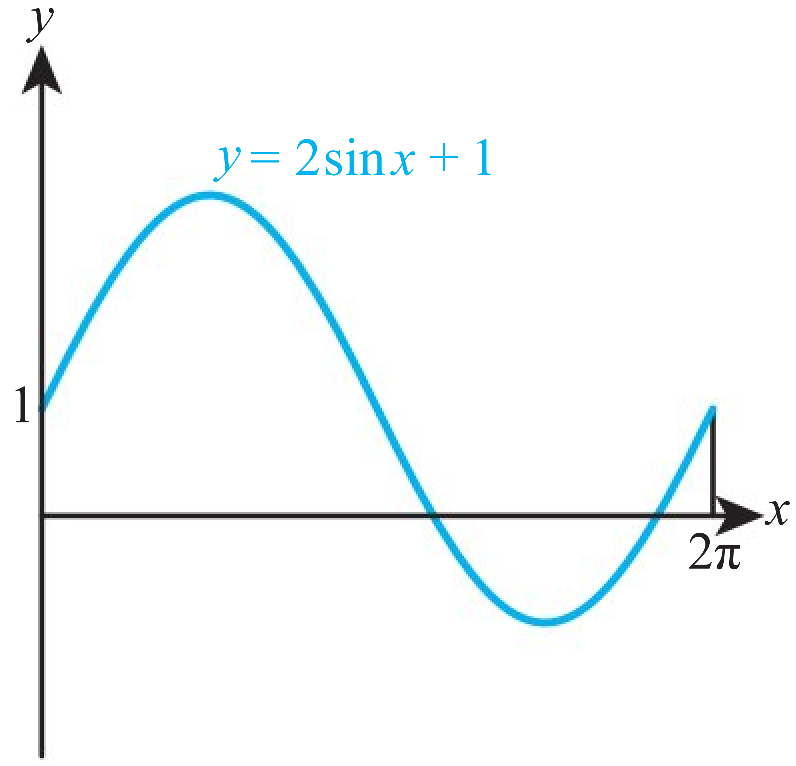
d 15.2 m

13 a $\begin{pmatrix} 4 \\ 0 \end{pmatrix}$ b $\begin{pmatrix} -8\sin 2t \\ 20\cos 4t \end{pmatrix}$

c $21.5\,\mathrm{m\,s^{-1}}$ d e.g. When $t = 0.785$

e $t = 3.14$

14 a



b $2.20\,\mathrm{m\,s^{-1}}$ c 6.28 m d 9.02 m

15 $-s(s-1)e^{-2s}$

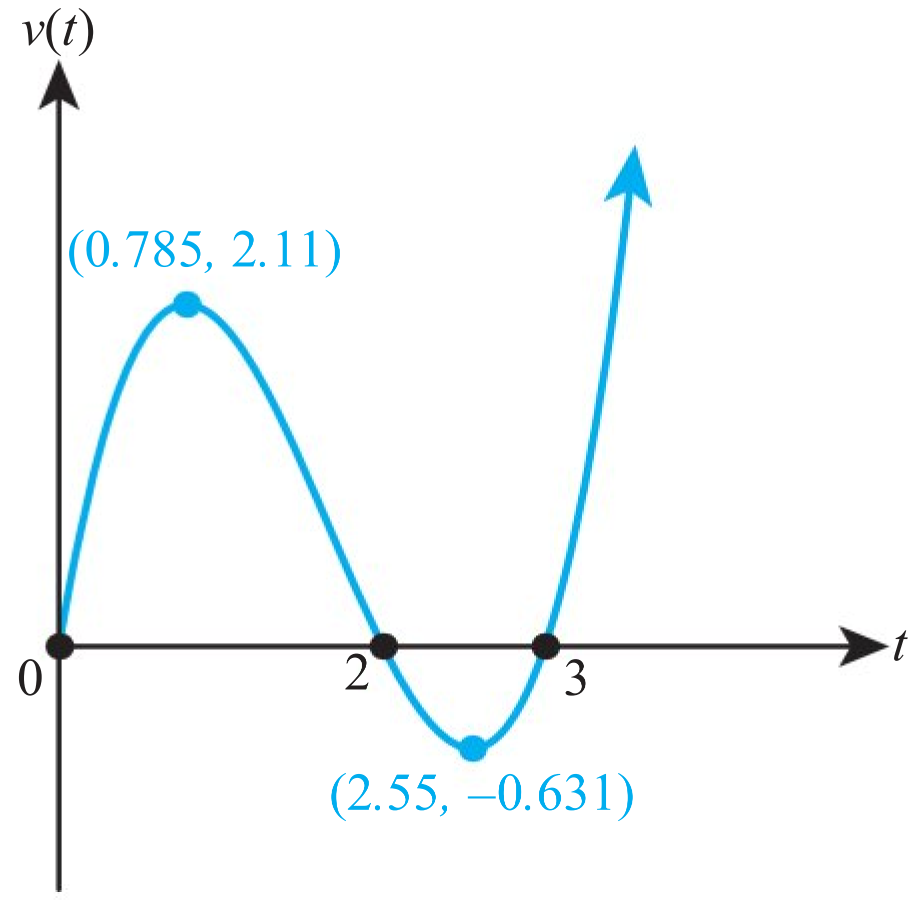
16 4 s

17 $17.6\,\mathrm{m\,s^{-1}}$

18 a Displacement = A, acceleration = B

b $t = 3$

19 a



b $0 \leqslant t < 0.785$ and $t > 2.55$

c $0 \leqslant t < 0.785$, $2 < t < 2.55$ and $t > 3$

d $x_A = \frac{1}{4}t^4 - \frac{5}{3}t^3 + 3t^2$

e $v_B = -20e^{-2t}$ f $t = 4.41$

20 a They meet when $t = 3$.

b 37.7°

21 a $v = 8 - 9.8t$, $s = 8t - 4.9t^2$

b 1.17 seconds

22 a $\mathbf{v} = \begin{pmatrix} 8 \\ 5 - 9.8t \end{pmatrix}$, $\mathbf{r} = \begin{pmatrix} 8t \\ 5t - 4.9t^2 \end{pmatrix}$

b 0.260 seconds

23 a $\mathbf{r} = \begin{pmatrix} 8\sin 2t \\ -8\cos 2t \end{pmatrix}$ b 2.17 seconds, 10.9 cm

24 a i $-10\,\mathrm{m\,s^{-2}}$ ii $-100\,\mathrm{m\,s^{-1}}$

b $\frac{dt}{dv} = \frac{1}{-10 - 5v}$ d $v = -2 - 98e^{-5(t-10)}$

e $s = -2t + \frac{98}{5}e^{-5(t-10)} + 500.4$

f $t = 250$

Chapter 13
Prior Knowledge

1 $\frac{1}{2}e^{2x} + c$

2 2

3 x^2

4 1035

Exercise 13A

1 a $\frac{db}{dt} = kb$ b $\frac{da}{dt} = k\sqrt{a}$

2 a $\frac{dr}{dt} = \frac{k}{r^2}$ b $\frac{dh}{dt} = \frac{k}{h}$

3 a $\frac{dF}{dt} = -k\sqrt{F}$ b $\frac{dv}{dt} = -\frac{k}{v}$

4 a $\frac{dI}{dt} = kI(N - I)$ b $\frac{dR}{dt} = \frac{kR(N - R)}{t}$

5 a $y = \frac{x^3}{3} + c$ b $y = 8\sqrt{x} + c$

6 a $y = \frac{e^{2t}}{2} + c$ b $y = 3\ln t + c$

7 a $s = -\frac{\cos 3x}{3} + c$ b $s = 2\tan t + c$

8 a $F = m - 3\ln m + c$ b $F = \frac{1}{m} + 3m + c$

9 a $y = Ae^{2x}$ b $y = Ae^{-y}$

10 a $y = Ae^{x} - 1$ b $y = Ae^{-x} + 1$

11 a $y = -\frac{3}{c + x^3}$ b $y = \sqrt{\frac{2}{c - x^4}}$

12 a $y = Ax$ b $y = \sqrt{x^2 + c}$

13 $y = -\frac{1}{x + c}$

14 $y = \sqrt[3]{3\sin x + c}$

15 $y = \ln\left(x^2 + c\right)$

16 $y = Ax$

17 $y = -\frac{1}{x^2 + c}$

18 a $y = Ae^{\tan x}$ b $y = 4e^{\tan x}$

19 $2y^2 = 3x^3 + 18$

20 $y = \sqrt[3]{\frac{9}{2}\left(x^2 + 2\right)}$

21 a $y = 1 + Ae^{x^2 + 4x}$

23 a $k = \frac{1}{5}$ b $m = 25e^{-\frac{t}{5}}$

c 3.47 seconds

24 a $\frac{1}{4}$ b 24 000

25 b $V = \sqrt{6000t + 90000}$

26 a $v = 100 - 100e^{-0.1t}$ b 40.8 m

27 $y = -2\sqrt{2 - e^{-2x}}$

28 $y = -\ln\left(c - e^{x}\right)$

29 $y = \frac{1}{2}\ln\left(4e^{x} - 3\right)$

30 $y = \sqrt{102 - 2\cos x}$

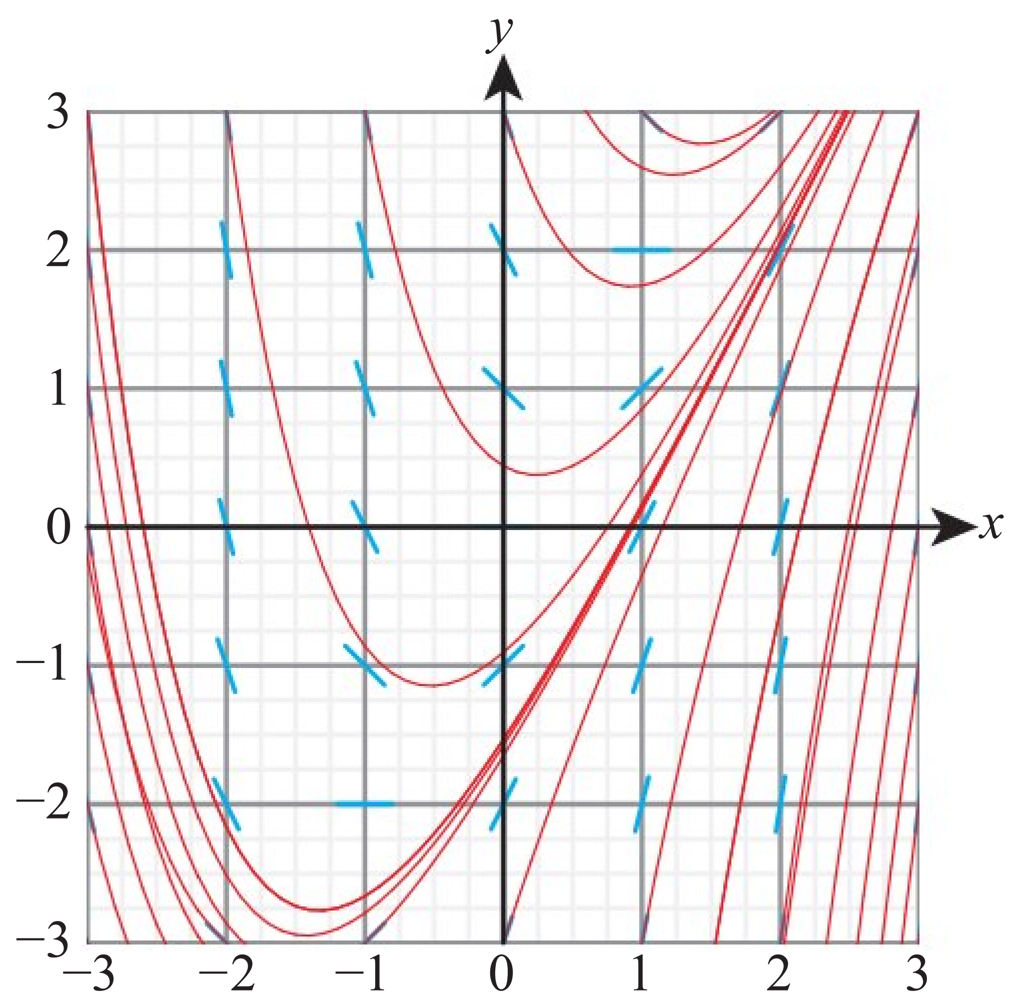
31 b $\pm\frac{\pi}{2}$

32 a $\frac{dr}{dt} = -0.0328$ b 15 minutes

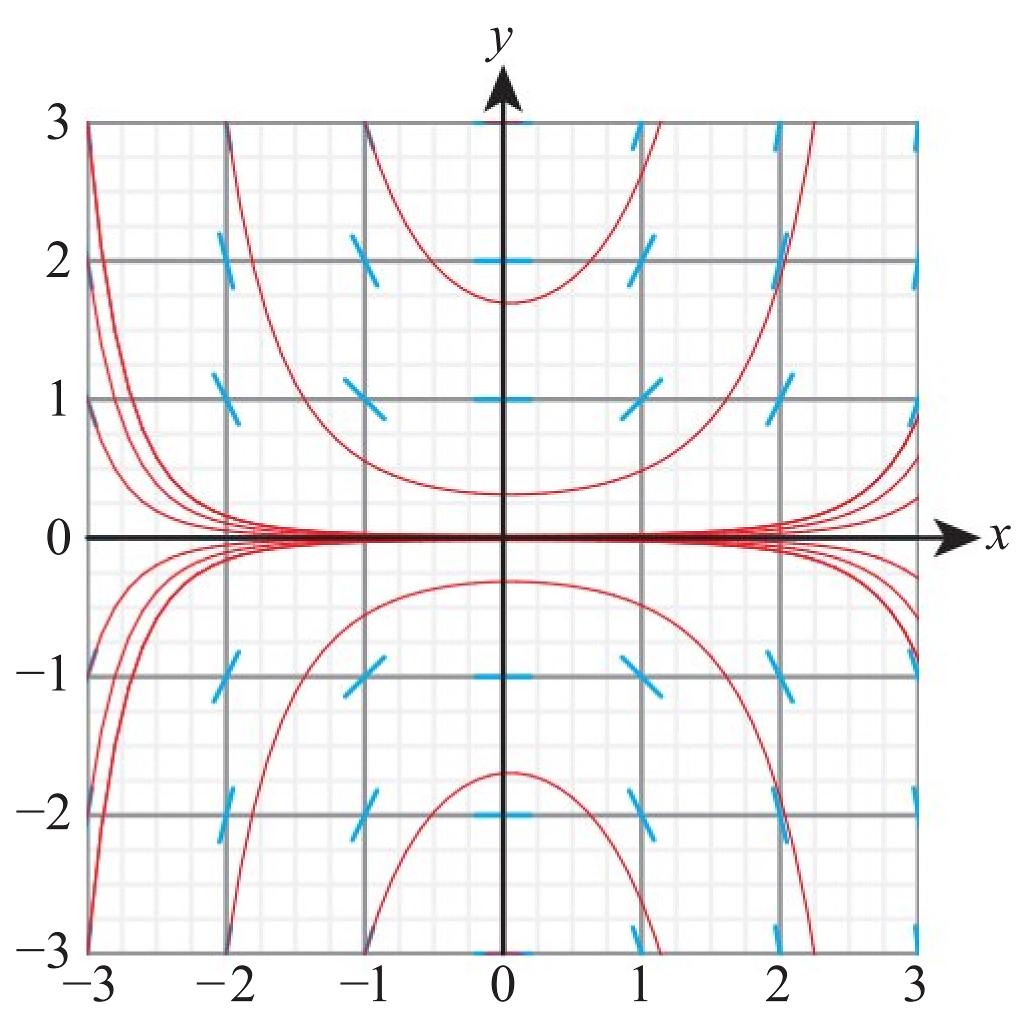
Exercise 13B

The slope fields shown in these answers are produced using technology. You would only be expected to sketch three or four trajectories to indicate behaviour.

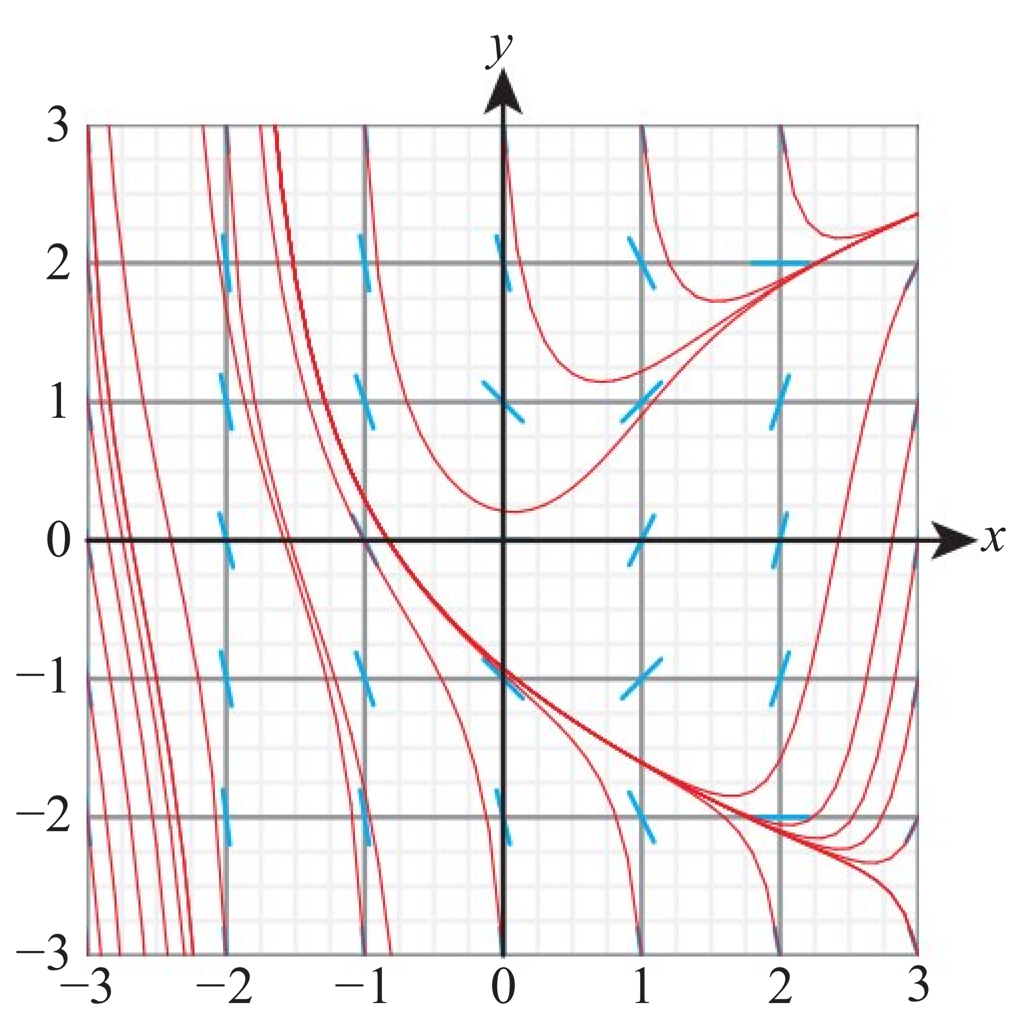
1 a



b



2 a



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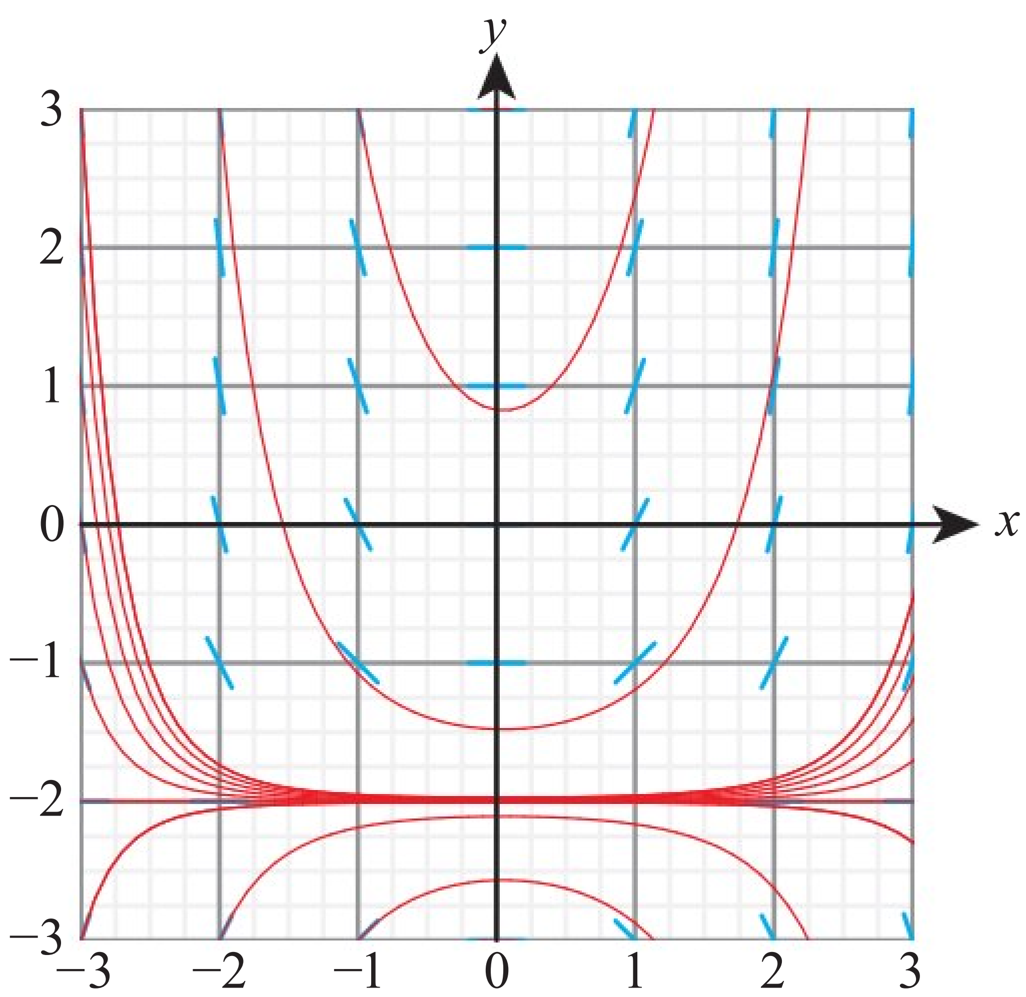
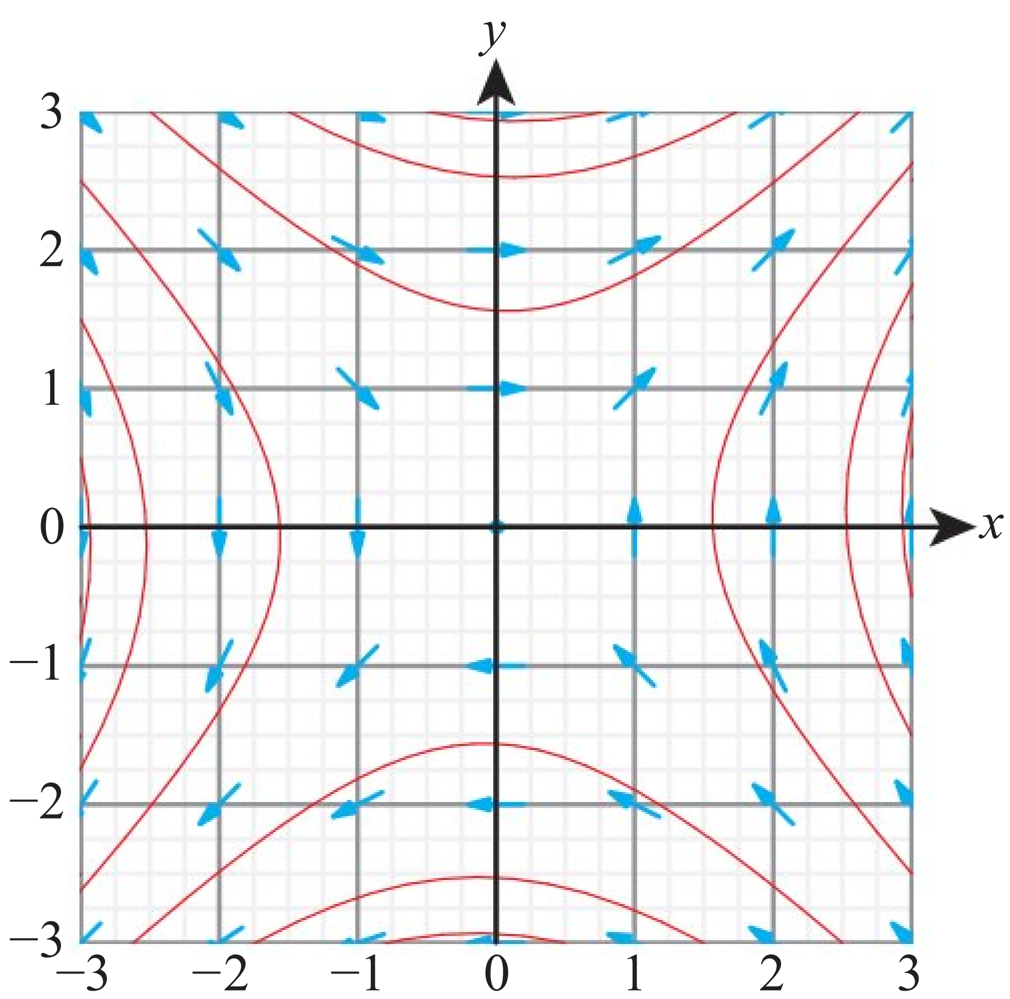
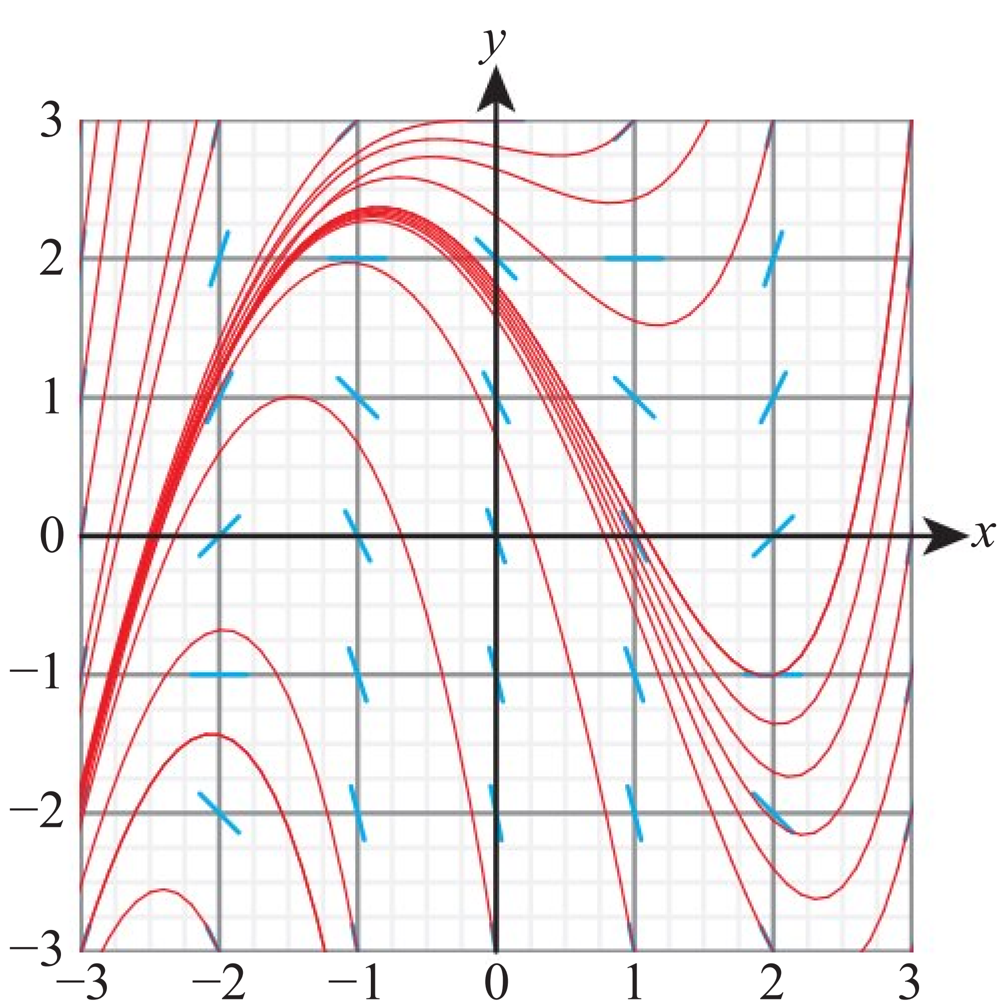


3 a



b



4 **a** 2.49 **b** 1.4

5 **a** 2.98 **b** 0.655

6 **a** 1.36 **b** 1.46

7 **a** 4.56 **b** 3.82

8 **a** $x = 0.224,\ y = -0.186$

b $x = -1.50,\ y = -3.16$

9 **a** $x = 0.271,\ y = -0.697$

b $x = -1.33,\ y = -0.458$

10 **a** $x = 1.41,\ y = 0.969$

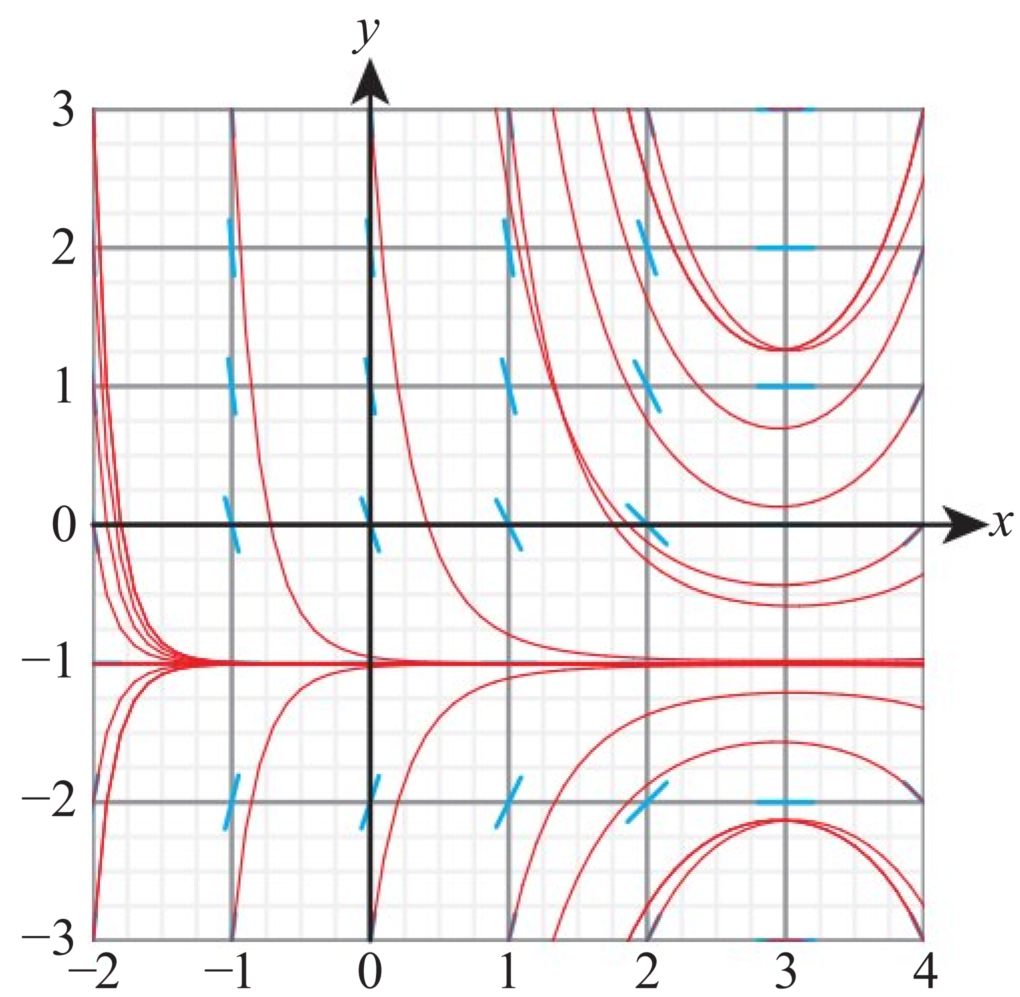
b $x = 0.667,\ y = 0.133$

11 **a** **i** 0.9 **ii** 3.8

b **i** 0.8 **ii** 3.6

c $y = x^2$, **b ii** is furthest away.

12 **a**



c Solution is constant (or in equilibrium).

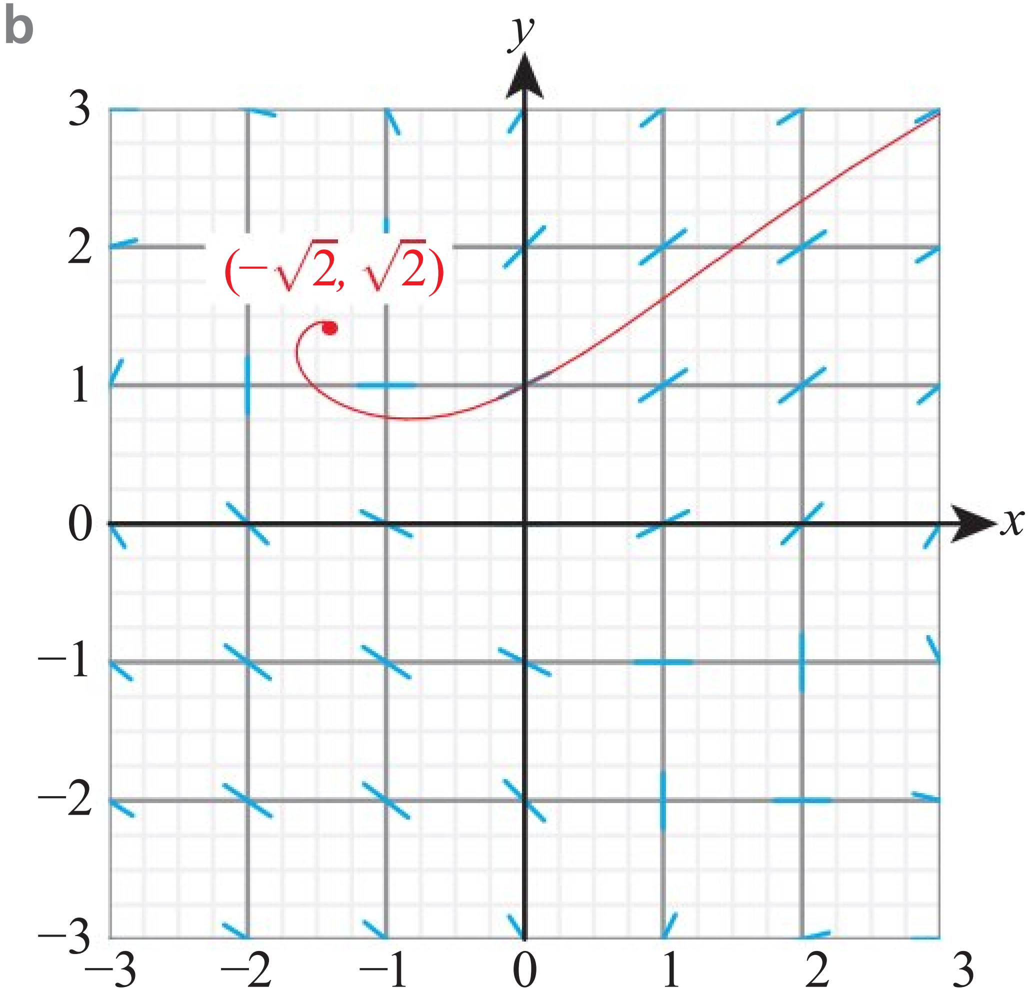
13 **a** 31.4

b Use a smaller step length.

14 157 rabbits, 37 foxes

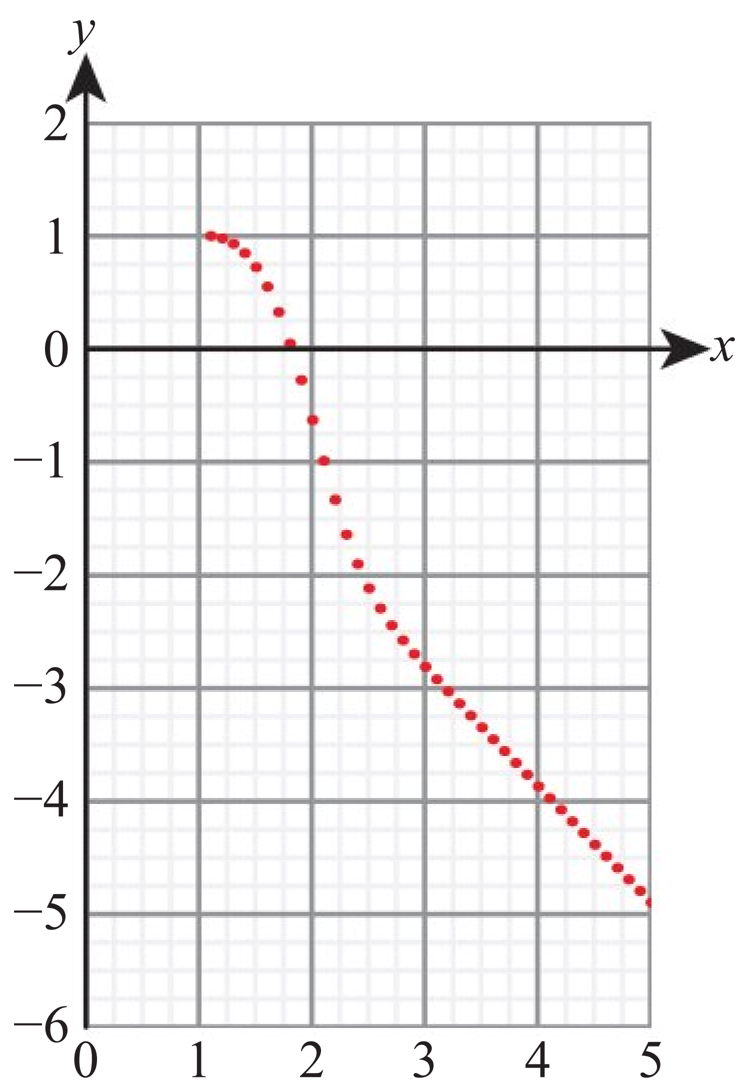
15 $x(0.25) \approx 8.12,\ y(0.25) = 9.46$

16 a, b

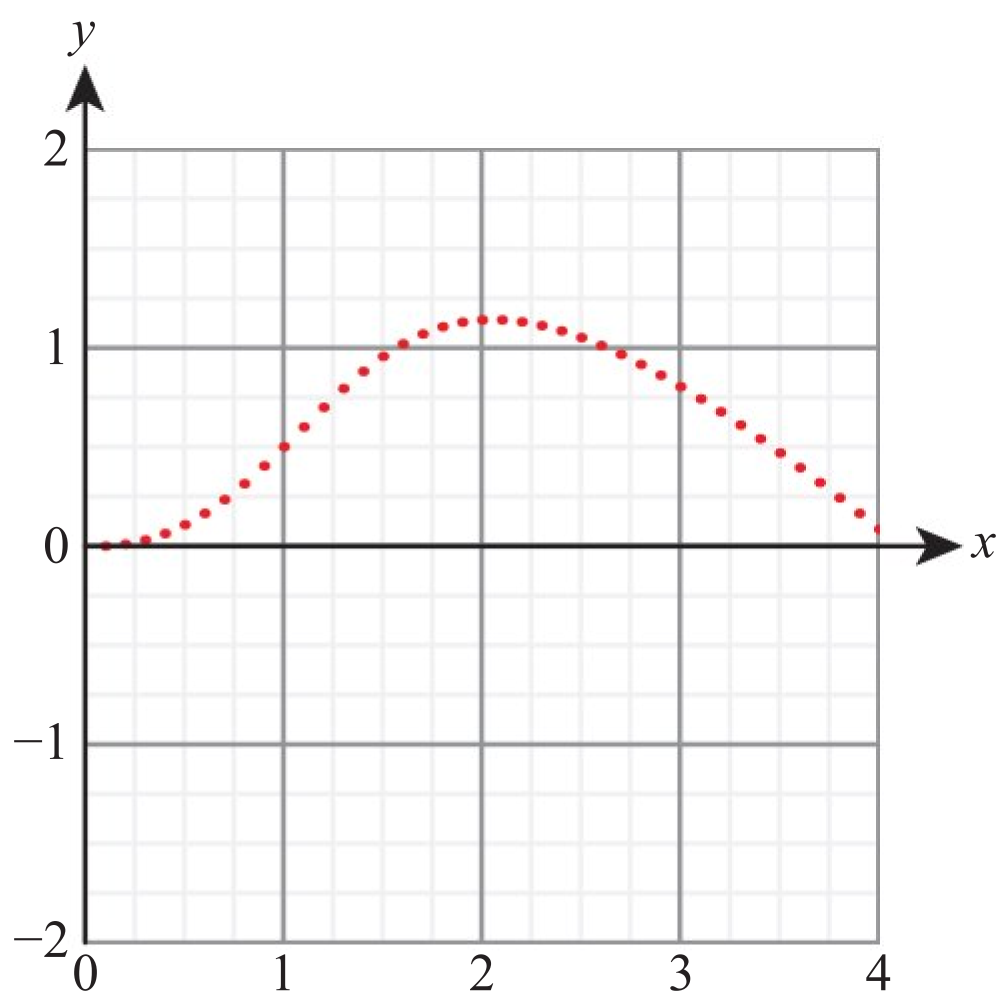


c 1.629

17



18 a



b 1.1

19 a 1.46 m b 3 seconds

20 a 0.825

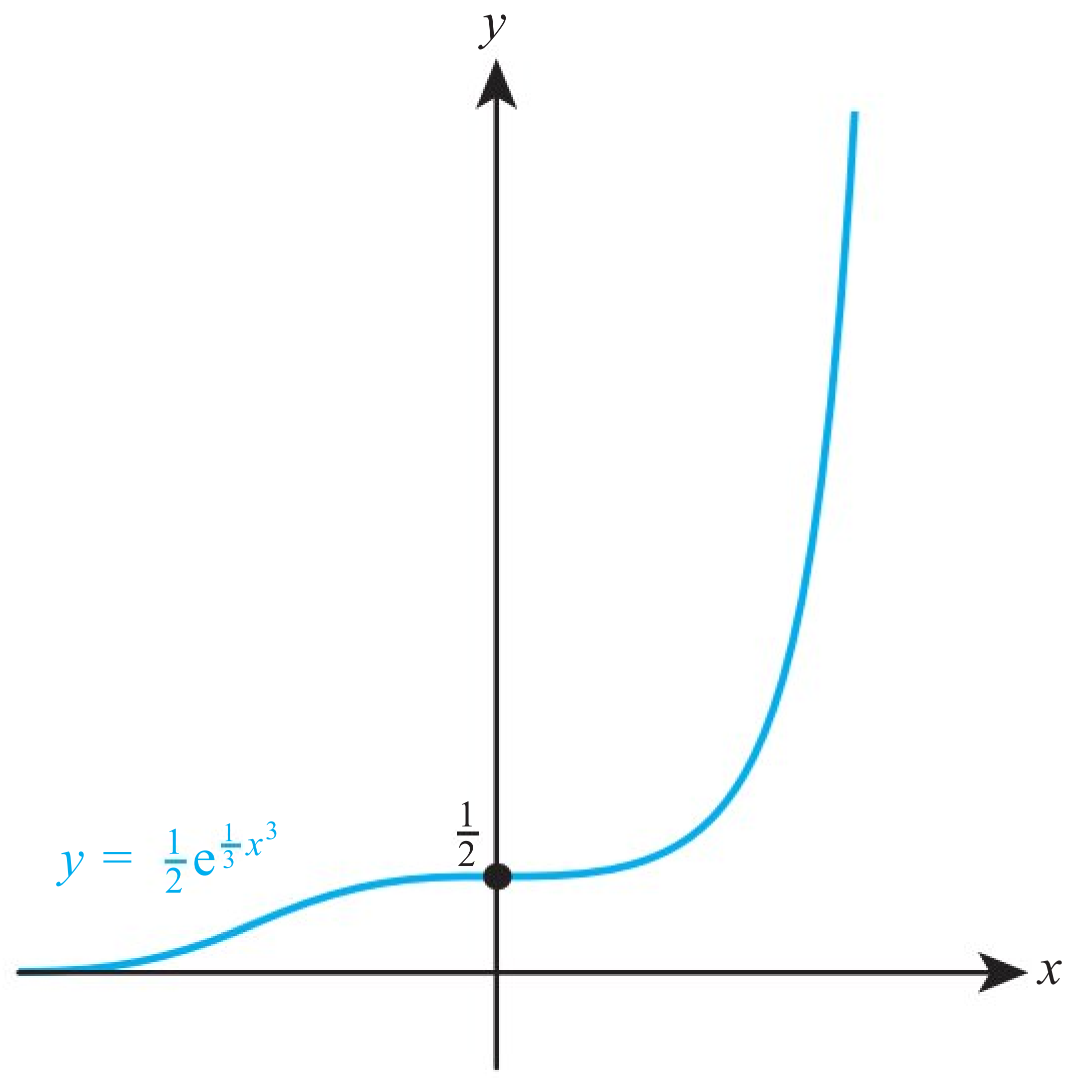
b $y = -\ln\left(-\frac{x^2}{2} + \frac{1}{2} + e^{-0.3}\right)$

c i 11.0% ii Take smaller h.

21 a i 0.615 ii Use smaller step length.

b $y = \frac{1}{2}e^{\frac{1}{3}x^3}$; $f(1) = 0.698$

c

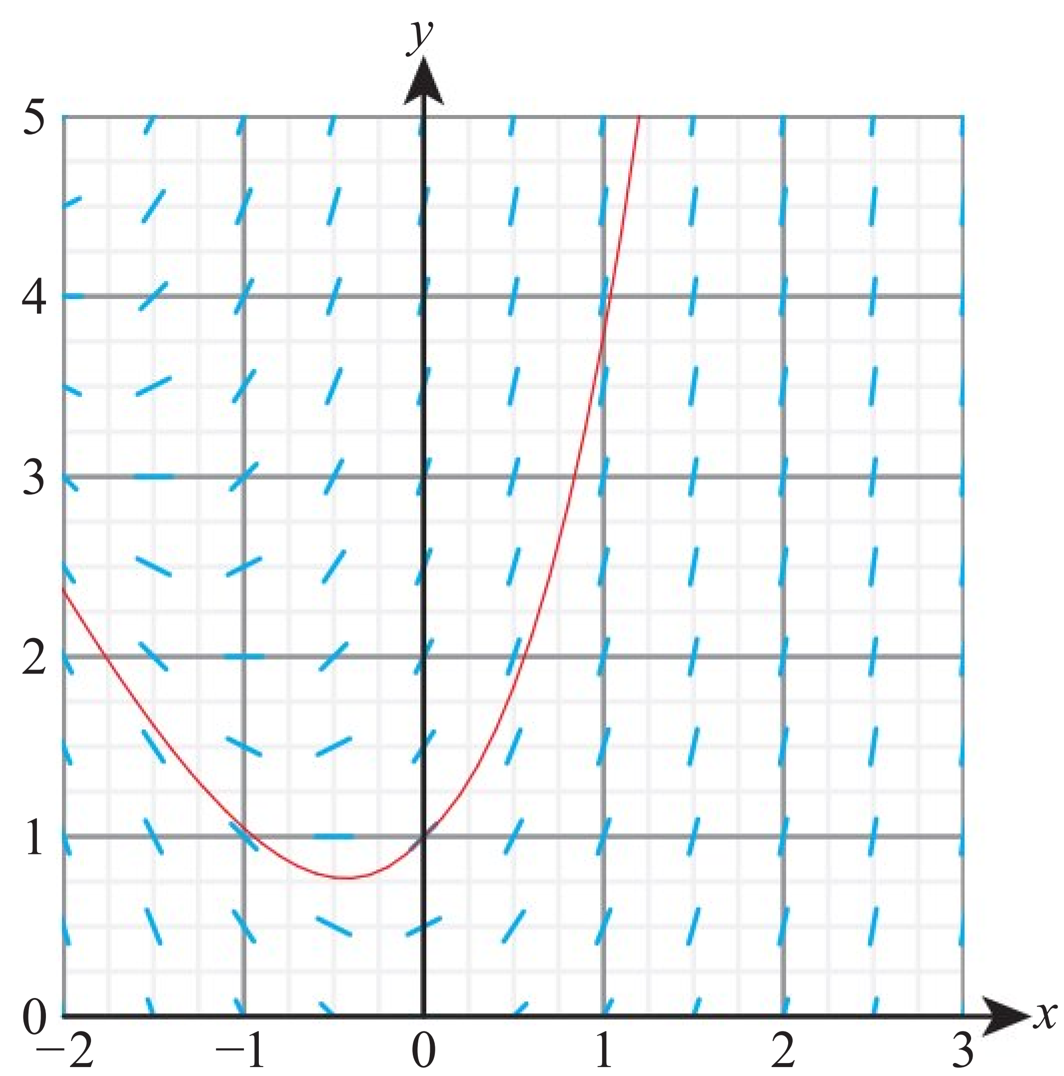


The tangent is always below the curve.

22 183 m

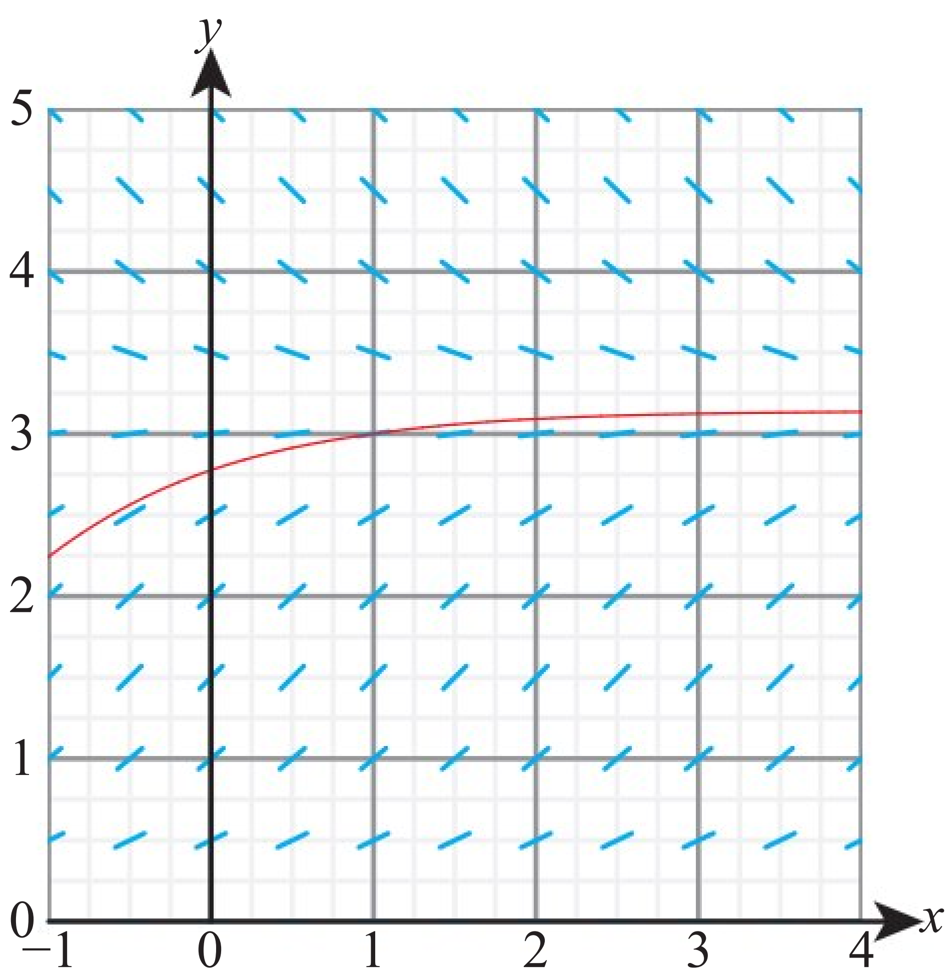
23 177 m

24 a



b 3.8

25 a



b 3.1

Exercise 13C

1 a $\begin{pmatrix} x \\ y \end{pmatrix} = Ae^{2t}\begin{pmatrix} 2 \\ 1 \end{pmatrix} + Be^{3t}\begin{pmatrix} 1 \\ 1 \end{pmatrix}$

b $\begin{pmatrix} x \\ y \end{pmatrix} = Ae^{2t}\begin{pmatrix} 3 \\ -5 \end{pmatrix} + Be^{4t}\begin{pmatrix} 1 \\ -1 \end{pmatrix}$

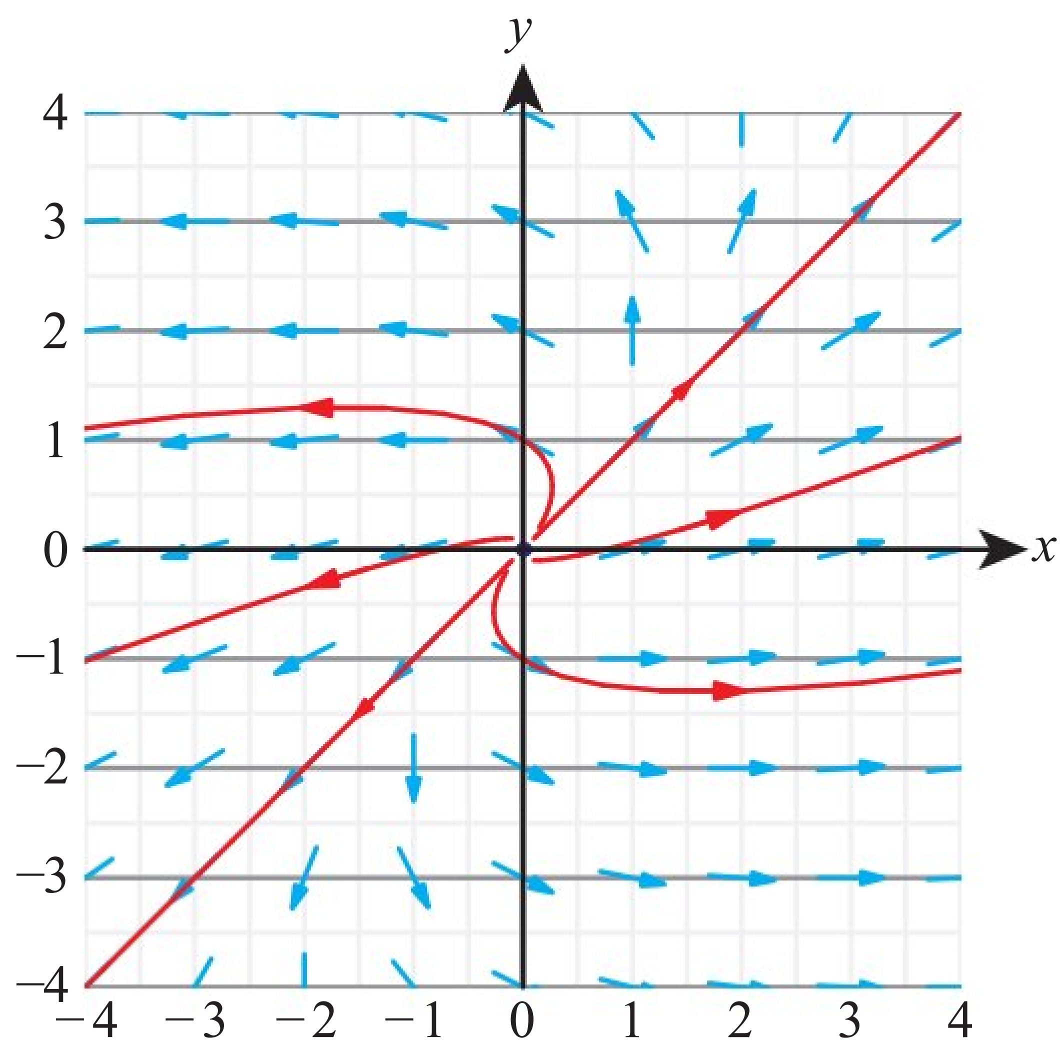
2 a $\begin{pmatrix} x \\ y \end{pmatrix} = Ae^{-t}\begin{pmatrix} 2 \\ 1 \end{pmatrix} + Be^{-11t}\begin{pmatrix} 4 \\ -3 \end{pmatrix}$

b $\begin{pmatrix} x \\ y \end{pmatrix} = Ae^{-t}\begin{pmatrix} 1 \\ 1 \end{pmatrix} + Be^{-2t}\begin{pmatrix} 4 \\ 3 \end{pmatrix}$

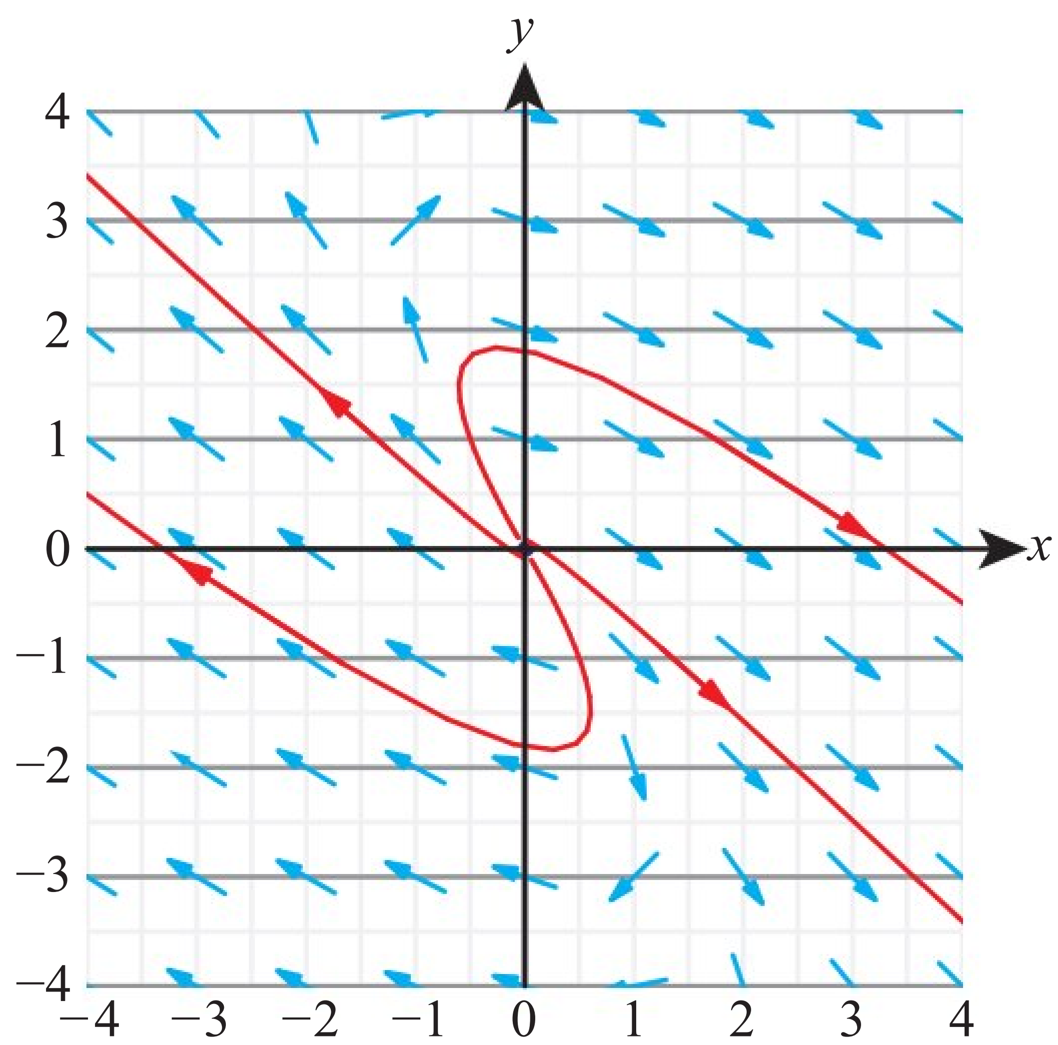
3 a $\begin{pmatrix} x \\ y \end{pmatrix} = Ae^{3t}\begin{pmatrix} 5 \\ 1 \end{pmatrix} + Be^{-t}\begin{pmatrix} 1 \\ 1 \end{pmatrix}$

b $\begin{pmatrix} x \\ y \end{pmatrix} = Ae^{2t}\begin{pmatrix} -5 \\ 1 \end{pmatrix} + Be^{-5t}\begin{pmatrix} -3 \\ 2 \end{pmatrix}$

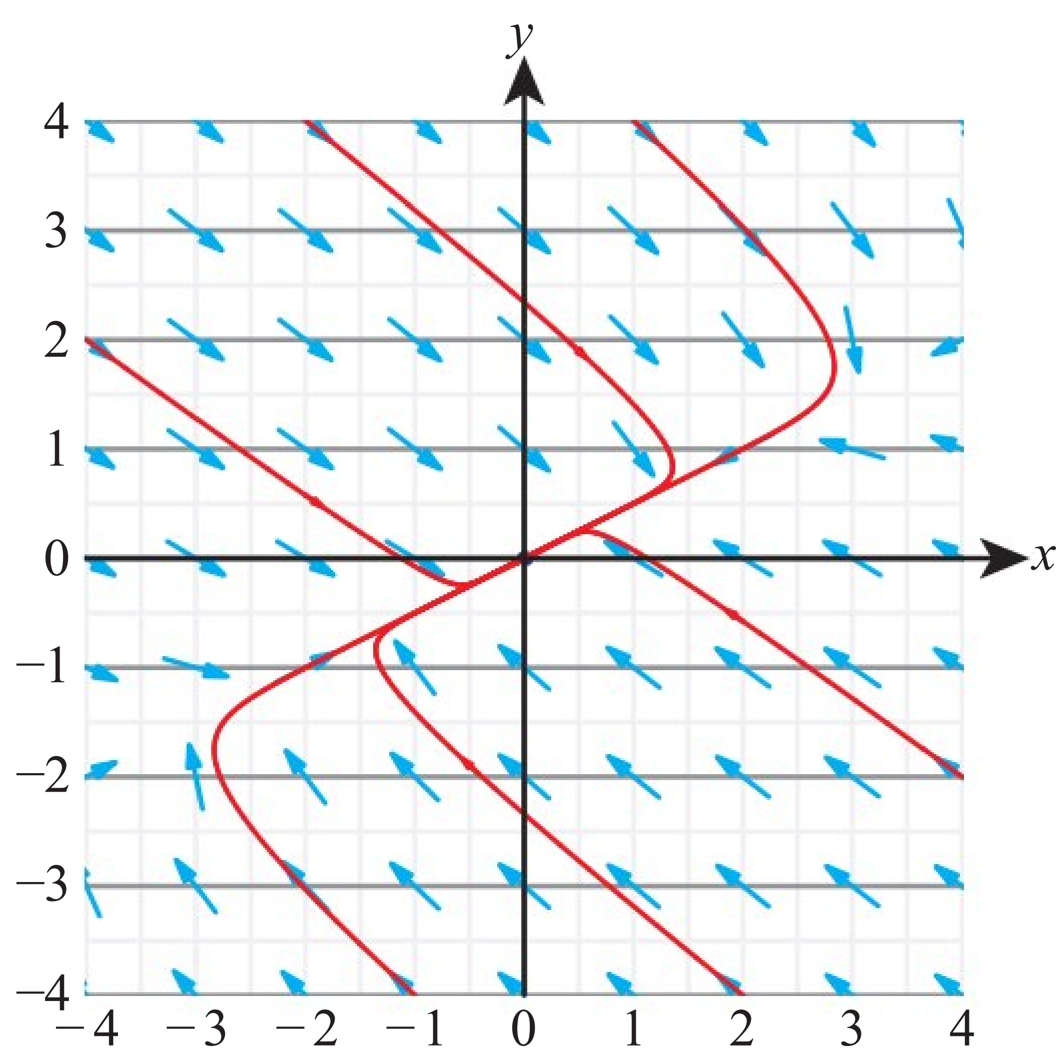
4 a



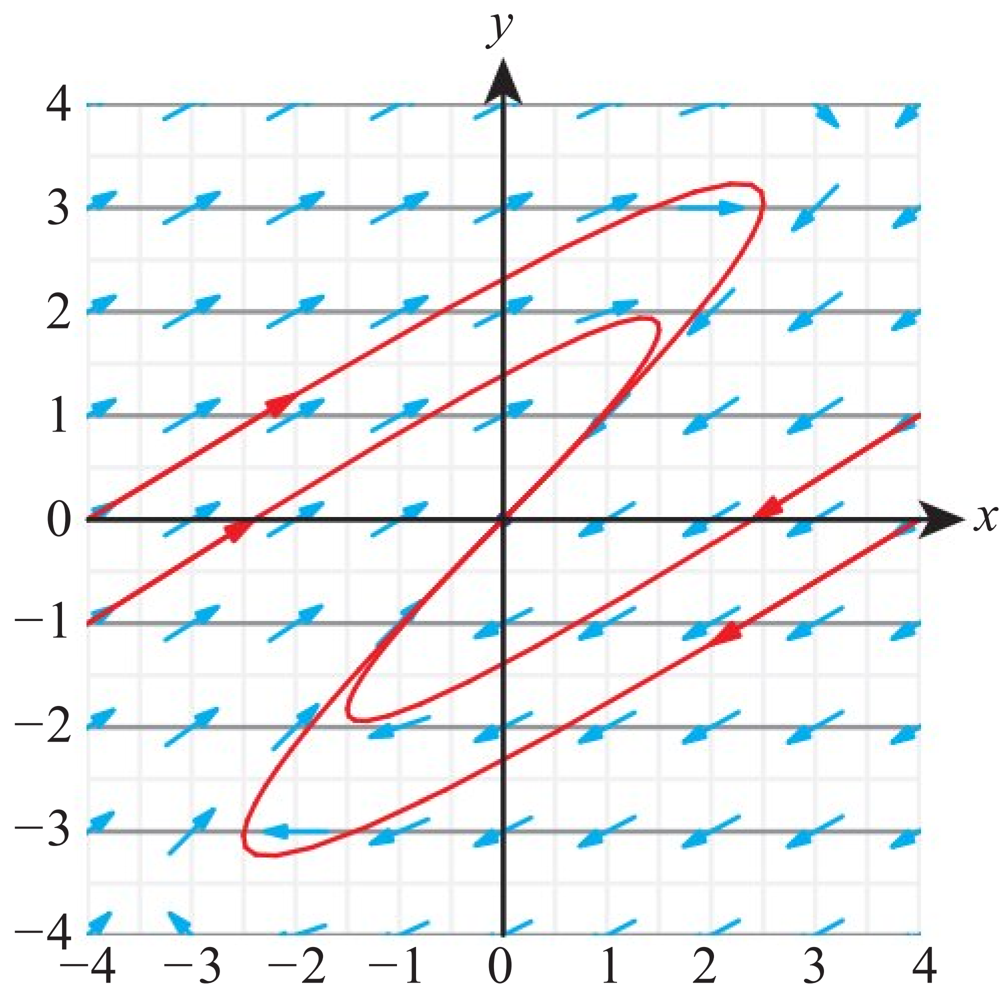
b



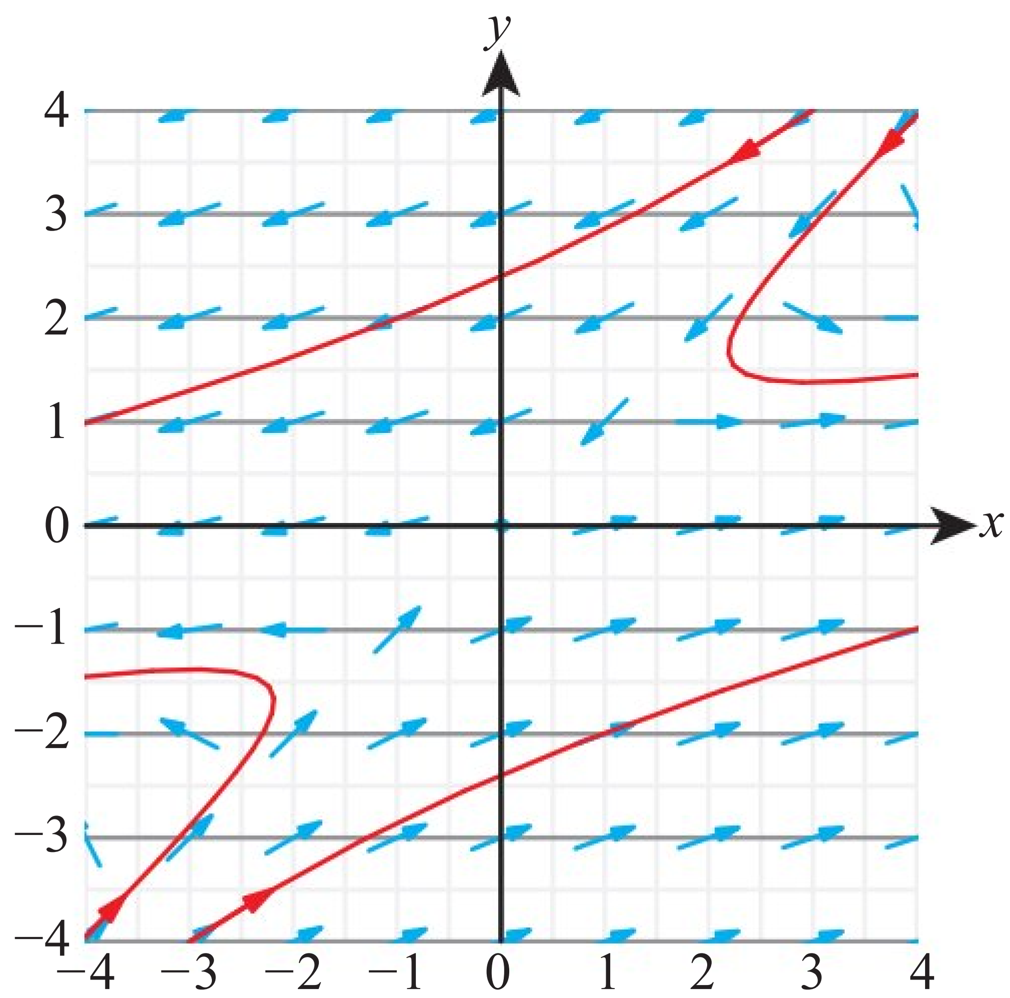
5 a



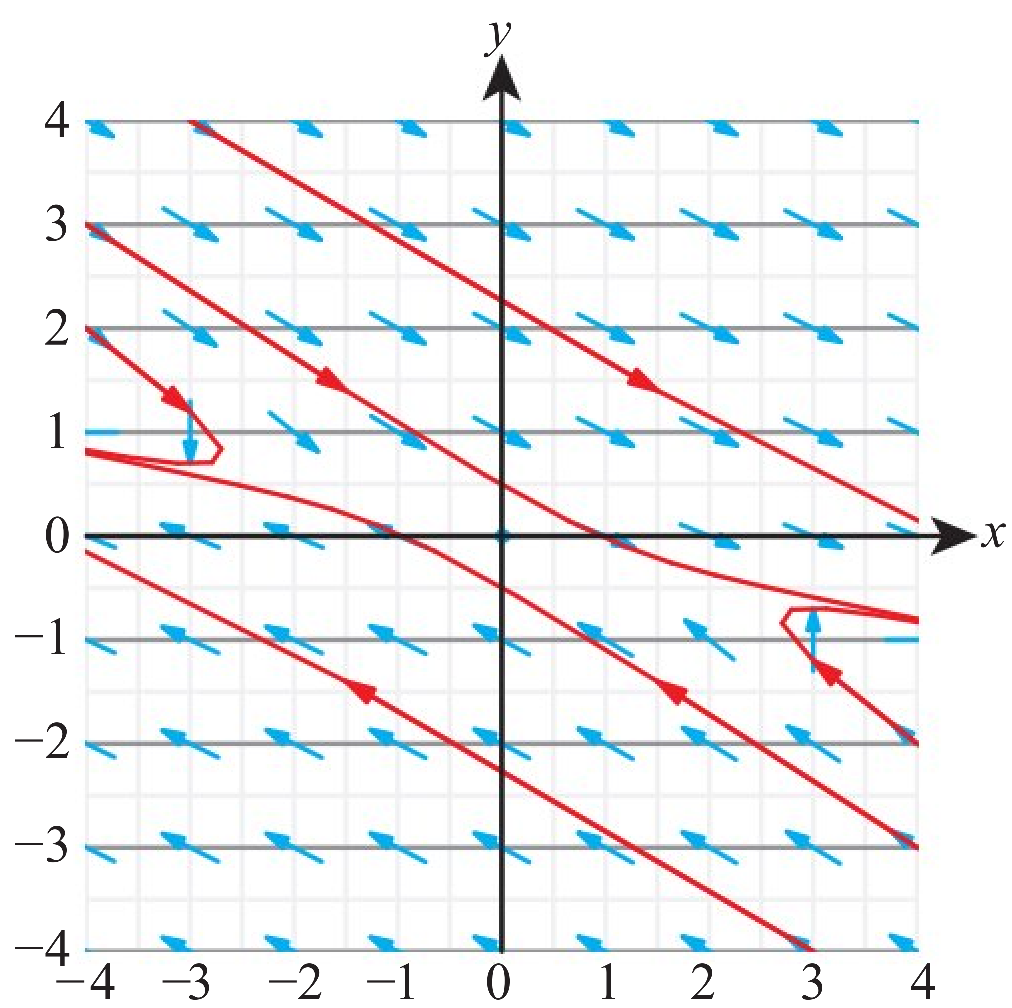
b


y
4
3
2
1
0
−1
−2
−3
−4
x
−4 −3 −2 −1 0 1 2 3 4

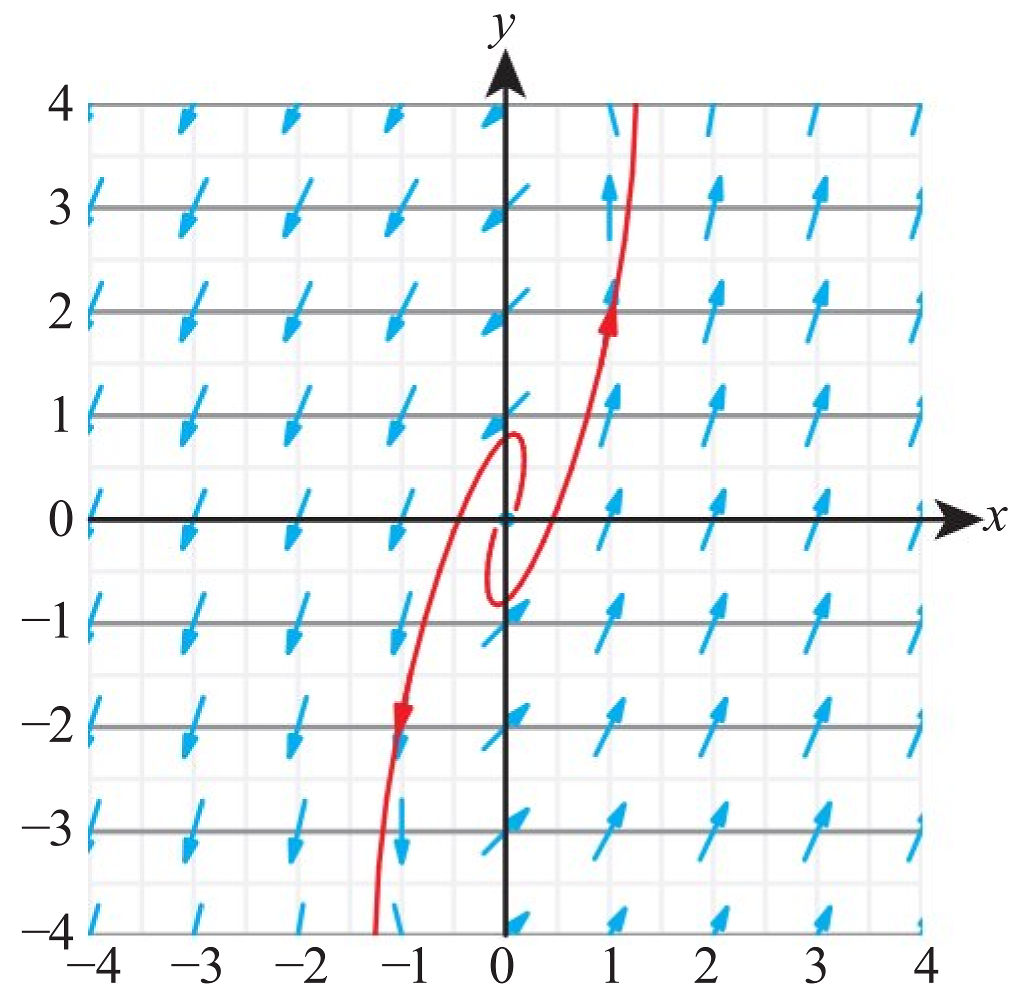
6 a


y
4
3
2
1
0
−1
−2
−3
−4
x
−4 −3 −2 −1 0 1 2 3 4

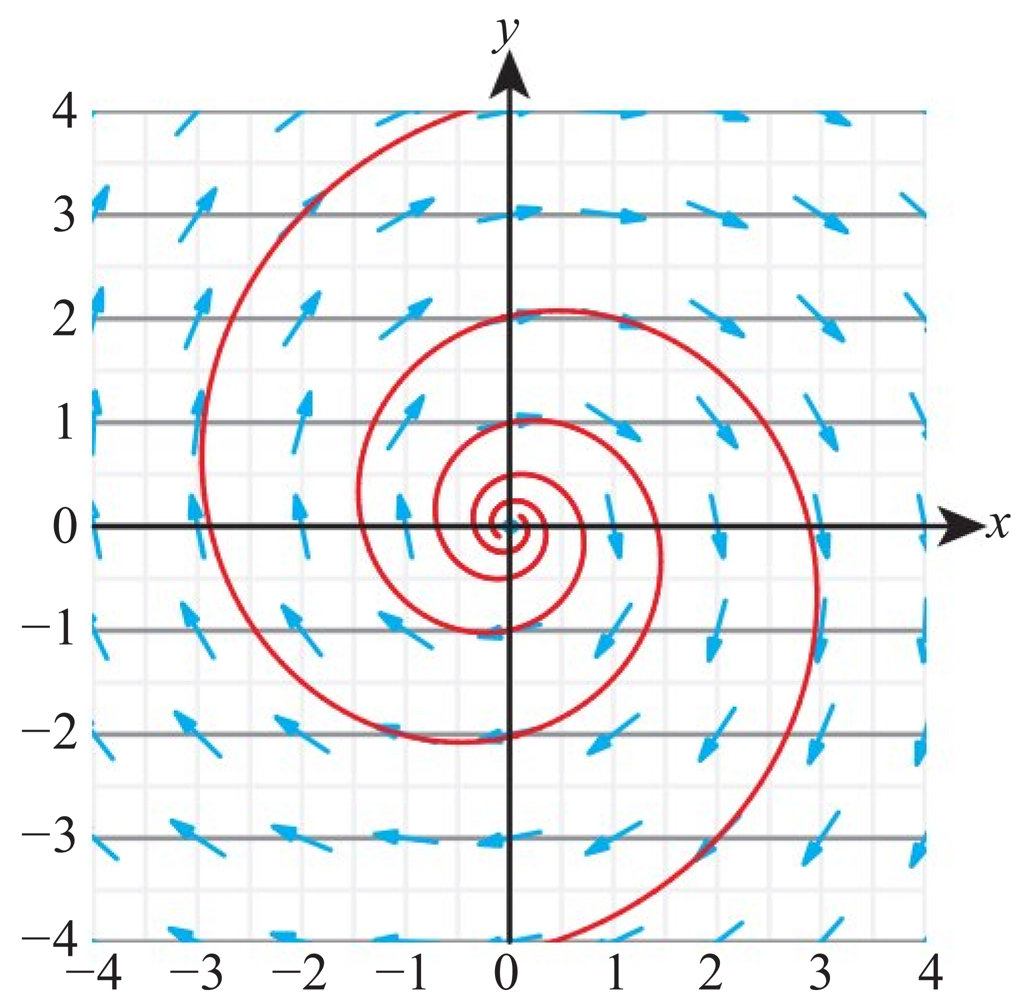
b


y
4
3
2
1
0
−1
−2
−3
−4
x
−4 −3 −2 −1 0 1 2 3 4

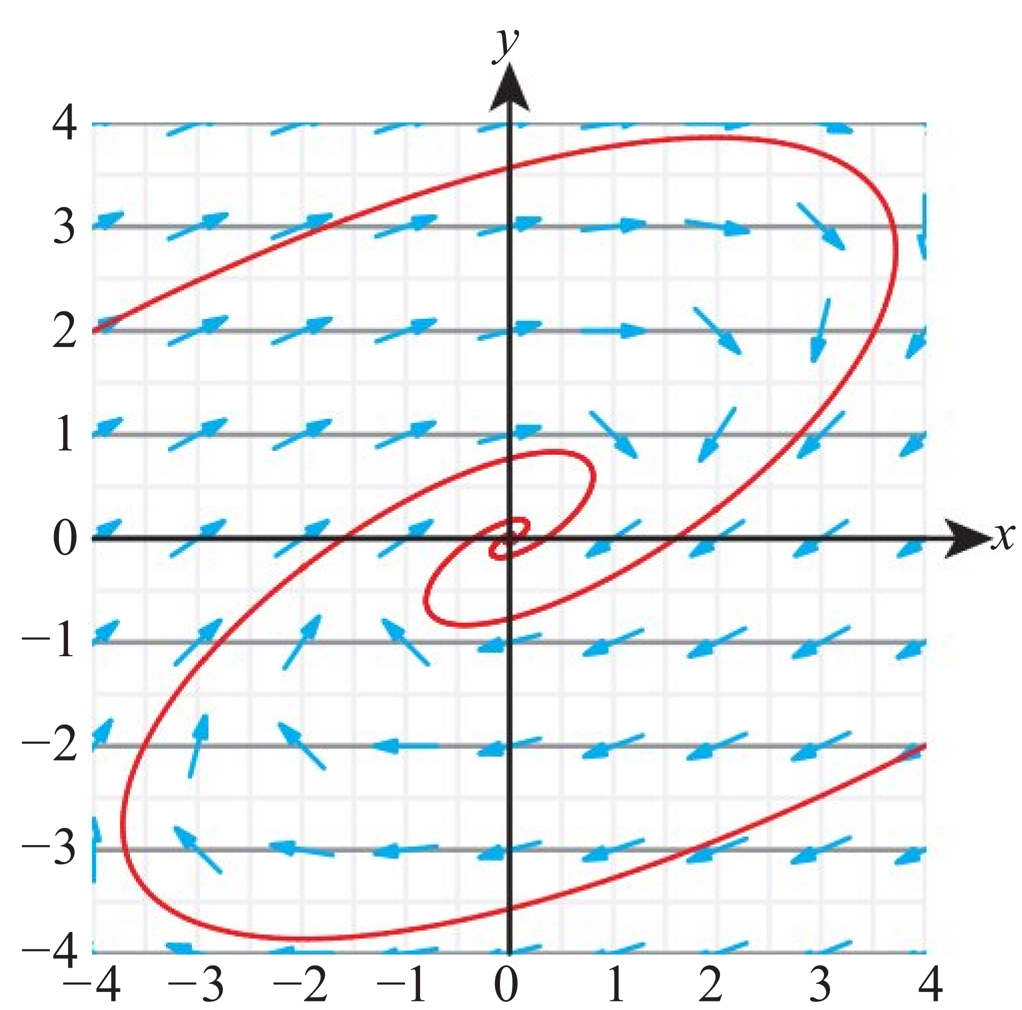
7 a


y
4
3
2
1
0
−1
−2
−3
−4
x
−4 −3 −2 −1 0 1 2 3 4

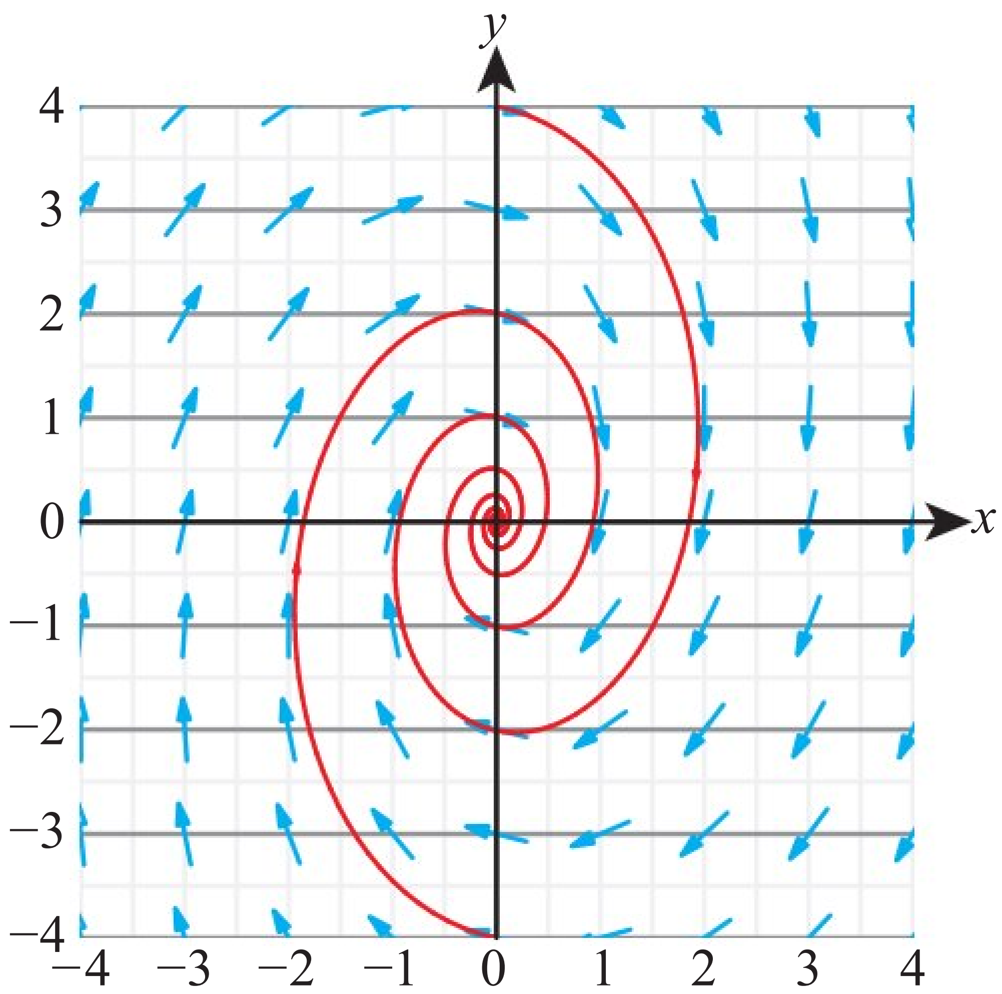
b


y
4
3
2
1
0
−1
−2
−3
−4
x
−4 −3 −2 −1 0 1 2 3 4

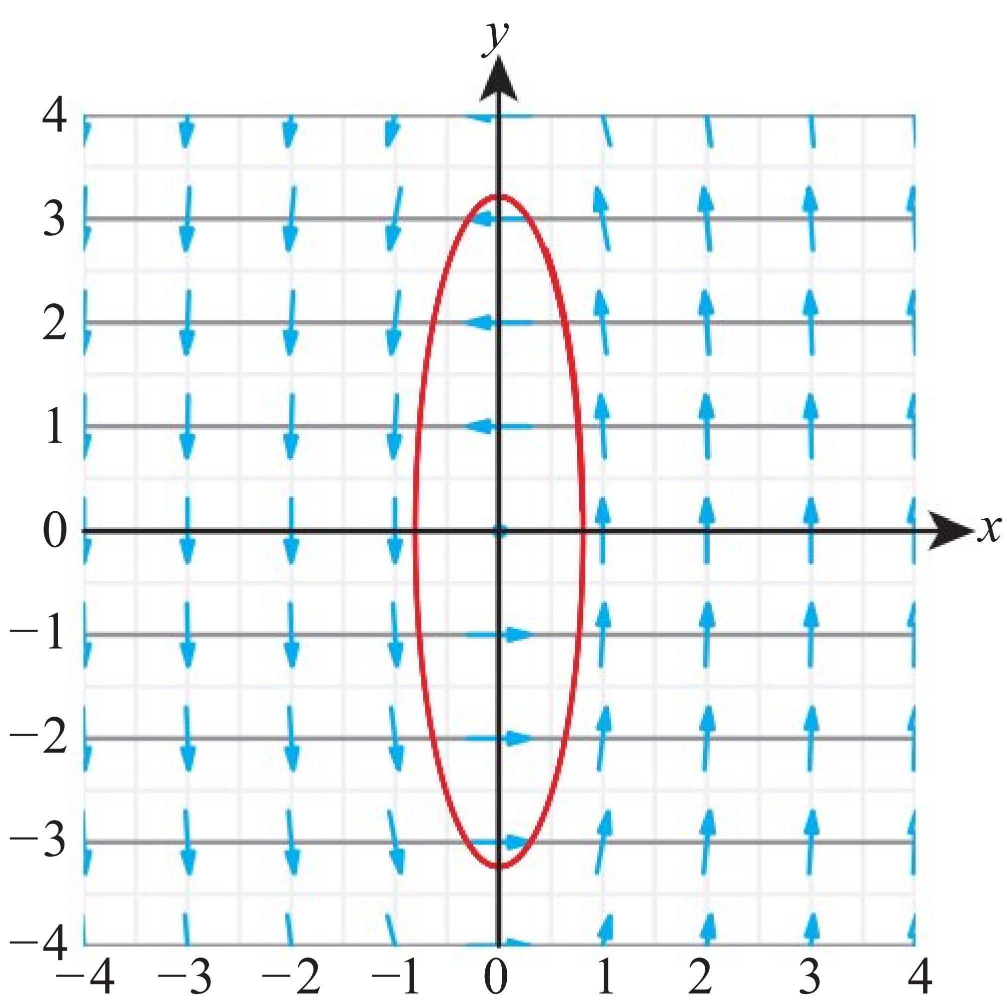
8 a


y
4
3
2
1
0
−1
−2
−3
−4
x
−4 −3 −2 −1 0 1 2 3 4

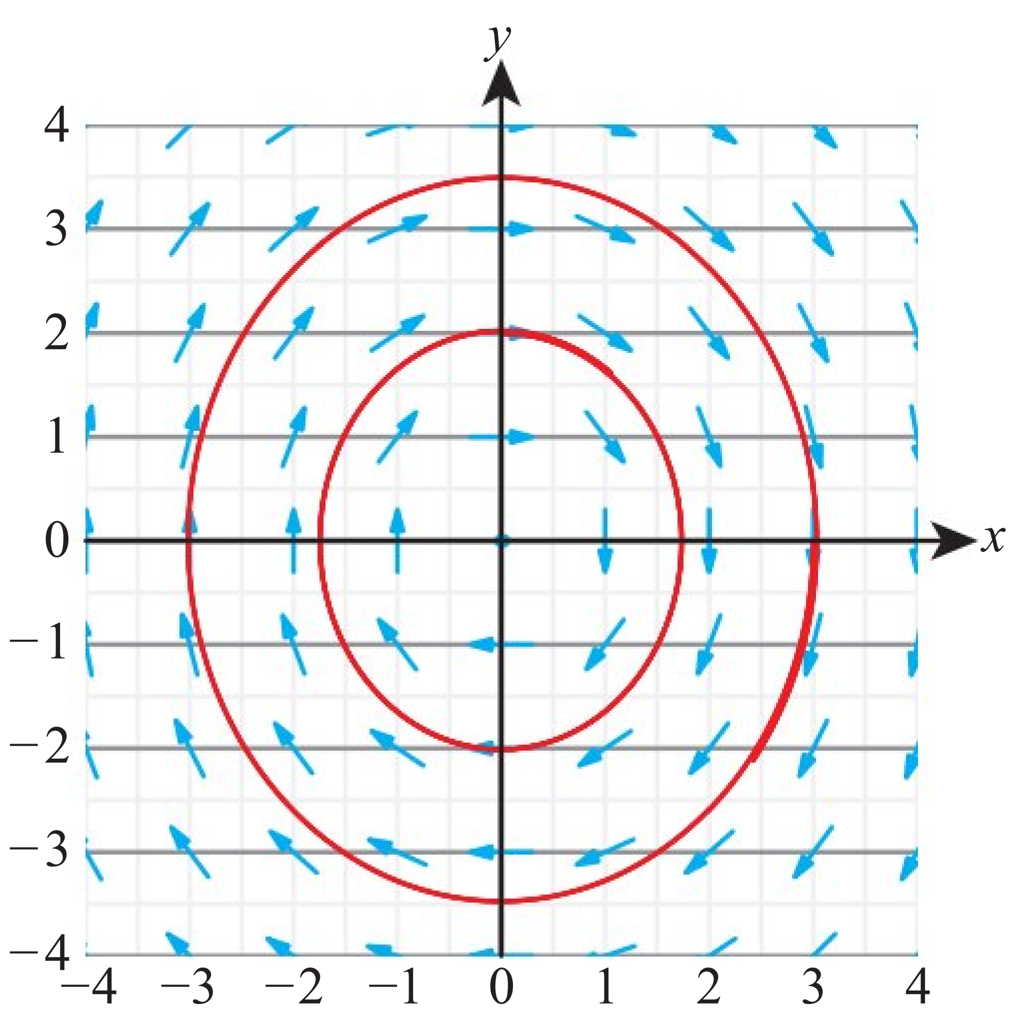
b



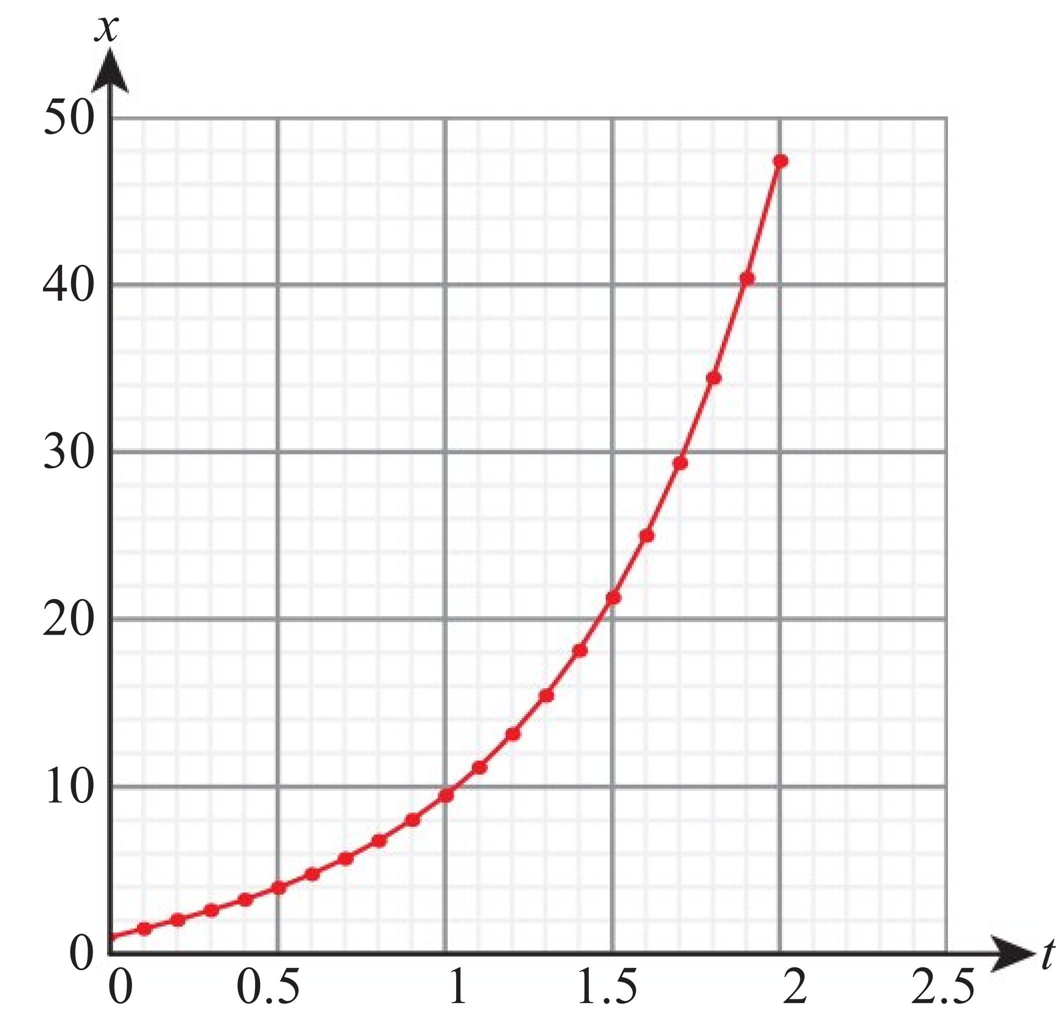
9 a



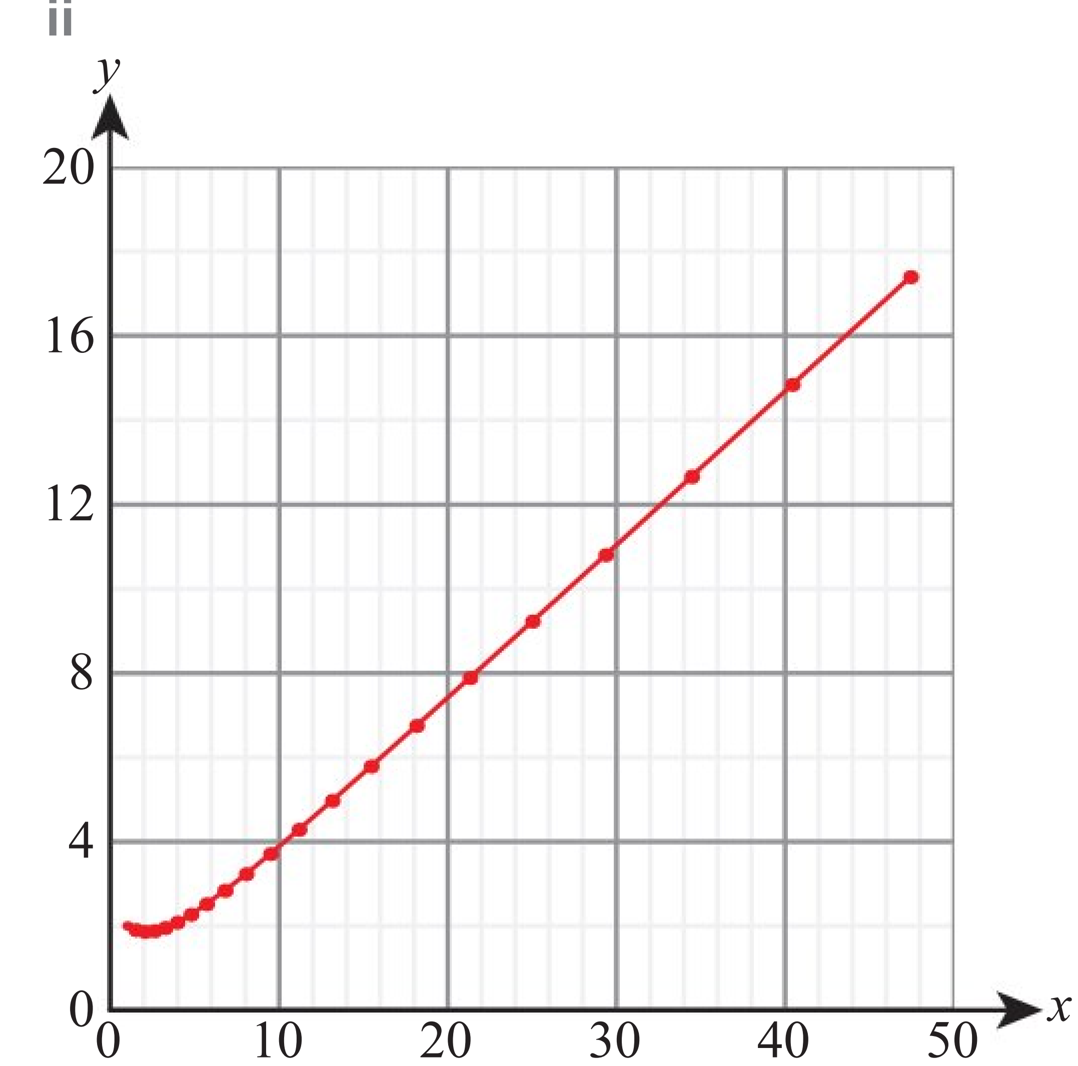
b



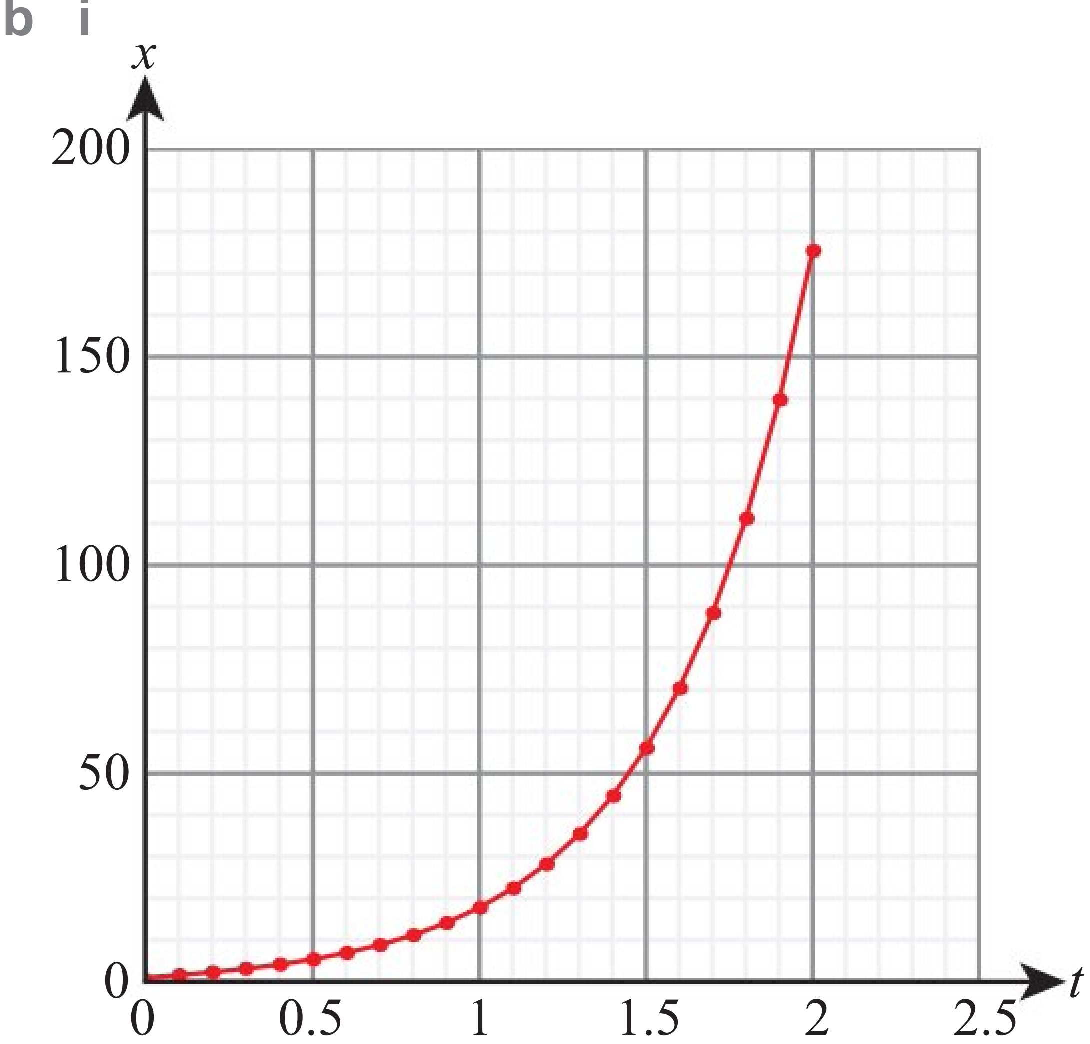
10 a i



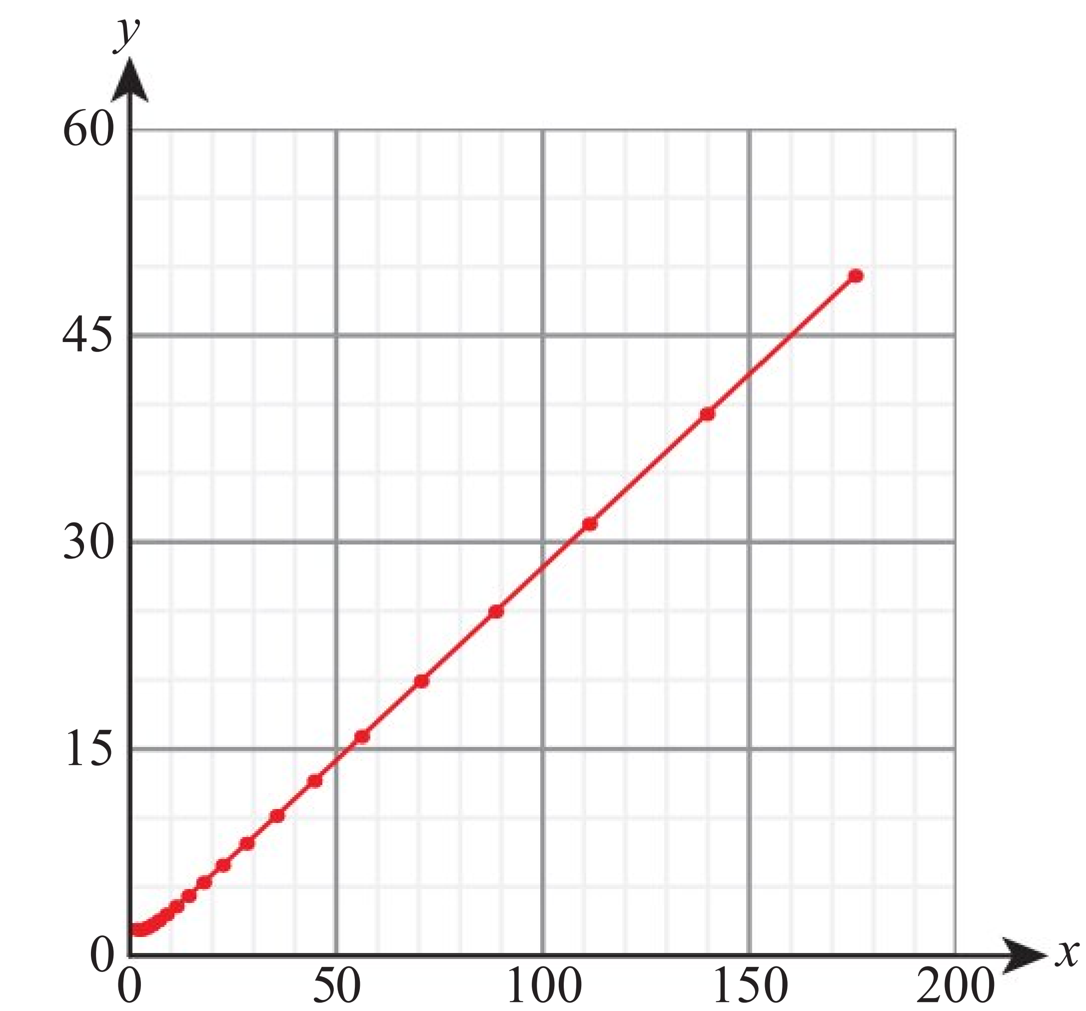
ii



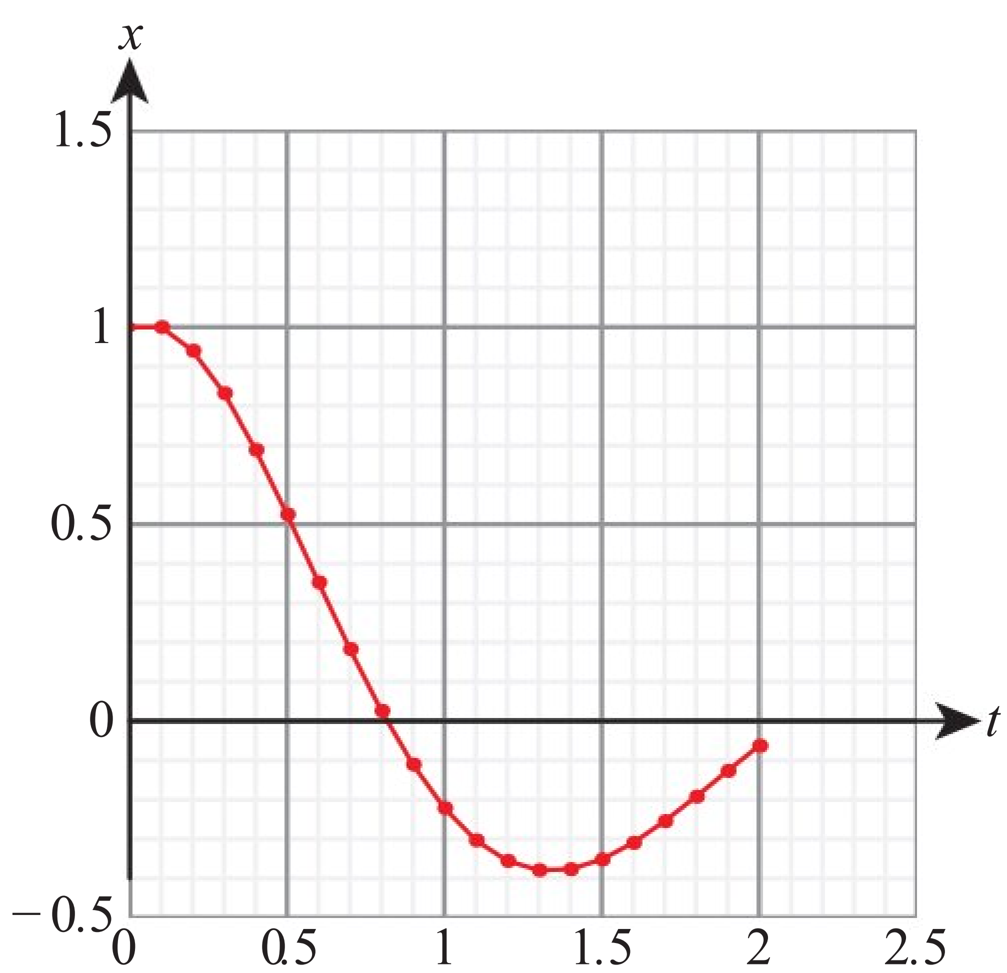
b i



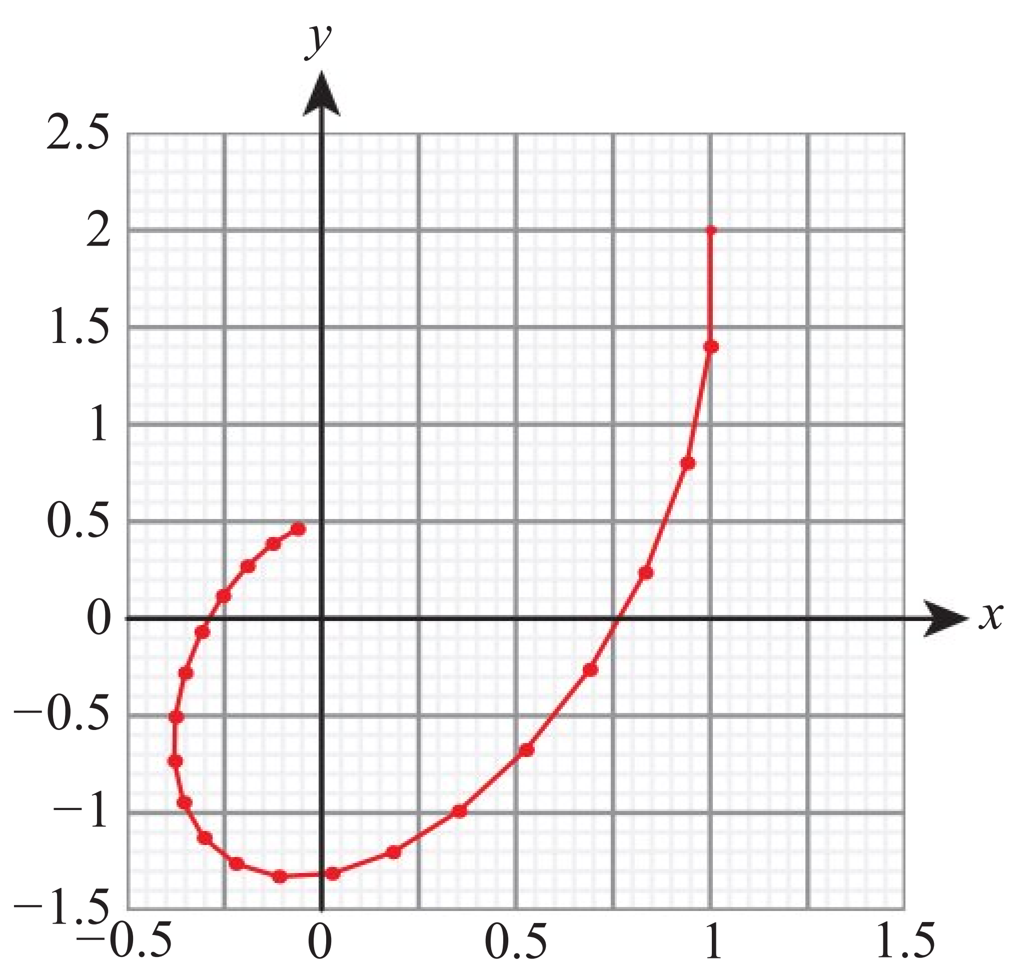
ii



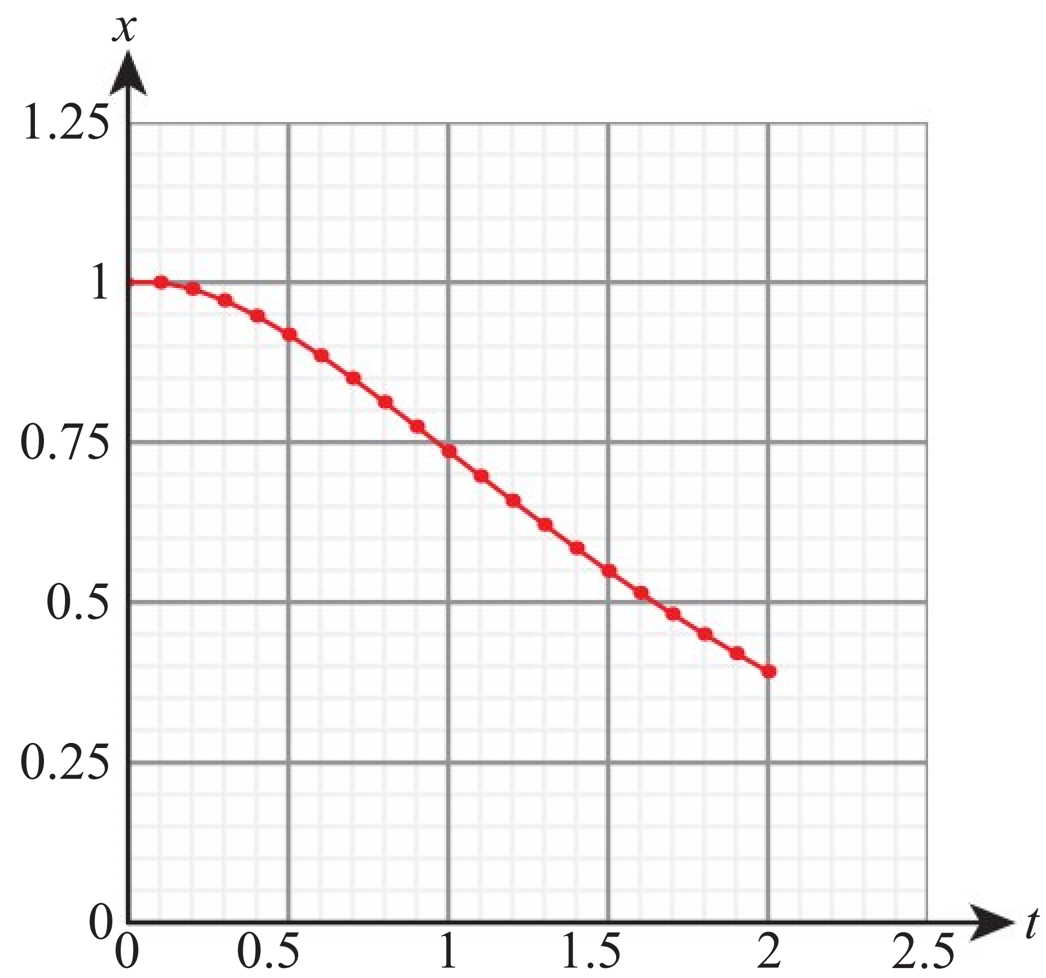
11 a i



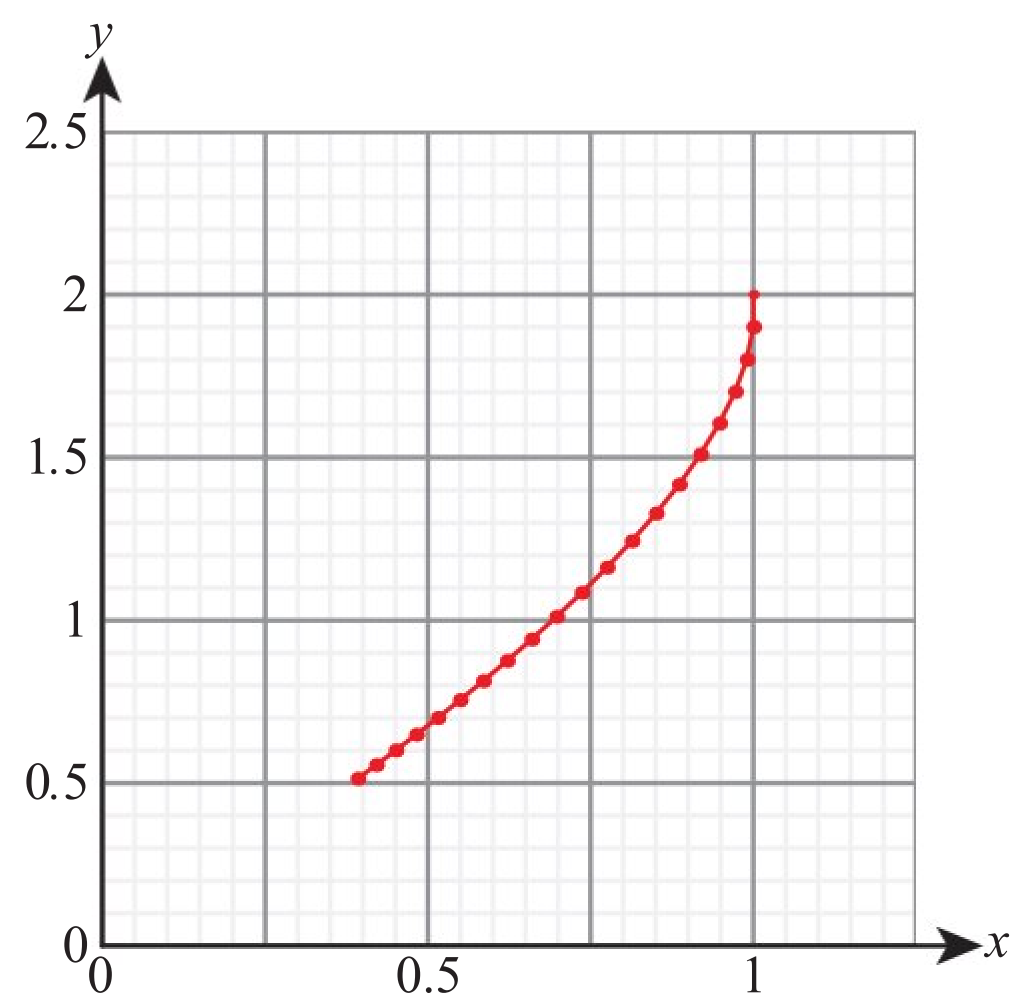
ii



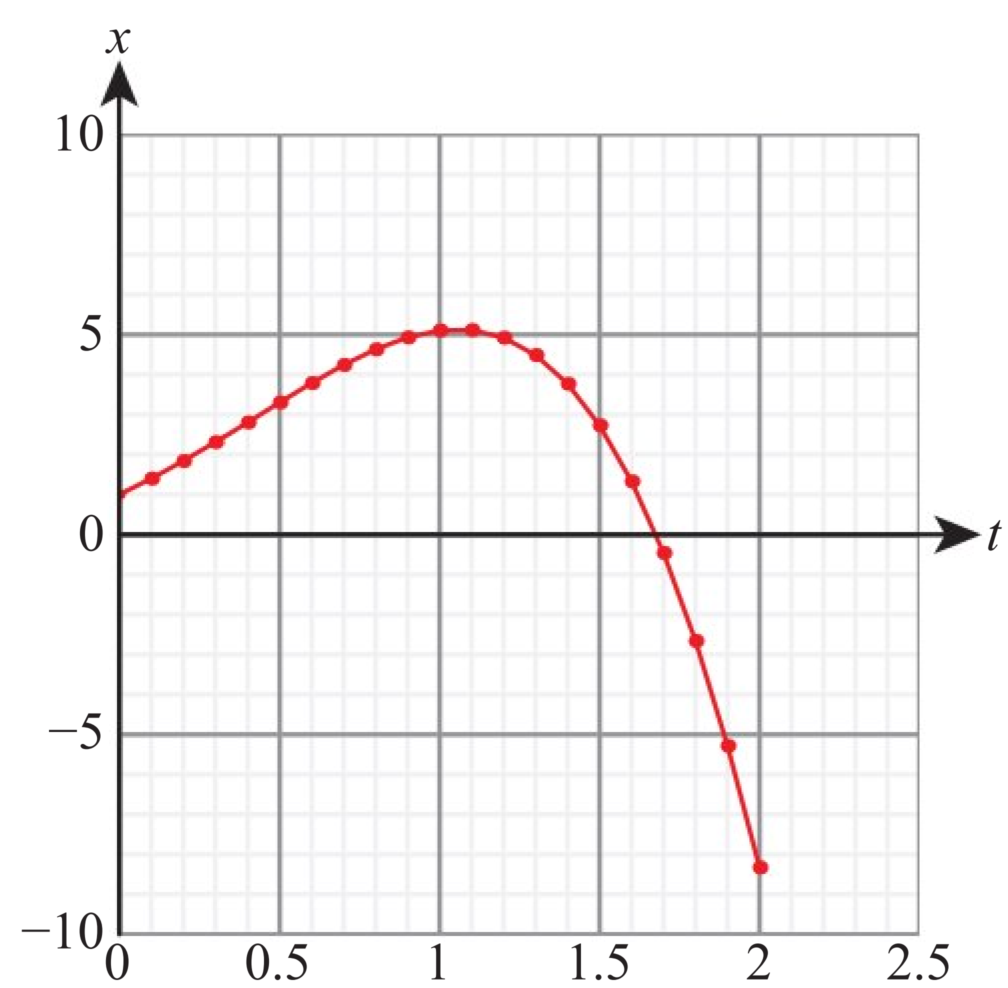
b i



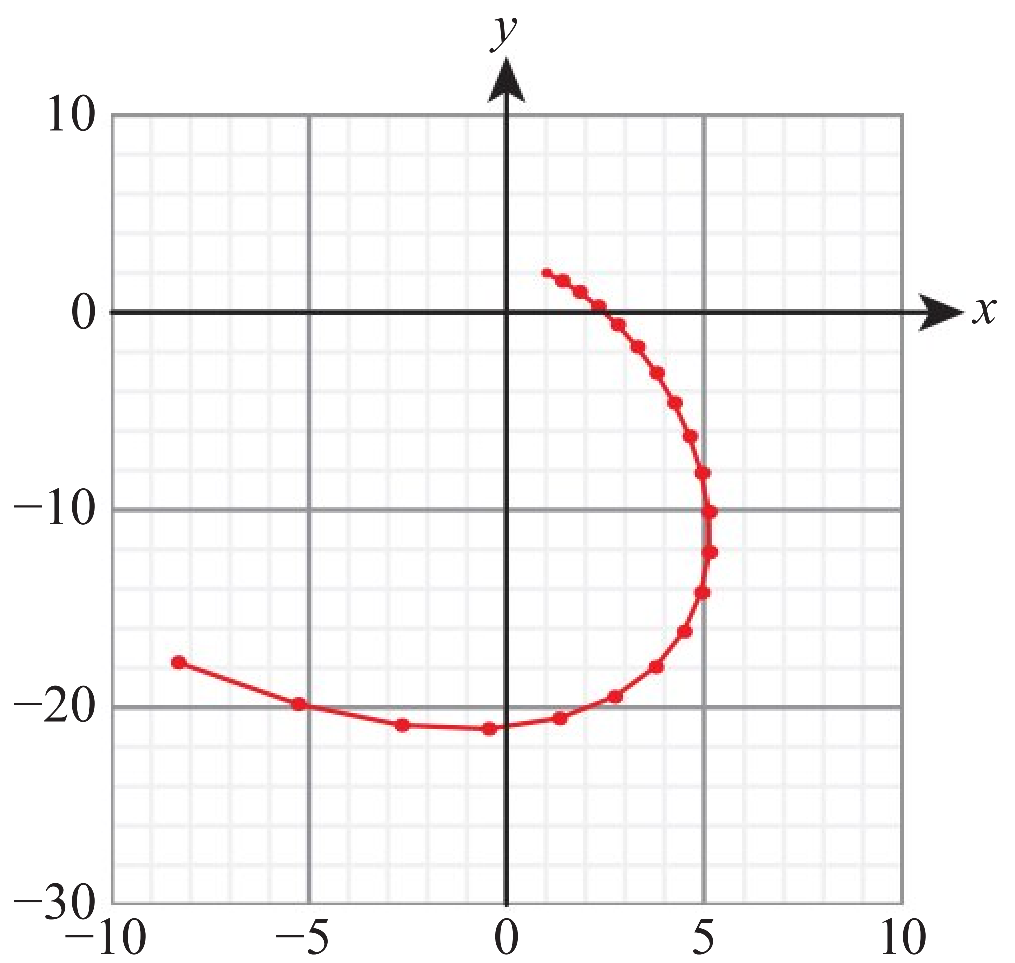
ii



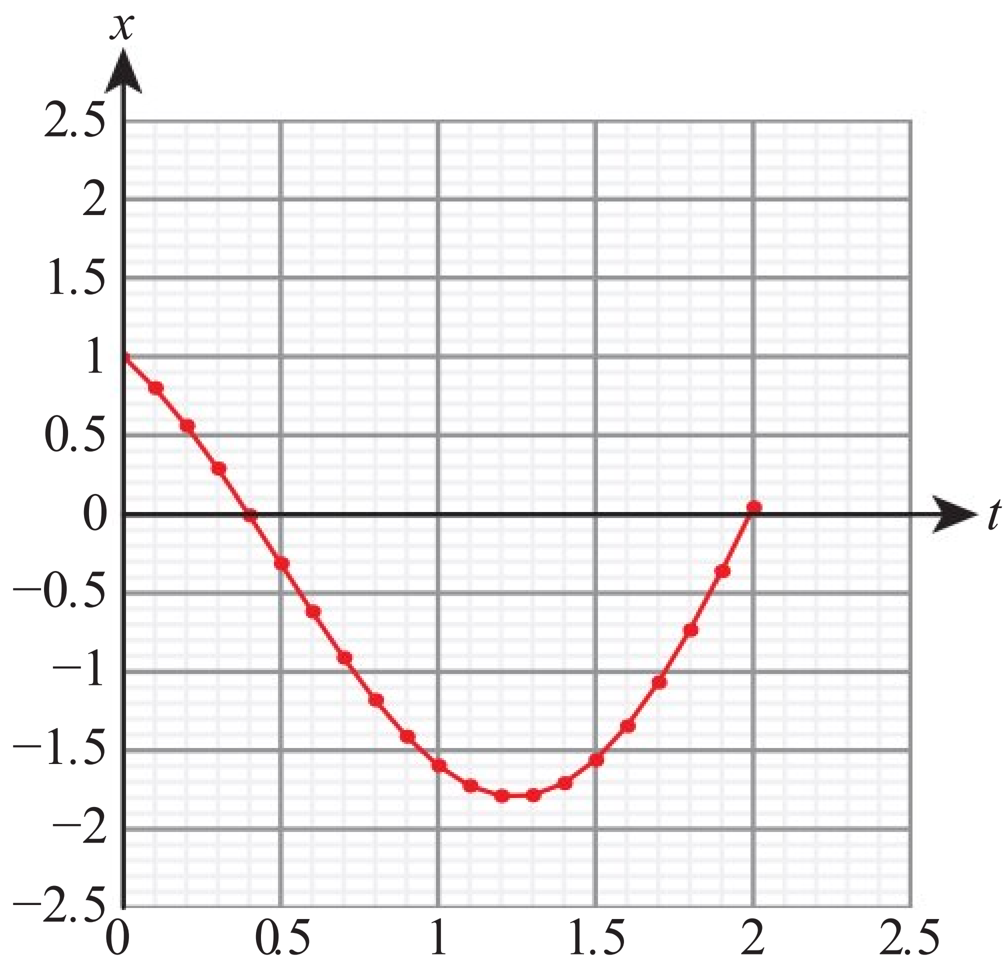
12 a i



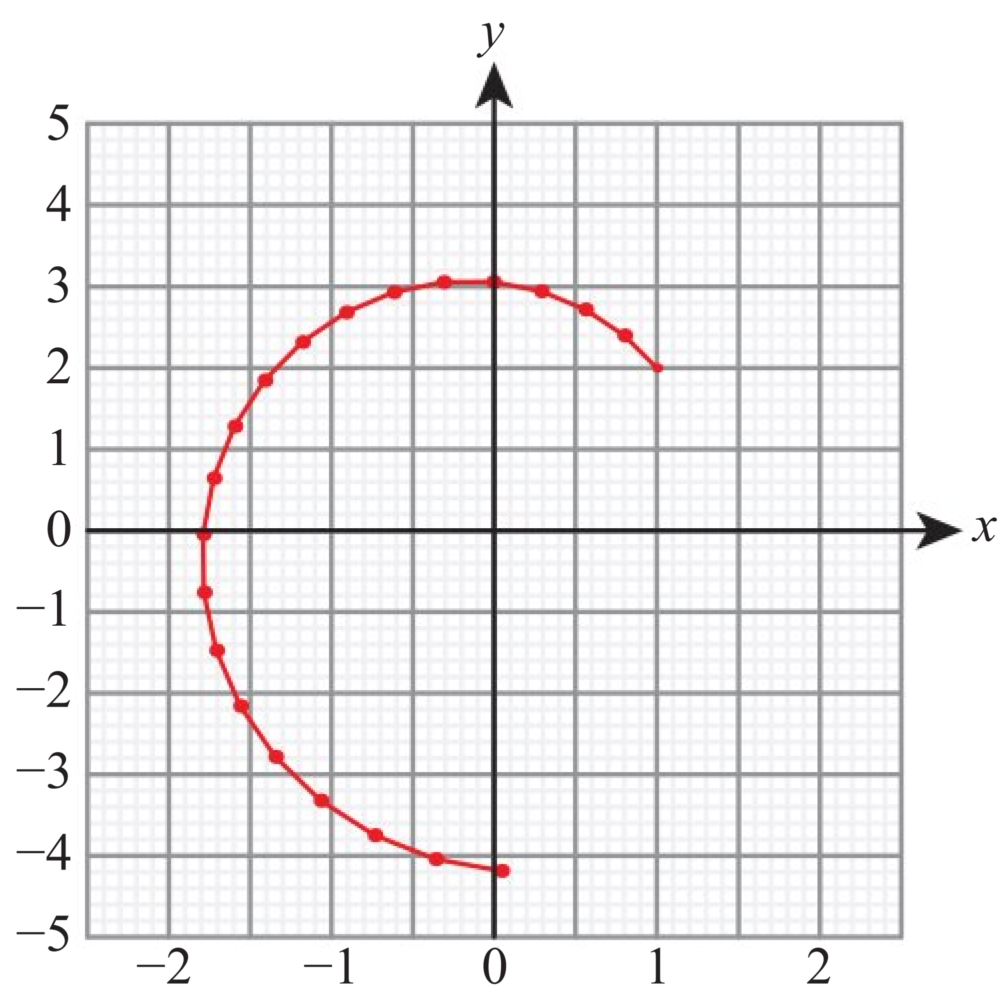
ii



b **i**



ii



13 **a** $-3, 5; \begin{pmatrix} 1 \\ -2 \end{pmatrix}, \begin{pmatrix} 1 \\ 2 \end{pmatrix}$

b $\begin{pmatrix} x \\ y \end{pmatrix} = Ae^{-3t}\begin{pmatrix} 1 \\ -2 \end{pmatrix} + Be^{5t}\begin{pmatrix} 1 \\ 2 \end{pmatrix}$

c $A = 1,\ B = 3\ \ x = e^{-3t} + 3e^{5t},\ y = -2e^{-3t} + 6e^{5t}$

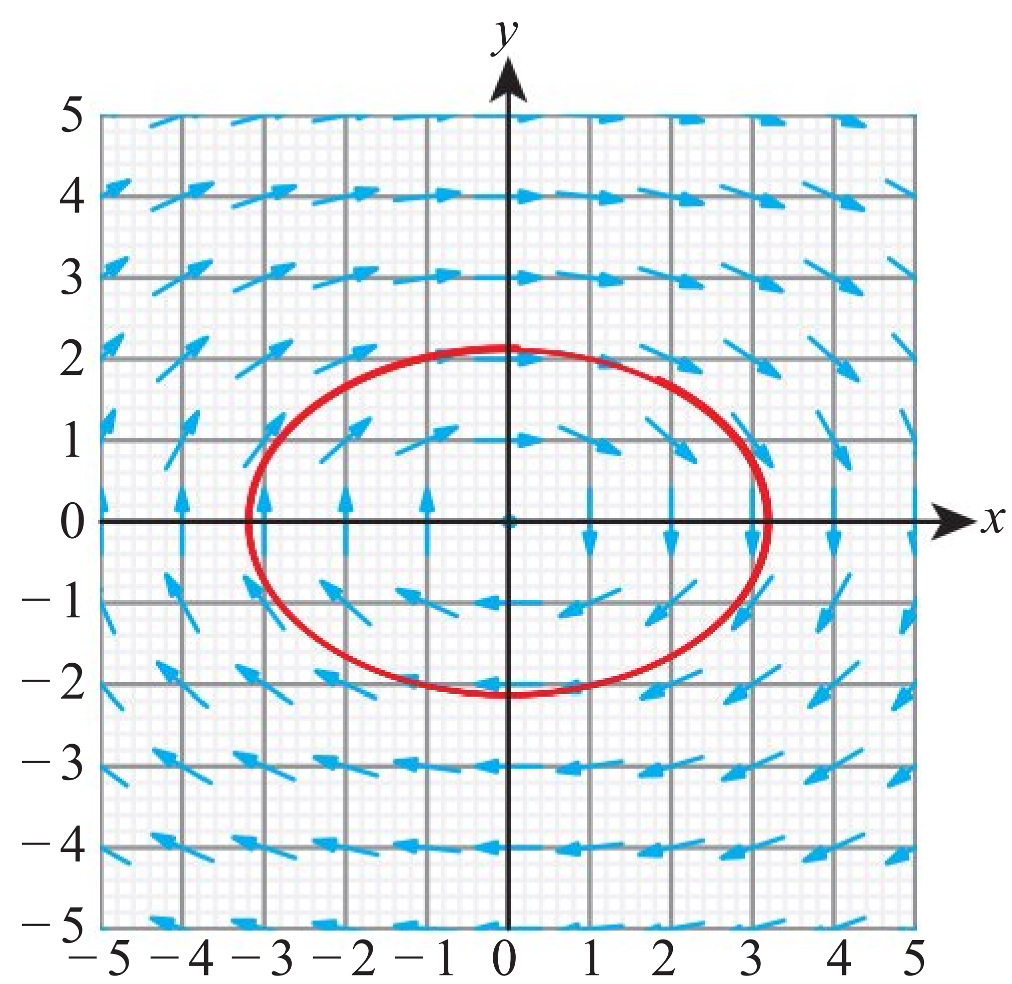
14 **a** $-1, 4$ **b** $x(t) = 3e^{-t} + e^{4t}$

Hint: You do not need to find the eigenvectors of the matrix.

15 **a** $\pm 0.6i$

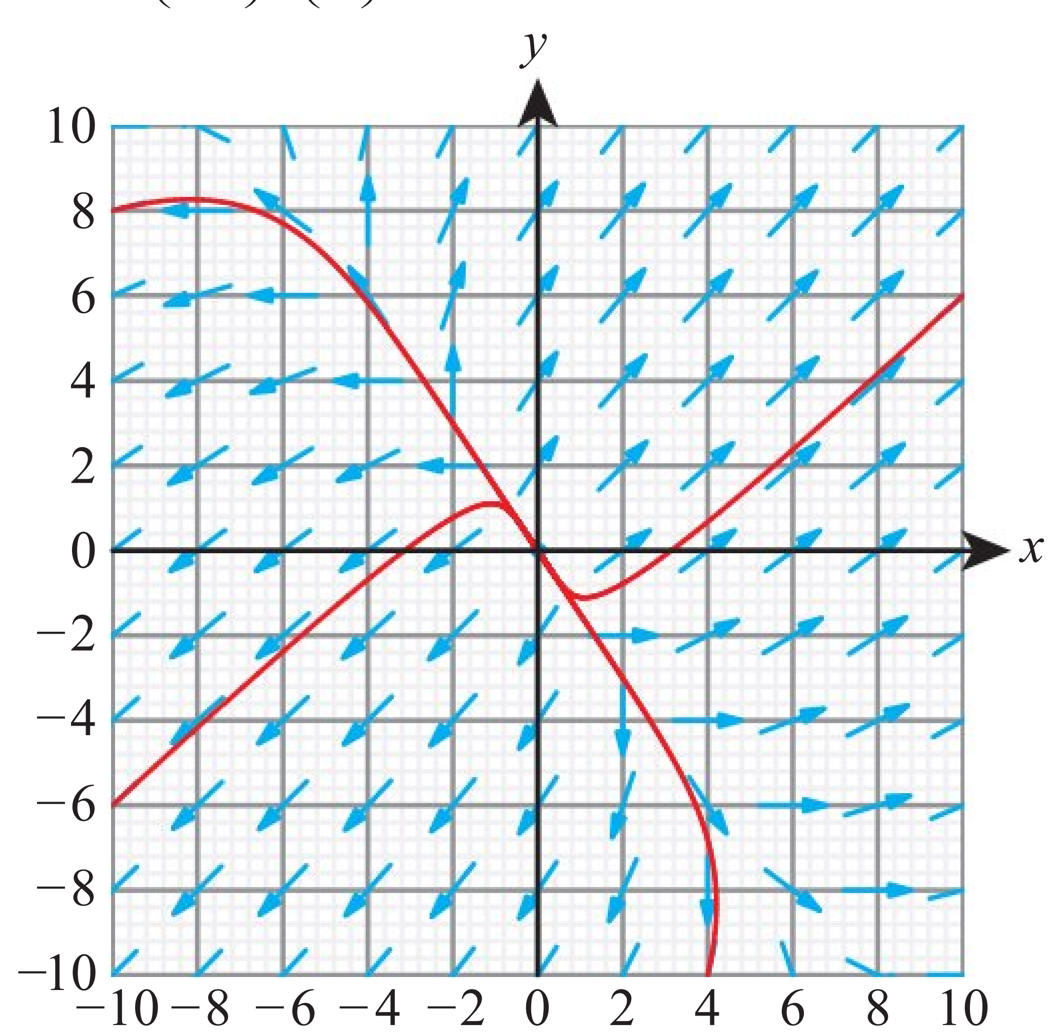
b **i** Increasing

ii



16 **a** $1, 6; \begin{pmatrix} 2 \\ -3 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \end{pmatrix}$

b



c $\begin{pmatrix} x \\ y \end{pmatrix} = Ae^{t}\begin{pmatrix} 2 \\ -3 \end{pmatrix} + Be^{6t}\begin{pmatrix} 1 \\ 1 \end{pmatrix}$;

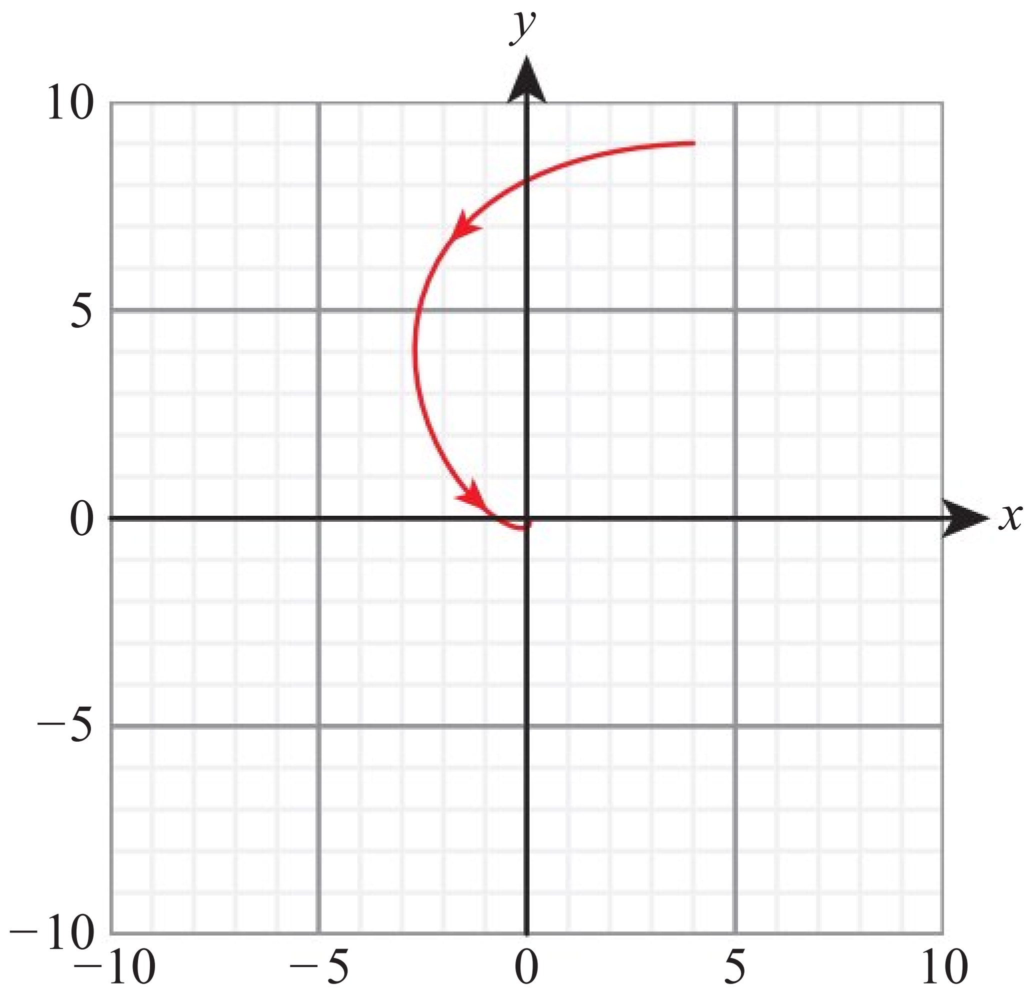
$x = 2e^{t} + 4e^{6t},\ y = -3Ae^{t} + 4e^{6t}$

d Not suitable, as it predicts indefinite growth.

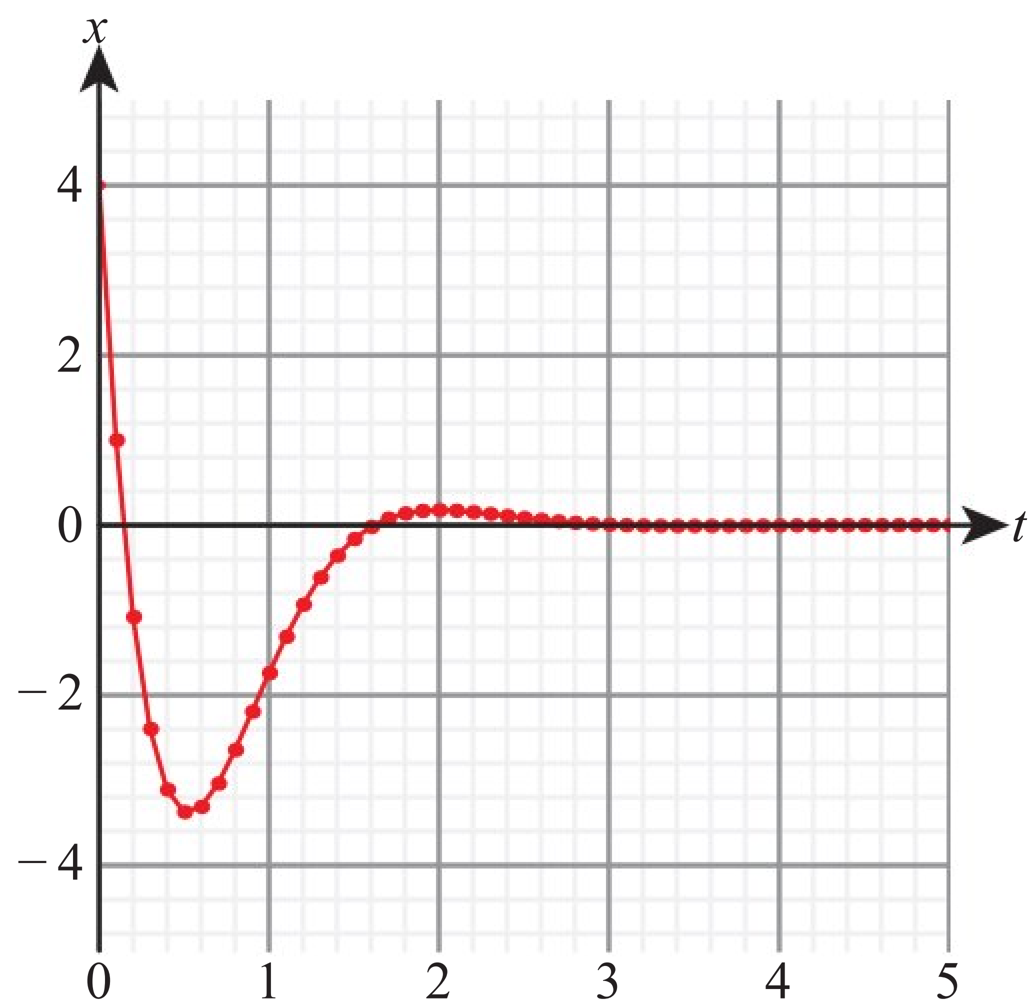
17 **a** 3.7 **b** (2, 2)

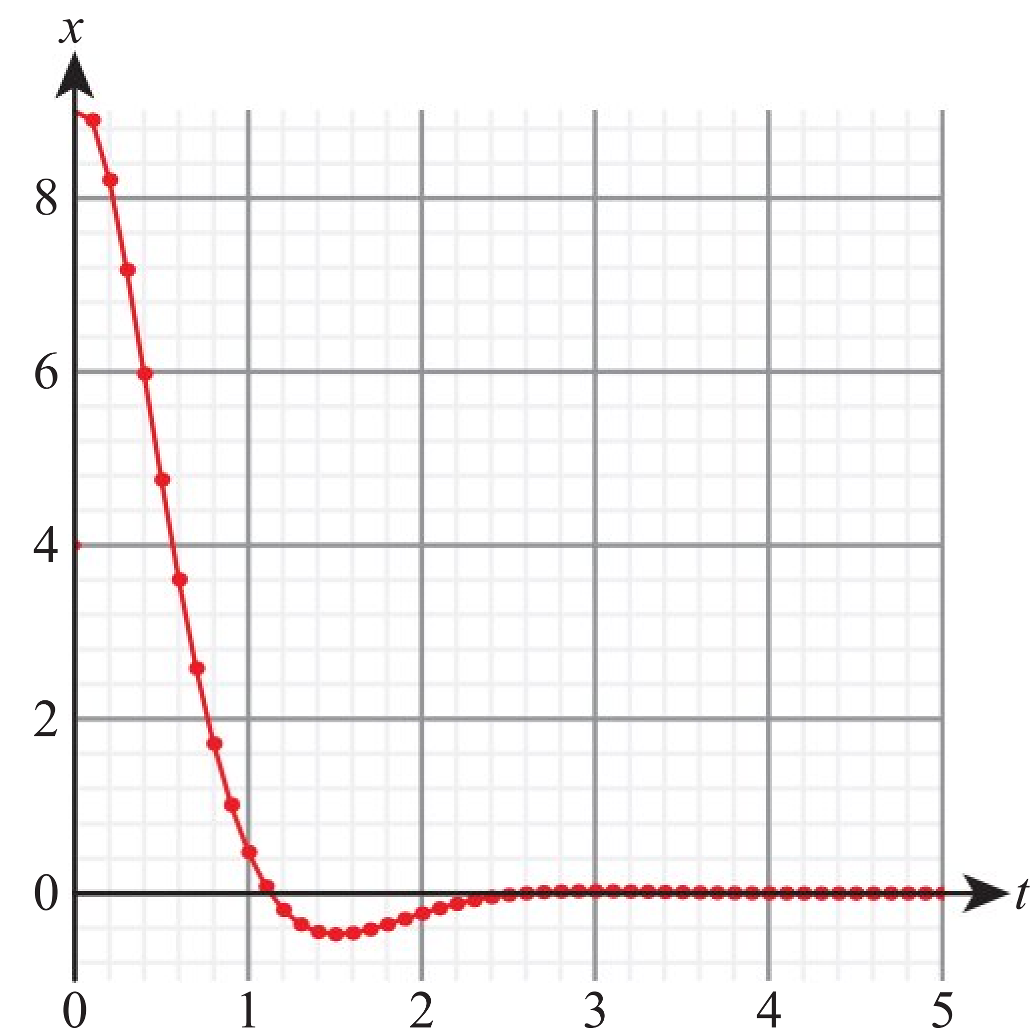
18 Options **b** and **c**

19 **a**



b



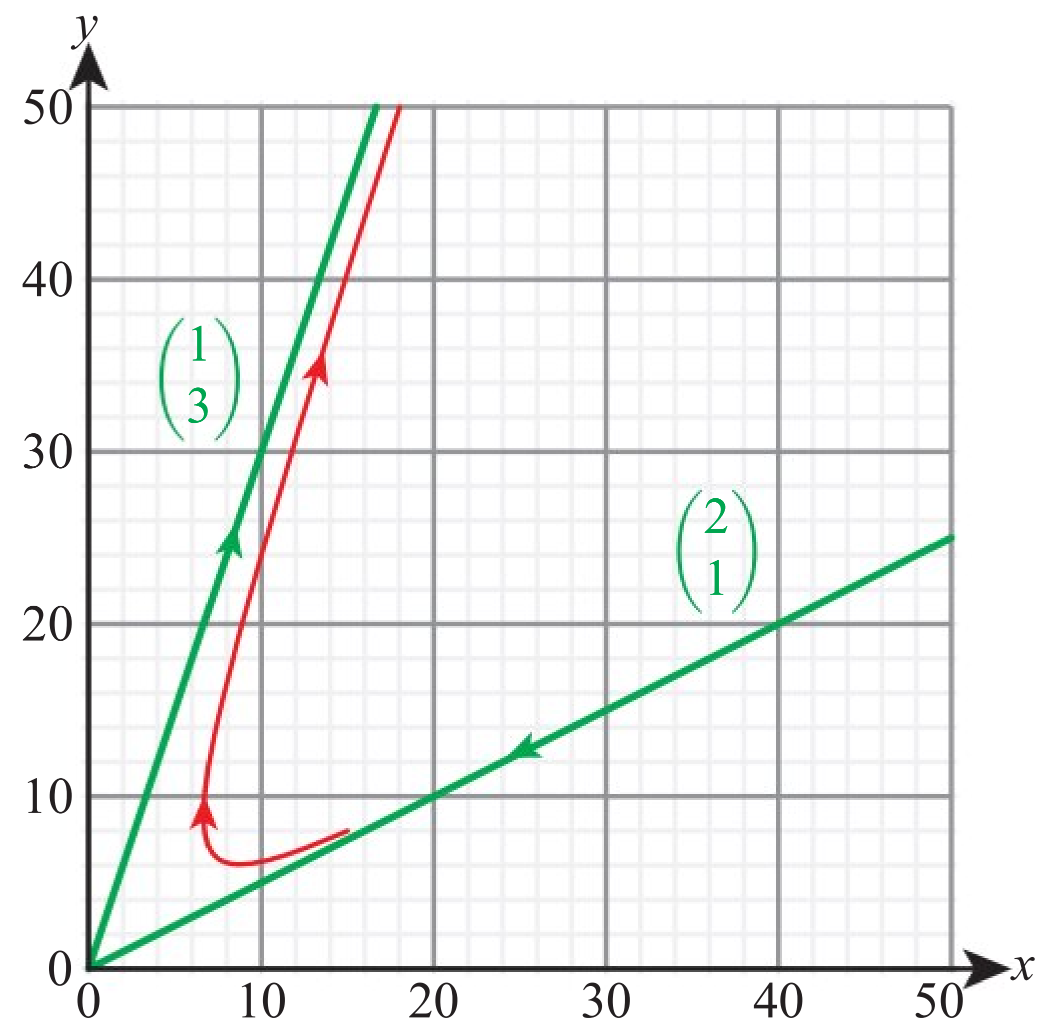


c Both oscillate with a decreasing amplitude.

20 **a** $10, -5, \begin{pmatrix} 1 \\ 3 \end{pmatrix}, \begin{pmatrix} 2 \\ 1 \end{pmatrix}$

b $x(t) = 0.2e^{10t} + 14.8e^{-5t},\ y(t) = 0.6e^{10t} + 7.4e^{-5t}$

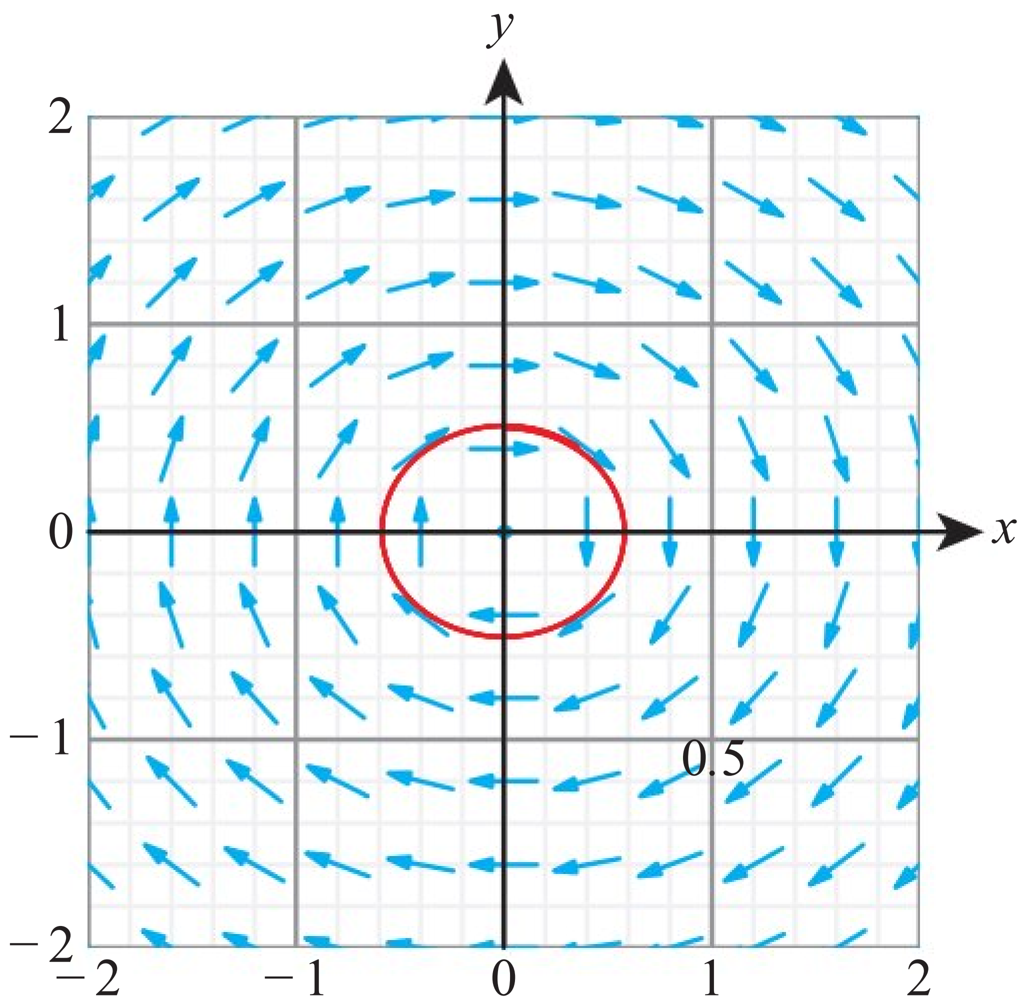
c



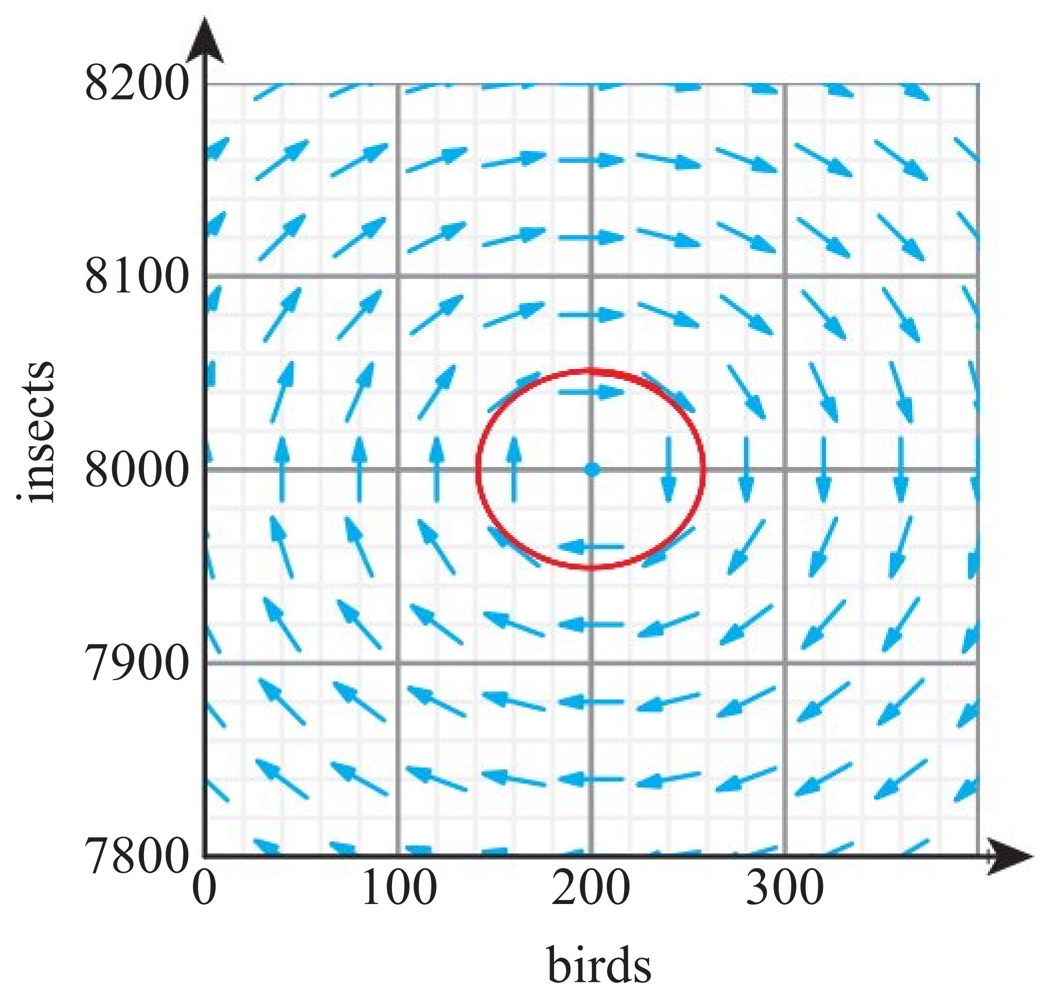
d 1 : 3

21 **a** $\pm\dfrac{2\sqrt{3}}{5}$

b 200 birds, 8050 insects; increasing c



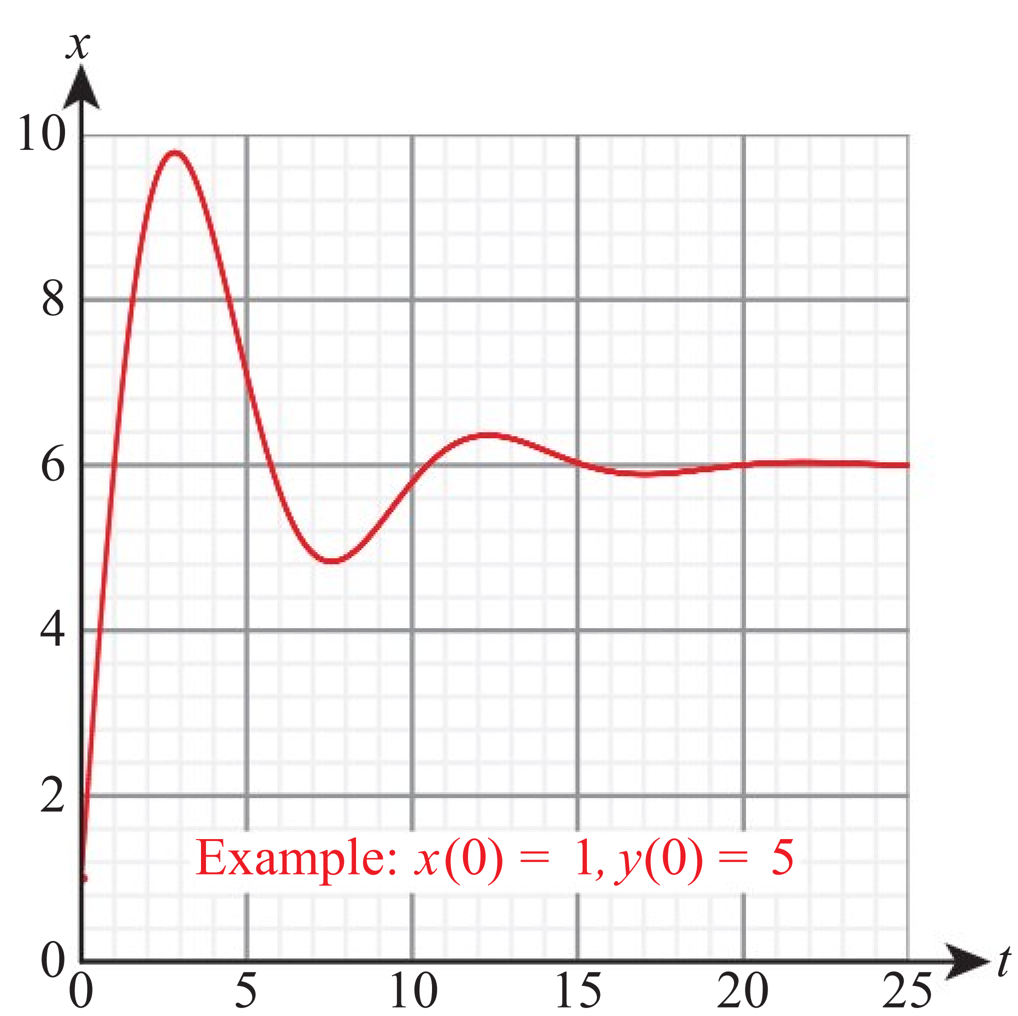
d



Both population sizes oscillate/change periodically.

22 a The number of spiders tends to 600 and the number of flies tends to 300.

b



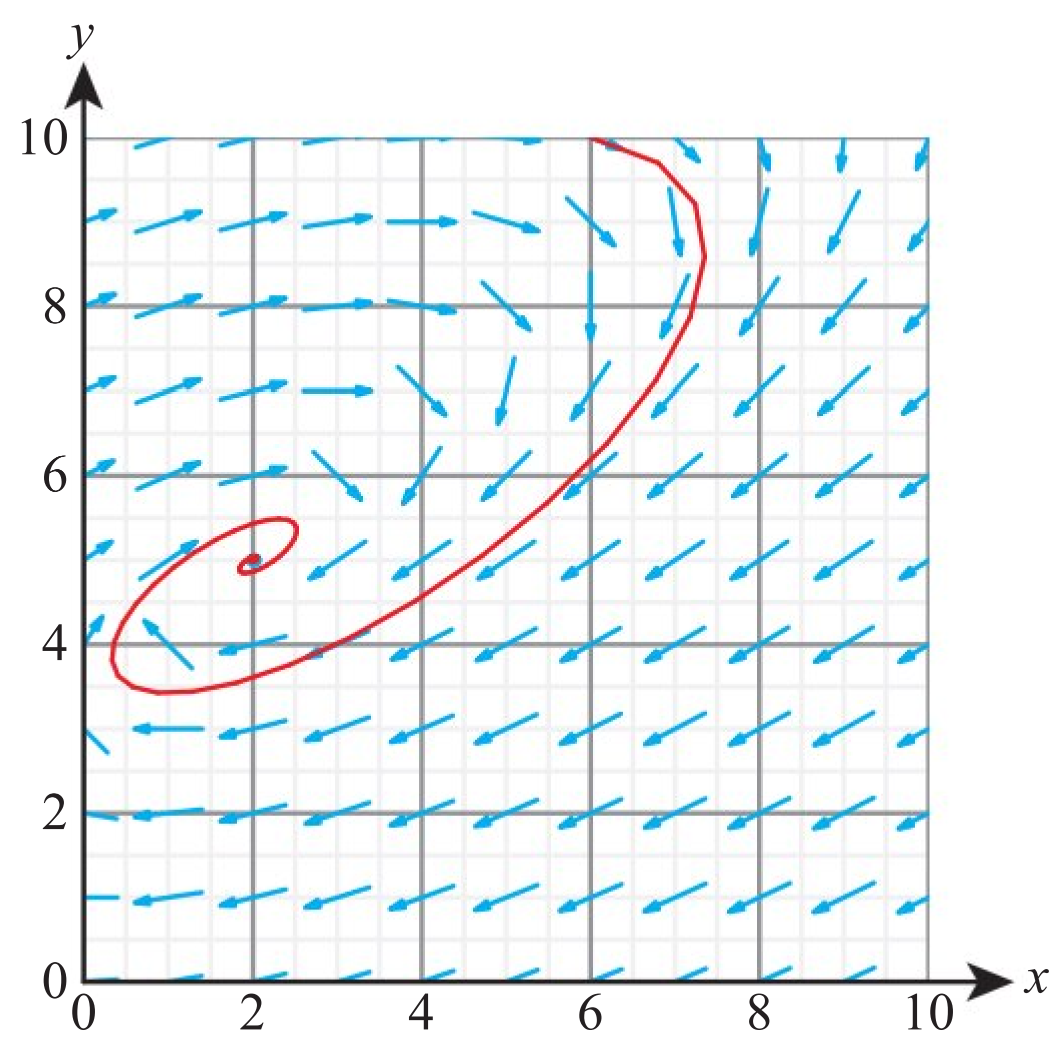
23 a x stays fairly constant, y increases.

b Option iii

24 a 200 sharks, 500 fish

b $A = \begin{pmatrix} -3 & 4 \\ -2 & 1 \end{pmatrix}$ c $-1 \pm 2i$

d



e The number of sharks oscillates around and tends to 200, the number of fish oscillates around and tends to 500.

Exercise 13D

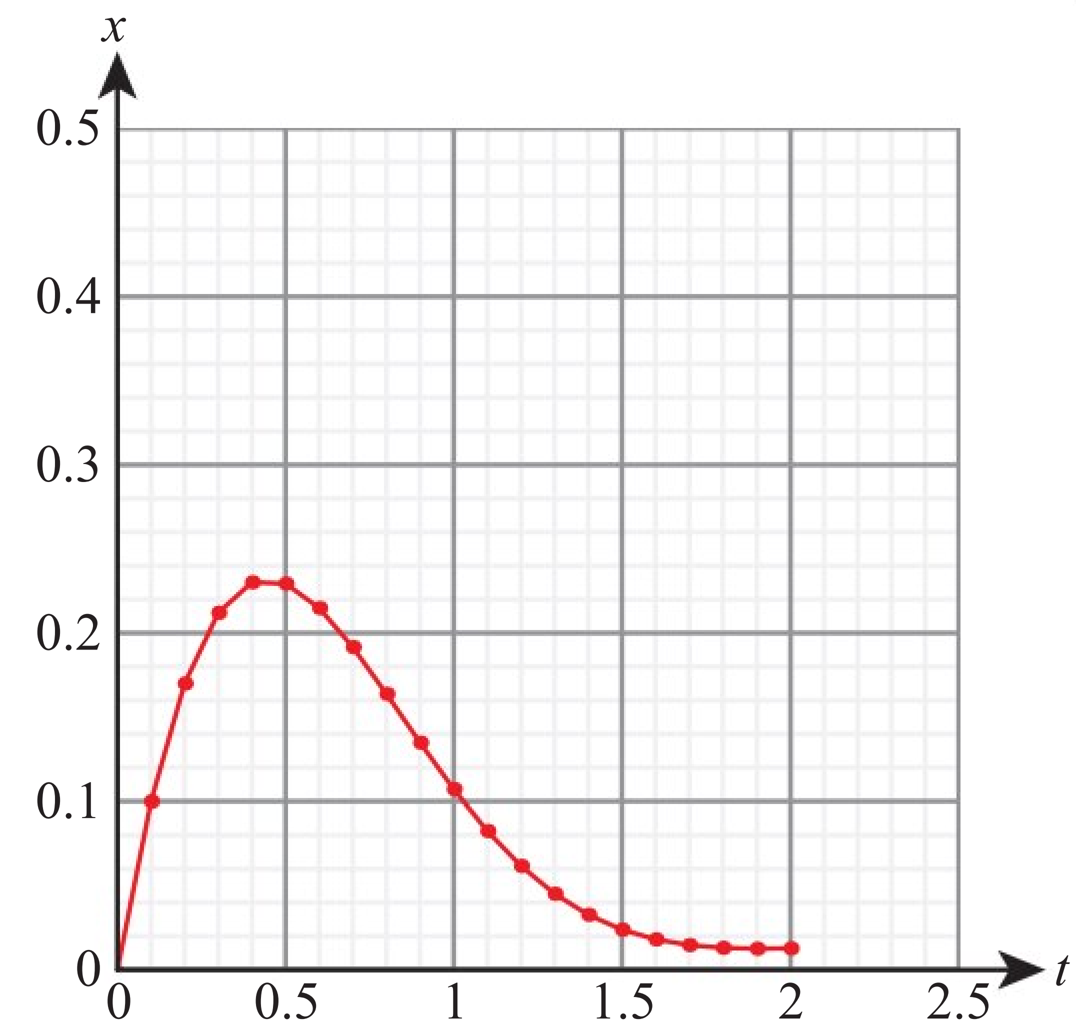
1	a Unstable	b Unstable	
2	a Stable	b Stable	
3	a Unstable	b Unstable	
4	a Stable	b Stable	
5	a Unstable	b Unstable	
6	a 9.42	b 0.121	
7	a −1.56	b −1.51	
8	a 0.346	b 0.144	
9	b i 699 000	ii 39 400	

c **b i** is more accurate because it has a smaller step length.

10 b −3.46

11 b 0.107

c



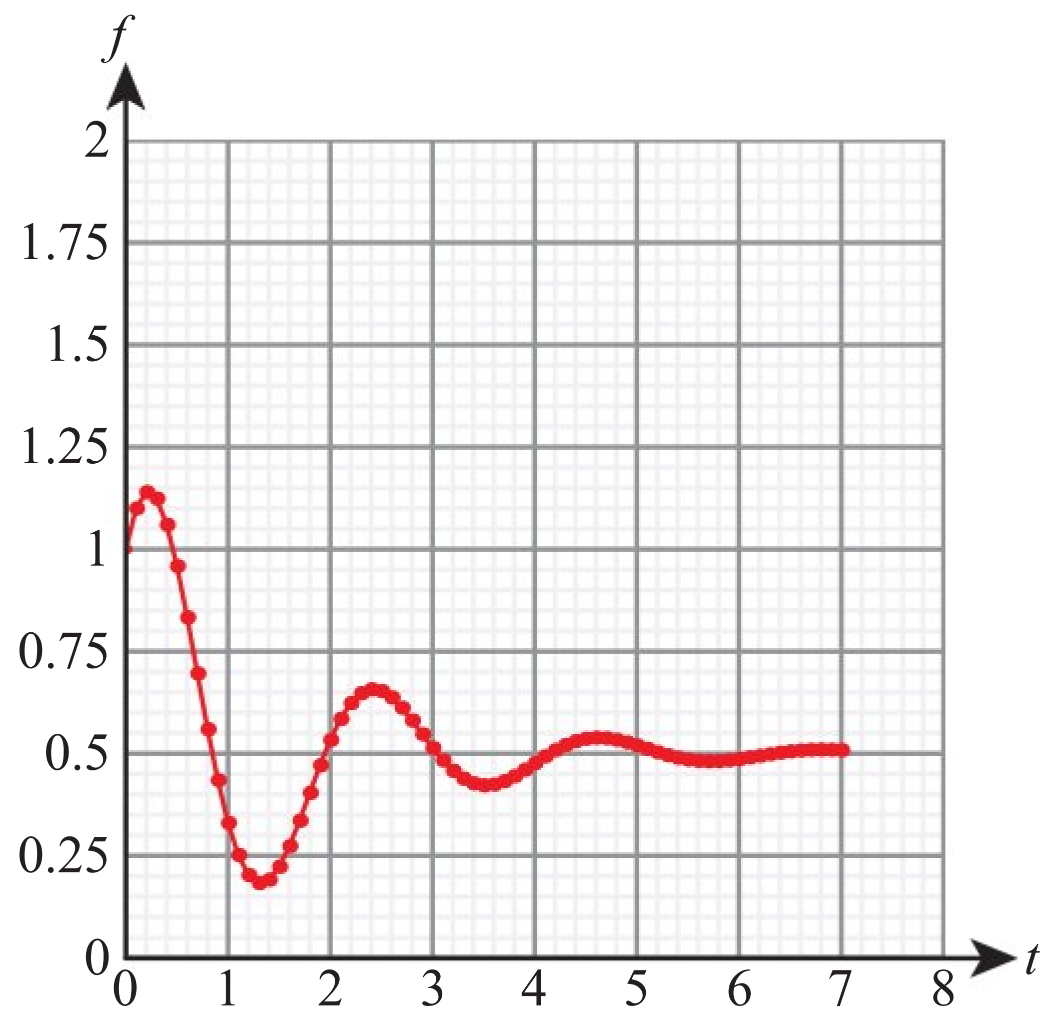
d 0.23

12 a 0.3 years b 1.5 years c \$1.9

d \$2

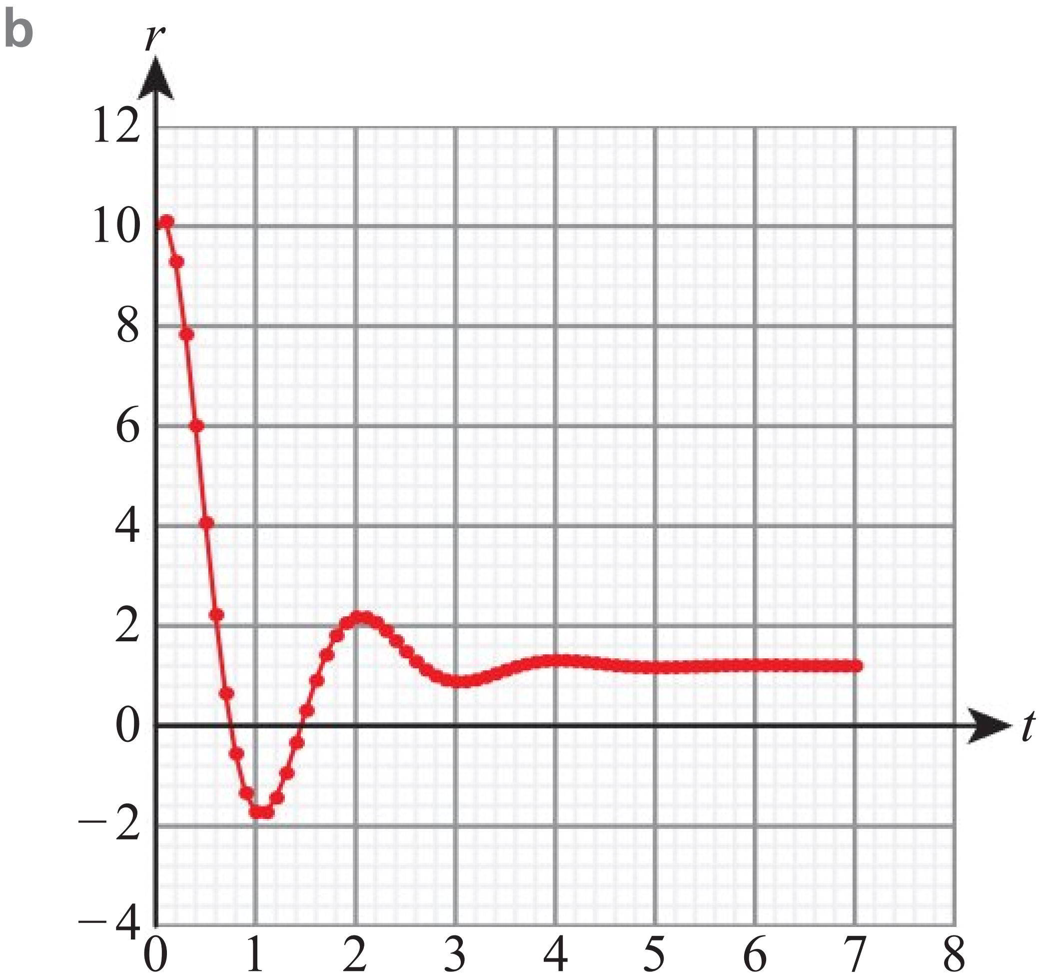
e For example, not taking into account random variation. Price seems too stable in the long term.

13 a



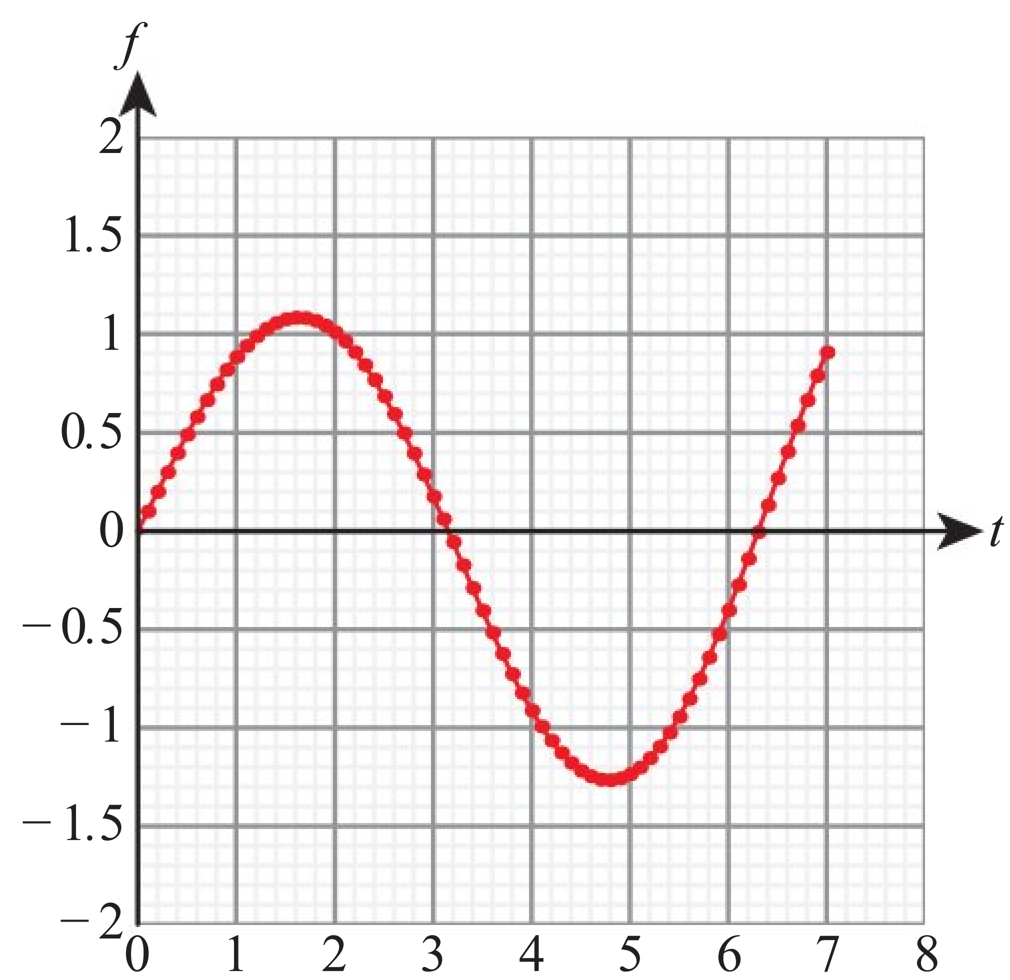
Oscillates with decreasing amplitude, levelling off around 500 foxes.

b



Includes a region with a negative population, which is impossible.

14 a i



ii 1.1 mm, 6.3 ms

b i 3.7 mm ii 0.4 mm

15 a i 1.64 ii 1.61

b $y = 2e^{0.3t} - e^{0.4t}$

c i 2.57% ii 0.753%

16 a i 0.875 ii 0.964

c i 4.02% ii 6.12%

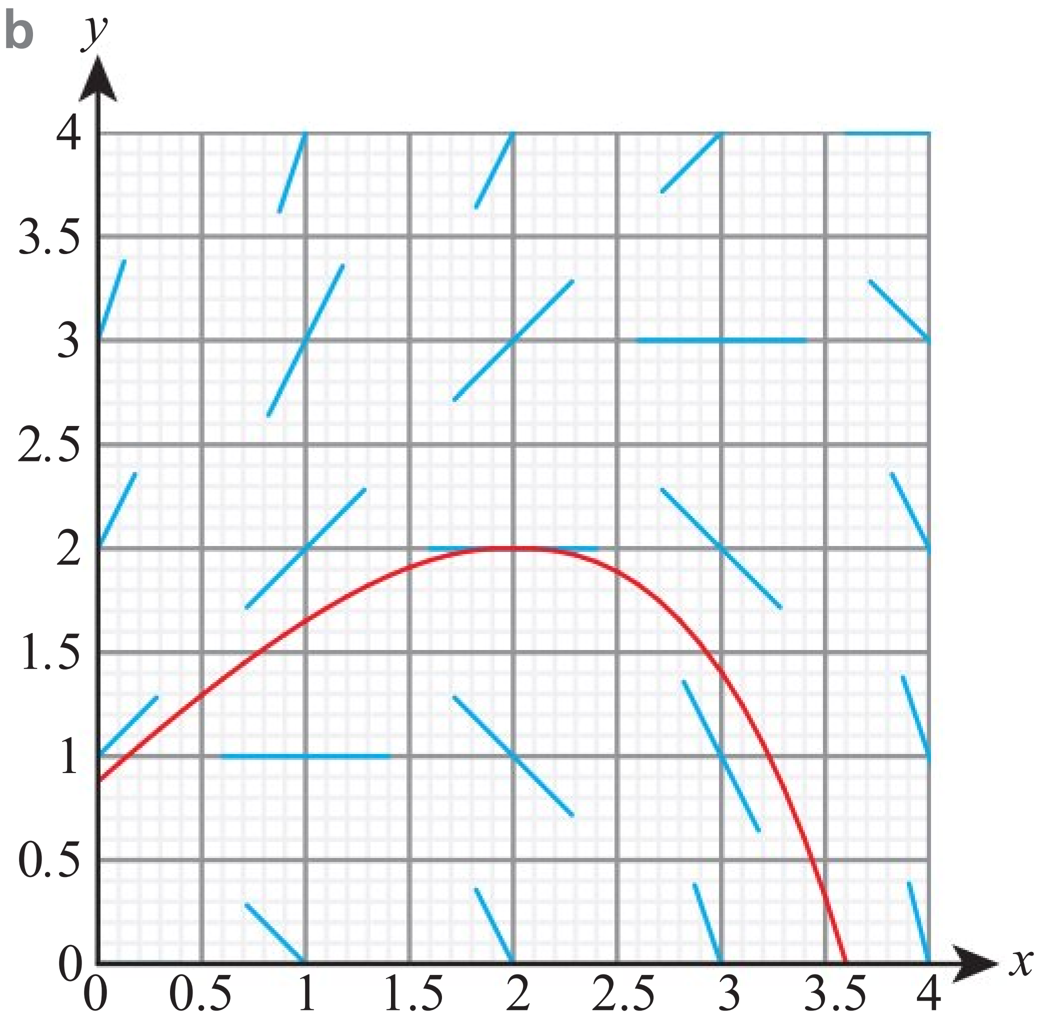
Chapter 13 Mixed Practice

1 **a** $y = Ae^{\frac{3}{2}\sin 2x}$ **b** $y = 5e^{\frac{3}{2}\sin 2x}$

2 $y = Ae^{x^3-2x}$

3 2.45

4 **a, b**



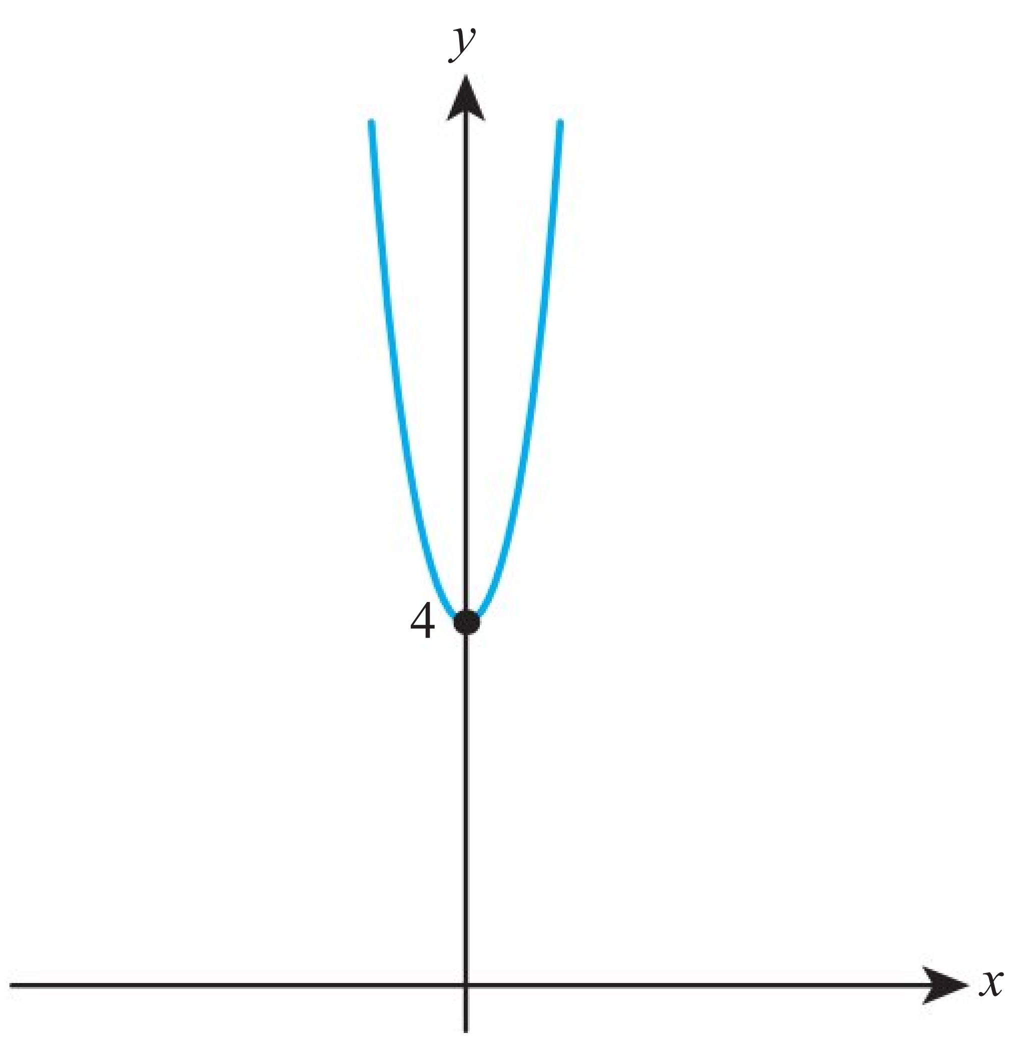
c 1.89

5 **a** 5.058 **b** $y = \sqrt[3]{\dfrac{3x^2 + 247}{2}}$

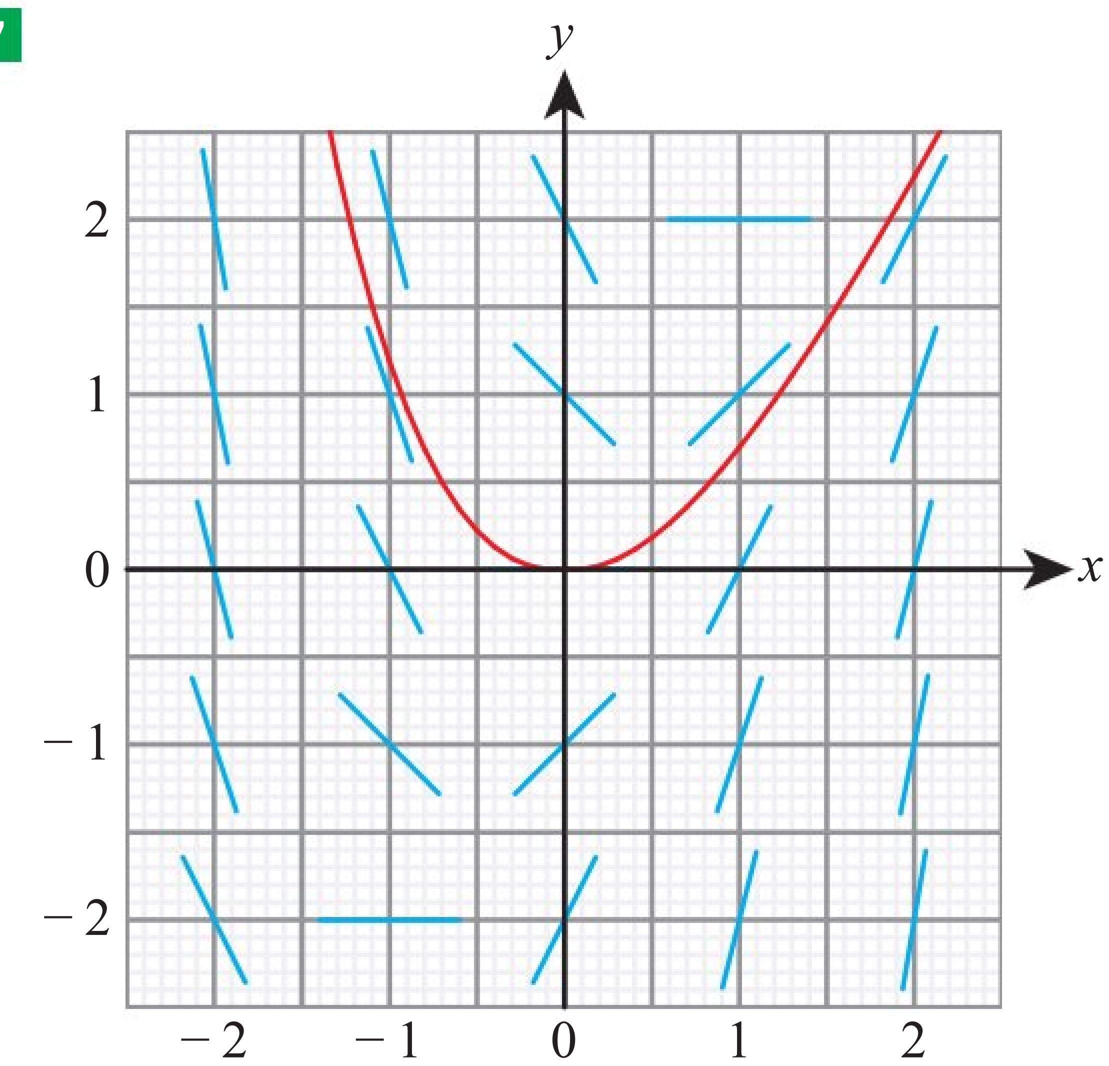
c 0.0256%

6 **a** $y = \left(x^2 + c\right)^2$

b



7

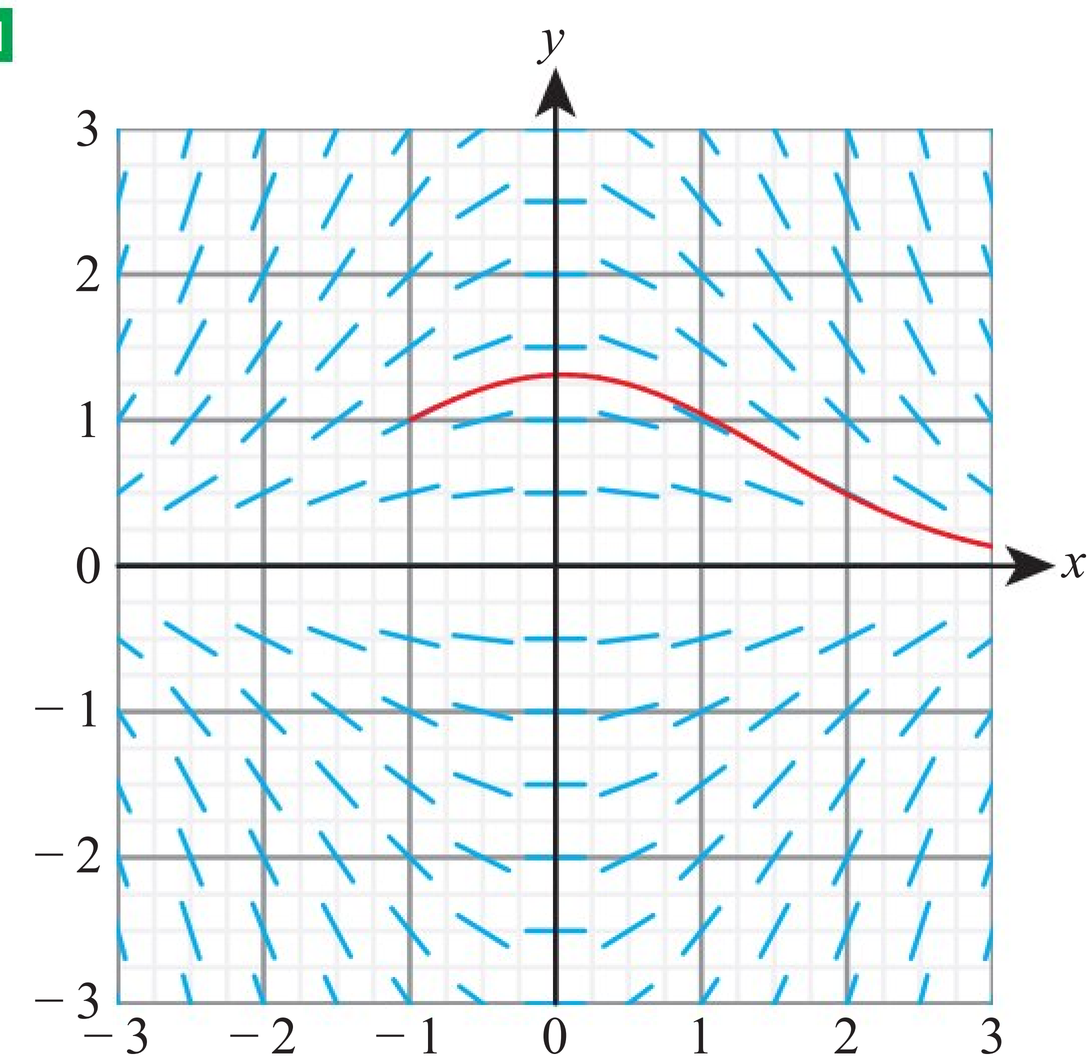


8 **b** $V = (3t + c)^2$ **c** 11.7 seconds

9 **b** $\begin{pmatrix} x \\ y \end{pmatrix} = Ae^{t}\begin{pmatrix} 1 \\ -2 \end{pmatrix} + Be^{5t}\begin{pmatrix} 1 \\ 2 \end{pmatrix}$

10 $x = -13.6,\ y = 25.4$

11

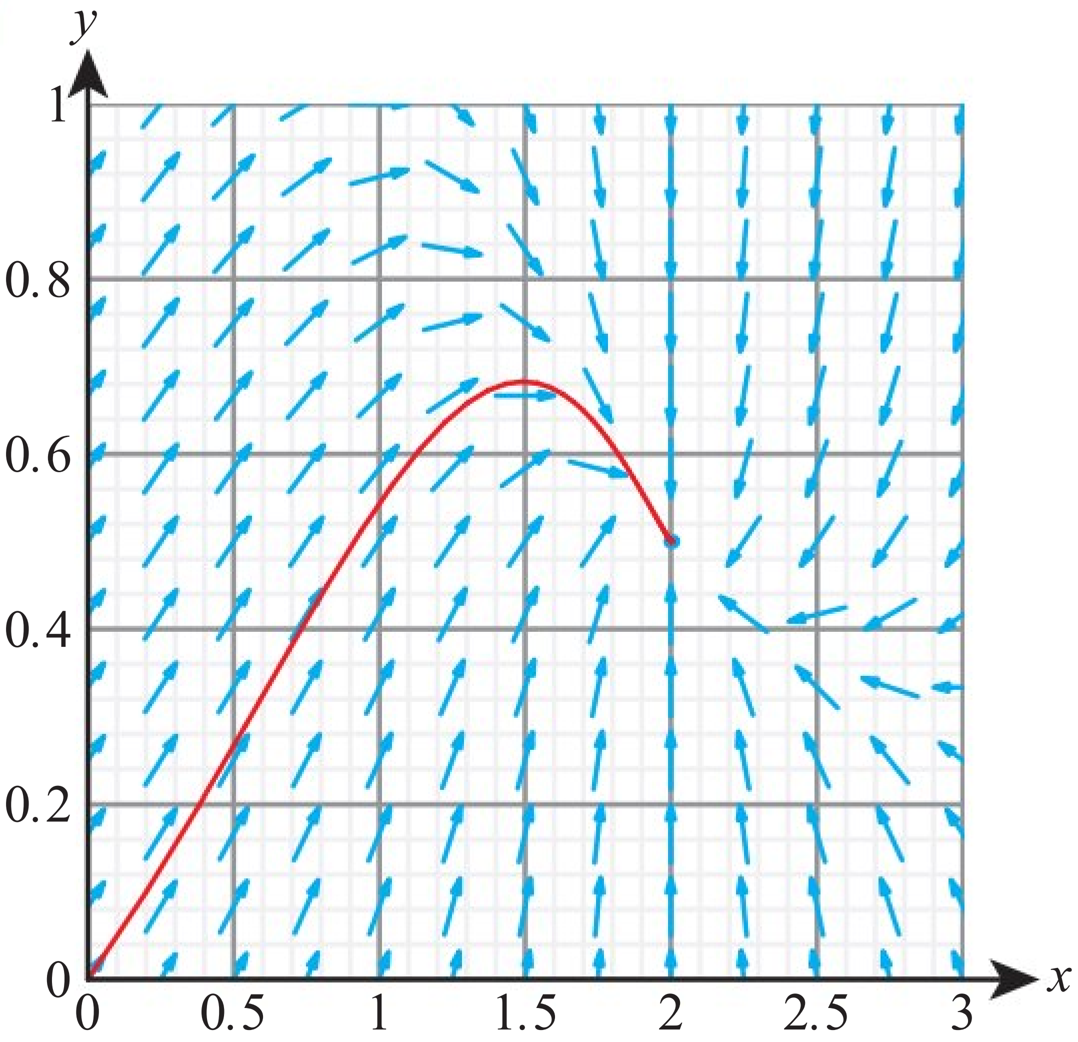


a 1.3 **b** 0

12 **a** 2.3 **b** 1.7

c Tends towards $x = 1,\ y = 1$

13



Hint: The phase plane is actually showing

$\frac{dx}{dt} = 2 - x$, $\frac{dy}{dt} = 1 - xy$

a 0.7 b 2

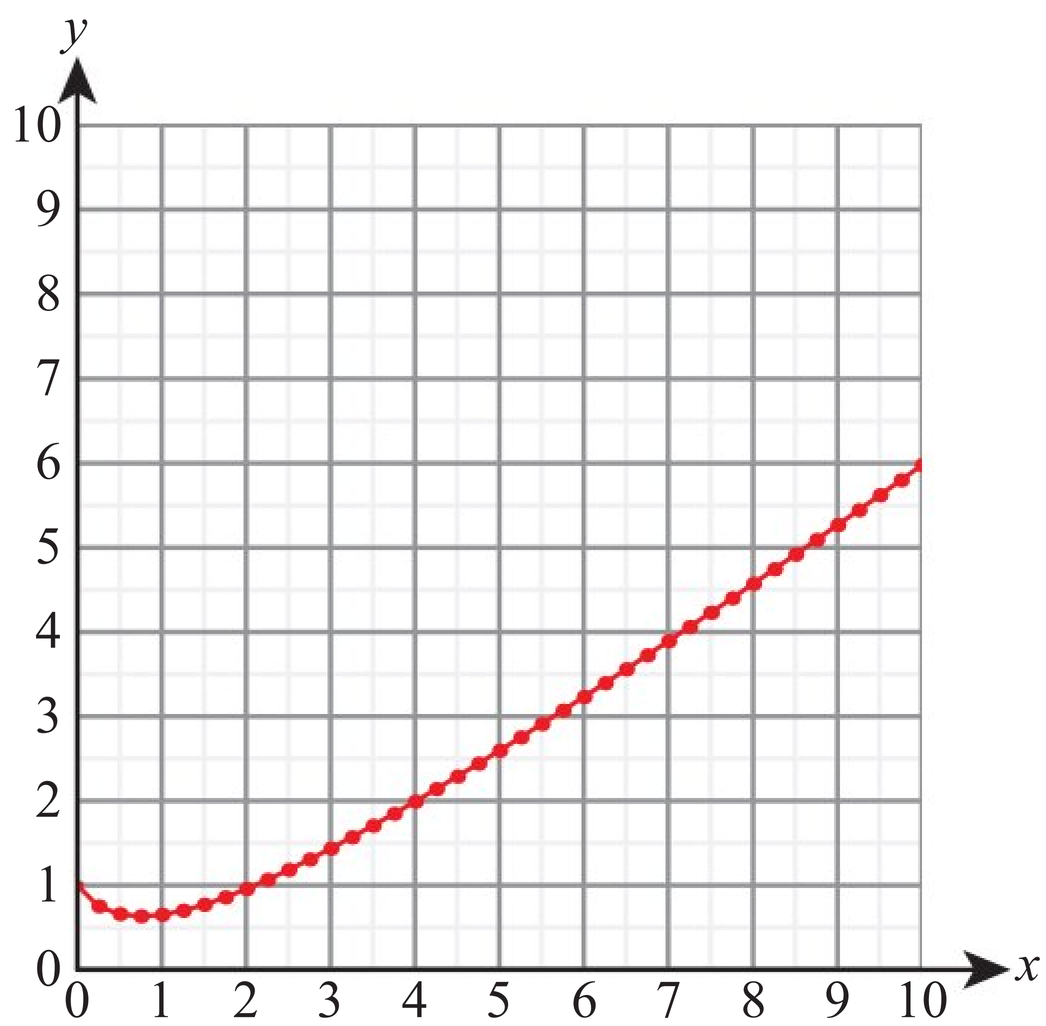
14 a 1.57

b Approximation is less than true value; within each step the gradient increases, but Euler's method uses a constant gradient.

15 a $R = R_0 e^{-kt}$ b $\frac{\ln 2}{k}$

c The time taken to go from $\frac{1}{2}$ to $\frac{1}{4}$ and from $\frac{1}{4}$ to $\frac{1}{8}$ etc. will be the same.

16 a



b 0.6

17 a −0.599

b $-\ln\left(e^{-x} + e - 1\right)$; 0.0178

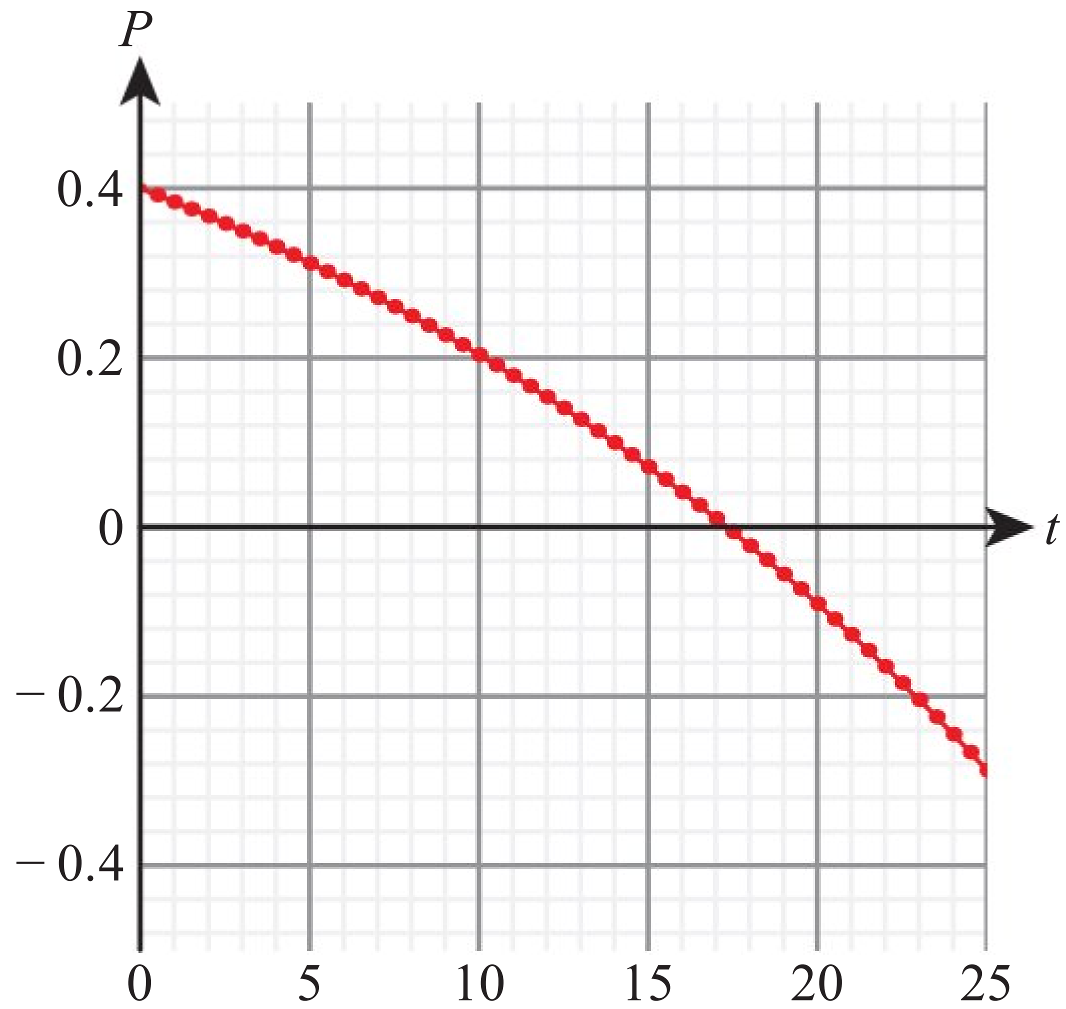
c Decrease the step length.

18 $N = 2e^{0.2\left(t - \frac{12}{\pi}\left(1 - \cos\left(\frac{\pi t}{6}\right)\right)\right)}$

19 a 3.92 b $y = 2e^{3 - \frac{3}{x}}$ c 1.90%

20 b i $1.31

ii



iii 17 years

21 a $m = Ae^{-kt}$ b 0.0990 c 23 years

22 a $y = Ae^{\frac{1}{2}(x-1)^2} - 2$ b 49.7

23 $y = \ln\left(3\sin 2x + c\right)$

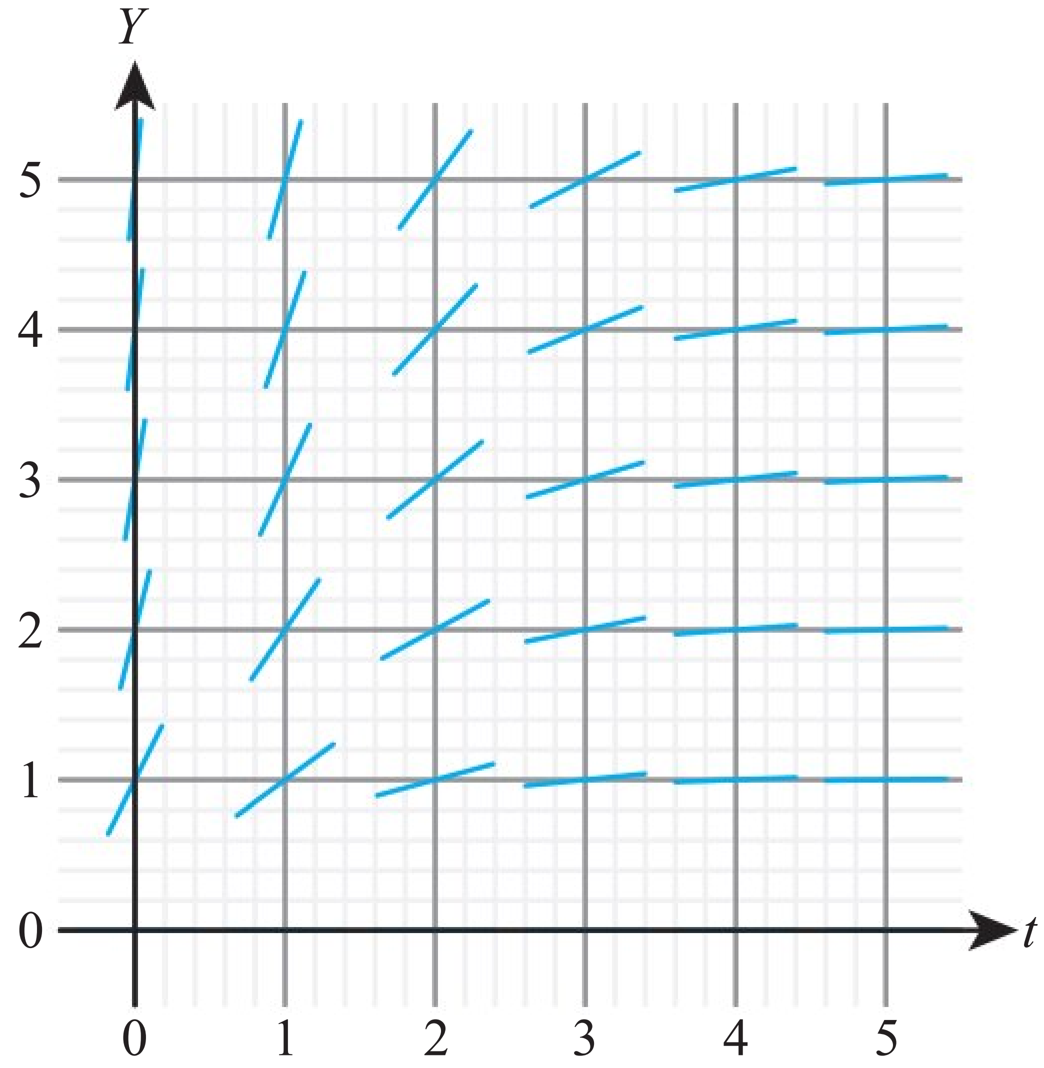
24 $(9.46, 3.71)$

25 a $f(x, t) = e^{-t} - 4x$ b −0.427

26 b 69.7°

27 a $\frac{dv}{dt} = 0.05\left(10 - 0.2v^2\right)$ b $0.994\,\mathrm{m\,s^{-1}}$

28 a



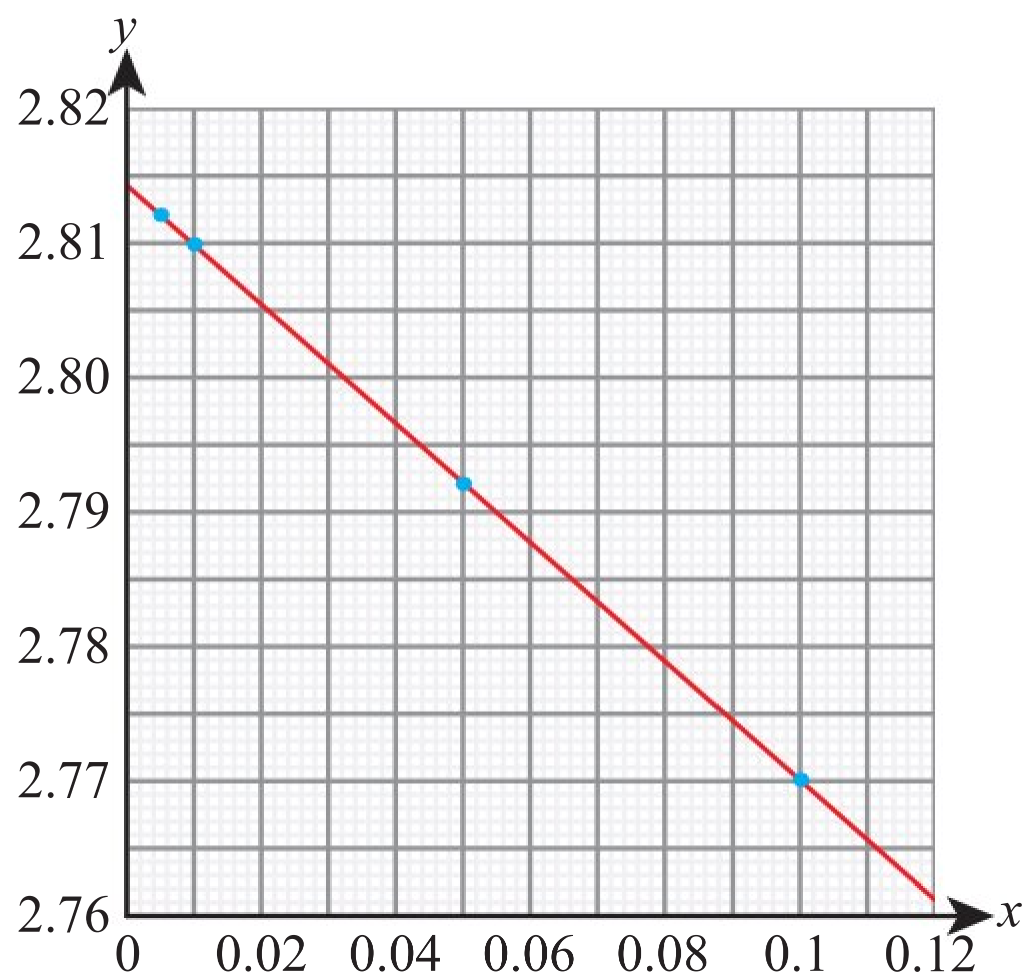
b $y = e^{(2-2e^{-t})}$

c $\ln\left(\dfrac{2}{2-\ln 2}\right) \approx 0.426$ years

d 738 000

29 a 2.7701

b

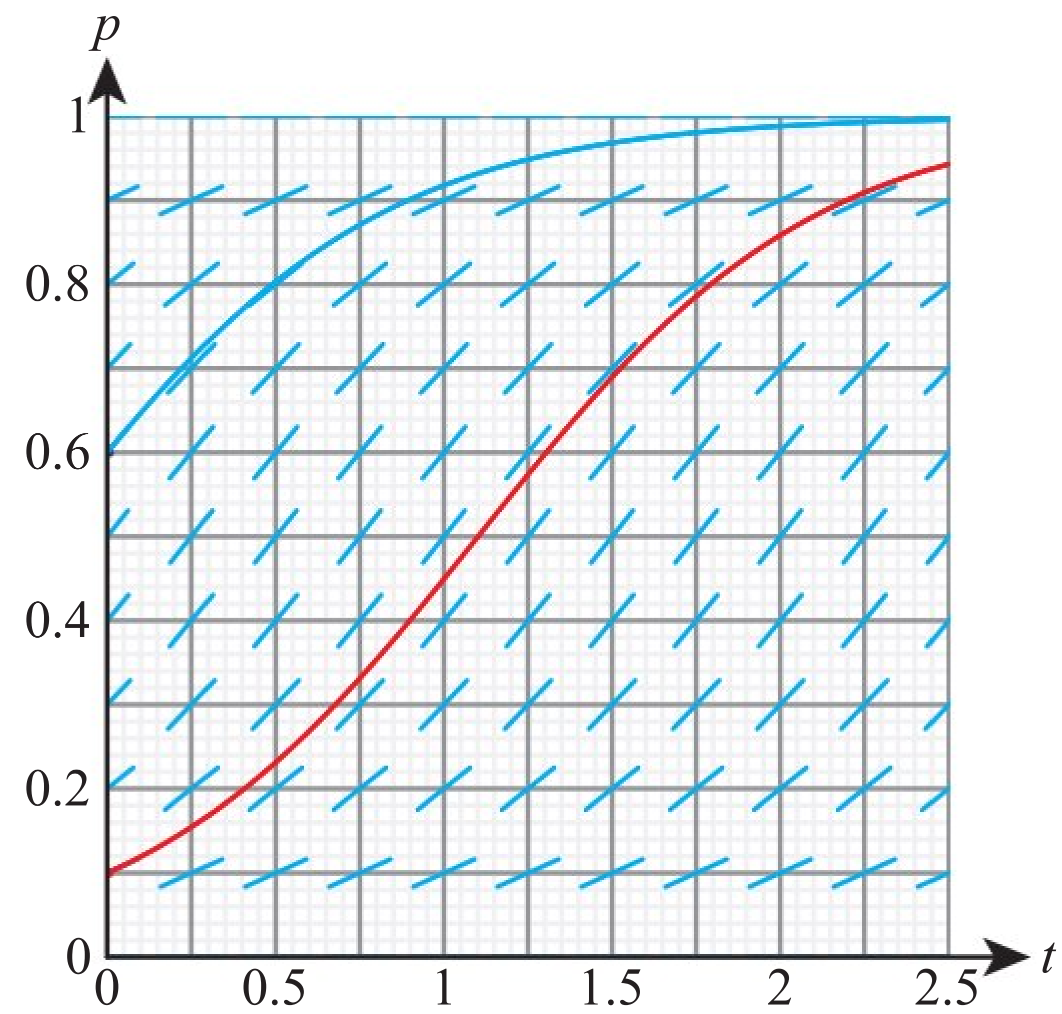


d 2.814

30 0.904

31 b 3.96

32 b

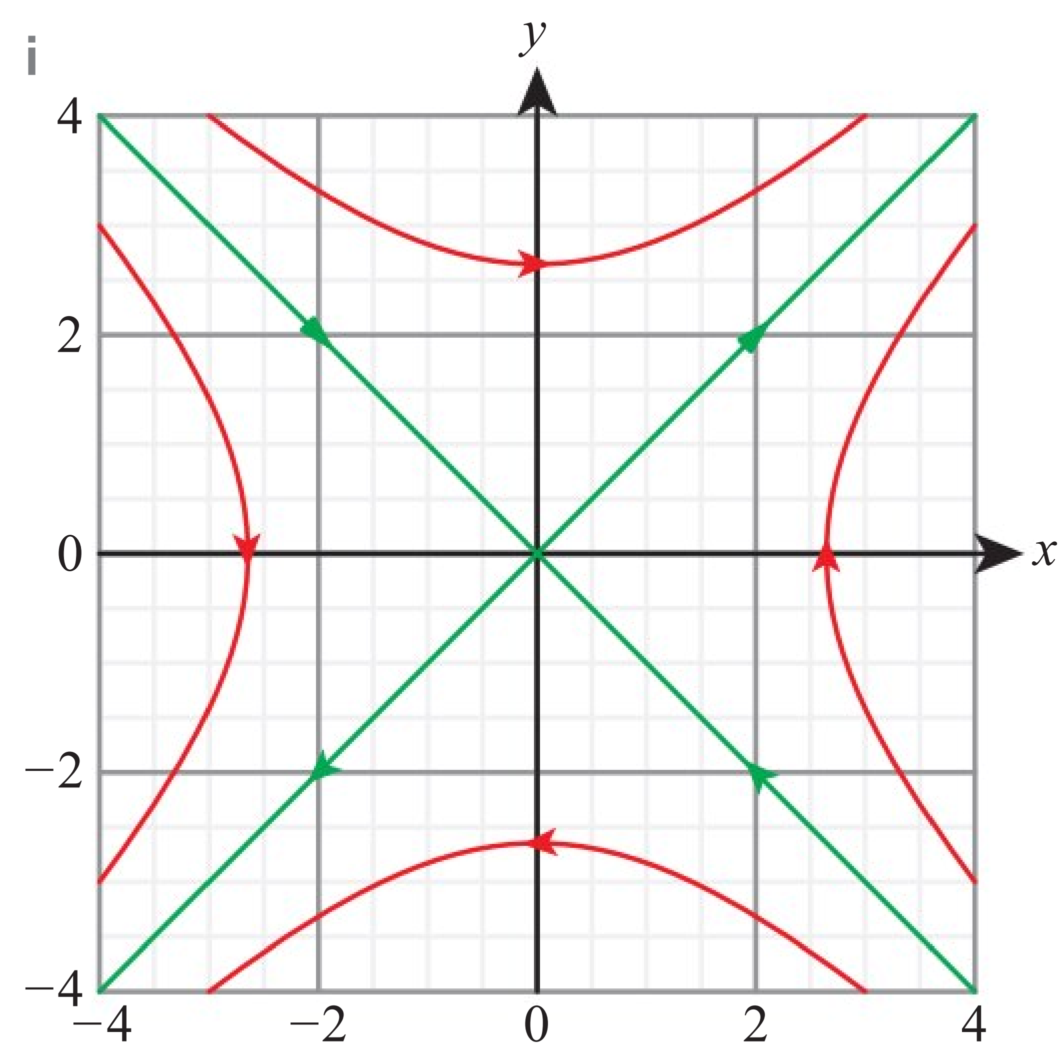


c i $p = \dfrac{e^{2t}}{9 + e^{2t}}$ ii $\ln 3 \approx 1.10$ weeks

33 a $\dfrac{dx}{dt} = y$

$\dfrac{dy}{dt} = x$

b i



ii Saddle

c i t values are not explicitly shown.

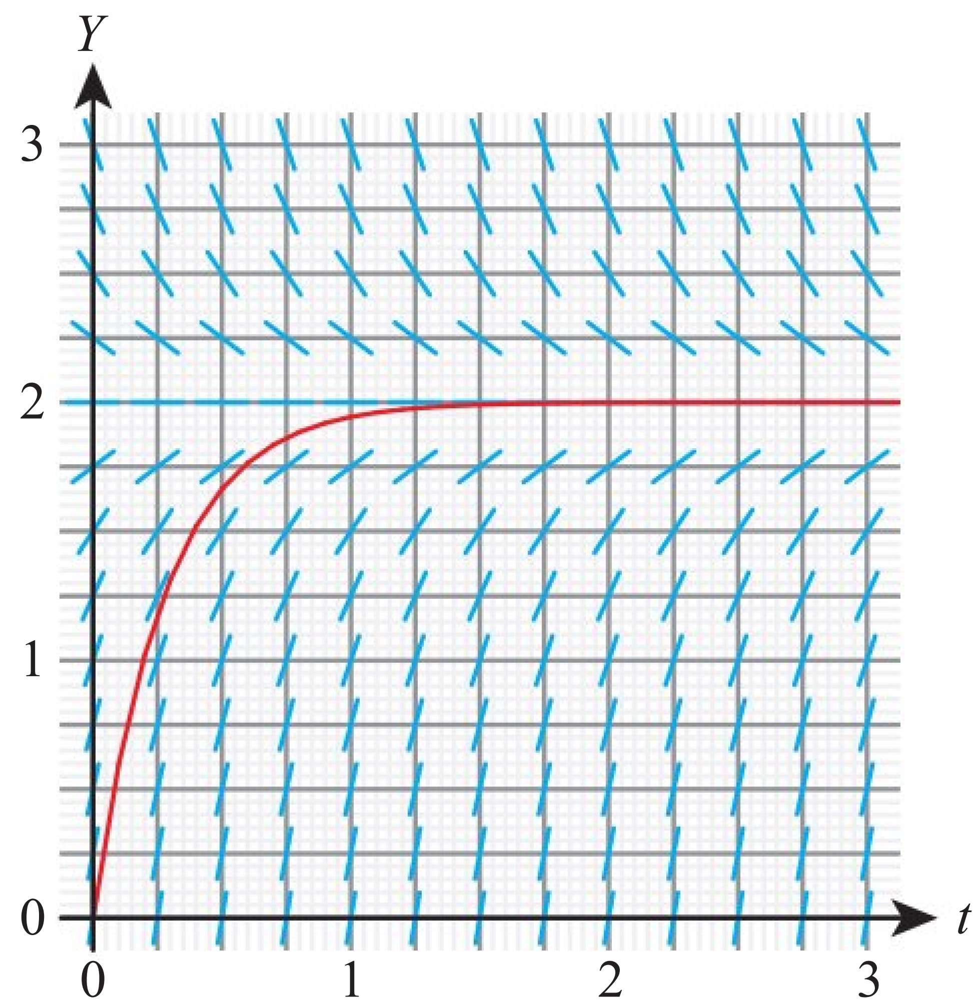
ii 6.08

iii Because the gradient is positive and increases with x, so the estimate of gradient at the beginning of the interval will be below the average value for the interval.

d i $x(t) = e^{x} - e^{-x}$ ii 16.1%

iii Smaller step length or better method e.g. Runge–Kutte.

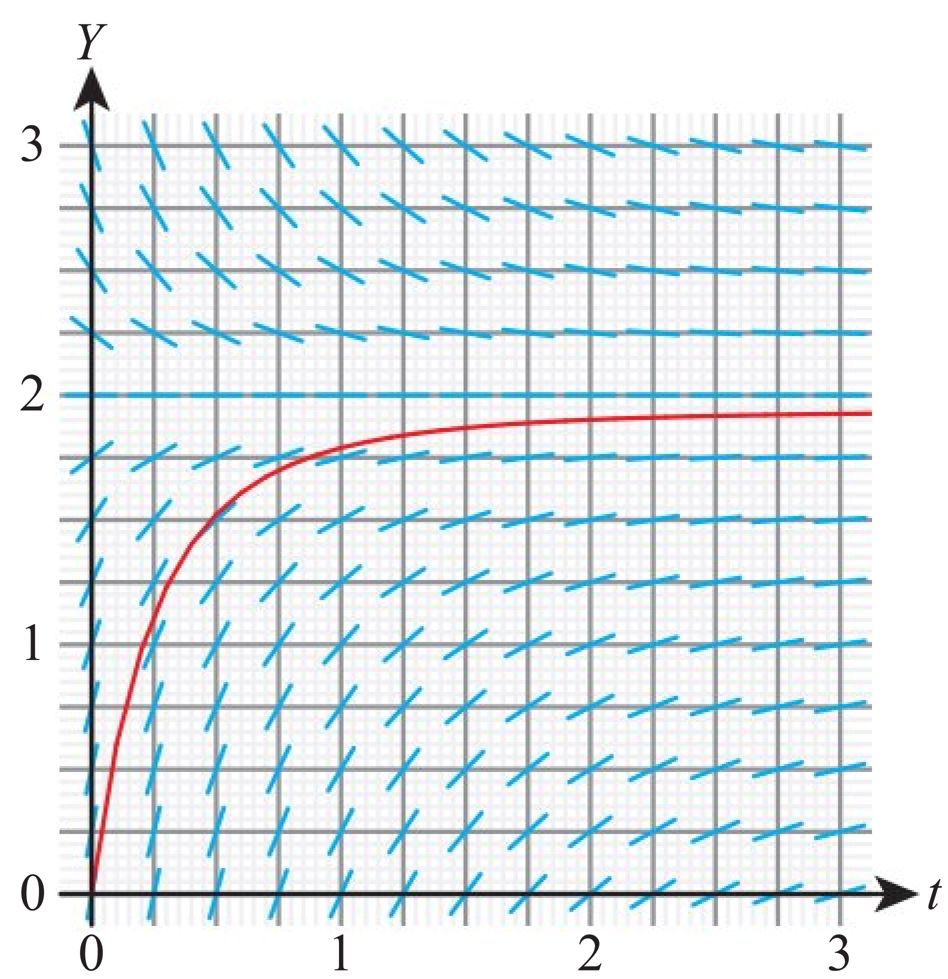
34 a i



iii 2000

b $y = 2 - 2e^{-3t}$

c i



ii $Y = 2 - 2e^{1.5\left(e^{-2t} - 1\right)}$

iii 1550

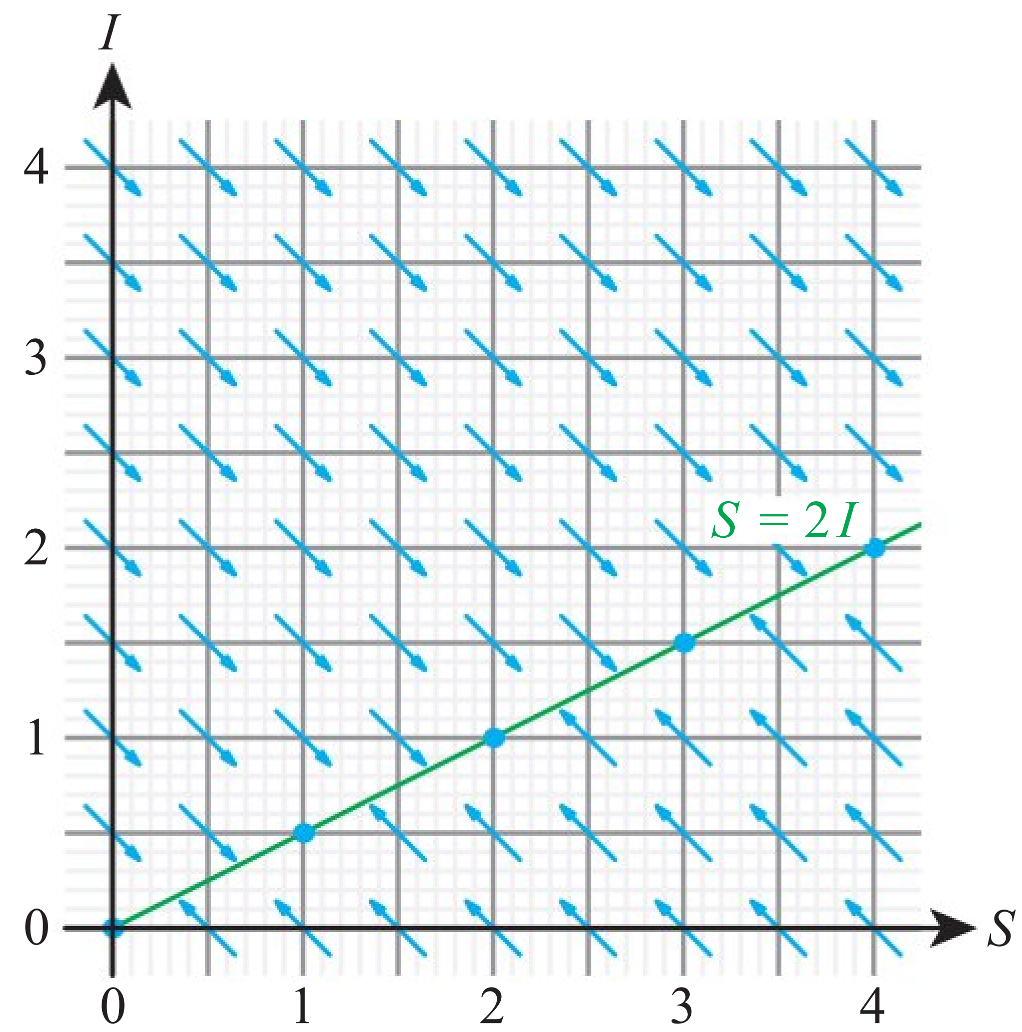
35 a $\frac{dS}{dt} = -0.2S + 0.4I$

$\frac{dI}{dt} = 0.2S - 0.4I$

b ii -1

c $S = 2I$

d



e i $S = 10 + 4e^{-6t}$

Hint: You could do this by finding eigenvalues and eigenvectors of a matrix, or you could substitute $I = 15 - S$ into the first differential equation. Which is easier?

ii 5 million

f e.g. Population stays constant, no death, no immune resistance, more infected people does not increase the infection rate.

Applications and interpretation HL: Practice Paper 1

1 a $\frac{1}{3}$ b $\frac{5}{16}$

2 a $2 + x + 2y$ b $\frac{x}{y}$

3 a i \$464 ii \$632

b No, predicts unlimited growth

4 a 21; -4 b 11

5 a $0 < f(t) \leqslant 16$ b $g(t) = -\frac{1}{3}\ln\left(\frac{t}{16}\right)$

c 0.231 years

6 a $\frac{(4 + 3i)}{5}$ b $p = \frac{2}{5}, q = -\frac{1}{5}$

7 a $\frac{1}{3}x^{-\frac{3}{2}}$ b $-\frac{1}{2}x^{-\frac{5}{2}}$

c $-\frac{2}{3}x^{-\frac{1}{2}} + c$

8 a $u_{n+1} = 1.04u_n + 100$

b 32 c 25.1%

9 $15.7\,\text{cm}^3$

10 a $\dfrac{-1 \pm i\sqrt{47}}{6}$ b 2.85

11 a 18°C b 63

c 0.11 d 30°C

12 a 18°C b (5.8, 2.2)

13 a 5.3623 b 6 c 10.6%

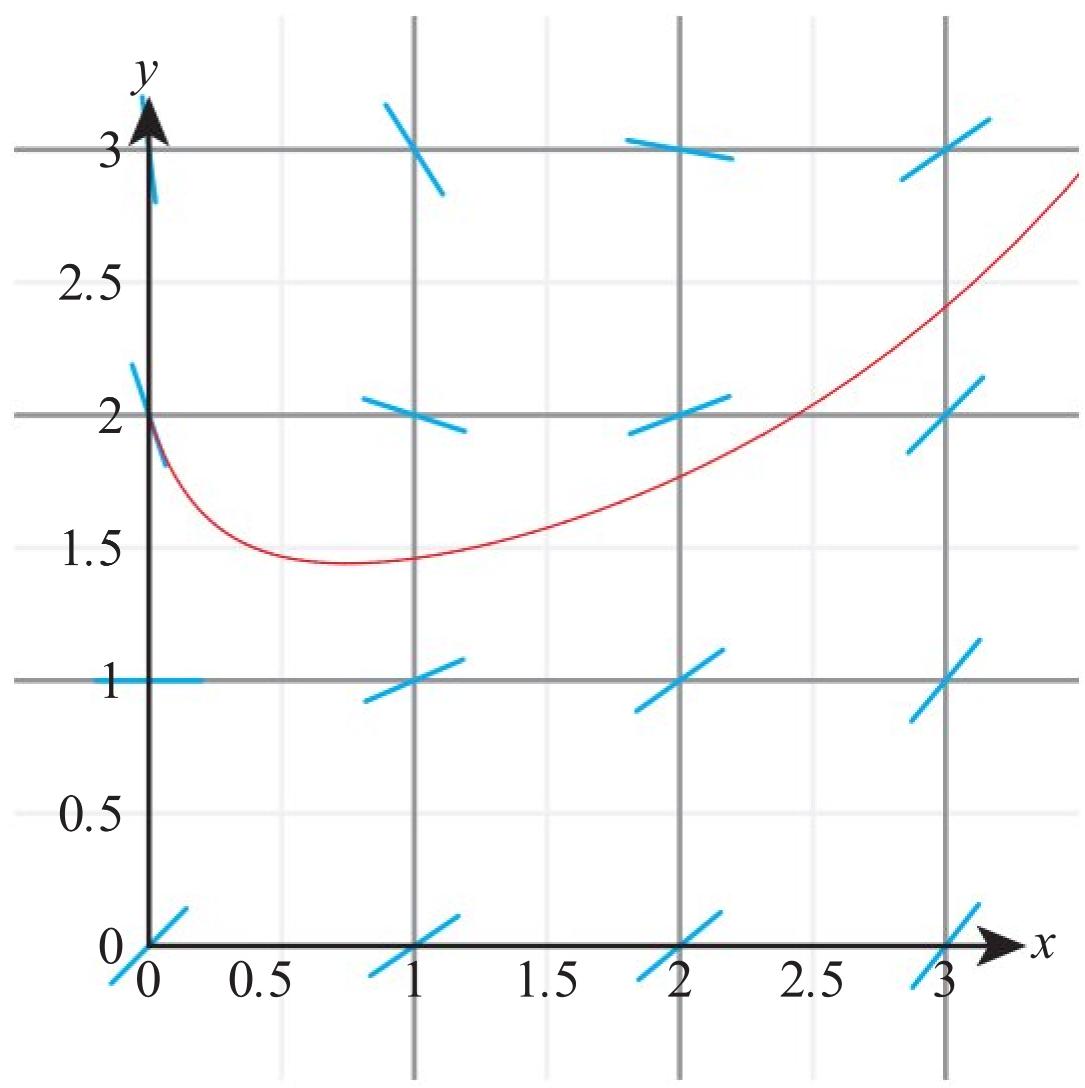
14 $v = \dfrac{t}{t+1} + \ln(t+1)$, $a = \dfrac{1}{(t+1)^2} + \dfrac{1}{t+1}$

15 a $A = 9.4$, $B = 1.8$, $C = 5.8$

b 8°C

16 a 13 b 43.7

17 a



b 1.47

18 a $y = 5 + 10e^{-0.03(t-1)^2}$

b Decreases to 500.

19 $V = 28.2\sin(30t - 0.748)$

Applications and interpretation HL: Practice Paper 2

1 a $320\,\text{m}^2$ b 28.7 m

c 151° d 45.4 m

e $5290\,\text{m}^2$

2 a $\bar{x} = 11.25$, $\sigma = 2.08$ b 120

c 0.64 d 0.520

e 0.6

3 a $\begin{pmatrix} 0.72 & 0.16 \\ 0.28 & 0.84 \end{pmatrix}$ b 0.300

c $\lambda = 1, \dfrac{14}{25}$

$v_1 = \begin{pmatrix} 4 \\ 7 \end{pmatrix}, v_2 = \begin{pmatrix} -1 \\ 1 \end{pmatrix}$

d $\mathbf{P} = \begin{pmatrix} 4 & -1 \\ 7 & 1 \end{pmatrix}$, $\mathbf{D} = \begin{pmatrix} 1 & 0 \\ 0 & \dfrac{14}{25} \end{pmatrix}$

e $\dfrac{4}{11} - \dfrac{4}{11}\left(\dfrac{14}{25}\right)^n$

f $\dfrac{4}{11}$

4 a Breakdowns occur at a constant rate and independently of each other.

b i 0.449 ii 0.953 iii 0.0357

c 0.00728

d $E(X) = \text{Var}(X) = 4.8$

e H_0: $\lambda = 4.8$; H_1: $\lambda > 4.8$

p-value $= 0.0558 > 0.05$

Don't reject H_0.

f 0.0251

g 0.869

5 a 3 km per minute

b $\begin{pmatrix} 4 \\ 10 \\ 10 \end{pmatrix}$ at $t = 3$ for A and $t = 6$ for B.

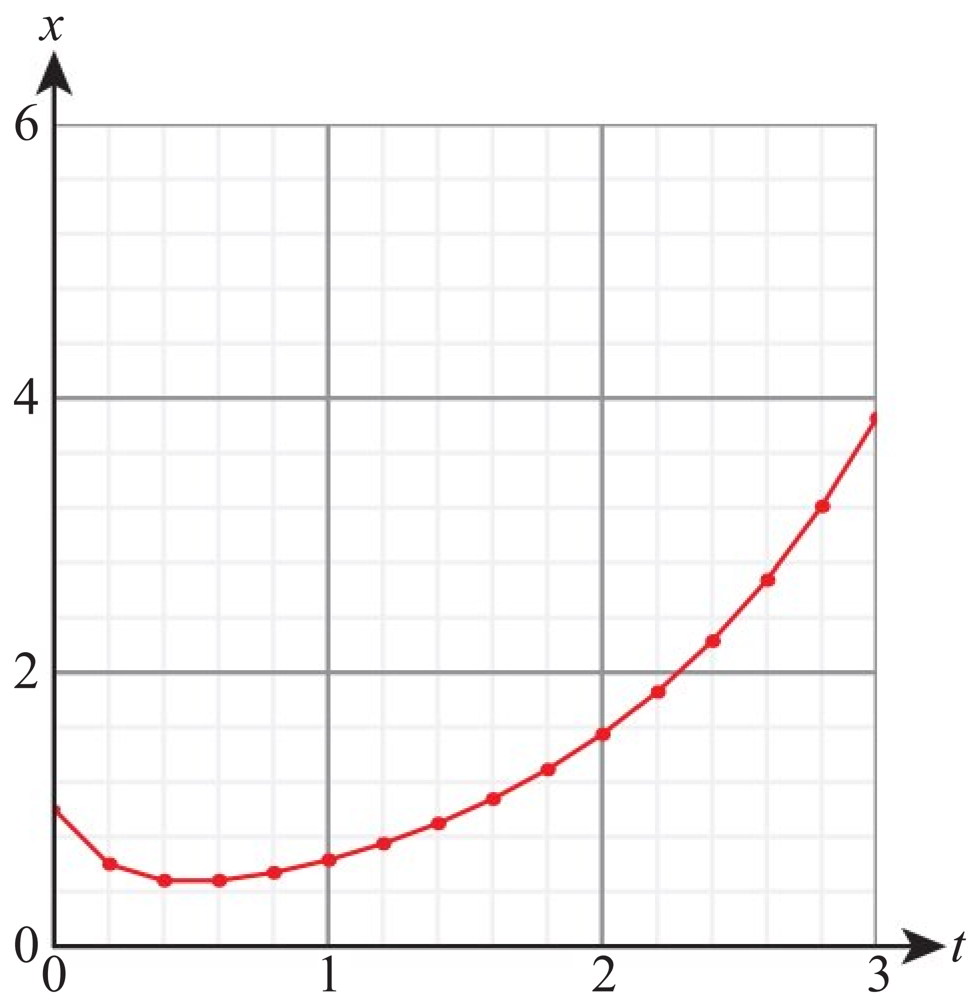
c $\sqrt{(6 - 0.5t)^2 + (3 + 0.5t)^2 + (-18 + 4t)^2}$

d No – closest distance is 6.45 km.

6 a $\frac{dx}{dt} = y, \frac{dy}{dt} = 3x - 2y$

b i 1.548

ii



c $x = \frac{1}{4}e^{t} + \frac{3}{4}e^{-3t}$

Unstable as $\lambda_1 > 0, \lambda_2 < 0$

d 16.3%

7 a $\frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}$

b $\frac{1}{2}\begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}$

c $\frac{1}{2}$

d $\frac{1}{4}$

e 1

Applications and interpretation HL: Practice Paper 3

1 a To reduce variation due to different ages.

b i $H_0: \mu_1 = \mu_2$; $H_1: \mu_1 \neq \mu_2$

ii $|t| = 0.69$, $p = 0.512 > 0.05$. No significant evidence of difference between the two groups.

iii e.g. Too strong a statement – no significant evidence of a difference rather than no difference.

The result in the test is not the same as speech ability.

The population being studied is not all children, but only those in a specific age range.

c i Experts' opinion

ii See if the results correlate with a more widely accepted test (concurrent validity) or future important measures of speech development (predictive validity).

d i 5.83, 4.29

ii $\chi^2 = 31.3$, $p = 2.61 \times 10^{-6} < 0.05$

iii $\chi^2 = 4.33$, $p = 0.115 > 0.05$

iv It supports the validity, since the data needs to be drawn from a normal distribution. However, it also needs to have equal variance which has not been tested.

e i It is not the same situation – there may have been natural improvement with increasing age.

ii $r_s = 0.885 > 0.649$, so there is evidence of correlation, suggesting that the test is reliable.

2 a i

$$\begin{array}{c} \\ A \\ B \\ C \\ D \\ E \\ F \end{array} \begin{array}{c} \begin{array}{cccccc} A & B & C & D & E & F \end{array} \\ \begin{pmatrix} 0 & 1 & 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 & 1 & 0 \\ 1 & 0 & 1 & 0 & 1 & 1 \\ 0 & 0 & 1 & 1 & 0 & 1 \\ 1 & 1 & 0 & 1 & 1 & 0 \end{pmatrix} \end{array}$$

ii 73

b i No, as some vertices have odd degrees.

ii 75 minutes (repeat AB and CE)

iii 126; all vertices have even degrees.

c i $a = 12$, $b = 9$

ii 37 ($FADECBF$)

iii 32

iv $34 \leqslant T \leqslant 37$

d F ($p = 0.191$)

Be the Examiner answers

1.1 Solution 3

1.2 Solution 2

2.1 Solution 3

2.2 Solution 2

2.3 Solution 3

3.1 Solution 2
4.1 Solution 1
4.2 Solution 2
5.1 Solution 3
5.2 Solution 2
5.3 Solution 2
6.1 Solution 1
8.1 Solution 3
8.2 Solution 3
8.3 Solution 2
9.1 Solution 2
10.1 Solution 2
10.2 All three solutions are correct
11.1 All three solutions are correct
13.1 Solution 1

Glossary

Acceleration The rate of change of velocity

Adjacency matrix A matrix that shows the number of edges between each pair of vertices

Adjacent Two vertices are called adjacent if they are joined by an edge. Two edges are called adjacent if they have a common vertex.

Amplitude Half the distance between the maximum and minimum values of a periodic function

Argand diagram Another term for the complex plane

Argument (of a complex number) The angle a number in the complex plane makes with the real axis, measured anticlockwise

Base vectors The vectors $\mathbf{i}$, $\mathbf{j}$ and $\mathbf{k}$, which are of magnitude 1 and parallel to the x, y and z axes respectively.

Cartesian form (of a complex number) A way of writing a complex number, z, in terms of its real and imaginary parts: $z = x + \mathrm{i}y$, where $x, y \in \mathbb{R}$

Central limit theorem (CLT) A result in statistics which states that, for a large sample, the sample mean approximately follows a normal distribution

Chain rule A rule for differentiating composite functions

Characteristic equation The equation $\det(\mathbf{M} - \lambda\mathbf{I}) = 0$, where λ is an eigenvalue of the matrix $\mathbf{M}$

Circuit A walk that starts and ends at the same vertex and has no repeated edges (so it is a closed trail)

Column vector A representation of a vector that lists its components parallel to the coordinate axes one above the other in a 2×1 or 3×1 matrix

Complete graph A graph in which every vertex is connected to every other vertex by exactly one edge

Complex conjugate If $z = x + \mathrm{i}y$, then the complex conjugate of z is $z^* = x - \mathrm{i}y$

Complex number A number that can be written in the form $x + \mathrm{i}y$, where $x, y \in \mathbb{R}$ and $\mathrm{i} = \sqrt{-1}$

Complex plane A Cartesian plane where the x-axis represents the real part of a complex number and the y-axis the imaginary part

Components The magnitude of a vector in a given direction, often parallel to the coordinate axes.

Concave-down The part(s) of a curve where the second derivative is negative

Concave-up The part(s) of a curve where the second derivative is positive

Confidence interval An interval that has a certain probability of containing the true mean

Connected (graph) A graph in which every two vertices are connected (directly or indirectly)

Critical region The set of all values which would lead to rejecting the null hypothesis

Critical value(s) The value(s) on the boundary of the critical region

Cross product Another term for vector product

Cycle A walk that starts and ends at the same vertex and has no other repeated vertices (so it is a closed path)

Definite integral An integral with limits. This results in a numerical answer (or an answer dependent on the given limits) and no constant of integration

Degree (of vertex) The number of edges coming out of a vertex

Determinant A numerical value calculated from the elements of the matrix, similar to the magnitude of a vector except that a determinant can be positive or negative

Diagonalization The process of expressing a matrix $\mathbf{M}$ in the form $\mathbf{M} = \mathbf{PDP}^{-1}$, where $\mathbf{D}$ is a diagonal matrix and $\mathbf{P}$ is a matrix whose columns are the eigenvectors of $\mathbf{M}$.

Digraph Another term for 'directed graph'

Directed graph A graph that has arrows on the edges to indicate the allowed direction

Direction vector (of a line) A vector parallel to a given line

Displacement Distance in a certain direction

Displacement vector A vector from one point to another point

Dot product Another term for scalar product

Edges The lines that connect vertices in a graph

Eigenvalue A scalar λ that satisfies $\mathbf{Mv} = \lambda\mathbf{v}$ for a matrix $\mathbf{M}$ and vector $\mathbf{v}$

Eigenvector A vector $\mathbf{v}$ that satisfies $\mathbf{Mv} = \lambda\mathbf{v}$ for a matrix $\mathbf{M}$ and scalar λ

Elements The entries in a matrix

Equilibrium point A point (a, b) Such that if $x(0) = a,\ y(0) = b$ then $x(t) = a,\ y(t) = b$ for all times t.

Eulerian circuit A circuit that traverses each edge exactly once

Eulerian graph A graph that has an Eulerian circuit

Eulerian trail A walk that uses each edge exactly once (without returning to the starting point)

Exponential (Euler) form A way of writing a complex number, z, in terms of its modulus, r, and argument, θ: $z = r\mathrm{e}^{\mathrm{i}\theta}$

Focus In a phase portrait, an equilibrium point with complex eigenvalues

Fractals Shapes formed by repeated application of a transformation

General solution (of differential equation) The solution containing an unknown constant

Graph A set of vertices and edges

Hamiltonian cycle A cycle visits each vertex exactly once

Hamiltonian graph A graph that has a Hamiltonian cycle

Hamiltonian path A path that visits each vertex exactly once

Identity matrix A square matrix with 1 as each element of the diagonal from top left to bottom right and 0 as every other element

Imaginary part If $z = x + \mathrm{i}y$, then the imaginary part of z is the real number y

In-degree In a digraph the number of edges directed into a vertex

Inverse matrix The inverse of a square matrix $\mathbf{M}$ is the matrix $\mathbf{M}^{-1}$ such that $\mathbf{M}\mathbf{M}^{-1} = \mathbf{M}^{-1}\mathbf{M} = \mathbf{I}$

Kruskal's algorithm An algorithm for finding a minimum spanning tree that adds edges, starting with the shortest, until the tree is connected

Limits (of integration) The lower and upper values used for a definite integral

Linearizing The process of writing the equation of a graph of the form $y = ax^n$ Or $y = ka^x$ In the form of a straight line graph by taking logs

Log–log graph A graph of $\log y$ against $\log x$

Markov chain A system consisting of two or more states in which the probability of being in any given state depends only on the previous state

Matrix A rectangular array of elements, which may be numerical or algebraic

Minimum spanning tree A tree of minimum total length (weight) which includes every vertex of the graph

Modulus (of complex number) The distance of a number from the origin in the complex plane

Modulus–argument (polar) form A way of writing a complex number, z, in terms of its modulus, r, and argument, θ: $z = r(\cos\theta + \mathrm{i}\sin\theta)$

Node In a phase portrait, an equilibrium point with real eigenvalues of the same sign (can be stable or unstable)

Order (of a matrix) A matrix with m rows and n columns has order $m \times n$

Out-degree In a digraph the number of edges directed out of a vertex

Parametric form (of equation of line) A form of the equation where x, y and z are expressed in terms of a parameter

Path A walk with no repeated vertices

Period The smallest value of x after which a function repeats

Phase portrait A diagram showing how x and y values change over time

Phase shift The horizontal shift between two periodic graphs

Point of inflection A point on a curve where the concavity changes

Poisson distribution A probability distribution used to model independent random events which happen at a constant average rate

Position vector A vector from the origin to a point

Prim's algorithm An algorithm for finding a minimum spanning tree that starts with a vertex and then adds the closest possible vertex at each stage

Product rule A rule for differentiating a product of two functions

Quotient rule A rule for differentiating a quotient of two functions

Radians An alternative measure of angle to degrees: 2π radians = 360°

Random walk Movement around a graph where at any vertex the decision of which edge to travel on next is made at random

Real part If $z = x + \mathrm{i}y$, then the real part of z is the real number x

Reflection in the y-axis (or x-axis) Multiplication of the x-values (or y-values) of all points on a curve by –1

Reliable The conclusions of a test are reliable if similar conclusions would be reached on each occasion the test is conducted in similar circumstances

Resultant vector The vector that results from the sum of two or more vectors

Saddle point In a phase portrait, an equilibrium point with real eigenvalues of opposite signs

Scalar A quantity that has only magnitude but no direction

Scalar product A scalar value given by $|\mathbf{a}||\mathbf{b}|\cos\theta$, where θ is the angle between $\mathbf{a}$ and $\mathbf{b}$

Semi-Eulerian A graph that has an Eulerian trail

Semi-log graph A graph of $\log y$ against x

Simple graph A graph that has no multiple edges and no vertex joined to itself.

Singular A matrix with zero determinant

Slope field A plot of the tangents at all points (x, y)

Solid of revolution A 3D shape formed by rotating part of a curve 360° around the x-axis (or y-axis)

Square matrix A matrix with the same number of rows and columns (of order $n \times n$)

Stable equilibrium point A point towards which solution curves move with time

Steady state The state $\mathbf{s}$ of a Markov chain such that $\mathbf{Ts} = \mathbf{s}$ for a transition matrix $\mathbf{T}$

Stretch Multiplication of the x-values (or y-values) of all points on a curve by a given scale factor

Strongly connected A digraph in which any two vertices are connected in both directions

Subgraph A new graph formed by using only some of the edges of the original graph

Subtends An arc subtends the angle at the centre of a circle formed between the two radii extending from each end of the arc to the centre

Sum of square residuals The sum of the squared differences between each data value and the corresponding prediction a regression model makes for that data value

Sum to infinity The sum of infinitely many terms of a geometric sequence

Survey Any method for collecting data for analysis

The Chinese postman problem To find the shortest path around the graph which uses each edge at least once and returns to the starting point

The travelling salesman problem To find the shortest route around a graph which visits each vertex at least once and returns to the starting point

Trail A walk with no repeated edges

Transition matrix (graph) A matrix of the probabilities of moving from one vertex to another in a graph

Transition matrix (Markov chain) A matrix representing the probabilities of changing from one state to another, or staying in the same state, in a Markov chain

Translation The addition of a constant to the x-values (or y-values) of all points on a curve

Tree A connected graph in which there are no closed paths (cycles)

Type I error Rejecting the null hypothesis when it was true

Type II error Failing to reject the null hypothesis when it was false

Unbiased estimator In statistics, an estimator calculated from a sample, whose expected value equals the population parameter

Unit square The square with vertices $(0, 0)$, $(1, 0)$, $(1, 1)$ and $(0, 1)$

Unit vector A vector of magnitude 1

Unstable equilibrium point An equilibrium point that is not stable

Upper and lower bounds (travelling salesman problem) The largest and smallest possible values for the length of the shortest Hamiltonian cycle

Valid A process is valid if it is measuring what it is intended to measure

Vector A quantity that has both magnitude and direction

Vector product A vector perpendicular to the two given vectors with magnitude $|\mathbf{a}||\mathbf{b}|\sin\theta$, where θ is the angle between $\mathbf{a}$ and $\mathbf{b}$

Velocity Speed in a certain direction

Velocity vector A vector that gives the direction of motion of an object and whose magnitude gives the speed with which the object is moving

Vertices The points of a graph

Volume of revolution The volume of a solid of revolution

Walk Any sequence of adjacent edges in a graph

Weight The number associated with each edge in a weighted graph

Weighted adjacency table A table that shows the weights of edges between each pair of vertices

Weighted graph A graph in which each edge has a number associated with it

Zero matrix A matrix whose elements are all 0

z-interval A confidence interval for the mean when you do know the population variance

t-interval A confidence interval for the mean when you do not know the population variance

Index

FOR THE
IB DIPLOMA
PROGRAMME

Mathematics
APPLICATIONS AND INTERPRETATION HL

Enable students to construct mathematical models by exploring challenging problems and the use of technology.

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About the authors

The authors are all University of Cambridge graduates and have a wide range of expertise in pure mathematics and in applications of mathematics, including economics, epidemiology, linguistics, philosophy and natural sciences.

Between them they have considerable experience of teaching IB Diploma Mathematics at Standard and Higher Level, and two of them currently teach at the University of Cambridge.

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